

EICHLER COHOMOLOGY OF GENERALIZED MODULAR FORMS OF REAL WEIGHTS

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ABSTRACT. In this paper, we prove the Eichler cohomology theorem of weakly parabolic generalized modular forms of real weights on subgroups of finite index in the full modular group. We explicitly establish the isomorphism for large weights by constructing the map from the space of cusp forms to the cohomology group.

1. INTRODUCTION

Eichler cohomology is the center of attention in this paper. In [13], we prove Eichler cohomology theorem for generalized modular forms of large integer weights using supplementary series. In [6], we use Stokes's theorem to prove similar results on integer weights with modifications on the multiplier system. However, when one tries to use the classical methods to derive Eichler cohomology theorems for real weights, obstacles with the definition of supplementary series arise. In [7], we extend the methods used in [6] to prove the isomorphism theorem for real weights without any restriction on the weight. Note that in [7], the isomorphism from the space of cusp forms to the cohomology group is not as naturally defined as the corresponding isomorphism in this paper. We use similar methods as in [4] to derive the same results for parabolic generalized modular forms of real weights.

Definition 1.1. A generalized modular form belonging to a subgroup Γ of finite index in the full modular group of real weight k and multiplier system v is a function $F(z)$ satisfying:

- (1) $F(z)$ is analytic in the upper half plane \mathbb{H}
- (2) $F(z)$ satisfies a transformation law

$$(1.1) \quad F(Mz) = v(M)(cz + d)^k F(z)$$

with $|v|$ is not necessarily 1 that depends only on the transformation M where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Note that the multiplier system $v : \Gamma \rightarrow \mathbb{C}$ satisfies the consistency condition

$$(1.2) \quad v(M_1 M_2)(c_3 z + d_3)^k = v(M_1)v(M_2)(c_1 M_2 z + d_1)^k (c_2 z + d_2)^k,$$

where

$$M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma$$

for $i = 1, 2, 3$ and $M_3 = M_1 M_2$.

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The k -th power in (1.1) is determined by the convention

$$(1.3) \quad w^k = |w|^k e^{ik \operatorname{arg} w}$$

where $-\pi \leq \operatorname{arg} w < \pi$, for $0 \neq w \in \mathbb{C}$.

We denote by $\{\Gamma, k, v\}$ the \mathbb{C} -vector space of generalized modular forms on the group Γ of real weight k and multiplier system v .

We shall assume that our generalized modular forms are weakly parabolic generalized modular forms which means that $|v(P)| = 1$ for all parabolic matrices P . It is important to mention that in [4], the multiplier system is assumed to be unitary, which means that $|v(M)| = 1$ for all matrices $M \in \Gamma$.

We rewrite (1.1) as

$$(1.4) \quad F(Mz)v(M)^{-1}(cz+d)^{-k} = F(z).$$

We introduce the stroke operator which is given by

$$(1.5) \quad (F|_v^k)(z) = F(Mz)v(M)^{-1}(cz+d)^{-k}.$$

Thus (1.4) becomes

$$(1.6) \quad F|_v^k = F.$$

It follows that

$$(1.7) \quad F|_v^k M_1 M_2 = (F|_v^k M_1)|_v^k M_2.$$

Let $S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 0$, generate the subgroup Γ_∞ of translations in Γ . Since F satisfies (1.1), then in particular

$$F(z + \lambda) = v(S)F(z) = e^{2\pi i \kappa} F(z)$$

with $0 \leq \kappa < 1$. Thus if F is meromorphic in \mathbb{H} and its poles do not accumulate at infinity, F has the Fourier expansion at ∞

$$(1.8) \quad F(z) = \sum_{m=-m_0}^{\infty} a_m e^{2\pi i(m+\kappa)z/\lambda}$$

valid for $y = \Im z > y_0$. Γ has also $t \geq 0$ inequivalent parabolic classes. Each of these classes corresponds to a cyclic subgroup of parabolic elements in Γ leaving fixed a parabolic cusp on the boundary of R , the fundamental region of Γ . We as well denote by \bar{R} , the closure of the fundamental region R .

Now let q_1, q_2, \dots, q_t be the inequivalent parabolic cusps (other than infinity) on the boundary of R and let Γ_i be the cyclic subgroup of Γ fixing q_j , $1 \leq j \leq t$. Suppose also that

$$Q_j = \begin{pmatrix} * & * \\ c_j & d_j \end{pmatrix}$$

is a generator of Γ_i ; Q_j is necessarily parabolic. For $1 \leq j \leq t$; put $v(Q_j) = e^{2\pi i \kappa_j}$, $0 \leq \kappa_j < 1$. Also F has the following Fourier expansion at q_j :

$$(1.9) \quad F(z) = (z - q_j)^{-k} \sum_{m=-m_j}^{\infty} a_m(j) e^{-2\pi i(m+\kappa_j)/\lambda_j(z-q_j)},$$

valid for $y = \Im z > y_j$. Here λ_j is a positive real number called the width of the cusp q_j and defined as follows. Let

$$A_j = \begin{pmatrix} 0 & -1 \\ 1 & -q_j \end{pmatrix},$$

so that A_j has determinant 1 and $A_j(q_j) = \infty$. Then $\lambda_j > 0$ is chosen so that

$$A_j^{-1} \begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix} A_j$$

generates Γ_j , the stabilizer of q_j . We let $C^+(\Gamma, k, v)$ denote the space of entire weakly parabolic generalized modular forms of real weight k and multiplier system v on Γ which in addition to being holomorphic in H , it has only terms with $m + \kappa \geq 0$ in (1.8) and $m + \kappa_j \geq 0$ in (1.9) for all $1 \leq j \leq t$. Also, let $C^0(\Gamma, k, v)$ denote the subspace of weakly parabolic generalized cusps forms which is a subspace of $C^+(\Gamma, k, v)$ but it has only terms with $m + \kappa > 0$ in (1.8) and $m + \kappa_j > 0$ in (1.9) for all $1 \leq j \leq t$.

2. EICHLER COHOMOLOGY FOR GENERALIZED MODULAR FORMS OF INTEGER WEIGHTS

We start with definitions inherited from the classical theory of Eichler cohomology. We now state what is known by Bol's identity [1]. With $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$,

$$(2.1) \quad \frac{d^{k+1}}{dz^{k+1}} \{(cz + d)^k F(Mz)\} = (cz + d)^{-k-2} F^{(k+1)}(Mz)$$

where $F^{(k+1)}$ denotes the $k + 1$ -th derivative of F . From (1.7), we see that if $F \in \{\Gamma, -k, v\}$ then $F^{(k+1)} \in \{\Gamma, k + 2, v\}$. The converse leads to what is called Eichler integrals (i.e. the polynomial periods). Actually Eichler integrals of weight $-k$ emerge when one takes the $(k + 1)$ -fold integral of a modular form of weight $k + 2$. As a result,

$$(2.2) \quad F|_v^{-k} M = F(z) + p_M(z)$$

where $p_M(z)$ is a polynomial of degree less than or equal to k .

Definition 2.1. $F(z)$ satisfying (2.2) is called an Eichler integral. Besides $\{p_M\}$ is called the system of period polynomials of f (or F).

Note that $\{p_M \mid M \in \Gamma\}$ occurring in (2.2) satisfy the following consistency condition

$$(2.3) \quad p_{M_1 M_2} = p_{M_1} |_v^{-k} M_2 + p_{M_2}$$

for all M_1 and M_2 in Γ .

Definition 2.2. If $C_1 = \{p_M \mid M \in \Gamma\}$ is any collection of polynomials of degree less than or equal to k such that (2.2) is satisfied, call $\{p_M \mid M \in \Gamma\}$ cocycle.

Definition 2.3. If $C_2 = \{p_M \mid M \in \Gamma\}$ is a set of polynomials of degree less than or equal to k such that there exists a polynomial p with $p_M = p |_v^{-k} M - p$, then $\{p_M \mid M \in \Gamma\}$ is called a coboundary.

Observe that the coboundary satisfies (2.3).

Definition 2.4. Let P_k denote the complex vector space of polynomials of degree less than k . Then $H_{-k,v}(\Gamma, P_k)$ is defined by C_1/C_2 .

Parabolic cohomology plays an important role in the theory of automorphic integrals; these are cocycles $\{p_V \mid V \in \Gamma\}$ which satisfy the following additional condition:

Let $Q_0 = S, Q_1, \dots, Q_t$ be a complete set of parabolic representatives for Γ . Then for each $h, 0 \leq h \leq t$, there exists a polynomial p_h of degree $\leq r$ such that $p_{Q_h} = p_h \mid Q_h - p_h$.

Definition 2.5. $\tilde{H}_{-k,v}(\Gamma, P_k)$ is a subgroup of $H_{-k,v}(\Gamma, P_k)$ defined as the space of parabolic cocycles modulo the coboundaries.

Eichler determined the structure of this space in terms of the space of classical automorphic forms. His theorem for classical modular forms [2] states that the vector space $C^+(\Gamma, k+2, v) \oplus C^0(\Gamma, k+2, \bar{v})$ is canonically isomorphic to the first cohomology group $H_{-k,v}(\Gamma, P_k)$.

In [6], we obtain a new result on generalized modular forms related to Eichler cohomology. We map the vector space $C^+(\Gamma, k+2, v) \oplus C^0(\Gamma, k+2, \hat{v})$ into a modified first cohomology group. Here the action of Γ on P_k is again by way of the slash operator in weight $-k$, but the multiplier system is modified to $\hat{v} = vv_E$, where v_E is the multiplier system of a nontrivial entire weakly parabolic generalized modular form of weight 0 and where v^* is a unitary multiplier system in weight $k+2$.

3. EICHLER COHOMOLOGY OF ARBITRARY REAL WEIGHT

Let k be an arbitrary real number and v a multiplier system for Γ of weight k . Let \mathbb{P} be a vector space of functions g holomorphic in \mathbb{H} which satisfy the growth condition

$$|g(z)| < K(|z|^\rho + y^{-\sigma}), \quad y = \Im z$$

for some positive constant K, ρ and σ . Since the weight here is not necessarily in \mathbb{Z} , polynomials of fixed degree cannot serve as the underlying space of functions in the definition of the Eichler cohomology groups we study. Instead, we employ as the underlying space the collection \mathbb{P} . This space was introduced in [4, p.612] in the context of the Eichler cohomology theory for unitary (i.e. the usual) modular forms of arbitrary real weight. It is worth mentioning that the space \mathbb{P} is preserved under the stroke operator.

N.B. Individual elements of \mathbb{P} need not be preserved under the stroke operator.

Definition 3.1. F is an automorphic integral of real weight $-k$ with respect to Γ provided F is meromorphic in \mathbb{H} , satisfies

$$(3.1) \quad F \mid_v^{-k} M - F \in \mathbb{P} \quad \text{for } M \in \Gamma,$$

and has left finite expansion at each parabolic point of R .

As a direct consequence of the consistency condition (1.2) for a MS v in weight k , the slash operator satisfies

$$(3.2) \quad F \mid_v^{-k} M_1 M_2 = (F \mid_v^{-k} M_1) \mid_v^{-k} M_2, \quad \text{for } M_1, M_2 \in \Gamma,$$

where F is any function defined on \mathbb{H} . In turn, (3.2) implies both (1.2) (put $F \equiv 1$) and the "additive cocycle condition"

$$(3.3) \quad \rho_{M_1 M_2} = \rho_{M_1} |v^{-k} M_2 + \rho_{M_2}, \quad \text{for } M_1, M_2 \in \Gamma,$$

where the ρ_M are the periods of F occurring in (3.1). The set $\{\rho_M : M \in \Gamma\}$ is called a *cocycle with respect to* $|v^{-k}$. A *coboundary* is a collection $\{\rho_M : M \in \Gamma\}$ of elements in \mathbb{P} such that

$$(3.4) \quad \rho_M = \rho |v^{-k} M - \rho, \quad \text{for } M \in \Gamma,$$

with ρ a *fixed* element of \mathbb{P} . Since every coboundary is a cocycle, we may define the *Eichler cohomology group* $H_{-k,v}(\Gamma, \mathbb{P})$ as the quotient space: cocycles/coboundaries.

We introduce the subspace $\tilde{H}_{-k,v}(\Gamma, \mathbb{P})$ of "parabolic" cohomology classes in $H_{-k,v}(\Gamma, \mathbb{P})$. A cocycle $\{\rho_M : M \in \Gamma\}$ in \mathbb{P} is called *parabolic* if there exists ρ_h in \mathbb{P} with the property

$$(3.5) \quad \rho_{Q_h} = \rho_h |v^{-k} Q_h - \rho_h, \quad 1 \leq h \leq t.$$

(Recall that $Q_h \in \Gamma$ is a parabolic element such that $\Gamma_{q_h} = \langle Q_h, -I \rangle$, where Γ_{q_h} is the stabilizer of the parabolic cusp q_h in Γ). A coboundary is clearly a parabolic cocycle, so we may form the quotient group: parabolic cocycles/coboundaries. The resulting subspace $\tilde{H}_{-k,v}(\Gamma, \mathbb{P})$ of $H_{-k,v}(\Gamma, \mathbb{P})$ is called the *parabolic Eichler cohomology group*.

Theorem 3.2. *For any real number k and for v a weakly parabolic multiplier system, we have*

$$(3.6) \quad H_{-k,v}(\Gamma, \mathbb{P}) = \tilde{H}_{-k,v}(\Gamma, \mathbb{P}).$$

Theorem 1 is a restatement of Theorem 2 in [4] but with a weakly parabolic multiplier system rather than a unitary one. For the proof of Theorem 2 in [4], Knopp mentions that the theorem is a consequence of a result of B.A. Taylor and the proof is given in [4, pp.627-628]. After careful examination of Knopp's proof for the above theorem for classical modular forms, it turns out that the same proof can be adopted for Theorem 1 above. This is due to the fact that $|v(P)| = 1$ for all parabolic matrices P and that the proof of Theorem 2 in [4] deals only with the parabolic elements of the group Γ considered.

The proof of one part of Theorem 4 is based on the construction of a certain holomorphic function Φ . Also the construction of Φ involves the introduction of the "Generalized Poincaré Series".

Definition 3.3. Let w be a multiplier system for Γ . Let k' be a positive even integer large enough and let $\{\phi_V\}$ be a parabolic cocycle of weight $-k$ which satisfies the additional condition that $\phi_S = 0$. Then the Generalized Poincaré Series $\Psi(\{\phi_V\}, k, w; z) = \Psi(z)$ is given by

$$(3.7) \quad \Psi(z) = \sum_{V \in \Theta} \phi_V(z) \bar{w}(V) (cz + d)^{-k'},$$

where Θ is a complete set of coset representatives for Γ/Γ_∞ and $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

Here $|w|$ is not necessarily 1. The estimation of the cocycle is done in the same way as in [4]. The only difference the multiplier system plays is in the convergence

of the Poincaré series. Recall that

$$(3.8) \quad |w(V)| \leq K\mu(V)^\alpha$$

where K is a positive constant, α is another constant depending on the modulus of the multiplier system at the generators of Γ and $\mu(V) = a^2 + b^2 + c^2 + d^2$ where a, b, c, d are the entries of V . Thus after carefully examining the bound for the cocycle in [[4], Lemma 6], a similar bound for the cocycle can be derived here by only taking the bound of the multiplier system into consideration. Thus

$$(3.9) \quad |\phi_V(z)\bar{w}(V)(cz+d)^{-k'}| \leq K_1^*(c^2+d^2)^{e+1+\alpha}(|z|^\eta + y^{-\eta})$$

where $z = x + iy$, $\eta = 6e - 2k$, $e = \max(\rho/2, \sigma + k/2)$ with ρ and σ constants independent of the particular generator involved in the estimation of the cocycles. It is important to mention that the Generalized Poincaré series converges absolutely for $k' > \psi = 2e + 4 + 2\alpha$.

Theorem 3.4. *For $k' > \psi$, the Generalized Poincaré Series (3.7) converges absolutely and, in fact, $\Psi \in \mathbb{P}$.*

Since by Theorem 2 the series converges absolutely for sufficiently large k' , it follows that

$$(3.10) \quad \Psi|_v^k M = w(M)(\gamma z + \delta)^{k'} \Psi(z) - w(M)(\gamma z + \delta)^{k'} g(z)\phi_M(z),$$

for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ where $g(z)$ is the generalized Eisenstein series

$$(3.11) \quad g(z) = \sum_{V \in \Theta} \bar{w}(V)(cz+d)^{-k'}.$$

We now state and prove a theorem that is similar to Theorem 3 in [4] but under the condition that v is a multiplier system associated with weakly parabolic generalized modular forms.

Theorem 3.5. *Let k be any real number, $k > \psi > 0$ and v a weakly parabolic multiplier system of weight k . Suppose $\{\phi_V | V \in \Gamma\}$ is a parabolic cocycle of weight $-k$ in \mathbb{P} ; that is, $\phi_V \in \mathbb{P}$, $\phi_{V_1 V_2} = \phi_{V_1}|_v^{-k} V_2 + \phi_{V_2}$, for all $V_1, V_2 \in \Gamma$, and for each j such that $0 \leq j \leq t$, there exists $\phi_{Q_j} = \phi_j|_v^{-k} Q_j - \phi_j$.*

Then there exists a function Φ , holomorphic in H , such that

$$(3.12) \quad \Phi|_v^{-k} V - \Phi = \phi_V$$

for all $V \in \Gamma$, and with expansions at the parabolic cusps $q_j, 0 \leq j \leq t$, of the form

$$(3.13) \quad \Phi(z) = \phi_j(z) + (z - q_j)^{-k} \sum_{m=-m_j}^{\infty} a_m(j) \exp\left\{\frac{-2\pi i(m + \kappa_j)}{\lambda_j(z - q_j)}\right\},$$

for $1 \leq j \leq t$,

$$(3.14) \quad \Phi(z) = \phi_0(z) + \sum_{m=-m_0}^{\infty} a_m(0) \exp\left\{\frac{2\pi i(m + \kappa)z}{\lambda}\right\}$$

for $j = 0$.

The construction of Φ involves Poincaré series defined above.

Proof. Let $\{\phi_V\}$ be a parabolic cocycle in \mathbb{P} for Γ of weight $-k$ and multiplier system v . For $V \in \Gamma$, put

$$(3.15) \quad \phi_V^* = \phi_V - (\phi_0 |_{v^{-k}} V - \phi_0).$$

Thus $\{\phi_V^*\}$ is a parabolic cocycle in \mathbb{P} . Notice that $\phi_S^* = \phi_0 | S - \phi_0 - (\phi_0 | S - \phi_0) = 0$. As a result we can form the Poincaré series $\Psi(\{\phi_V^*\}, k', w; z) = \Psi^*(z)$; and $\Psi^*(z) \in \mathbb{P}$. Now let $F^*(z) = -\Psi(z)/g(z)$ then $F^* |_{v^{-k}} M - F^* = \phi_M^*$, for $M \in \Gamma$. Define $F(z) = F^*(z) + \phi_0(z)$, we have

$$(3.16) \quad F |_{v^{-k}} M - F = \phi_M^* + \phi_0 | M - \phi_0 = \phi_M$$

for $M \in \Gamma$. Following [4], we see that $g(z) \neq 0$ and thus F is meromorphic in \mathbb{H} .

Since g has finitely many zeroes in $\bar{R} \cap H$, F has finitely many poles in $\bar{R} \cap H$. For each j , $0 \leq j \leq t$, consider the function $F_j = F - \phi_j$. Then we have

$$F_j | Q_j - F_j = (F | Q_j - F) - (\phi_j | Q_j - \phi_j) = \phi_{Q_j} - \phi_{Q_j} = 0;$$

and thus it follows that

$$F_j(z) = (z - q_j)^{-k} \sum_{m=-\infty}^{\infty} a_m(j) \exp \left\{ \frac{-2\pi i(m + \kappa_j)}{\lambda_j(z - q_j)} \right\},$$

for $1 \leq j \leq t$,

$$F_0(z) = \sum_{m=-\infty}^{\infty} a_m(0) \exp \left\{ \frac{2\pi i(m + \kappa)z}{\lambda} \right\}$$

for $j = 0$.

Also by the same argument as in [4], $F(z)$ has expansions of the form (3.13), (3.14).

Since F may have poles in H we need to modify it to obtain the function Φ of Theorem 3. It follows from [11] that there exists $f \in \{\Gamma, k, v_u\}$ with poles at given principal parts at finitely many points of $\bar{R} \cap H$ and is otherwise holomorphic in R with possible exceptions at the cusps. In [8], Knopp and Mason proved that given a multiplier system of a generalized modular form v , then v can be written as $v = v_E v_u$ where v_E is the multiplier system of an entire generalized modular form E of weight 0 and v_u is unitary. Note that $E(z)$ is an entire generalized modular form of dimension 0 that has no zeroes or poles [8]. The existence of such a function $E(z)$ can be found in [5]. As a result, fE is a generalized modular form of weight $-k$ and multiplier system v that has its poles and zeroes as f . We form now $\Phi = F - fE$. Notice that by results from Petersson in [11], there exists such an $f \in \{\Gamma, -k, v\}$ which have poles whose principal parts agree with those of F in $\bar{R} \cap H$. Since $fE |_{v^{-k}} M = fE$ for $M \in \Gamma$, we still have $\Phi |_{v^{-k}} M - \Phi = \phi_M$. Since fE has a pole of finite order at each parabolic cusps, then Φ has the expansions (3.13) and (3.14) at those cusps. Finally, Φ is holomorphic in $\bar{R} \cap H$. \square

We now state the main theorem in this paper. Notice that the result here follows from Theorem A* in [7]. However, in the present paper, we are explicitly constructing the isomorphism which seems to arise naturally and which actually fails for small weights due to convergence issues of the generalized Poincaré series.

Theorem 3.6. The Main Theorem *Let ψ be an integer computable depending on k and v . If $k \leq -2$ or $k > \psi > 0$ with v a multiplier system of weight k , then $H_{-k,v}(\Gamma, \mathbb{P}) = \tilde{H}_{-k,v}(\Gamma, \mathbb{P})$ are isomorphic to $C^0(\Gamma, k + 2, \bar{v})$.*

Case 1 $k \leq -2$

By Mittag-Leffler theorem for automorphic forms, there exist $G' \in \{\Gamma, -k, v_u\}$ such that G' is holomorphic in \mathbb{H} and G' has a principal part at each cusp that agrees precisely with the principal part at the cusp of the expansion of Φ . As in the proof of Theorem 3 above and due to the result of Knopp and Mason in [8], we can write $v = v_u v_E$ where E is again an entire generalized modular form of weight 0. Now let $G = G'E$, thus $G \in \{\Gamma, -k, v\}$, holomorphic in \mathbb{H} and has a principal part at each cusp as that of G' . So put $\Phi^* = \Phi - G$, then Φ^* is holomorphic in \mathbb{H} and has expansion (3.13) at each cusp q_j in which no negative power in the local parameter appears and as a result, $|\Phi^*(z)| < K(|y|^\rho + y^{-\sigma})$ for constants K, ρ, σ and for $z = x + iy \in \mathbb{H} \cap \bar{R}$. Also

$$\Phi^*|_v^{-k} - \Phi^* = \phi_M, \quad M \in \Gamma.$$

Notice also as in [4], if we get that $|\Phi^*(z)| < K(y^\alpha + y^{-\beta})$ for $z \in \bigcup_{V \in \Gamma/\Gamma_\infty} V(\bar{R}) \cap \mathbb{H}$ then $\Phi^* \in \mathbb{P}$ and hence $H_{-k, v}(\Gamma, \mathbb{P}) = 0$. To prove this, we follow the same proof as in [4, p.622] by constructing $f(z) = y^{-k/2} |\Phi^*(z)|$, $y = \Im z > 0$. Then for $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$,

$$f(Mz) = |v(M)| f(z) + y^{-k/2} |v(M)| |\phi_M(z)|.$$

The only difference here from that of [4] is that $|v(M)|$ is not necessarily equal to 1 for non-parabolic elements M of Γ . As a result, we get the same inequality with a change in the estimate as in [4, p.623] and after carefully checking that Lemma 8 is still valid for our purposes.

$$|\Phi^*(z)| \leq y^{k/2} \left(\frac{1}{my} + y \right)^{-k/2} |v(V)| \left\{ K_R \left(\frac{1}{my} + y \right)^\rho + y^{-\sigma} + K_1(y^{\rho_0} + y^{-\sigma_0}) \right\}$$

where $K_R, K_1, \sigma_0, \rho_0$ and m are positive constants.

Case 2 $k > \psi$

The proof of this case follows exactly as the proof of Theorem 1 in [4] for the case $r > 0$ with two differences that will be mentioned below.

1. The first difference is the weakly parabolic multiplier system that is considered here. It turned out that the multiplier system will not impact the proof if we make sure that the theorems used by Knopp continue to work here. Specifically, in the proof of Theorem 1 in [4], Knopp used three theorems from Niebur's [9, 10]. Thus showing that those theorems will also work for the case of weakly parabolic multiplier systems will be sufficient to deduce the proof of Theorem 4 for the case $k > \psi$ in the present paper.

2. The other difference is the existence of a function $f \in \{\Gamma, r, v\}$ in [4, p. 623] (the case $r > 0$) which has poles of prescribed principal parts at each of the cusps q_1, \dots, q_t and is holomorphic in \mathbb{H} . However, the existence of this function is also guaranteed for weakly parabolic multiplier by multiplication of f with an entire generalized modular form E of weight 0. This method is used in the present paper in the proof of Theorem 3 and also repeated in the proof of this theorem for the case $k \leq -2$.

We now list the extensions of the three theorems of Niebur that were mentioned above in the context of weakly parabolic multiplier systems. The Generalization of Theorem N_1 in the context of weakly parabolic generalized modular forms is

proved as Theorem 1 in [14]. In fact, in [14], we generalize two theorems from [9, 10] (mentioned in [4] as Theorems $N_1 N_2$) to the case of weakly parabolic generalized modular forms.

Generalization of Theorem N_1 . Let $k > \psi > 0$ where ψ is an integer computable depending on k and v . Let m_0 be a nonnegative integer and a_{-1}, \dots, a_{-m_0} complex numbers. Then there exists F , an automorphic integral of weight $-k$ with respect to Γ , which is holomorphic in \mathbb{H} and at finite cusps q_1, \dots, q_t and which has an expansion (3.13) with principal part

$$(3.17) \quad a_{-m_0} e^{2\pi i(-m_0+\kappa)z/\lambda} + \dots + a_{-1} e^{2\pi i(-1+\kappa)z/\lambda}$$

at $q = \infty$. The function F has the transformation properties

$$(3.18) \quad F|_v^{-k} V = F, \quad V \in \Gamma_\infty$$

and

$$(3.19) \quad \overline{F|_v^{-k} - F} = \int_{V^{-1}\infty}^{i\infty} G(\tau)(\tau - \bar{z})^k d\tau, \quad V \in \Gamma - \Gamma_\infty,$$

where $G \in C^0(\Gamma, k+2, \bar{v})$ is determined by a_{-m_0}, \dots, a_{-1} and the path of integration is a vertical line. Furthermore, $F|V - F$ is a parabolic cocycle in \mathbb{P} .

The following theorem is a generalization of a theorem from [9] that was mentioned in [4] as Theorem N_2 . It is actually the converse of Generalization of Theorem N_1 .

Generalization of Theorem N_2 . Let ψ be an integer computable depending on k and v . Let $k > \psi$ and let $G \in C^0(\Gamma, k+2, \bar{v})$ then there exists an automorphic integral F satisfying (3.18) and (3.19) such that F is holomorphic in \mathbb{H} and at q_1, \dots, q_t .

The following theorem is again a generalization of Theorem N_3 from [4] but for weakly parabolic multiplier systems. It follows as well from the main result of [12] where relations between Fourier coefficients of generalized modular forms were determined using the circle method.

Generalization of Theorem N_3 . Let ψ be an integer computable depending on k and v . For $k > \psi$ If there exists $\tilde{F} \in \{\Gamma, -k, v\}$ which is holomorphic in \mathbb{H} and at q_1, \dots, q_t , and which has principal part (3.17) at $q_0 = \infty$, then the function F of Theorem A is in $\{\Gamma, -k, v\}$ and in fact $F = \tilde{F}$. In this case the cusp form G of (3.19) is $\equiv 0$.

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