NORMS INEQUALITIES FOR SQUARE FUNCTIONS IN SOME MORREY'S SUBSPACES

JUSTIN FEUTO

Abstract. We prove that the intrinsic square function and the intrinsic Littlewood-Paley g_λ^*-function as defined by Wilson, are bounded in a family of weighted subspaces of Morrey spaces. The corresponding commutators generated by bounded mean oscillation functions are also considered.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional euclidean space equipped with the euclidean norm $|\cdot|$ and the Lebesgue measure dx. For $1 \leq p, q \leq \infty$, the amalgam of $L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ is the space $(L^q, L^p)(\mathbb{R}^n)$ of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ which are locally in $L^q(\mathbb{R}^n)$ and such that the function $y \mapsto \left\| f\chi_{B(y,1)} \right\|_q$ belongs to $L^p(\mathbb{R}^n)$, where for $r > 0$, $B(y, r) = \{x \in \mathbb{R}^n / |x - y| < r\}$ is the open ball centered at y with radius r, $\chi_{B(y,r)}$ denoting the characteristic function of the ball $B(y, r)$ and $\|\cdot\|_q$ the usual Lebesgue norm in $L^q(\mathbb{R}^n)$.

Amalgams arise naturally in harmonic analysis and were introduced by N. Wiener in 1926. But its systematic study goes back to the work of Holland [9]. We refer the reader to the survey paper of Fournier and Steward [7] for more information about these spaces. We recapitulate some of their properties in the following proposition.

Proposition 1.1. *Let $1 \leq q, p \leq \infty$.*

(1) $(L^q, L^q)(\mathbb{R}^n) = L^q(\mathbb{R}^n)$
(2) $L^q(\mathbb{R}^n) \cup L^p(\mathbb{R}^n) \subset (L^q, L^q)(\mathbb{R}^n)$ if $q \leq p$,
(3) $(L^q, L^q)(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ if $p \leq q$,
(4) The map $f \mapsto \|f\|_{q,p}$, where

$$\|f\|_{q,p} := \left(\int_{\mathbb{R}^n} \left\| f\chi_{B(y,1)} \right\|_q^p \right)^{\frac{1}{p}} \tag{1.1}$$

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with the usual modification when $p=\infty$, is a norm on $(L^q,L^p)(\mathbb{R}^n)$ (if we identify functions that differ only on null subset of $\mathbb{R}^n$) under which it is a Banach space.

As we observe in the above proposition, the amalgam spaces $(L^q,L^p)(\mathbb{R}^n)$ are interesting especially when $q\le p$. This will be a general assumption throughout this work.

In the Lebesgue space $L^q(\mathbb{R}^n)$, it is well known that for $r>0$ and $x\in\mathbb{R}^n$, the dilation operator $\delta_r^q : f\mapsto r^{\frac{d}{q}}f(r\cdot)$ and the translation operators $\tau_x : f\mapsto f(\cdot - x)$ are isometries. We use the usual convention that $\frac{1}{\infty}=0$. When we consider the amalgam space $(L^q,L^p)(\mathbb{R}^n)$, only translation operators conserve this property, which is just a consequence of the one in Lebesgue spaces. But it is easy to see that $f\in(L^q,L^p)$ if and only if we have

$$\left\|\delta_r^\alpha f\right\|_{q,p}<\infty, \tag{1.2}$$

for all $r>0$ and all $\alpha>0$. Notice that for $r>0$ and $\alpha>0$, we have

$$\begin{aligned}\left\|\delta_r^\alpha f\right\|_{q,p} &= r^{n(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\left(\int_{\mathbb{R}^n}\left\|f\chi_{B(y,r)}\right\|_q^p dy\right)^{\frac{1}{p}} \\ &\approx \left[\int_{\mathbb{R}^n}\left(|B(y,r)|^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\left\|f\chi_{B(y,r)}\right\|_q\right)^p dy\right]^{\frac{1}{p}},\end{aligned} \tag{1.3}$$ [1]

where $|B(y,r)|$ stands for the Lebesgue measure of the ball $B(y,r)$. This bring us to consider the subspace $(L^q,L^p)^\alpha(\mathbb{R}^n)$ of $(L^q,L^p)(\mathbb{R}^n)$ that consists in measurable functions f such that $\left\|f\right\|_{q,p,\alpha}<\infty$, where for $1\le q,p,\alpha\le\infty$,

$$\left\|f\right\|_{q,p,\alpha} := \sup_{r>0}\left\|\delta_r^\alpha f\right\|_{q,p}. \tag{1.4}$$

Taking into consideration Relation (1.3), we can generalize these spaces in the context of space of homogeneous type in the sense of Coifman and Weiss (see [5]).

As proved by Fofana in [6], where these spaces were first considered, the spaces $(L^q,L^p)^\alpha(\mathbb{R}^n)$ are non trivial if and only if $q\le\alpha\le p$. In this case it is proved in [6, 1], that for $1\le q<\alpha$ fixed and p going from α to ∞, they form a chain of distinct Banach spaces beginning with Lebesgue spaces $L^\alpha(\mathbb{R}^n)$ and ending by the classical Morrey's space $L^{q,d(1-\frac{q}{\alpha})}(\mathbb{R}^n)=(L^q,L^\infty)^\alpha(\mathbb{R}^n)$. It is proved [6] that for $q<\alpha<p<\infty$, the weak Lebesgue space $L^{\alpha,\infty}(\mathbb{R}^n)$ is continuously embedded in the space $(L^q,L^p)^\alpha(\mathbb{R}^n)$,i.e., there exists $C>0$ such that

$$\left\|f\right\|_{q,p,\alpha}\le C\left\|f\right\|_{\alpha,\infty}^*,\ \text{ for all } f\in L^{\alpha,\infty}. \tag{1.5}$$

[1]Hereafter we propose the following abbreviation $\mathbf{A}\approx\mathbf{B}$ for the inequalities $C^{-1}\mathbf{A}\le\mathbf{B}\le C\mathbf{A}$, where C is a positive constant independent of the main parameters.

We recall that a measurable function f belongs to the weak-Lebesgue space $L^{p,\infty}$ if

$$\|f\|_{p,\infty}^{*} := \sup_{\lambda>0} \lambda^{\frac{1}{p}} \left|\left\{x \in \mathbb{R}^{d} : |f(x)| > \lambda\right\}\right| < \infty.$$

These spaces and many of their properties have been extended in the context of homogeneous groups by the author in his thesis [3] (see also [4]).

In the rest of this work, we will always assume that $1 \leq q \leq \alpha \leq p \leq \infty$.

We recall that many classical results established in the context of Lebesgue spaces in Fourier analysis have been extended to the setting of $(L^q, L^p)^{\alpha}(\mathbb{R}^n)$ spaces. For example, Hölder and Young inequalities are just a consequence of their analog in Lebesgue spaces [6]. The Hardy-Littlewood-Sobolev inequality for fractional integrals has been generalized to this case in [1, 2]. In this work, we are interested in the norm inequalities involving some intrinsic functions (see [12]).

For $0 < \eta \leq 1$, we denote by $\mathcal{C}_\eta$ the family of function φ defined on $\mathbb{R}^n$ with support in the closed unit ball $\mathbb{B} = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and vanishing integral, i.e., $\int_{\mathbb{R}^n} \varphi(x)dx = 0$, and such that for all $x, x' \in \mathbb{R}^n$, $|\varphi(x) - \varphi(x')| \leq |x - x'|^{\eta}$. Let $\mathbb{R}_{+}^{n+1} = \mathbb{R}^n \times (0, \infty)$ and $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. For all $(y, t) \in \mathbb{R}_{+}^{n+1}$ and $f \in L_{loc}^{1}(\mathbb{R}^n)$, we set

$$A_\eta f(y, t) = \sup_{\varphi \in \mathcal{C}_\eta} |f * \varphi_t(y)| .$$

The intrinsic square function of f (of order η) is defined by the formula

$$S_\eta(f)(x) = \left(\int_{\Gamma(x)} A_\eta(f)(y, t)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where for $x \in \mathbb{R}^n$, $\Gamma(x)$ denote the usual "cone of arperture one",

$$\Gamma(x) = \left\{(y, t) \in \mathbb{R}_{+}^{n+1} : |x - y| < t\right\} .$$

We also define the intrinsic Littlewood-Paley g-function $g_\eta(f)$ and g_λ^*-function $g_{\lambda,\eta}^*(f)$ by

$$g_\eta(f)(x) = \left(\int_0^\infty (A_\eta(f)(x, t))^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

and

$$g_{\lambda,\eta}^*(f)(x) = \left(\int_{\mathbb{R}_{+}^{n+1}} \left(\frac{t}{t + |x - y|}\right)^{\lambda n} (A_\eta(f)(y, t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

respectively. Wilson in [13] proved that for $1 < q < \infty$, the operators S_η for $0 < \eta \leq 1$ are bounded in the weighted Lebesgue space L_w^q, namely the space consisting in measurable functions f satisfying

$$\|f\|_{q_w} := \left(\int_{\mathbb{R}^n} |f(x)|^q \, w(x)dx \right)^{\frac{1}{q}} < \infty,$$

whenever the weight w fulfilled the $\mathcal{A}_q$ condition of Muckenhoupt (see Section 2 for the definition). This result has been extended by Wang (see Theorem 1.1 [11]) to weighted Morrey spaces $L^{q,\kappa}_w(\mathbb{R}^n)$. We recall that for $0 < \kappa < 1$ the space $L^{q,\kappa}_w(\mathbb{R}^n)$ consists of measurable functions f such that $\|f\|_{L^{q,\kappa}_w} < \infty$, where

$$\|f\|_{L^{q,\kappa}_w} := \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^q \, w(x)dx \right)^{\frac{1}{q}}.$$

The boundedness of the operators $g^*_{\lambda,\eta}$ in the weighted Morrey space $L^{q,\kappa}_w(\mathbb{R}^n)$ is also proved in Theorem 1.3 of [11], while in Theorem 1.2 and 1.4 the author consider the boundedness of the commutator operators $[b, S_\eta]$ and $\left[b, g^*_{\lambda,\eta}\right]$ as defined by Relations (2.2) and (2.3).

In this paper we give norm inequalities of these operators in weighted version of the space $(L^q, L^p)^\alpha(\mathbb{R}^n)$ say $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$, for $1 < q \leq \alpha \leq p \leq \infty$ (see the next section for the definition).

The remaining of this paper is organized as follows:

In the second section, we recall the definition of $(L^q_w, \ell^p)^\alpha$ spaces and we state our main results. Section three is devoted to the prove of the main results.

Throughout the paper, the letter C is used for non-negative constants that may change from one occurrence to another. The notation $\mathbf{A} \lesssim \mathbf{B}$ will always mean that the ratio $\mathbf{A}/\mathbf{B}$ is bounded away from zero by a constant independent of the relevant variables in $\mathbf{A}$ and $\mathbf{B}$. For $\alpha > 0$ and a ball $B \subset \mathbb{R}^n$, we write αB for the ball with same center as B and with radius α times radius of B. We denote by $\mathbb{N}^*$ the set of all positive integers.

2. Definitions and statement of the main results

A weight w on $\mathbb{R}^n$,i.e., a positive locally integrable function on $\mathbb{R}^n$, is of class $\mathcal{A}_p$ or belongs to $\mathcal{A}_p$ for $1 < p < \infty$ if there exists a constant $C > 0$ such that for all balls $B \subset \mathbb{R}^n$ we have

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w^{\frac{-1}{p-1}}(x)dx \right)^{p-1} \leq C. \tag{2.1}$$

We put $[w]_{\mathcal{A}_p} = \inf \{C \in \mathbb{R}^n : (2.1) \text{ holds}\}$. Let w be a weight on $\mathbb{R}^n$ and $1 \leq q, p, \alpha \leq \infty$. We define the space $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$ as the space of all measurable functions f satisfying $\|f\|_{q_w,p,\alpha} < \infty$, where for $r > 0$, we put

$$ {}_r\|f\|_{q_w,p,\alpha} := \left[\int\limits_{\mathbb{R}^n} \left(w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|f\chi_{B(y,r)}\right\|_{q_w} \right)^p dy \right]^{\frac{1}{p}}, $$

with $w(B(y,r)) = \int_{B(y,r)} w(x)dx$ and the usual modification when $p = \infty$, and

$$ \|f\|_{q_w,p,\alpha} := \sup_{r>0} {}_r\|f\|_{q_w,p,\alpha}. $$

When $w \equiv 1$, we recover the space $(L^q, L^p)^\alpha(\mathbb{R}^n)$, while for $q < \alpha$ and $p = \infty$, the space $(L^q_w, L^\infty)^\alpha(\mathbb{R}^n)$ is noting but the weighted Morrey space $L^{q,\kappa}_w(\mathbb{R}^n)$, with $\kappa = \frac{1}{q} - \frac{1}{\alpha}$.We are now ready to state our main results. The first result giving the boundedness of the operators S_η, is an extension of Theorem 1.1 of [11].

Theorem 2.1. *Let $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. The operators S_η are bounded in $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$. More precisely,*

$$ \|S_\eta(f)\|_{q_w,p,\alpha} \lesssim \|f\|_{q_w,p,\alpha}. $$

The next result is an extension of Theorem 1.2 of [11] to the space $(L^q_w, \ell^p)^\alpha(\mathbb{R}^n)$. We first recall some definitions. The commutator $[b, S_\eta]$ of a locally integrable function b and S_η is defined by

$$ [b, S_\eta](f)(x) = \left(\int\limits_{\Gamma(x)} \sup_{\varphi\in\mathcal{C}_\eta} \left| \int\limits_{\mathbb{R}^n} (b(x) - b(z))\varphi_t(y - z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}. \tag{2.2} $$

In [11], it is proved that the commutator $[b, S_\eta]$ is bounded in the weighted Morrey space $L^{q,\kappa}_w(\mathbb{R}^n)$ whenever the weight w fulfills the $\mathcal{A}_q$ condition, and b belongs to $BMO(\mathbb{R}^n)$ (bounded mean oscillation functions) space, i.e.,

$$ \|b\|_{BMO(\mathbb{R}^n)} := \sup_{B:\text{ ball}} \frac{1}{|B|} \int\limits_B |b(x) - b_B|\, dx < \infty. $$

We have the following result in the case of our spaces.

Theorem 2.2. *Let $0 < \eta \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. Suppose that $b \in BMO(\mathbb{R}^n)$, then there exists a constant $C > 0$ not depending on f such that*

$$ \|[b, S_\eta](f)\|_{q_w,p,\alpha} \leq C \|f\|_{q_w,p,\alpha}, $$

for all $f \in (L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

When considering the intrinsic g-function of Littlewood-Paley, we have the following extension of Theorem 1.3 of [11].

Theorem 2.3. *Let $0 < \eta \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. If $\lambda > \max\{q, 3\}$ then there exists a constant $C > 0$ such that*

$$\left\| g^*_{\lambda,\eta}(f) \right\|_{q_w,p,\alpha} \leq C \left\| f \right\|_{q_w,p,\alpha},$$

for all $f \in (L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

For any locally integrable function b, the commutator $\left[b, g^*_{\lambda,\eta}\right]$ is the operator defined by

(2.3)

$$\left[b, g^*_{\lambda,\eta}\right](f)(x) = \left(\int\limits_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\eta} \left| \int\limits_{\mathbb{R}^n} (b(x) - b(z)) \varphi_t(y - z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Using the same argument as in Theorem 2.1 and 2.2, we can prove the following.

Theorem 2.4. *Let $0 < \eta \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. If $b \in BMO(\mathbb{R}^n)$ and $\lambda > \max\{q, 3\}$ then there exists a constant $C > 0$ such that*

$$\left\| \left[b, g^*_{\lambda,\eta}\right](f) \right\|_{q_w,p,\alpha} \leq C \left\| f \right\|_{q_w,p,\alpha},$$

for all $f \in (L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

This result is an extension of Theorem 1.4 of [11]. Since for any $0 < \eta \leq 1$ the functions $S_\eta(f)$ and $g_\eta(f)$ are pointwise comparable as we can see in [12], as an immediate consequence of Theorem 2.1 and 2.2 we have the following results.

Corollary 2.5. *Let $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. The operator g_η is bounded in $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$. More precisely,*

$$\left\| g_\eta(f) \right\|_{q_w,p,\alpha} \lesssim \left\| f \right\|_{q_w,p,\alpha}.$$

Corollary 2.6. *Let $0 < \eta \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. Suppose that $b \in BMO(\mathbb{R}^n)$, then there exists a constant $C > 0$ not depending on f such that*

$$\left\| [b, g_\eta](f) \right\|_{q_w,p,\alpha} \leq C \left\| f \right\|_{q_w,p,\alpha},$$

for all $f \in (L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

3. Proof the main results

Proof of Theorem 2.1. Let $f \in (L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

We fix $r > 0$ and let $B = B(y, r)$ for some $y \in \mathbb{R}^n$. As in [11], we write $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$. Since S_η is a sub-additive operator, we have

(3.1) $$\left\| S_\eta(f)\chi_B \right\|_{q_w} \leq \left\| S_\eta(f_1)\chi_B \right\|_{q_w} + \left\| S_\eta(f_2)\chi_B \right\|_{q_w}.$$

We are going to estimate each of the terms of the second member of (3.1). For the term in f_1, we have

$$\|S_\eta(f_1)\chi_B\|_{q_w} \lesssim \|f\chi_{2B}\|_{q_w} \tag{3.2}$$

as an immediate consequence of the boundedness of S_η in $L^q_w(\mathbb{R}^n)$. Our attention will be focused now on the second term.

Let $\varphi \in \mathcal{C}_\eta$, and $t > 0$. Since the family $\mathcal{C}_\eta$ is uniformly bounded with respect to the L^∞-norm, we have

$$|f_2 * \varphi_t(u)| \lesssim t^{-n} \int\limits_{(2B)^c\cap\tilde{B}(u,t)} |f(z)|\,dz, \tag{3.3}$$

for all $u \in \mathbb{R}^n$, where $\tilde{B}(u,t) := \{z \in \mathbb{R}^n / \, |z-u| \leq t\}$. Thus for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |S_\eta(f_2)(x)| &\lesssim \left[\int\limits_{\Gamma(x)} \left(t^{-n} \int\limits_{(2B)^c\cap\tilde{B}(u,t)} |f(z)|\,dz\right)^2 \frac{dudt}{t^{n+1}}\right]^{\frac{1}{2}} \\ &\lesssim \sum_{k=1}^{\infty} \int\limits_{2^{k+1}B\setminus 2^kB} |f(z)| \left[\int\limits_0^\infty \left(\int\limits_{B(x,t)} \chi_{\tilde{B}(z,t)}(u)du\right) \frac{dt}{t^{3n+1}}\right]^{\frac{1}{2}} dz \end{aligned}$$

where the last control is an application of Minkowski's integral inequality.

We suppose $x \in B(y,r)$. For $k \in \mathbb{N}^*$, $z \in 2^{k+1}B \setminus 2^kB$ and $t > 0$, $\int_{B(x,t)} \chi_{\tilde{B}(z,t)}(u)du \neq 0$ implies that there exists $u_0 \in B(x,t) \cap B(z,t)$. It follows that

$$2t \geq |x-u_0| + |z-u_0| \geq |x-z| \geq |y-z| - |x-y| \geq 2^{k-1}r. \tag{3.4}$$

Thus for $x \in B = B(y,r)$,

$$\begin{aligned} |S_\eta(f_2)(x)| &\lesssim \sum_{k=1}^{\infty} \int\limits_{2^{k+1}B\setminus 2^kB} |f(z)| \left(\int\limits_{2^{k-2}r}^\infty \int\limits_{B(x,t)} du \frac{dt}{t^{3n+1}}\right)^{\frac{1}{2}} dz \\ &\lesssim \sum_{k=1}^{\infty} \int\limits_{2^{k+1}B\setminus 2^kB} |f(z)| \left(\int\limits_{2^{k-2}r}^\infty \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} dz \lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int\limits_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz. \end{aligned}$$

But we have for every $k \in \mathbb{N}^*$

$$\frac{1}{|2^{k+1}B|} \int\limits_{2^{k+1}B} |f(z)|\,dz \lesssim \|f\chi_{2^{k+1}B}\|_{q_w} w(2^{k+1}B)^{-\frac{1}{q}}, \tag{3.5}$$

since $w \in \mathcal{A}_q$. Hence,

$$\left\|S_\eta(f_2)\chi_{B(y,r)}\right\|_{q_w} \lesssim \sum_{k=1}^{\infty} \|f\chi_{2^{k+1}B}\|_{q_w} \left(\frac{w(B)}{w(2^{k+1}B)}\right)^{\frac{1}{q}}. \tag{3.6}$$

The weight w being in $\mathcal{A}_q$ with $1 < q < \infty$, there exists $1 < s < \infty$ such that

$$\frac{w(B(y,r))}{w(B(y,2^{k+1}r))} \lesssim \frac{1}{2^{nk(1-\frac{1}{s})}}. \tag{3.7}$$

Multiplying both Inequalities (3.2) and (3.6) by $w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}$ and taking into consideration Relation (3.7), we have

$$\begin{aligned} w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|S_\eta(f)\chi_{B(y,r)}\right\|_{q_w} &\lesssim w(B(y,2r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|f\chi_{B(y,2r)}\right\|_{q_w} \\ &+ \sum_{k=1}^{\infty} w(B(y,2^{k+1}r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|f\chi_{B(y,2^{k+1}r)}\right\|_{q_w} \frac{1}{2^{nk(\frac{1}{s'\alpha}-\frac{1}{s'p})}}. \end{aligned} \tag{3.8}$$

Since (3.8) is true for all $y \in \mathbb{R}^n$, the L^p norm of both sides led to

$${}_r\left\|S_\eta(f)\right\|_{q_w,p,\alpha} \lesssim \|f\|_{q_w,p,\alpha}\,,\ r>0,$$

and the result is obtained by taking the supremum over all $r > 0$. □

For the proof of the next result on commutator, we use the following characterization of BMO (see [10]). Let b be a locally integrable function. If $b \in BMO(\mathbb{R}^n)$, then for every $1 < p < \infty$, we have

$$\|b\|_{BMO(\mathbb{R}^n)} \approx \sup_{B:\text{ ball}} \left(\frac{1}{|B|}\int_B |b(x)-b_B|^p\,dx\right)^{\frac{1}{p}} \tag{3.9}$$

Proof of Theorem 2.2. Fix $r > 0$ and let $B = B(y,r)$ be a ball in $\mathbb{R}^n$. As above, we split f into two parts f_1 and f_2 such that $f_1 = f\chi_{2B}$ and $f = f_1 + f_2$. It comes from the sub-additivity of the commutator that

$$\begin{aligned} w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|[b,S_\eta](f)\chi_B\right\|_{q_w} &\leq w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|[b,S_\eta](f_1)\chi_B\right\|_{q_w} \\ &+ w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|[b,S_\eta](f_2)\chi_B\right\|_{q_w} \end{aligned} \tag{3.10}$$

Let us estimate each of the term in the right hand sides of (3.10). For the term in f_1, we have

$$w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|[b,S_\eta](f_1)\chi_{B(y,r)}\right\|_{q_w} \lesssim w(B(y,2r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|f\chi_{B(y,2r)}\right\|_{q_w}, \tag{3.11}$$

according to the boundedness of the commutator on $L^q_w(\mathbb{R}^n)$ (Theorem 3.1 [11]) and the doubling character of w. For the second term, we proceed almost as in [11].

Let $x\in\mathbb{R}^n$. For $u\in\mathbb{R}^n$, we have

$$\begin{aligned}\sup_{\varphi\in\mathcal{C}_\eta}\left|\int_{\mathbb{R}^n}(b(x)-b(z))\varphi_t(u-z)f_2(z)dz\right| &\leq |b(x)-b_B|\sup_{\varphi\in\mathcal{C}_\eta}\left|\int_{\mathbb{R}^n}\varphi_t(u-z)f_2(z)dz\right|\\ &+ \sup_{\varphi\in\mathcal{C}_\eta}\left|\int_{\mathbb{R}^n}(b(z)-b_B)\varphi_t(u-z)f_2(z)dz\right|,\end{aligned}$$

so that

(3.12)

$$\begin{aligned}\left|[b,S_\eta]\,f_2(x)\right| &\leq |b(x)-b_B|\,S_\eta(f_2)(x)\\ &+ \left(\int_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\eta}\left|\int_{\mathbb{R}^n}(b(z)-b_B)\varphi_t(u-z)f_2(z)dz\right|^2\frac{dudt}{t^{n+1}}\right)^{\frac{1}{2}}.\end{aligned}$$

If $x\in B=B(y,r)$, then

$$|S_\eta(f_2)(x)|\lesssim\sum_{k=1}^{\infty}\left\|f\chi_{2^{k+1}B}\right\|_{q_w}w(2^{k+1}B)^{-\frac{1}{q}},$$

as we can see in the proof of Theorem 2.1. It follows that

$$\begin{aligned}&w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\left\|\left|b-b_B\right|S_\eta(f_2)\chi_B\right\|_{q_w}\\ &\quad\lesssim\|b\|_{BMO}\sum_{k=1}^{\infty}\left(\frac{1}{2^{nk}}\right)^{\frac{1}{s'}\left(\frac{1}{\alpha}-\frac{1}{p}\right)}w(2^{k+1}B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\left\|f\chi_{2^{k+1}B}\right\|_{q_w},\end{aligned}$$

where we use the following estimation

$$\left(\frac{1}{w(B)}\int_B|b(x)-b_B|^q\,w(x)dx\right)^{\frac{1}{q}}\lesssim\|b\|_{BMO}, \tag{3.13}$$

which is satisfied whenever $w\in\mathcal{A}_q$. This comes from the fact that for $w\in\mathcal{A}_q$, with $1\leq q<\infty$, there exist two reals constants $C>0$ and $s>1$ depending only on n, q and $[w]_{\mathcal{A}_q}$ such that for all balls B, we have the following Reverse Hölder condition (Theorem 9.2.2 [8])

$$\left(\frac{1}{|B|}\int_B w^s(z)dz\right)^{\frac{1}{s}}\leq\frac{C}{|B|}\int_B w(z)dz,$$

and Relation (3.9). The second term on the right hand sides of Relation (3.12) is controlled by

$$\begin{aligned}&\left[\int_{\Gamma(x)}\left(t^{-n}\int_{(2B)^c\cap\tilde{B}(u,t)}\left|b(z)-b_{2^{k+1}B}\right|\left|f(z)\right|dz\right)^2\frac{dudt}{t^{n+1}}\right]^{\frac{1}{2}}\\ &+\left[\int_{\Gamma(x)}\left(t^{-n}\int_{(2B)^c\cap\tilde{B}(u,t)}\left|b_{2^{k+1}B}-b_B\right|\left|f(z)\right|dz\right)^2\frac{dudt}{t^{n+1}}\right]^{\frac{1}{2}},\end{aligned} \tag{3.14}$$

according to the uniformly bounded property of the family $\mathcal{C}_\eta$. Using once more the Minkowski's inequality for integrals and Inequality (3.4), we have

$$\left[\int_{\Gamma(x)}\left(t^{-n}\int_{(2B)^c\cap\tilde{B}(u,t)}|b(z)-b_{2^{k+1}B}|\,|f(z)|\,dz\right)^2\frac{dudt}{t^{n+1}}\right]^{\frac{1}{2}} \tag{3.15}$$
$$\lesssim\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B\backslash 2^kB}|b(z)-b_{2^{k+1}B}|\,|f(z)|\,dz,$$

for all $x\in B(y,r)$. But then Hölder inequality allow to write

$$\begin{aligned}\int_{(2^{k+1}B\backslash 2^kB)}|b(z)-b_{2^{k+1}B}|\,|f(z)|\,dz \;&\leq\; \left(\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|^{q'}\,w(z)^{-\frac{q'}{q}}dz\right)^{\frac{1}{q'}}\\ &\times\;\left(\int_{2^{k+1}B}|f(z)|^q\,w(z)dz\right)^{\frac{1}{q}},\end{aligned}$$

so that using the fact that the weight $v(z)=w(z)^{-\frac{q'}{q}}$ belongs to $\mathcal{A}_{q'}$ whenever $w\in\mathcal{A}_q$ and Relation (3.13), we obtain

$$\int_{(2^{k+1}B\backslash 2^kB)}|b(z)-b_{2^{k+1}B}|\,|f(z)|\,dz\lesssim\left\|f\chi_{2^{k+1}B}\right\|_{q_w}\left|2^{k+1}B\right|w(2^{k+1}B)^{-\frac{1}{q}}\left\|b\right\|_{BMO}. \tag{3.16}$$

Finally, taking (3.16) into (3.15) yield

$$\left[\int_{\Gamma(x)}\left(t^{-n}\int_{(2B)^c\cap\tilde{B}(u,t)}|b(z)-b_{2^{k+1}B}|\,|f(z)|\,dz\right)^2\frac{dudt}{t^{n+1}}\right]^{\frac{1}{2}} \tag{3.17}$$
$$\lesssim\left\|b\right\|_{BMO}\sum_{k=1}^{\infty}\left\|f\chi_{2^{k+1}B}\right\|_{q_w}w(2^{k+1}B)^{-\frac{1}{q}}.$$

for all $x\in B$. Thus the $L^q_w(B)$ norm of the first term of (3.14) is controlled by

$$\left\|b\right\|_{BMO}\sum_{k=1}^{\infty}\left\|f\chi_{2^{k+1}B}\right\|_{q_w}\left(\frac{w(B)}{w(2^{k+1}B)}\right)^{\frac{1}{q}}.$$

For the second term of (3.14), we use the fact that

$$|b_{2^{k+1}B}-b_B|\;\lesssim(k+1)\left\|b\right\|_{BMO}.$$

It follows that for $x \in B$, we have

$$\left[\int_{\Gamma(x)} \left(t^{-n} \int_{(2B)^c \cap \tilde{B}(u,t)} |b_{2^{k+1}B} - b_B| \, |f(z)| \, dz \right)^2 \frac{dudt}{t^{n+1}} \right]^{\frac{1}{2}}$$
$$\lesssim \|b\|_{BMO} \left(\sum_{k=1}^{\infty} (k+1) \left\| f\chi_{2^{k+1}B} \right\|_{q_w} w(2^{k+1}B)^{-\frac{1}{q}} \right),$$

according to Minkowski inequality and Relation (3.5). Thus the $L^q_w(B)$ norm of the second term of (3.14) is majored by an absolute constant times

$$\|b\|_{BMO} \left(\sum_{k=1}^{\infty} (k+1) \left\| f\chi_{2^{k+1}B} \right\|_{q_w} \left(\frac{w(B)}{w(2^{k+1}B)} \right)^{\frac{1}{q}} \right).$$

Hence,

$$\begin{aligned} & w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| [b, S_\eta] (f_2) \chi_{B(y,r)} \right\|_{q_w} \\ \text{(3.18)} \qquad & \lesssim \|b\|_{BMO} \left(\sum_{k=1}^{\infty} \frac{k+2}{2^{\frac{2nk}{s'}(\frac{1}{\alpha}-\frac{1}{p})}} w(B(y, 2^{k+1}r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{B(y,2^{k+1}r)} \right\|_{q_w} \right) \end{aligned}$$

for all $y \in \mathbb{R}^n$. Taking Estimation (3.11) and (3.18) in (3.10) yield,

(3.19)

$$\begin{aligned} & w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| [b, S_\eta] (f) \chi_{B(y,r)} \right\|_{q_w} \\ & \lesssim \|b\|_{BMO} \left(\sum_{k=1}^{\infty} \frac{k+2}{2^{\frac{2nk}{s'}(\frac{1}{\alpha}-\frac{1}{p})}} w(B(y, 2^{k+1}r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{B(y,2^{k+1}r)} \right\|_{q_w} \right) \\ & + w(B(y,2r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{B(y,2r)} \right\|_{q_w} \end{aligned}$$

for all $y \in \mathbb{R}^n$. Thus taking the L^p-norm of both sides of (3.19), we have

$$_r \left\| [b, S_\eta] (f) \right\|_{q_w,p,\alpha} \lesssim (1 + \|b\|_{BMO}) \, \|f\|_{q_w,p,\alpha},$$

for all $r > 0$, since the series $\sum_{k=1}^{\infty} \frac{k+2}{2^{\frac{2nk}{s'}(\frac{1}{\alpha}-\frac{1}{p})}}$ converges. We end the proof by taking the supremum over all $r > 0$. □

For the proof of Theorem 2.3, we will need the following varying-aperture versions of S_η. For $0 < \eta \leq 1$ and $\beta > 0$, we define $S_{\eta,\beta}(f)$ by

$$S_{\eta,\beta}(f)(x) = \left(\int_{\Gamma_\beta(x)} A_\eta(f)(y,t)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}, \tag{3.20}$$

where $\Gamma_\beta(x) = \left\{ (x,t) \in \mathbb{R}^{n+1}_+ / \, |x-y| < \beta t \right\}$.

Proof of Theorem 2.3. As we can see in the proved of Theorem 1.3 in [11], for all $x \in \mathbb{R}^n$, we have

$$g^*_{\lambda,\eta}(f)(x)^2 \lesssim S_\eta(f)(x)^2 + \sum_{j=1}^{\infty} 2^{-j\lambda n} S_{\eta,2^j}(f)(x)^2.$$

Let $r > 0$. For any ball $B = B(y,r)$ include in $\mathbb{R}^n$, it comes from the above inequality that

(3.21)
$$\begin{aligned} w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| g^*_{\lambda,\eta}(f)\chi_B \right\|_{q_w} &\lesssim w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_\eta(f)\chi_B \right\|_{q_w} \\ &+ \sum_{j=1}^{\infty} 2^{-j\lambda n/2} w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f)\chi_B \right\|_{q_w}. \end{aligned}$$

By Theorem 2.1, we have that the L^p norm of the first term of (3.21) is controlled by $\|f\|_{q_w,p,\alpha}$. For the terms under the summation, we proceed as in Theorem 2.1. Let $j \in \{1,2,\ldots\}$, we write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$, It follows that

$$\begin{aligned} w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f)\chi_B \right\|_{q_w} &\leq w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f_1)\chi_B \right\|_{q_w} \\ &+ w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f_2)\chi_B \right\|_{q_w}. \end{aligned} \tag{3.22}$$

Applying Lemma 4.1-4.3 and Theorem A of [11], we obtain that

$$w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f_1)\chi_B \right\|_{q_w} \lesssim (2^{jn} + 2^{jnq/2}) w(2B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{2B} \right\|_{q_w} \tag{3.23}$$

Let us estimate know the term in f_2. Using estimation (3.3) and (3.5) and proceed as in [11], we obtain that

$$\begin{aligned} \left| S_{\eta,2^j}(f)(x) \right| &\lesssim 2^{3jn/2} \sum_{k=1}^{\infty} \frac{1}{\left|2^{k+1}B\right|} \int_{2^{k+1}B \setminus 2^k B} |f(z)|\, dz \\ &\lesssim 2^{3jn/2} \sum_{k=1}^{\infty} \left\| f\chi_{B(y,2^{k+1}r)} \right\|_{q_w} w(B(y,2^{k+1}r))^{-\frac{1}{q}}, \end{aligned}$$

for all $x \in B(y,r)$. Thus

(3.24)
$$w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f_2)\chi_B \right\|_{q_w} \lesssim 2^{3jn/2} \sum_{k=1}^{\infty} \frac{w(2^{k+1}B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}}{2^{nk(1-\frac{1}{s})(\frac{1}{\alpha}-\frac{1}{p})}} \left\| f\chi_{2^{k+1}B} \right\|_{q_w}.$$

If we take Estimations (3.23) and (3.24) in (3.22), we have

$$\begin{aligned} w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| S_{\eta,2^j}(f)\chi_{B(y,r)} \right\|_{q_w} &\lesssim (2^{jn} + 2^{jnq/2}) w(B(y,2r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{B(y,2r)} \right\|_{q_w} \\ &+ 2^{3jn/2} \sum_{k=1}^{\infty} \frac{w(B(y,2^{k+1}r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}}{2^{nk(1-\frac{1}{s})(\frac{1}{\alpha}-\frac{1}{p})}} \left\| f\chi_{B(y,2^{k+1}r)} \right\|_{q_w}, \end{aligned}$$

for all $x \in \mathbb{R}^n$, so that taking the L^p-norm of both sides of the above inequality, we obtain

$$_r \left\| S_{\eta,2^j}(f) \right\|_{q_w,p,\alpha} \lesssim (2^{jn} + 2^{jnq/2}) \|f\|_{q_w,p,\alpha} + \|f\|_{q_w,p,\alpha} 2^{3jn/2}, \tag{3.25}$$

Therefore the L^p norm of the (3.21) gives

(3.26)

$$\begin{aligned} {}_r\left\|g^*_{\lambda,\eta}(f)\right\|_{q_w,p,\alpha} &\lesssim \|f\|_{q_w,p,\alpha}\left(1+\sum_{j=1}^{\infty}2^{-j\lambda n/2}(2^{jn}+2^{jnq/2}+2^{3jn/2})\right)\\ &\lesssim \|f\|_{q_w,p,\alpha}. \end{aligned}$$

where the convergence of the series is due to the fact that $\lambda > \max\{q, 3\}$. By taking the supremum over all $r > 0$, we conclude the proof. □

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Laboratoire de Mathématiques Fondamentales, UFR Mathématiques et Informatique, Université de Cocody, 22 B.P 1194 Abidjan 22. Côte d'Ivoire

E-mail address: justfeuto@yahoo.fr