The equal tangents property

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Abstract

Let M be a C^2-smooth strictly convex closed surface in $\mathbb{R}^3$ and denote by H the set of points x in the exterior of M such that all the tangent segments from x to M have equal lengths. In this note we prove that if H is either a closed surface containing M or a plane, then M is an Euclidean sphere. Moreover, we shall see that the situation in the Euclidean plane is very different.

1 In the Euclidean plane

Let K be a strictly convex body in the plane. The following fact is well known: *if the two tangent segments to K from every point $x \notin K$ have equal lengths then K is an Euclidean disc* (see, for instance, [4]). This statement is easily proved by elementary geometry. The result was extended to the case of Minkowski planes by S. Wu [9], and Z. Lángi [3] also gave a characterization of the ellipsoid among centrally symmetric convex bodies in terms of tangent segments of equal Minkowski length.

In the Euclidean plane, one may obtain the same conclusion with considerably weaker assumptions. Namely, one has the following characterization of a circle in terms of equal tangent segments.

Lemma 1 *Let γ be a strictly convex closed curve in the plane, and let ℓ be a tangent line through a point $p \in \gamma$. Suppose that the two tangent segments to γ from every point $x \in \ell$ have equal lengths. Then γ is a circle (see, e.g., [2] or Section 3).*

Thus it is natural to ask whether the same conclusion remains true if the locus of points from which the tangent segments to γ have equal lengths is a line ℓ that is not tangent to γ. We consider the two cases separately: first, when ℓ intersects γ, and second, when ℓ is disjoint from γ.

Example 2 Let us construct a non-circular curve with the desired equitangent property.

Consider two circles and their radical axis ℓ (that is, the set of points having equal power with respect to both of them). Then the tangent segments to both circles from the points of ℓ are equal. Let $x, y \in \ell$ be two points in the exterior of the convex hull of these circles. Draw the tangents xa, xb, yc, and yd as shown in Figure 1, and also draw the two arcs of the circles tangent to xa, xb at a and b, and to yc, yd at c and d, respectively. Then the union of the arcs $\widehat{ab}, \widehat{bd}, \widehat{dc}$, and $\widehat{ca}$ is a C^1-smooth and strictly convex curve γ with the property that for every point $p \in \ell \setminus \mathrm{conv}(\gamma)$, the two tangent segments to γ from p have equal lengths.

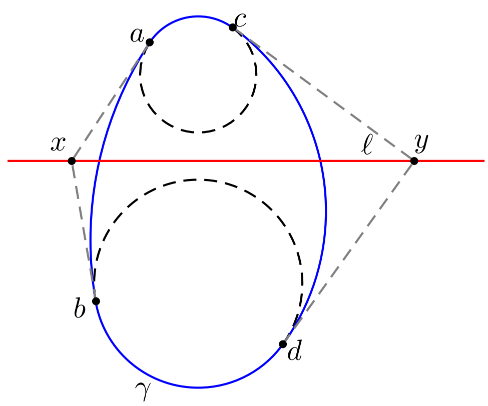

Figure 1

A characterization in terms of hyperbolic geometry. In the case when ℓ does not intersect γ, we have a complete characterization of such curves.

Lemma 3 *Assume that ℓ is the horizontal axis and that γ lies in the upper half plane. Then the tangent segments to γ from every point of ℓ are equal if and only if γ is a curve of constant width in the hyperbolic metric, considered in the upper half plane model.*

Proof. The two tangent segments from a point $x \in \ell$ have equal lengths if and only if the circle centered at x is orthogonal to γ at both intersection points (more precisely, is orthogonal to support lines to γ at these points). See Figure 2.

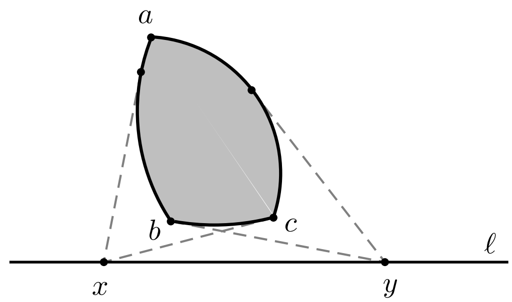

Figure 2.

The circles centered at points of ℓ are the geodesics of the hyperbolic plane, and the upper half plane model is conformal. Thus a geodesic segment can make a full circuit inside γ, remaining orthogonal to it at both end points. This property characterizes convex bodies of constant width, see [1]. $\square$

Next we construct pairs of curves in the plane, Γ and γ, such that Γ encloses γ and, for every point $x \in \Gamma$, the two tangent segments from x to γ have equal lengths. Compare with [7] where a pair of curves Γ and γ is constructed such that, for every point $x \in \Gamma$, the tangent segments from x to Γ have unequal lengths.

Example 4 Consider a regular convex n-gon, with odd $n \geq 5$, and make the classical construction of a body of constant width. Concretely, let $V_1V_2V_3V_4V_5$ be a regular pentagon and let λ be the length of its diagonals. Fix $\varepsilon \geq 0$. Draw the lines V_1V_3, V_1V_4, V_5V_2, V_5V_3, and V_2V_4. Now, draw the arcs of the circles centered at V_1 and the radii ε and $\lambda + \varepsilon$ from line V_3V_1 to line V_4V_1, see Figure 3. Do the same at the remaining vertices. We obtain a C^1-smooth convex curve γ of constant width $\lambda + 2\varepsilon$.

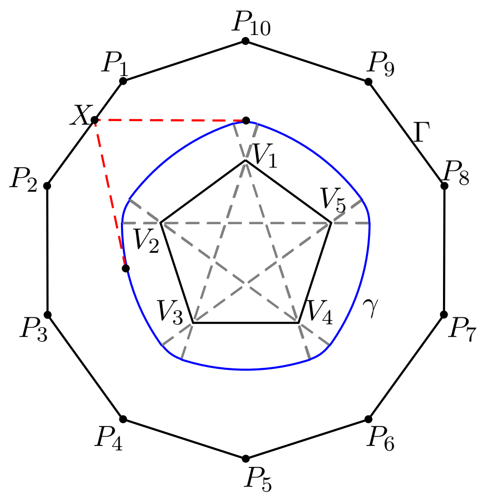

Figure 3.

Let $\Gamma = P_1P_2 \ldots P_{10}$ be the regular decagon constructed in the following way: the segment P_1P_2 is contained in the radical axis of the circles with centers V_1 and V_5 and the radii ε and $\lambda + \varepsilon$, respectively; this axis is orthogonal to the side V_1V_5. Likewise, the segment P_2P_3 is contained in the radical axis of the circles with centers V_4 and V_3 and the radii ε and $\lambda + \varepsilon$, respectively, etc. Then the tangent segments to γ from every point of Γ are equal.

Remark 5 It is interesting to investigate what happens if one imposes additional assumptions on the curve Γ. For example, is it true that if Γ is a circle then γ also must be a circle? We do not know the answer to this question.

We remark that the existence of planar bodies (different from the circle) floating in equilibrium in all positions, [8], imply that there exist non-trivial pairs of smooth strictly convex curves Γ and γ with the desired equal tangent property, and moreover, the length of the tangent segments is constant for all points of Γ. In this setting, one can prove that if Γ is the boundary of a body which floats in equilibrium in all positions and γ (the boundary of its floating body) is homothetic to Γ then the curves are concentric circles. We do not dwell on the proof here.

2 In Euclidean space

In this section we shall see that the situation in Euclidean 3-space is very different from the plane.

Let M be a C^2-smooth strictly convex closed surface in $\mathbb{R}^3$. Denote by H the set of points x in the exterior of M such that all the tangent segments from x to M have equal lengths. The following theorem states that M is a sphere, provided H is large enough.

Theorem 6 *Suppose that H is*
(i) a closed surface containing M in its interior;
(ii) a plane;
(iii) the union of three distinct lines.
Then M is the sphere.

Proof. Let x be a point outside of M. Denote by γ_x the curve on M consisting of the contact points between the tangents to M from x and M. Since all the tangent segments from x to M have the same length, the curve γ_x belongs to a sphere $S(x)$ centered at x. Hence γ_x is a line of curvature of $S(x)$. The surfaces M and $S(x)$ are orthogonal along the curve γ_x. Therefore, by Joachimstahl's theorem[1], γ_x is also a line of curvature of M.

The idea of the proof is to show that almost every (and then, by continuity, every) point of M is umbilic. Through a non-umbilic point there pass exactly two lines of curvature, so if one has three such lines through a point then this point is umbilic.

To prove (i) and (ii), pick a point $p \in M$. Consider the intersection curve of the tangent plane T_pM with the surface H. Choose three points x_1, x_2, x_3 on this curve. Then the curves $\gamma_{x_i},\ i = 1, 2, 3$, are different lines of curvature on M through point p. Hence p is umbilic.

[1]Let two surfaces intersect along a curve γ, and the angle between the surfaces along γ is constant. If γ is a line of curvature on one surface then it is also a line of curvature on the other one.

Likewise, in case (iii), let $p \in M$ be such a point that T_pM intersects each of the three lines that constitute H at a single point. Almost every point p satisfies this condition. Denoting the intersection points by x_1, x_2, x_3, we repeat the argument from the previous paragraph. $\square$

3 Further results

The following optical (or billiard) property of ellipses is well known (see, e.g., [6]). Let $\mathcal{E}$ be an ellipse with the foci P and Q, and let X be a point outside of $\mathcal{E}$. Let ℓ_1 and ℓ_2 be the tangent lines to $\mathcal{E}$ from X. Then the angles between the pairs of lines ℓ_1 and XP, and ℓ_2 and XQ, are equal.

One has the following converse characterization of ellipses, somewhat in the spirit of Lemma 1.

Proposition 7 *Let ℓ be a line tangent to a convex body K in the plane, and let P and Q be two points in the interior of K. For every point X in ℓ consider the other tangent line, L_X, to K. Suppose that the angle between ℓ and XP is equal to the angle between L_X and XQ. Then, K is an ellipse with foci P and Q. See Figure 4.*

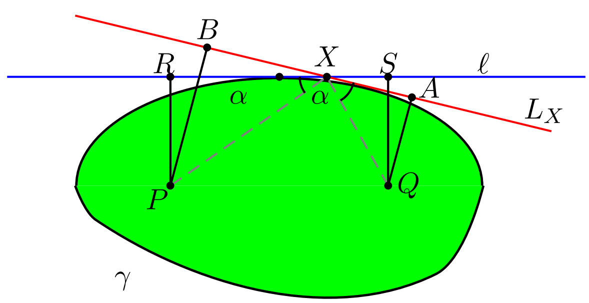

Figure 4.

Proof. Suppose that the angle between L_X and XQ is smaller than the angle between L_X and XP. Let A and B be the projections of Q and P on the line L_X and let R and S be the projections of P and Q on the line ℓ. We conclude from the hypothesis of the proposition that the right triangles $\triangle XQA$ and

$\triangle XPR$ are similar, and so are the right triangles $\triangle XQS$ and $\triangle XPB$. From these similarities we conclude that

$$RP \cdot SQ = PB \cdot QA.$$

Let $\mathcal{E}$ be the ellipse with foci P and Q, tangent to ℓ. Let L'_X be the tangent line to $\mathcal{E}$, parallel to L_X, such that the ray QA intersects L'_X. Let A' and B' be the projections of Q and P on L'_X. Using the optical property of ellipses, one concludes that $RP \cdot QS = PB' \cdot QA'$. It follows that L'_X coincides with L_X. Thus the tangent lines to $\mathcal{E}$ and K from all points of the line ℓ coincide, and hence $K = \mathcal{E}$. $\square$

In conclusion, we remark that Lemma 1 holds in all three classical geometries: elliptic, Euclidean, and hyperbolic; we give a proof that works in all three cases, cf. [5, 7].

In the argument below, a "circle" means a curve of constant curvature. In Euclidean and elliptic geometry this is indeed a circle; in hyperbolic geometry this may be a circle, a horocycle, or an arc with both endpoints at infinity, depending on the value of the curvature.

Let x be a point of $\ell = T_p\gamma$, and let $xy,\ y \in \gamma$, be the other tangent segment to γ from x. Then $|xy| = |xp|$ if and only if there exists a "circle" tangent to γ at p and y.

Consider the family of "circles", tangent to ℓ at point p. In the complement of p, these curves form a smooth foliation $\mathcal{F}$. Since the two tangent segments to γ from every point $x \in \ell$ have equal lengths, the curve γ is everywhere tangent to the leaves of the foliation $\mathcal{F}$. It follows that γ coincides with a leaf, that is, γ is a "circle". Since γ is a closed curve, it is indeed a circle.

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