A HAMILTON-JACOBI THEORY FOR SINGULAR LAGRANGIAN SYSTEMS IN THE SKINNER AND RUSK SETTING

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Abstract. We develop a Hamilton-Jacobi theory for singular lagrangian systems in the Skinner-Rusk formalism. Comparisons with the Hamilton-Jacobi problem in the lagrangian and hamiltonian settings are discussed.

Contents

1. Introduction

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the **principal function**) such that

$$\frac{\partial S}{\partial t} + h(q^A, \frac{\partial S}{\partial q^A}) = 0, \tag{1.1}$$

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where $h = h(q^A, p_A)$ is the hamiltonian function of the system. If we put $S(t, q^A) = W(q^A) - tE$, where E is a constant, then W satisfies

$$h(q^A, \frac{\partial W}{\partial q^A}) = E; \tag{1.2}$$

W is called the **characteristic function**.

Equations (1.1) and (1.2) are indistinctly referred as the **Hamilton-Jacobi equation** (see [1, 2, 23]).

This theory works for classical mechanical systems, where the lagrangian function is usually the kinetic energy corresponding to a Riemannian metric on the configuration manifold minus a potential energy. This is the case of the so-called regular lagrangian systems, that have a well-defined hamiltonian counterpart. The theory has been recently reformulated in a geometrical setting (see [3, 4, 5]) that has permitted its extension to nonholomic mechanical systems [13, 15], and even classical field theories [14, 18].The procedure is based on the comparison of the hamiltonian vector field X_h on the cotangent bundle T^*Q and its projection onto Q via a closed 1-form γ on Q; the result says that both vector fields are γ-related if and if the Hamilton-Jacobi equations $d(h \circ \gamma) = 0$ holds.

On the other hand, a Hamilton-Jacobi theory for singular lagrangian systems is far to be accomplished. There were several attempts ([20, 21, 22]), based on the homogeneization of the given lagrangian, which leads to a new lagrangian system with null energy such that it is possible to discuss the Hamilton-Jacobi equation for the constraints themselves. The main problem is that, due to the integrability condition for the resultant partial differential equation, one can only consider first class constraints. Therefore, the treatment of the cases when second class constraints appear should be developed by *ad hoc* arguments (as in [22], for instance). Thus, in [20] and [21] the authors only discuss the case of primary constraints.

A more modern discussion on this subject can be found in [3, 12], but these authors only consider the case of primary constraints. More recently, in [17] it is proposed a Hamilton-Jacobi theory for arbitrary singular systems that works even if the system exhibit secondary constraints. The strategy is to apply the geometric procedure described above in combination with the constraint algoritm developed by M.J. Gotay and J.M. Nester [7, 8, 9, 10] and that geometrizes the well-known Dirac theory of constraints [6].

In the present paper we take a different approach, and consider the Skinner and Rusk setting to treat with singular lagrangians [24, 25]. Skinner and Rusk have considered a geometrized framework where the velocities and the momenta are independent coordinates. To do this, they considered the dynamics on the Withney sum of TQ (the space of velocities) and T^*Q (the phase space).

Given a lagrangian function $L : TQ \to \mathbb{R}$ (singular or regular, no matter) one considers the bundle $TQ \oplus T^*Q$ with canonical projections $pr_1 : TQ \oplus T^*Q \to TQ$ and $pr_2 : TQ \oplus T^*Q \to T^*Q$ onto the first and second factors. We then define a function $D : TQ \oplus T^*Q \longrightarrow \mathbb{R}$ by $D(X_p, \alpha_p) = \alpha_p(X_p) - L(X_p)$. In bundle coordinates (q^A, v^A, p_A), D is given by $D(q^A, v^A, p_A) = v^A p_A - L(q^A, v^A)$, and it is sometimes refered as the Pontryagin hamiltonian or generalized energy (see [26]). We can also define a 2-form Ω on $TQ \oplus T^*Q$ by $\Omega = pr_2^*(\Omega_Q)$, where Ω_Q denotes the canonical symplectic 2-form of T^*Q.

Then, one discuss the presymplectic system $(TQ \oplus T^*Q, \Omega, dD)$ and obtain the corresponding sequence of constraint submanifolds, which, of course, have a close relation with those obtained by Gotay and Nester on the lagrangian and hamiltonian sides. It should be noticed that this algorithm includes the SODE condition just from the very beginning.

We apply the Hamilton-Jacobi geometric procedure to this presymplectic system and develop the corresponding Hamilton-Jacbi theory. The relation with the Hamilton-Jacobi problems on the lagrangian and hamiltonian sides are extensively discussed.

2. Notation and background

In this work all manifolds are assumed to be finite dimensional and C^∞. Given a function f, the differential at a point p will be indistinctly denoted by $d_p f$ or $df(p)$.

We refer to [19] for a detailed description of lagrangian and hamiltonian mechanical systems.

Let Q be a differentiable manifold and denote by TQ and T^*Q the tangent and cotangent bundles, and by $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ the respective canonical projections on Q.

We introduce two canonical structures on the tangent bundle of a manifold: the vertical endomorfism S, and the Liouville vector field Δ. In bundle coordinates, (q^A, v^A), they are respectively given by

$$S = dq^A \otimes \frac{\partial}{\partial v^A},$$
$$\Delta = v^A \frac{\partial}{\partial v^A}.$$

Let now $L : TQ \to \mathbb{R}$ be a lagrangian on TQ; we can define the Poincaré-Cartan 2-form and the energy function of L by

$$\Omega_L = -d\theta_L, \text{ where } \theta_L = S^*(dL),$$
$$E_L = \Delta(L) - L,$$

which in local coordinates read as

$$\begin{aligned} \theta_L &= \frac{\partial L}{\partial v^A}\, dq^A, \\ \Omega_L &= dq^A \wedge d\frac{\partial L}{\partial v^A}, \\ E_L &= v^A \frac{\partial L}{\partial v^A} - L(q,v). \end{aligned}$$

We look for vector fields ξ which simultaneously satisfy the equations

$$i_\xi\, \Omega_L = dE_L \tag{2.1}$$

$$S\,\xi = \Delta. \tag{2.2}$$

If the lagrangian L is regular, that is, $det(\frac{\partial^2 L}{\partial v^A \partial v^B}) \neq 0$, then the form Ω_L is symplectic (Ω_L has maximal rank) and there exists a unique vector field ξ on TQ which satisfies the equation (2.1). This vector field automatically satisfies the SODE condition (2.2).

If the lagrangian is not regular, then Ω_L is no longer symplectic and equation (2.1) has no solution in general and even if there is a solution it is not necesary a SODE. Therefore for a singular lagrangian L, Ω_L is a presymplectic form (that is, the rank is not maximal, althought, for simplicity, it is assumed that it is constant).

We define the Legendre transformation associated to L as the mapping

$$\begin{array}{rccl} FL: & TQ & \longrightarrow & T^*Q \\ & (q^A, v^A) & \to & FL(q^A, v^A) = (q^A, \frac{\partial L}{\partial v^A}(q^A, v^A)). \end{array}$$

From a direct inspection in local coordinates we know that the Legendre transformation is a local diffeomorfism if and only if L is regular.

We can apply the Gotay-Nester-Hinds algorithm of constraints, see [7, 8, 9], to the presymplectic system $(TQ,\ \Omega_L,\ dE_L)$ and hence we obtain a sequence of constraint submanifolds

$$\cdots P_k \hookrightarrow \cdots \hookrightarrow P_2 \hookrightarrow P_1 = TQ.$$

Assume that the algorithm stabilizes at some step k, say $P_{k+1} = P_k$, which is called the final constraint submanifold, denoted by $P_f = P_k$.

In this paper we will only consider almost regular lagrangians $L : TQ \to \mathbb{R}$, that is:

(i) $M_1 =$Im$(\mathbb{F}L)$ is a submanifold of T^*Q, and
(ii) $FL : TQ \to$ Im$(\mathbb{F}L)$ is a surjective submersion of connected fibers.

Under these assumptions, the energy E_L is projected onto a function $h_1 : M_1 \to \mathbb{R}$ such that $h_1 \circ FL = E_L$

$$\begin{array}{ccc} TQ & \xrightarrow{FL} & T^*Q \\ & \searrow^{FL_1} & \uparrow{\scriptstyle j_1} \\ & & M_1 = \mathrm{Im}(FL) \end{array}$$

Here FL_1 is the restriction of FL to its image, and $j_1 : M_1 \to T^*Q$ is the canonical inclusion.

Next, study the presymplectic system given by $(M_1,\ \Omega_1 = j_1^*\Omega_Q,\ dh_1)$, where Ω_Q is the canonical symplectic form on T^*Q. Therefore, we consider the equation

$$i_Y \, \Omega_1 = dh_1. \tag{2.3}$$

As above we can apply the presymplectic algorithm and we obtain a sequence of constraint submanifolds

$$\cdots M_k \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow T^*Q.$$

It is obvious that

$$FL(P_i) = M_i, \text{ for any } i,$$

and, furthermore, the induced mappings

$$FL_i = FL_{|P_i} : P_i \to M_i$$

are surjective submersions, for all i.

Hence, both algorithms stabilizes at the same step, say k, and then

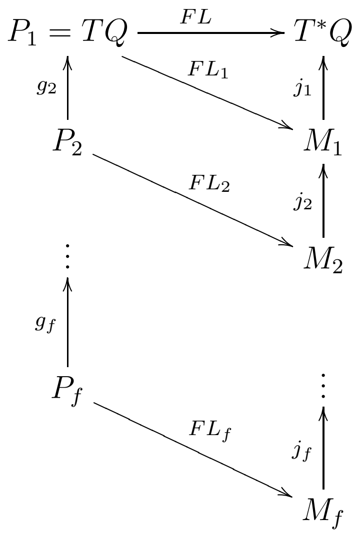
$$FL(P_f) = M_f,$$

and

$$FL_f : P_f \to M_f$$

is a surjective submersion (with the obvious notations).

The following diagram summarizes the above discussion.



where g_i and j_i denote the natural inclusions.

The relation between equations (2.1) and (2.3) is given by the following theorem.

Proposition 2.1. *If* $\xi \in T_pTQ$ *satisfies* (2.1), *then* $TFL(\xi) \in T_{FL(p)}M_1$ *satisfies* (2.3). *Therefore, if* ξ *is a* FL_f*-projectable solution of* (2.1), *then its projection* $TFL_f(\xi)$ *is a solution of* (2.3).

Conversely, if Y *is a solution of* (2.1), *then any* FL_f *projectable vector field on* P_f *which projects on* Y, *is a solution of* (2.3).

Next, we shall discuss the SODE problem as it was stated by M.J. Gotay and J.M. Nester [7, 8].

The results can be summarized in the following result.

Theorem 2.2.

(i) *If* ξ *is a* FL_f*-projectable vector field on* P_f *then for any* $p \in M_f$ *there exists a unique point in each fiber* $FL_f^{-1}(p)$, *denoted by* $\eta_\xi(p)$ *at which* ξ *is a SODE. The point* $\eta_\xi(p)$ *is given by*

$$\eta_\xi(p) = T\tau_Q(\xi(p)).$$

(ii) *The map*

$$\begin{array}{rccl} \beta_\xi : & M_f & \longrightarrow & P_f \\ & p & \to & \beta_\xi(p) = \eta_\xi(p) \end{array}$$

is a section of $FL_f : P_f \to M_f$ *and on* $Im(\beta_\xi)$ *there exists a unique vector field, denoted by* Y_ξ, *which simultaneously satisfies the equations*

$$i_{Y_\xi}\Omega_L = dE_L, \quad SY_\xi = \Delta.$$

We will now recall the construction of the solution of the dynamical equation which simultaneously satisfies the SODE condition. If $Y = (FL_f)_*(\xi)$, then Y is a vector field on M_f satisfying $i_Y \Omega_1 = dh_1$. The vector field Y_ξ described in (ii) is given by

$$Y_\xi(\beta_\xi(p)) = T\beta_\xi(Y(p)), \text{ for all } p \in M_f.$$

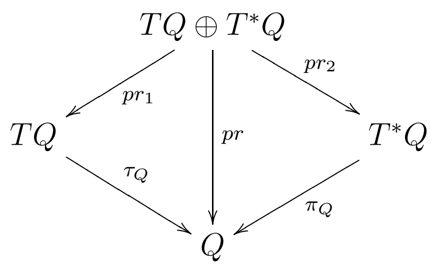
A detailed discussion can be found in [19, 7, 8, 9, 11].

3. The Skinner and Rusk formalism

Skinner and Rusk, [24, 25], have considered a geometrized framework where the velocities and the momenta are independent coordinates. Indeed, they considered the dynamics on the Withney sum of TQ (the space of velocities) and T^*Q (the phase space).

In this section we will briefly recall the Skinner and Rusk formalism.

Let Q be a differentiable manifold and $L : TQ \to \mathbb{R}$ a lagrangian. We can consider the bundle $TQ \oplus T^*Q$ given by the Withney sum of $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$. We will denote by $pr_1 : TQ \oplus T^*Q \to TQ$ and $pr_2 : TQ \oplus T^*Q \to T^*Q$ the projections onto the first and second factors, and by $pr : TQ \oplus T^*Q \to Q$ the projection onto Q. We then have the following commutative diagram



(3.1)

We can define a function

$$\begin{array}{rccl} D: & TQ \oplus T^*Q & \longrightarrow & \mathbb{R} \\ & (X_p, \alpha_p) & \to & D(X_p, \alpha_p) = \alpha_p(X_p) - L(X_p). \end{array}$$

In bundle coordinates (q^A, v^A, p_A), D is given by $D(q^A, v^A, p_A) = v^A p_A - L(q^A, v^A)$. The function D is sometimes refered as the Pontryagin hamiltonian or generalized energy (see [26]).

We can define a 2-form Ω on $TQ \oplus T^*Q$ by $\Omega = pr_2^*(\Omega_Q)$, where Ω_Q denotes the canonical symplectic 2-form of T^*Q.

Next, we can consider the presymplectic system given by $(W_0 = TQ \oplus T^*Q,\ \Omega,\ dD)$ and study the equation

$$i_X \Omega = dD, \tag{3.2}$$

applying the Gotay-Nester-Hinds algorithm of constraints. Hence, we obtain

$$W_1 = \{x \in W_0 \text{ such that there exists } X \in T_x W_0 \text{ satisfying } i_X \Omega = dD\}.$$

In canonical coordinates (q^A, v^A, p_A), we have

$$\begin{aligned} \Omega &= dq^A \wedge dp_A, \\ dD &= -\frac{\partial L}{\partial q^A} dq^A + (p_A - \frac{\partial L}{\partial v^A}) dv^A + v^A dp_A. \end{aligned}$$

So, given a tangent vector $X = a^A \frac{\partial}{\partial q^A} + b^A \frac{\partial}{\partial v^A} + c^A \frac{\partial}{\partial p_A} \in T_{(q^A, v^A, p_A)} W_0$ we deduce that

$$i_X \Omega = -c^A dq^A + a^A dp_A$$

and (3.2) is equivalent to the following conditions

$$\begin{aligned} a^A &= v^A, \\ c^A &= -\frac{\partial L}{\partial q^A}, \\ p^A - \frac{\partial L}{\partial v^A} &= 0, \quad 1 \leq A \leq n. \end{aligned} \tag{3.3}$$

Next, we should restrict the dynamics to $W_1 = \{(q^A, v^A, p_A) \in W_0$ such that $p_A = \frac{\partial L}{\partial v^A}\}$, that is, $W_1 = \text{graph}(FL)$, where $FL : TQ \to T^*Q$ has been defined in section 2.

Accordingly with the Gotay-Nester-Hinds algorithm, a solution X must be tangent to W_1. Assume that such X has the local expression

$$X = \overline{a}^A \frac{\partial}{\partial q^A} + \overline{b}^A \frac{\partial}{\partial v^A} + \left(\frac{\partial^2 L}{\partial v^A \partial q^B} \overline{a}^B + \frac{\partial^2 L}{\partial v^A \partial v^B} \overline{b}^B\right) \frac{\partial}{\partial p_A} \tag{3.4}$$

Then, taking into account (3.3) and (3.4),we deduce

$$\begin{aligned} &\overline{a}^A = v^A \\ &\frac{\partial^2 L}{\partial v^A \partial q^B} v^B + \frac{\partial^2 L}{\partial v^A \partial v^B} \overline{b}^B = -\frac{\partial L}{\partial q^A}. \end{aligned} \tag{3.5}$$

If there exists such a vector field X tangent to W_1, satisfying the above conditions, we have done, and the final constraint manifold W_f is just W_1. For instance, if the lagrangian is regular, $det(\frac{\partial^2 L}{\partial v^B \partial v^A}) \neq 0$, we can compute $\overline{b}^A$ explicitly. If we denote by C_{AB} the matrix $C_{AB} = \left(\frac{\partial^2 L}{\partial v^B \partial v^A}\right)$ and C^{AB} its inverse, then

$$\overline{b}^A = -C^{AB} \left(v^A \frac{\partial^2 L}{\partial v^B \partial q^A} - \frac{\partial L}{\partial q^A} \right).$$

Otherwise, we need to continue the process, and then we obtain a sequence of submanifolds

$$\ldots \hookrightarrow W_k \hookrightarrow \ldots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = TQ \oplus T^*Q.$$

If the algorithm stabilizes, that is, there exists k such that $W_k = W_{k+1}$, then W_k is called the final constraint submanifold and denoted by W_f.

4. A Hamilton-Jacobi theory in the Skinner-Rusk setting

In this section we will develop a Hamilton-Jacobi theory in the Skinner-Rusk formalism. We will use the same notation introduced in the previous sections and discuss separately the regular and the singular cases.

4.1. **The regular case.** Assume that we begin with a regular lagrangian $L : TQ \to \mathbb{R}$. Then, $W_f = W_1$.

A section of $TQ \oplus T^*Q$ is given by $\sigma = (Z, \gamma)$ where Z and γ are a vector field and a 1-form on Q, respectively. Assume that σ satisfies the following conditions

(i) $\mathrm{Im}(\sigma) \subset W_1 = \mathrm{graph}(FL)$, and
(ii) $d(pr_2 \circ \sigma) = d\gamma = 0$.

Then, by the regularity of L, we know that there exists a unique vector field on W_1, say X, satisfying

$$i_X \Omega = dD,$$

and then we can define a vector field on Q by

$$X^\sigma(p) = Tpr(X(\sigma(p))), \text{ for all } p \in Q.$$

Now we have the following proposition.

Proposition 4.1. *Under the previous conditions, $d(D \circ \sigma) = 0$ if and only if the vector fields X and X^σ are σ-related.*

Proof.
"$\Rightarrow$"

Assume that $d(D \circ \sigma) = 0$ holds, then we will prove first that $(i_{(X - T\sigma(X^\sigma))}\Omega = 0)_{|\mathrm{Im}(\sigma)}$.

It is clear that if $x \in \mathrm{Im}(\sigma)$ then $T_x(TQ \oplus T^*Q) = T_x\mathrm{Im}(\sigma) + V$, where V denotes the vertical bundle of the projection $pr : TQ \oplus T^*Q \to Q$. We will show that $i_{(X - T\sigma(X^\sigma))}\Omega$ anihilates $T_x\mathrm{Im}(\sigma)$ and V. Indeed, by the definition of Ω, it is obvious that Ω vanishes acting on two elements of V. Since $X - T\sigma(X^\sigma)$ is vertical, we have

$$\left(i_{(X - T\sigma(X^\sigma))}\Omega\right)(V) = 0.$$

Given $p \in Q$, since X is a solution on W_1, we get

$$(i_{X(p)}\Omega) \circ T\sigma(p) = TD(\sigma(p)) \circ T\sigma(p) = T(D \circ \sigma)(p).$$

On the other hand, $(i_{T\sigma(X^\sigma(p))}\,\Omega)\circ T\sigma(p)=0$ since for any $Y\in T_pQ$ we have

$$
\begin{aligned}
&(i_{T\sigma(X^\sigma(p))}\,\Omega)\,(T\sigma(p)(Y)) = \Omega(T\sigma(X^\sigma(p)),T\sigma(Y))\\
&=\Omega(T\sigma(X^\sigma(p)),T\sigma(Y)) = pr_2^*(\Omega_Q)(T\sigma(X^\sigma(p)),T\sigma(Y))\\
&=(\Omega_Q)(Tpr_2\circ T\sigma(X^\sigma(p)),Tpr_2\circ T\sigma(Y)) = (\Omega_Q)(T\gamma(X^\sigma(p)),T\gamma(Y))\\
&=-d\gamma(T\gamma(X^\sigma(p)),T\gamma(Y))\\
&=0
\end{aligned}
$$

and so, we conclude that

$$\left(i_{(X-T\sigma(X^\sigma))}\Omega\right)(T\mathrm{Im}(\sigma))_{|\mathrm{Im}(\sigma)}=0,$$

which implies

$$
\begin{aligned}
&\left(i_{(X-T\sigma_f(X^\sigma))}\Omega\right)(V+T\mathrm{Im}(\sigma))_{|\mathrm{Im}(\sigma)}\\
&=\left(i_{(X-T\sigma_f(X^\sigma))}\Omega\right)(T(TQ\oplus T^*Q))_{|\mathrm{Im}(\sigma)}=0.
\end{aligned}
$$

Therefore $(X-T\sigma(X^\sigma))\in\ker(\Omega)$. This means that $i_{(X-T\sigma(X^\sigma))}\,\Omega=0$, and hence $\iota^*\left(i_{(X-T\sigma(X^\sigma))}\,\Omega\right)=i_{(X-T\sigma(X^\sigma))}\,(i^*\Omega)=0$, where $\iota: W_1\to W_0$ is the inclusion.

It is not hard to see, that if L is regular then $i^*\Omega$ is symplectic and so $(X=T\sigma(X^\sigma))_{|\mathrm{Im}(\sigma)}$.

"$\Leftarrow$" Since $\left((i_{(X-T\sigma(X^\sigma))}\Omega)\circ T\sigma=d(D\circ\sigma)\right)$, if $X=T\sigma(X^\sigma)$, then $d(D\circ\sigma)=0$. □

4.2. **The singular case.** Assume now that $L:TQ\to\mathbb{R}$ is an almost regular singular lagrangian.

Suppose that the algorithm of Gotay-Nester-Hinds applied to $(W_0=TQ\oplus T^*Q,\ \Omega,\ dD)$ stabilizes at a final constraint submanifold W_f. By construction, there exists at least one vector field X on W_f such that

$$(i_X\,\Omega=dD)_{|W_f}$$

We need some regularity conditions, thus we will also assume that $Q_i=pr(W_i)$ are submanifolds and that $pr_i=pr_{|W_i}:W_i\to Q_i$ are submersions.

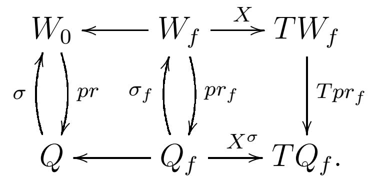
A section of $pr:TQ\oplus T^*Q\to Q$ is given by $\sigma=(Z,\gamma)$, where Z and γ are respectively a vector field and a 1-form on Q. We will denote by σ_f the restriction of σ to $Q_f=pr(W_f)$ of σ. Suppose that σ verifies the following conditions:

(i) $\mathrm{Im}(\sigma)\subset W_1$.
(ii) $\mathrm{Im}(\sigma_f)\subset W_f$.
(iii) $d(pr_2\circ\sigma)=d\gamma=0$, that is, γ is closed.

Using σ we can define a vector field on Q_f by

$$X^\sigma(p)=Tpr(X(\sigma_f(p))),\quad p\in Q_f.$$

The construction is illustrated in the following diagram



The relation between $T\sigma_f(X^\sigma)$ and X is shown in the following theorem.

Proposition 4.2. *The conditions*

$$d(D \circ \sigma)_{|Q_f} = 0$$

and

$$(X - T\sigma_f(X^\sigma) \in \ker(\Omega))_{|Im(\sigma_f)}$$

are equivalent.

Proof. The proof follows by similar arguments as in Proposition 4.1. □

Definition 4.3. *A section σ of $TQ \oplus T^*Q$, $\sigma = (Z, \gamma)$, satisfying the following conditions*

(i) $Im(\sigma) \subset W_1$.
(ii) $Im(\sigma_f) \subset W_f$.
(iii) $d(pr_2 \circ \sigma) = d\gamma = 0$.
(iv) $d(D \circ \sigma)_{|Q_f} = 0$

will be called a solution of ***the Hamilton-Jacobi problem for the lagrangian*** L ***in the Skinner-Rusk setting****.*

Remark 4.4. The last proposition says that $T\sigma_f(X^\sigma)$ is a vector field along $\mathrm{Im}(\sigma_f)$ which is also a solution of the equation (3.2). So if we find an integral curve $c(t)$ of X^σ on Q_f, then $(\sigma_f \circ c)(t)$ is an integral curve of a solution of (3.2). ⋄

Remark 4.5. The natural question is if X and X^σ are σ_f-related in the singular case, as it happens in the standard Hamilton-Jacobi theory, see [17]. The answer is that, as we discussed later (section 6), in some cases the fields are not necessarily σ_f-related. ⋄

5. Comparison with the Hamiltonian and lagrangian settings

In the previous section we have developed a Hamilton-Jacobi theory in the Skinner-Rusk setting. The Skinner-Rusk formalism unifies lagrangian and hamiltonian formalisms, so we would like to relate the present Hamilton-Jacobi theory to the corresponding ones for the two formalisms (see [17]).

5.1. **The hamiltonian setting.**

5.1.1. *The* ***regular case****.* If the lagrangian, L, is regular, that is, FL is a local diffeomorfism, then we can define locally a hamiltonian function $h : T^*Q \to \mathbb{R}$ by $h = E_L \circ FL^{-1}$. Let us now assume that the lagrangian is hyperregular, that is, FL is a global diffeomorfism and h is globally defined. Denote by X_h the corresponding hamiltonian vector field

$$i_{X_h} \Omega_Q = dh.$$

Let γ be a closed 1-form on Q; then we can define a vector field on Q by

$$X^\gamma(p) = T\pi_Q(X_h(\gamma(p))) \text{ for all } p \in Q.$$

Then we have the following Hamilton-Jacobi theorem.

Proposition 5.1. *The vector fields X and X^γ are γ-related if and only if $d(h \circ \gamma) = 0$.*

Proof. For a proof see [1]. □

5.1.2. *The singular case.* Since we are considering an almost regular lagrangian $L : TQ \to \mathbb{R}$, then we can apply the Dirac theory of constraints developed in Section 2.

We have to study the presymplectic system given by $(M_1,\ \Omega_1 = j_1^*\Omega_Q,\ dh_1)$, where $j_1 : M_1 \to T^*Q$ is the inclusion and h_1 is defined implicitly by $h_1 \circ FL = E_L$.

If we apply the Gotay-Nester-Hinds algorithm, we obtain a sequence

$$\cdots M_k \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow T^*Q;$$

assume that we obtain a final constraint submanifold, denoted by M_f. We also assume that $Q_i = \pi_Q(M_i)$ are submanifolds and that $\pi_i = \pi_{Q|M_i} : M_i \to Q_i$ are submersions.

Remark 5.2. It is important to notice that the algorithm of Gotay-Nester-Hinds applied to the same lagrangian in the Skinner-Rusk setting and in the corresponding hamiltonian setting does not necessary stop at the same level. For example, the lagrangian given by $L(q^1, q^2, v^1, v^2) = v^1 q^2$ produces the two presymplectic systems (M_1, Ω_1, dh_1) and $(W_0 = TQ \oplus T^*Q, \Omega, dD)$. The first algorithm stabilizes in $k = 1$, but the second one does in $k = 2$.

◇

Let γ be a 1-form on Q satisfying the following conditions:

(i) $\mathrm{Im}(\gamma) \subset M_1$.
(ii) $\mathrm{Im}(\gamma_f) \subset M_f$, where γ_f denotes the restriction to Q_f of γ.
(iii) $d\gamma = 0$.

Then, if Y is a vector field on M_f solving the equation $i_Y\,\Omega_1 = dh_1$, we can construct the vector field Y^γ on Q_f biven by

$$Y^\gamma(p) = T\pi_Q(Y(\gamma_f(p))), \qquad \text{for each } p \in Q_f$$

and obtain an analogous of theorem 4.2 (notice that in this case we can ensure that the vector fields are γ_f-related, see [17] for the details).

Proposition 5.3. *We have*

$$d(h_1 \circ \gamma)_{|Q_f} = 0 \Leftrightarrow Y \text{ and } Y^\gamma \text{ are } \gamma_f\text{-related.}$$

Proof. Given $q \in Q_f$, we have

$$\left(i_{\left(Y(\gamma(q)) - T_q\gamma_f(Y^\gamma(q))\right)}\,\Omega_1\right) \circ T_q\gamma = i_{Y(\gamma(q))}\,\Omega_1 \circ T_q\gamma - i_{T_q\gamma_f(Y^\gamma(q))}\,\Omega_1 \circ T_q\gamma$$
$$= d_{\gamma_f(q)}h_1 \circ T_q\gamma = d_q(h_1 \circ \gamma)$$

where we have $T_q\gamma_f(Y^\gamma) = T_q\gamma(Y^\gamma)$ and

$$i_{T_q\gamma_f(Y^\gamma(q))}\,\Omega_1 \circ T_q\gamma(Y(q)) = \Omega_1(T_q\gamma(Y^\gamma), T_q\gamma(Y(q)))$$
$$= (\gamma^*\Omega_1)(Y^\gamma(q), Y(q)) = d\gamma(Y^\gamma(q), Y(q)) = 0,$$

for all $Y_q \in T_qQ$.

The previous discussion can be applied to every point $q \in Q_f$; therefore, taking into account that Ω_1 vanishes acting on two vertical tangent vectors, we can deduce the following

$$Y - T\gamma_f(Y^\gamma) \in \ker(\Omega_1) \Leftrightarrow d(h_1 \circ \gamma)_{|Q_f} = 0.$$

As we did before, we will see that Y and Y^γ are γ_f related.

Remember that for any point p of M_1 we have a decomposition

$$T_p(T^*Q) = T_pM_1 + V_p(T^*Q),$$

where $V(T^*Q)$ denotes as above the space of vertical tangent vectors on p.

Since $Y - T\gamma_f(Y^\gamma)$ is vertical at the points of $\mathrm{Im}(\gamma_f)$, given any $U \in V_p$, $p \in \mathrm{Im}(\gamma_f)$, then

$$\Omega_Q(Y - T\gamma(Y^\gamma), U) = 0$$

Now, given $U \in T_pM_1$ we get

$$\Omega_Q(Y - T\gamma_f(Y^\gamma), U) = \Omega_1(Y - T\gamma_f(Y^\gamma), U) = 0$$

because $(Y - T\gamma_f(Y^\gamma)) \in \ker(\Omega_1)$, and hence $\Omega_Q(Y - T\gamma_f(Y^\gamma), Z) = 0$ for any tangent vector $Z \in T_p(T^*Q)$ at any point of $\mathrm{Im}(\gamma_f)$. Since Ω_Q is non-degenerate, we deduce that $Y = T\gamma_f(Y^\gamma)$ along $\mathrm{Im}(\gamma_f)$.

□

Definition 5.4. *A* 1*-form* γ *satisfying the previous conditions will be called a solution of the* ***Hamilton-Jacobi problem for*** L ***in the hamiltonian setting****.*

We are now going to relate the Hamilton-Jacobi problem in the Skinner-Rusk setting and the corresponding one in the hamiltonian setting. First, the following result gives the relation between W_i and M_i, and also a relation between solutions of equations (2.3) and (3.2).

Lemma 5.5.
(i) *If* $X \in T_pW_1$ *satisfies* $i_X\Omega = dD$, *then* $X_2 = Tpr_2(X) \in T_{pr_2(p)}M_1$ *satisfies* $i_{X_2}\Omega = dh_1$.
(ii) *For each step* k *of the constraint algorithms applied to the presymplectic systems* $(M_1,\ \Omega_1,\ dh_1)$ *and* $(W_0 = TQ \oplus T^*Q,\ \Omega,\ dD)$ *we have*

$$pr_2(W_k) \subset M_k,$$

and, if we denote the respective final constraint submanifolds by W_f *and* M_f, *then*

$$pr_2(W_f) = M_f.$$

(iii) *We have* $pr(W_f) = \pi_Q(M_f) = Q_f$.

Proof.
(i) Recall that a vector $\xi \in T_{(q^A,v^A)}TQ$, $\xi = u^A\frac{\partial}{\partial q^A} + w^A\frac{\partial}{\partial v^A}$ satisfies $i_\xi\Omega_L = dE_L$ iff

$$\frac{\partial^2 L}{\partial v^A \partial v^B}(v^B - u^B) = 0$$
$$\frac{\partial^2 L}{\partial v^A \partial v^B}u^B + \frac{\partial^2 L}{\partial v^A \partial q^B}w^B - \frac{\partial L}{\partial q^A} = \frac{\partial^2 L}{\partial v^B \partial q^A}(v^B - u^B)$$

If $X \in T_pW_1$ verifies $i_X\Omega = dD$, then X has the expression (3.4) and satisfies (3.5). So, it is clear that $X_1 = Tpr_1(X)$ satisfies $i_{X_1}\Omega_L = dE_L$. Since X is tangent to W_1, $X_2 = Tpr_2(X) = TFL \circ Tpr_1(X) = TFL(X_1)$ and using Proposition 2.1 we can conclude that $i_{X_2}\Omega_1 = dh_1$.

(ii) It will be proved by induction.

For $k = 1$ we have $pr_2(W_1) = M_1$ since $W_1 = \mathrm{graph}(FL)$.

Assume that $pr_2(W_k) \subset M_k$. Then

$$W_{k+1} = \{x \in W_k \text{ such that there exists } X \in T_xW_k \text{ satisfying } i_X\Omega = dD\}$$
$$M_{k+1} = \{y \in M_k \text{ such that there exists } Y \in T_yM_k \text{ satisfying } i_Y\Omega_1 = dh_1\}.$$

If $x \in W_{k+1}$, then there exists $X \in T_xW_k$, satisfying $i_X\Omega = dD$. Since $pr_2(W_k) \subset M_k$, $Tpr_2(X) \in TM_k$ and by (i) $i_{Tpr_2(X)}\Omega_1 = dh_1$. Thus, we have proved that $pr_2(x) \in M_{k+1}$ and that $pr_2(W_k) \subset M_k$.

To prove that $pr_2(W_f) = M_f$, take a solution Y of equation 2.3 on M_f. Then we can construct a vector field ξ on P_f which is FL_f-related with Y, and using Theorem 2.2 we obtain a vector field Y_ξ along the image of the section β_ξ which satisfies (2.1) and (2.2). We can construct the map

$$\begin{array}{rccc} \overline{\beta}_\xi : & M_f & \longrightarrow & TQ \oplus T^*Q \\ & (q^A, p^A) & \to & (\beta_\xi(q^A, p_A), (q^A, p_A)). \end{array}$$

It is easy to see, that the vector field $T\overline{\beta}_\xi(Y)$ on $\mathrm{Im}(\overline{\beta}_\xi)$ is a solution of (3.2). By the maximality of the final constraint manifold W_f, we can conclude that $\mathrm{Im}(\overline{\beta}_\xi) \subset W_f$, but $M_f = pr_2(\mathrm{Im}(\overline{\beta}_\xi)) \subset pr_2(W_f) \subset M_f$ and then the result follows.

(iii) It is a direct consequence of (ii) and the commutativity of diagram (3.1). □

A solution of the Hamilton-Jacobi problem as stated in the previous section is given by a section σ of $TQ \oplus T^*Q$, so $\sigma = (Z, \gamma)$, where Z and γ are a vector field and a 1-form on Q, respectively.

We will see that γ satisfies the Hamilton-Jacobi problem in the hamiltonian sense.

From the fact that σ is a solution of the Hamilton-Jacobi problem in the Skinner-Rusk setting, we deduce:

(i) Since $\mathrm{Im}(\sigma) \subset W_1$, then $\mathrm{Im}(\gamma) = pr_2(Im(\sigma)) \subset pr_2(W_1) = M_1$.
(ii) Since $\mathrm{Im}(\sigma_f) \subset W_f$, then $\mathrm{Im}(\gamma_f) = pr_2(Im(\sigma_f)) \subset pr_2(W_f) = M_f$.
(iii) Since $d(pr_2 \circ \sigma) = d\gamma = 0$, then γ is closed.
(iv) Since $\mathrm{Im}(\sigma) \subset W_1$, then $D \circ \sigma = h_1 \circ \gamma$ and then, using that $d(D \circ \sigma)_{|Q_f} = 0$, we finally get $d(h_1 \circ \gamma)_{|Q_f} = 0$.

On the other hand, given a vector field X on W_f which is a solution of (3.2), we can obtain a solution of (2.3) along $\mathrm{Im}(\gamma_f)$ by defining

$$X_2(\gamma_f(p)) = Tpr_2(X(\sigma_f(p))), \text{ for all } p \in Q_f.$$

Now, from Lemma 5.5 it follows that X_2 is a solution of (2.3).

As above we can construct the projected vector field on Q_f, by putting

$$X_2^\gamma(p) = T\pi_f(X_2(\gamma_f(p))), \text{ for all } p \in Q_f.$$

Remark 5.6. By the commutativity of the diagram (3.1) we deduce that $pr = \pi_Q \circ pr_2$, and in consequence we have

$$X^\sigma(p) = Tpr(X(\sigma_f(p))) = T\pi_Q \circ pr_2(X(\sigma_f(p))) = T\pi_f(X_2(\gamma_f(p)))$$

for all $p \in Q_f$, and so, $X^\sigma = X_2^\gamma$. ◇

Summarizing the above discussion, we can conclude that it is possible to relate the Hamilton-Jacobi theory in the Skinner-Rusk setting to the Hamilton-Jacobi theory on T^*Q. In this case the vector fields X_2 and X_2^γ are γ_f-related.

5.2. **The lagrangian setting.** In this section we will relate the Hamilton-Jacobi theory developed in the Skinner-Rusk setting with the corresponding one on the lagrangian side

5.2.1. *The regular case.* If the lagrangian L is regular, then we have a symplectic system given by $(TQ,\ \Omega_L,\ E_L)$. Then there exists a unique solution ξ of the equation 2.1 which automatically satisfies the SODE condition.

Given Z a vector field on Q such that $Z^*\Omega_L = 0$ we can define the following vector field on Q

$$\xi^Z(p) = T\tau_Q(\xi(Z(p))) \text{ for all } p \in Q$$

and obtain the following result.

Proposition 5.7. *Under the previous conditions, the vector fields ξ and ξ^Z are Z-related if and only if $d(E_L \circ Z) = 0$.*

Proof. The proof is a consequence of Proposition 5.1. □

5.2.2. *The singular case.* In this case, we will discuss the presymplectic system given by (TQ, Ω_L, dE_L). Applying the Gotay-Nester-Hinds algorithm we obtain a sequence of submanifolds

$$\cdots P_k \hookrightarrow \cdots \hookrightarrow P_2 \hookrightarrow P_1 = TQ.$$

We also assume that $Q_i = \tau_Q(P_i)$ are submanifolds and that $\tau_i = pr_{Q|P_i} : P_i \to Q_i$ are submersions, for any index i.

Remember that the algorithm of Gotay-Nester-Hinds applied to the presymplectic systems $(M_1,\ \Omega_1,\ dh_1)$ and $(TQ,\ \Omega_L,\ dE_L)$ stop at the same step, so we will denote the final constraint manifold of the system (TQ, Ω_L, dE_L) by P_f.

Let Z be a vector field on Q satisfying the following properties:

(i) $\mathrm{Im}(Z_f) \subset P_f$, where Z_f denotes the restriction of Z to Q_f.
(ii) $Z^*\Omega_L = 0$.

Then, if ξ is a vector field on P_f solving the equation $i_\xi\,\Omega_L = dE_L$, we can construct the vector field ξ^Z on Q_f by

$$\xi^Z(p) = T\tau_Q(\xi(Z_f(p))), \text{ for all } p \in Q_f.$$

Now, we can develop the corresponding Hamilton-Jacobi theory in the lagrangian setting.

Proposition 5.8. *Under the above hypothesis for Z we have*

$$d(E_L \circ Z)_{|Q_f} = 0 \Leftrightarrow \big(\xi - TZ_f(\xi^Z)\big) \in \ker(\Omega_L).$$

Proof. "⇒"

Assume that $d(E_L \circ Z)_{|Q_f} = 0$ holds, then we will prove that

$$(i_{\left(\xi - TZ_f(\xi^Z)\right)}\Omega_L = 0)_{|\mathrm{Im}(Z_f)}.$$

For any $x \in \mathrm{Im}(Z_f)$ we have the decomposition $T_x(TQ) = T_x\mathrm{Im}(Z) + V(TQ)$, where $V(TQ)$ denotes the vertical bundle of the projection $\tau_Q : TQ \to Q$.

Since Ω_L vanishes acting on two elements of $V(TQ)$ and $\xi - TZ_f(\xi^Z)$ is vertical, we have

$$\left(i_{\left(\xi - TZ_f(\xi^Z)\right)}\Omega\right)(V(TQ)) = 0$$

Since ξ is a solution along $\mathrm{Im}(Z_f)$, we have

$$\left(i_{\xi(p)}\,\Omega_L\right) \circ T\sigma(p) = d_{Z(p)}E_L \circ TZ_f(p) = d_p(E_L \circ Z)$$

for any $p \in Q_f$.

On the other hand, $(i_{TZ_f(\xi^Z(p))}\,\Omega_L) \circ TZ(p) = 0$, since for any $Y \in T_pQ$ we get

$$\begin{aligned}&\left(i_{\left(\xi - TZ_f(\xi^Z)\right)}\,\Omega_L\right) \circ TZ(p)(Y) = \Omega_L(TZ_f(\xi^Z(p)), TZ(Y))\\ &= (Z^*\Omega_L)\left((\xi^Z, Y\right) = -d\gamma(\xi^Z, Y) = 0\end{aligned}$$

and so we can conclude that

$$\left(i_{\left(\xi - TZ_f(\xi^Z)\right)}\Omega_L\right)(T\mathrm{Im}(Z)) = 0$$

"$\Leftarrow$"

Since $i_{\left(\xi - TZ_f(\xi^Z)\right)}\Omega_L = d(E_L \circ Z)$, if $(\xi - TZ_f(\xi^Z)) \in \ker(\Omega_L)$, then $d(E_L \circ Z)_{|Q_f} = 0$ □

Definition 5.9. *A vector field on Q, Z satsifying the previous conditions will be called a solution of the* ***Hamilton-Jacobi problem for*** L ***in the lagrangian setting****.*

The vector fields ξ and ξ^Z are not necessarily related as the next example shows.

Example 5.10. Let $L : T\mathbb{R}^2 \to \mathbb{R}$ be the lagrangian given by

$$L(q^1, q^2, v^1, v^2) = q^1\,v^2 + q^2\,v^1$$

We have

$$\begin{aligned}&FL(q^1, q^2, v^1, v^2) = (q^1, q^2, q^2, q^1),\\ &E_L(q^1, q^2, v^1, v^2) = q^1\,v^2 + q^2\,v^1 - q^1\,v^2 - q^2\,v^1 = 0,\\ &\Omega_L = 0,\end{aligned}$$

so every vector field ξ on $T\mathbb{R}^2$ satifies

$$i_\xi\,\Omega_L = dE_L.$$

Therefore, the algorithm of Gotay-Nester-Hinds stabilizes at the first step, and $P_f = P_1 = TQ$.

Moreover, every vector field Z on $\mathbb{R}^2$ is a solution of the Hamilton-Jacobi problem, since $E_L \circ Z = 0$ and $Z^*\Omega_L = 0$.

Let ξ be the solution satisfying the SODE condition given by

$$\xi(q^1, q^2, v^1, v^2) = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + \frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^2}$$

Let Z be

$$Z(q^1, q^2) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}$$

An easy computation shows that

$$TZ(Z(q^1, q^2)) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2},$$

but

$$\xi(Z(q^1, q^2)) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + \frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^2} \neq TZ(Z(q^1, q^2)).$$

Thus, the vector fields ξ and ξ^Z are not Z-related.

Next we will show that a solution of the Hamilton-Jacobi problem in the Skinner-Rusk formalism induces a solution of the Hamilton-Jacobi theory in the lagrangian setting.

The following lemma is analogous to Lemma 5.5.

Lemma 5.11.

(i) *If $X \in T_pW_1$ satisfies $i_X \Omega = dD$, then $X_2 = Tpr_1(X)$ satisfies $i_{X_1} \Omega_L = dE_L$ and the SODE condition* (2.2).

(ii) *For each step k of the constraint algorithm applied to the presymplectic systems $(M_1,\ \Omega_1,\ dh_1)$ and $(W_0 = TQ \oplus T^*Q,\ \Omega,\ dD)$, we have*

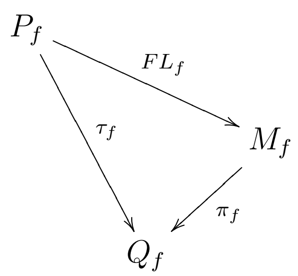
$$pr_2(W_k) \subset P_k$$

(iii) *We have $pr(W_f) = \tau_Q(P_f)$*

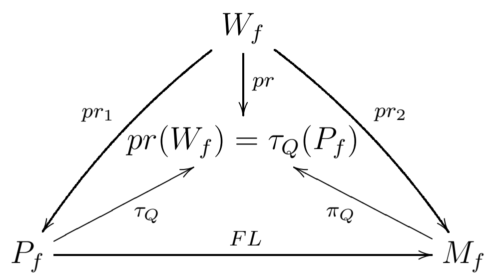
Proof.

(i) and (ii) are proved using similar arguments to that in Lemma 5.5.

(iii) Since the following diagram



is commutative, and FL_f is a surjective submersion, we deduce that $\pi_Q(M_f) = \tau_Q(P_f)$. By Lemma 5.5 (iii), we obtain $\pi_Q(M_f) = pr(W_f)$, and the result follows. The situation can be summarized in the following commutative diagram



□

If $\sigma = (Z, \gamma)$ is a solution of the Hamilton-Jacobi problem, we deduce the following results:

(i) Since $\mathrm{Im}(\sigma_f) \subset W_f$, then $pr_1(\mathrm{Im}(\sigma_f)) \subset pr_1(W_f) \subset P_f$.
(ii) We have $Z^*\Omega_L = 0$, since $Z^*\Omega_L = Z^*(d\theta_L) = d(Z^*\theta_L) = d(FL(Z)) = d\gamma = 0$.
(iii) Since $\mathrm{Im}(\sigma) \subset W_1$, then $D \circ \sigma(p) = E_L \circ Z(p)$ and, because $d(D \circ \sigma)_{|Q_f} = 0$, then $d(E_L \circ Z)_{|Q_f} = 0$.

Now, given a solution X of (3.2), we can obtain a solution of (2.1) along $\mathrm{Im}(Z_f)$ using Lemma 5.11, and putting

$$X_1(Z_f(p)) = Tpr_1(X(\sigma(p))), \text{ for all } p \in Q_f$$

We can also define the vector field on Q_f given by

$$X_1^Z(p) = T\tau_Q(X_1(Z_f(p))).$$

The vector fields X_1 and X_1^Z are not Z_f-related in general, as we have proved in example 5.10.

Remark 5.12. By the commutativity of diagram (3.1) we have $pr = \tau_Q \circ pr_1$ and hence

$$X^\sigma(p) = Tpr(X(\sigma_f(p))) = T\tau_Q \circ pr_1(X(Z_f(p))) = T\tau_Q(X_1(Z_f(p))),$$

for all $p \in Q_f$, and so $X^\sigma = X_1^Z$.

Moreover, since X_1 satisfies the SODE condition, then

$$X_1^Z(p) = T\tau_Q(X_1(Z(p))) = \tau_{TQ}(X_1(Z(p))) = Z(p) = Z_f(p),$$

and we have

$$X^\sigma = X_1^Z = X_2^\gamma = Z_f.$$

Note that this means that we only need to compute X_2^γ to obtain Z_f. ◇

6. Final considerations

In the last section we show that a solution of the Hamilton-Jacobi problem in the Skinner-Rusk setting, $\sigma = (Z, \gamma)$, gives a solution of the Hamilton-Jacobi problem in the lagrangian and hamiltonian settings (Z and γ respectively). A solution of the equation 3.2 along $\mathrm{Im}(\sigma)$ can be also projected to solutions of 2.1 and 2.3 along $\mathrm{Im}(Z)$ and $\mathrm{Im}(\gamma)$, denoted respectively by X_1 and X_2.

If we take a vector field X solution of the equation (3.2) on W_f, using σ we can compute X^σ. Now we can easily conclude that the vector fields X and X^γ are σ_f related iff the corresponding vector fields X_1 and X_1^Z are Z_f related in the lagrangian setting.

To illustrate the above results we revisite example 5.10 in the Skinner-Rusk setting and apply the corresponding Hamilton-Jacobi theory.

Example 6.1. Consider the lagrangian given in Example 5.10

$$L(q^1, q^2, v^1, v^2) = q^1\, v^2 + q^2\, v^1 .$$

Then, on $T\mathbb{R}^2 \oplus T^*\mathbb{R}^2$ we have

$$D(q^1, q^2, v^1, v^2, p_1, p_2) = v^1 p_1 + v^2 p_2 + v^1 q^2 + v^2 q^1 ,$$

and hence

$$\begin{aligned} dD(q^1, q^2, v^1, v^2, p_1, p_2) = \quad & -v^2 dq^1 - v^1 dq^2 + (p_1 - q^2) dv^1 \\ & +(p_1 - q^1) dv^2 + v^1 dp_1 + v^2 dp_2 \end{aligned} \tag{6.1}$$

Recall that we must compute

$$\begin{aligned} W_1 = \quad & \{(q^A, v^A, p_A) \text{ such that there exists } X \in T_{(q^A, v^A, p_A)} T\mathbb{R}^2 \oplus T^*\mathbb{R}^2 \\ & \text{satisfying } i_X\, \Omega = dD\}. \end{aligned}$$

If

$$X = a^1 \frac{\partial}{\partial q^1} + a^2 \frac{\partial}{\partial q^2} + b^1 \frac{\partial}{\partial v^1} + b^2 \frac{\partial}{\partial v^2} + c^1 \frac{\partial}{\partial p_1} + c^2 \frac{\partial}{\partial p_2} \tag{6.2}$$

then

$$i_X\, \Omega = -c^1 dq^1 - c^2 dq^2 + a^1 dp_1 + a^2 dp_2 \tag{6.3}$$

and so

$$a^1 = v^1, \quad a^2 = v^2, \quad c^1 = v^2, \quad c^2 = v^1, \quad p_1 - q^2 = 0, \quad p_2 - q^1 = 0 \tag{6.4}$$

must hold.

Therefore, $W_1 = \{(q^1, q^2, v^1, v^2, q^2, q^1) \text{ such that } q^A,\ v^A \in \mathbb{R}\} = \mathrm{graph}(FL)$.

Next, we compute

$$\begin{aligned} W_2 = \quad & \{(q^1, q^2, v^1, v^2, q^2, q^1) \in W_1 \text{ such that there exists} \\ & X \in T_{(q^1, q^2, v^1, v^2, q^2, q^1)} W_1 \text{ satisfying } i_X\, \Omega = dD\} \end{aligned}$$

If $X \in TW_1$ then X can be locally expressed as

$$\begin{aligned} X &= a^1 \tfrac{\partial}{\partial q^1} + a^2 \tfrac{\partial}{\partial q^2} + b^1 \tfrac{\partial}{\partial v^1} + b^2 \tfrac{\partial}{\partial v^2} \\ &+ (\tfrac{\partial^2 L}{\partial v^1 \partial q^1} a^1 + \tfrac{\partial^2 L}{\partial v^1 \partial q^2} a^2 + \tfrac{\partial^2 L}{\partial v^1 \partial v^1} + \tfrac{\partial^2 L}{\partial v^2 \partial v^1}) \tfrac{\partial}{\partial p_1} \\ &+ (\tfrac{\partial^2 L}{\partial v^2 \partial q^1} a^1 + \tfrac{\partial^2 L}{\partial v^2 \partial q^2} a^2 + \tfrac{\partial^2 L}{\partial v^2 \partial v^1} b^1 + \tfrac{\partial^2 L}{\partial v^2 \partial v^2} b^2) \tfrac{\partial}{\partial p_2} \\ &= a^1 \tfrac{\partial}{\partial q^1} + a^2 \tfrac{\partial}{\partial q^2} + b^1 \tfrac{\partial}{\partial v^1} + b^2 \tfrac{\partial}{\partial v^2} + a^2 \tfrac{\partial}{\partial p_1} + a^1 \tfrac{\partial}{\partial p_2} \end{aligned} \tag{6.5}$$

Taking into account (6.4) and (6.5), for every point $(q^1, q^2, v^1, v^2, q^2, q^1) \in W_1$ we obtain

$$X = v^1 \frac{\partial}{\partial q^2} + v^2 \frac{\partial}{\partial q^2} + b^1 \frac{\partial}{\partial v^1} + b^2 \frac{\partial}{\partial v^2} + v^2 \frac{\partial}{\partial p_1} + v^1 \frac{\partial}{\partial p_2}$$

for arbitrary b^1, b^2, and so $W_2 = W_1$ and therefore the final constraint submanifold is W_1; consequently, $Q_f = Q$.

Now, a solution of the Hamilton-Jacobi problem in the Skinner-Rusk setting is given by $\sigma = (Z, \gamma)$ such that

(i) $\mathrm{Im}(\sigma) \subset W_1$.
(ii) $\mathrm{Im}(\sigma_f) \subset W_f$.
(iii) $d(pr_2 \circ \sigma) = d\gamma = 0$, that is, γ is closed.
(iv) $d(D \circ \sigma)_{|Q_f} = 0$

It is easy to see that every pair given by a vector field Z and its image by the Legendre transformation, that is $(Z, \gamma = FL(Z))$ is a solution of the problem. In fact, by construction $\mathrm{Im}(\sigma) \subset W_1$ and $D_{|W_1} = 0 \Rightarrow D \circ \sigma = 0$. Following the argument in example 5.10 we can take $Z(q^1, q^2) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}$, and so

$$\sigma(q^1, q^2) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + q^2 dq^1 + q^1 dq^2$$

If we consider the solution

$$X(q^1, q^2, v^1, v^2) = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + \frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^2} + v^2 \frac{\partial}{\partial p_1} + v^1 \frac{\partial}{\partial p_2}$$

then

$$X^\sigma(q^1, q^2) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}$$

and

$$T\sigma(X^\sigma(q^1, q^2)) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}.$$

A direct inspection shows that

$$T\sigma(X^\sigma(q^1, q^2)) \neq X(\sigma(q^1, q^2) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + \frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^2} + \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}.$$

We can also obtain information of the Hamilton-Jacobi problem in the Skinner-Rusk setting from a solution of the Hamilton-Jacobi problem in the hamiltonian side.

If γ is a solution of the Hamilton-Jacobi problem in the hamiltonian setting and Y a vector field on M_f wich is a solution of equation (2.3), then we can define Y^γ as before.

We can also define a section $\tilde{\sigma}$ of $pr_f : W_f \to Q_f$ given by $\tilde{\sigma}(p) = (Y^\gamma(p), \gamma(p))$ for all $p \in Q_f$. An easy computation shows that $T\tilde{\sigma}(Y^\gamma)$ is a vector field along $\mathrm{Im}(\tilde{\sigma})$ which solves (3.2). Moreover if we find a vector field Z on Q such that $FL \circ Z = \gamma$ and $Z_f = Y^\gamma$, then the pair (Z, γ) is a solution of the Hamilton-Jacobi problem in the Skinner-Rusk setting.

7. Appendix: The Gotay-Nester-Hinds algorithm of constraints

In this section we will briefly review the constraint algorithm of constraints for presymplectic systems (see [11, 7]).

Let M_1 be a manifold, Ω a presymplectic structure on M_1, i.e., Ω is a closed 2-form, and α a 1-form on M_1. We will call $(M_1,\ \Omega,\ \alpha)$ a presymplectic system.

Gotay *et al.* developed an algorithm to find N, a submanifold of M_1 where we can solve the equation

$$i_X \Omega = \alpha \tag{7.1}$$

with X tangent to N.

The previous equation could not hold for every point of M_1, because α could not be in the range of Ω. So it is necesary to introduce the following set

$$M_2 = \{p \in M_1 \text{ such that there exists } X \in T_pM_1 \text{ satisfying } i_X \Omega = \alpha\},$$

and it is assumed that M_2 is a submanifold.

At the points of M_2 there exists solution to equation (7.1) but in an algebraic sense, that is, the solution could not be tangent to M_2. This forces a further restriction to

$$M_3 = \{p \in M_2 \text{ such that there exists } X \in T_pM_2 \text{ satisfying } i_X \Omega = \alpha\},$$

which is also assumed to be a submanifold.

Proceeding as above, the algorithm will produce a sequence of submanifolds

$$\cdots M_3 \ldots \hookrightarrow^{j_3} M_2 \hookrightarrow^{j_2} M_1$$

where

$$M_{l+1} = \{p \in M_l \text{ such that there exists } X \in T_pM_l \text{ satisfying } i_X \Omega = \alpha\},$$

and j_l denote the inclusions.

There are three possibilities:

(i) There exists k such that $M_k = \emptyset$.
(ii) There exists k such that $M_k = M_{k+1}$.
(iii) The algorithm does not end.

In the second case the submanifold M_k is called the final constraint submanifold and is denoted by M_f. By construction there exists a vector field on M_f such that is solution of equation (7.1). The third case is only possible in the infinite dimensional setting. In this case, the final constraint submanifold is defined by $M_f = \cap_{i=1} M_i$.

Note that the final constraint submanifold is maximal in the sense that if R is submanifold of M_1 where there exists a tangent solution of equation (7.1), then $R \subset M_f$.

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