A STUDY ON MULTIPLE ZETA VALUES FROM THE VIEWPOINT OF ZETA-FUNCTIONS OF ROOT SYSTEMS

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Abstract. We study multiple zeta values (MZVs) from the viewpoint of zeta-functions associated with the root systems which we have studied in our previous papers. In fact, the r-ple zeta-functions of Euler-Zagier type can be regarded as the zeta-function associated with a certain sub-root system of type C_r. Hence, by the action of the Weyl group, we can find new aspects of MZVs which imply that Zagier's well-known formula for MZVs coincides with Witten's volume formula associated with the above sub-root system of type C_r. Also, from this observation, we can prove some new formulas which especially include the parity results of double and triple zeta values. As another important application, we give certain refinement of restricted sum formulas, which gives restricted sum formulas among MZVs of an arbitrary depth r which were previously known only in the cases of depth $2, 3, 4$. Furthermore, considering a sub-root system of type B_r analogously, we can give relevant analogues of Zagier's formula, parity results and restricted sum formulas.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ be the set of positive integers, non-negative integers, rational integers, rational numbers, real numbers, and complex numbers, respectively.

We define the Euler-Zagier r-ple zeta-function (simply called the Euler-Zagier sum) by

$$\zeta_r(s_1,\ldots,s_r)=\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_r^{s_r}}, \tag{1.1}$$

where $s_1,\ldots,s_r$ are complex variables. When $(s_1,s_2,\ldots,s_r)\in\mathbb{N}^r$ $(s_r>1)$, this is called the multiple zeta value (MZV) of depth r first studied by Hoffman [7] and Zagier[38]. Though the opposite order of summation in the definition of $\zeta_r(s_1,\ldots,s_r)$ is also used recently, we use the order in (1.1) through this paper because it is natural in our study. In the research of MZVs, the main target is to give non-trivial relations among them, in order to investigate the structure of the algebra generated by them (for the details, see Kaneko [11]).

In our previous papers [13]-[20] and [28], as more general multiple series, we defined and studied multi-variable zeta-functions associated with root systems of type X_r $(X = A,B,C,D,E,F,G)$ denoted by $\zeta_r(s_1,\ldots,s_n;X_r)$ where n is the number of positive roots of type X_r (see definition (2.1)). In particular when $s_1=\cdots=s_r=s$, $\zeta_r(s,\ldots,s;X_r)$ essentially coincides with the Witten zeta-function (see Witten [36] and Zagier[38]). An important fact is

$$\zeta_r(2k,2k,\ldots,2k;X_r)\in\mathbb{Q}\cdot\pi^{2kn}\quad(k\in\mathbb{N}), \tag{1.2}$$

which is a consequence of Witten's volume formula given in [36]. Since we considered multi-variable version of Witten zeta-function, we were able to determine the rational coefficients in (1.2) explicitly in a generalized form (see [19, Thoerem 4.6]).

Recently, in our previous paper [21], we regarded MZVs as special values of zeta-functions of root systems of type A_r, and clarified the structure of the shuffle product procedure for MZVs from this viewpoint. In fact, we showed that the shuffle product procedure can be described in terms of partial fraction decompositions of zeta-functions of root systems of type A_r.

The main idea in the present paper is to regard (1.1) as a specialization of zeta-functions of root systems of type C_r (see below). It is essential in our theory that C_r is not simply-laced. In fact, there exists a subset of the root system of type C_r so that the Euler-Zagier sum (1.1) is the zeta-function associated with this subset (see Section 4). This subset itself is a root system, and hence the Weyl group naturally acts on (1.1). General fundamental results will be stated in Section 3, and their proofs will be given in Section 9. As a consequence, it can be shown that a kind of formula (1.2) corresponding to this sub-root system implies Zagier's well-known formula

$$\zeta_r(2k,2k,\ldots,2k)\in\mathbb{Q}\cdot\pi^{2kr}\quad(k\in\mathbb{N})$$

(see Corollary 4.2).

Furthermore, based on this observation in the cases when $r=2,3$, we will give explicit formulas for double series and for triple series (see Proposition 5.1 and Theorem 5.2) which include what is called the parity results for double and triple zeta values (see Corollary 5.3).

Similarly we can consider analogues of those results corresponding to a sub-root system of type B_r. In fact, we can define a B_r-type analogue of $\zeta_r(\mathbf{s})$ by

$$\zeta_r^\sharp(\mathbf{s})=\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{i=1}^{r}\frac{1}{(2\sum_{j=r-i+1}^{r-1}m_j+m_r)^{s_i}},\tag{1.3}$$

which is a "partial sum" of the series of $\zeta_r(\mathbf{s})$ (see Section 6). From the viewpoint of root systems, we see that this has some properties similar to those of $\zeta_r(\mathbf{s})$, because the root system of type B_r is a dual of that of type C_r. Actually we can obtain an analogue of Zagier's formula for this series (see Corollary 6.2). We also prove a formula between the values of $\zeta_2^\sharp(\mathbf{s})$ and the Riemann zeta values (see Theorem 6.4), which gives the parity result corresponding to type B_r (see Theorem 6.4). This result plays an important role in a recent study on the dimension of the linear space spanned by double zeta values of level 2 given by Kaneko and Tasaka (see [12]).

The fact that parity results hold in those classes implies that those are "nice" classes. In Section 8 we will study those classes from the analytic point of view, and prove that those classes, as well as the subclass of zeta-functions of root systems of type A_r introduced in [21], are "closed" in a certain analytic sense.

Another important consequence of our fundamental theorem in Section 3 is the "refined restricted sum formulas" for the values of $\zeta_r(\mathbf{s})$ and $\zeta_r^\sharp(\mathbf{s})$, which are embodied in Corollaries 4.1 and 6.1. One of the famous formulas among MZVs is the sum formula, which is, in the case of double zeta values, written as

$$\sum_{j=2}^{K-1}\zeta_2(K-j,j)=\zeta(K)\quad(K\in\mathbb{Z}_{\geq 3}).\tag{1.4}$$

Gangl, Kaneko and Zagier [6] obtained the following formulas, which "divide" (1.4) for even K into two parts:

$$\begin{aligned}\sum_{\substack{a,b\in\mathbb{N}\\a+b=N}}\zeta_2(2a,2b)&=\frac{3}{4}\zeta(2N)\in\mathbb{Q}\cdot\pi^{2N}\quad(N\in\mathbb{Z}_{\geq 2}),\\ \sum_{\substack{a,b\in\mathbb{N}\\a+b=N}}\zeta_2(2a-1,2b+1)&=\frac{1}{4}\zeta(2N)\in\mathbb{Q}\cdot\pi^{2N}\quad(N\in\mathbb{Z}_{\geq 2}),\end{aligned}\tag{1.5}$$

which are sometimes called the restricted sum formulas. More recently, Shen and Cai [33] gave restricted sum formulas for triple and fourth zeta values (see (7.1) and (7.2)). As we will discuss in Section 7, our Corollaries 4.1 and 6.1 give more refined restricted sum formulas for $\zeta_r(\mathbf{s})$ and for $\zeta_r^\sharp(\mathbf{s})$ of an arbitrary depth r. From these refined formulas we can deduce the restricted sum formulas for an arbitrary depth r, actually in a generalized form involving a parameter d (see Theorems 7.1 and 7.3).

A part of the results in the present paper has been announced in [22].

2. Zeta-functions of root systems and root sets

In this section, we recall the definition of zeta-functions of root systems studied in our papers [13]-[19]. For the details of basic facts about root systems and Weyl groups, see [5, 8, 9].

Let V be an r-dimensional real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. The dual space V^* is identified with V via the inner product of V. Let Δ be a finite irreducible reduced root system, and $\Psi=\{\alpha_1,\ldots,\alpha_r\}$ its fundamental system. We fix Δ_+ and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta=\Delta_+\coprod\Delta_-$. Let $Q=Q(\Delta)$ be the root lattice, $Q^\vee$ the coroot lattice, $P=P(\Delta)$ the weight lattice, $P^\vee$ the coweight lattice, and P_{++} the set of integral strongly dominant weights respectively defined by

$$Q=\bigoplus_{i=1}^r\mathbb{Z}\,\alpha_i,\qquad Q^\vee=\bigoplus_{i=1}^r\mathbb{Z}\,\alpha_i^\vee,$$
$$P=\bigoplus_{i=1}^r\mathbb{Z}\,\lambda_i,\qquad P^\vee=\bigoplus_{i=1}^r\mathbb{Z}\,\lambda_i^\vee,\qquad P_{++}=\bigoplus_{i=1}^r\mathbb{N}\,\lambda_i,$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of $\Psi^\vee$ and Ψ satisfying $\langle\alpha_i^\vee,\lambda_j\rangle=\delta_{ij}$ (Kronecker's delta) and $\langle\lambda_i^\vee,\alpha_j\rangle=\delta_{ij}$ respectively.

Let $\sigma_\alpha:V\to V$ be the reflection with respect to a root $\alpha\in\Delta$ defined by

$$\sigma_\alpha: v\mapsto v-\langle\alpha^\vee,v\rangle\alpha.$$

For a subset $A\subset\Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha\in A$. In particular, $W=W(\Delta)$ is the Weyl group, and $\{\sigma_j:=\sigma_{\alpha_j}\,|\,1\leq j\leq r\}$ generates W. For $w\in W$, denote $\Delta_w=\Delta_+\cap w^{-1}\Delta_-$. The zeta-function associated with Δ is defined by

$$\zeta_r(\mathbf{s},\mathbf{y};\Delta)=\sum_{\lambda\in P_{++}}e^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},\tag{2.1}$$

where $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}\in\mathbb{C}^{|\Delta_+|}$ and $\mathbf{y}\in V$. This can be regarded as a multi-variable version of Witten zeta-functions formulated by Zagier [38] based on the work of Witten [36].

Let Δ^* be a subset of Δ_+. We call Δ^* a *root set* (or a *root subset* of Δ_+) if, for any λ_j $(1\leq j\leq r)$, there exists an element $\alpha\in\Delta^*$ for which $\langle\alpha,\lambda_j\rangle\neq0$ holds. We define the zeta-function associated with a root set Δ^* by

$$\zeta_r(\mathbf{s},\mathbf{y};\Delta^*)=\sum_{\lambda\in P_{++}}e^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta^*}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}.\tag{2.2}$$

In the case $\mathbf{y}=\mathbf{0}$, this zeta-function was introduced in [15]. When the root system is of type X_r, we write $\Delta=\Delta(X_r)$, $\Delta^*=\Delta^*(X_r)$, and so on.

Remark 2.1. The notion of $\zeta_r(\mathbf{s},\mathbf{y};\Delta^*)$ depends not only on Δ^*, but also on Δ_+, because the summation on (2.2) runs over all strongly dominant weights associated with Δ_+.

3. Fundamental formulas

In this section, we state several fundamental formulas which are certain extensions of our previous results given in [15, 16, 19]. Proofs of theorems stated in this section will be given in Section 9.

Let $\mathscr{V}$ be the set of all bases $\mathbf{V}\subset\Delta_+$. Let $\mathbf{V}^*=\{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$ be the dual basis of $\mathbf{V}^\vee=\{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let $L(\mathbf{V}^\vee)=\bigoplus_{\beta\in\mathbf{V}}\mathbb{Z}\beta^\vee$. Then we have $|Q^\vee/L(\mathbf{V}^\vee)|<\infty$. Fix $\phi\in V$ such that $\langle\phi,\mu_\beta^{\mathbf{V}}\rangle\neq0$ for all $\mathbf{V}\in\mathscr{V}$ and $\beta\in\mathbf{V}$. If the root system Δ is of A_1 type, then we choose

$\phi=\alpha_1^\vee$. We define a multiple generalization of the fractional part as

$$\{\mathbf{y}\}_{\mathbf{V},\beta}=\begin{cases}\{\langle\mathbf{y},\mu^{\mathbf{V}}_\beta\rangle\} & (\langle\phi,\mu^{\mathbf{V}}_\beta\rangle>0),\\ 1-\{-\langle\mathbf{y},\mu^{\mathbf{V}}_\beta\rangle\} & (\langle\phi,\mu^{\mathbf{V}}_\beta\rangle<0),\end{cases}$$

where the notation $\{x\}$ on the right-hand sides stands for the usual fractional part of $x\in\mathbb{R}$. Let $\mathbf{T}=\{t\in\mathbb{C}\mid |t|<2\pi\}^{|\Delta_+|}$.

Definition 3.1. For $\mathbf{t}=(t_\alpha)_{\alpha\in\Delta_+}\in\mathbf{T}$ and $\mathbf{y}\in V$, we define

$$\begin{aligned}F(\mathbf{t},\mathbf{y};\Delta)&=\sum_{\mathbf{V}\in\mathscr{V}}\Bigl(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}}t_\beta\langle\gamma^\vee,\mu^{\mathbf{V}}_\beta\rangle}\Bigr)\\&\quad\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\Bigl(\prod_{\beta\in\mathbf{V}}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigr)\\&=\sum_{\mathbf{k}\in\mathbb{N}_0^{|\Delta_+|}}\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)\prod_{\alpha\in\Delta_+}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}\end{aligned}\tag{3.1}$$

which is independent of choice of ϕ.

Remark 3.2. In [16], $F(\mathbf{t},\mathbf{y};\Delta)$ is defined in a different way. The above is [16, Theorem 4.1]. In particular when $\Delta=\Delta(A_1)$, we see that

$$F(\mathbf{t},\mathbf{y};\Delta(A_1))=\frac{te^{t\{y\}}}{e^t-1},$$

which is the generating function of ordinary Bernoulli periodic functions $\{B_k(\{y\})\}$.

Let

$$S(\mathbf{s},\mathbf{y};\Delta)=\sum_{\lambda\in P\setminus H_{\Delta^\vee}}e^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},\tag{3.2}$$

where $H_{\Delta^\vee}=\{v\in V\mid\langle\alpha^\vee,v\rangle=0\text{ for some }\alpha\in\Delta\}$ is the set of all walls of Weyl chambers. For $\mathbf{s}\in\mathbb{C}^{|\Delta_+|}$, we define $(w\mathbf{s})_\alpha=s_{w^{-1}\alpha}$, where if $w^{-1}\alpha\in\Delta_-$ we use the convention $s_{-\alpha}=s_\alpha$.

Proposition 3.3 ([19, Theorem 4.4],[16, Proposition 3.2])**.**

$$\begin{aligned}S(\mathbf{k},\mathbf{y};\Delta)&=\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};\Delta)\\&=(-1)^{|\Delta_+|}\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr)\end{aligned}\tag{3.3}$$

for $k_\alpha\in\mathbb{Z}_{\geq2}$ $(\alpha\in\Delta_+)$.

Remark 3.4. It should be noted that the formula (3.3) holds in the cases $k_\alpha=1$ for some $\alpha\in\Delta_+$, while it does not hold in the cases $k_\alpha=0$ for any $\alpha\in\Delta_+$.

For $\mathbf{v}\in V$, and a differentiable function f on V, let

$$(\partial_{\mathbf{v}}f)(\mathbf{y})=\lim_{h\to0}\frac{f(\mathbf{y}+h\mathbf{v})-f(\mathbf{y})}{h}$$

and for $\alpha\in\Delta_+$,

$$\mathfrak{D}_\alpha=\frac{\partial}{\partial t_\alpha}\Big|_{t_\alpha=0}\partial_{\alpha^\vee}.$$

Let $\Delta^*\subset\Delta_+$ be a root set and let $A=\Delta_+\setminus\Delta^*=\{\nu_1,\ldots,\nu_N\}\subset\Delta_+$, and define

$$\mathfrak{D}_A=\mathfrak{D}_{\nu_N}\cdots\mathfrak{D}_{\nu_1}.$$

Similarly we define

$$\mathfrak{D}_{\alpha,2} = \frac{1}{2}\frac{\partial^2}{\partial t_\alpha^2}\bigg|_{t_\alpha=0} \partial^2_{\alpha^\vee}, \tag{3.4}$$

$$\mathfrak{D}_{A,2} = \mathfrak{D}_{\nu_N,2}\cdots\mathfrak{D}_{\nu_1,2}. \tag{3.5}$$

Further, let $A_j = \{\nu_1,\ldots,\nu_j\}$ $(1 \le j \le N-1)$, $A_0 = \emptyset$, and

$$\mathscr{V}_A = \{\mathbf{V}\in\mathscr{V} \mid \nu_{j+1} \notin \mathrm{L.h.}[\mathbf{V}\cap A_j]\ (0\le j\le N-1)\},$$

where L.h.[·] denotes the linear hull (linear span). Let $\mathscr{R}$ be the set of all linearly independent subsets $\mathbf{R} = \{\beta_1,\ldots,\beta_{r-1}\} \subset \Delta$ and

$$\mathfrak{H}_{\mathscr{R}} := \bigcup_{\substack{\mathbf{R}\in\mathscr{R}\\ q\in Q^\vee}} (\mathrm{L.h.}[\mathbf{R}^\vee] + q). \tag{3.6}$$

Remark 3.5. It is to be noted that $\mathbf{y}\in\mathfrak{H}_{\mathscr{R}}$ if and only if $\langle \mathbf{y}+q, \mu_\beta^{\mathbf{V}}\rangle \in \mathbb{Z}$ for some $\mathbf{V}\in\mathscr{V}, \beta\in\mathbf{V}, q\in Q^\vee$. In fact, if $\mathbf{y}\in\mathfrak{H}_{\mathscr{R}}$ then we can write $\mathbf{y} = \sum_{j=1}^{r-1} a_j\beta_j^\vee + q$ $(a_j\in\mathbb{R})$. We can find an element $\beta_r\in\Delta$ such that $\mathbf{V} = \{\beta_1,\ldots,\beta_r\}\in\mathscr{V}$. Then $\langle \mathbf{y}-q, \mu_{\beta_r}^{\mathbf{V}}\rangle = 0 \in\mathbb{Z}$. Conversely, assume $\langle \mathbf{y}+q,\mu_\beta^{\mathbf{V}}\rangle = c\in\mathbb{Z}$. Write $\mathbf{V} = \{\beta_1,\ldots,\beta_{r-1},\beta\}$. Since this is a basis, we may write $\mathbf{y}+q = \sum_{j=1}^{r-1} a_j\beta_j^\vee + a\beta^\vee$ with $a_j, a\in\mathbb{R}$. Then $c = \langle\mathbf{y}+q,\mu_\beta^{\mathbf{V}}\rangle = a$, especially $a\in\mathbb{Z}$. Therefore $a\beta^\vee - q\in Q^\vee$, which implies $\mathbf{y}\in\mathfrak{H}_{\mathscr{R}}$.

Definition 3.6. For $\Delta_+\setminus\Delta^* = A = \{\nu_1,\ldots,\nu_N\}\subset\Delta_+$, $\mathbf{t}_{\Delta^*} = \{t_\alpha\}_{\alpha\in\Delta^*}$ and $\mathbf{y}\in V$, we define

$$\begin{aligned} F_{\Delta^*}(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta) &= \sum_{\mathbf{V}\in\mathscr{V}_A} (-1)^{|A\setminus\mathbf{V}|} \\ &\quad\times\Bigl(\prod_{\gamma\in\Delta_+\setminus(\mathbf{V}\cup A)} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}\setminus A} t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle}\Bigr) \\ &\quad\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\Bigl(\prod_{\beta\in\mathbf{V}\setminus A}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigr). \end{aligned} \tag{3.7}$$

Theorem 3.7. *For $\Delta_+\setminus\Delta^* = A = \{\nu_1,\ldots,\nu_N\}\subset\Delta_+$, $\mathbf{t}_{\Delta^*} = \{t_\alpha\}_{\alpha\in\Delta^*}$ and $\mathbf{y}\in V\setminus\mathfrak{H}_{\mathscr{R}}$. we have*

$$\bigl(\mathfrak{D}_A F\bigr)(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta) = \bigl(\mathfrak{D}_{A,2}F\bigr)(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta) = F_{\Delta^*}(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta), \tag{3.8}$$

and hence is independent of choice of the order of A. The function $F_{\Delta^*}(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta)$ is the continuous extension of $\bigl(\mathfrak{D}_AF\bigr)(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta)$ in $\mathbf{y}$ in the sense that $\bigl(\mathfrak{D}_AF\bigr)(\mathbf{t}_{\Delta^*},\mathbf{y}+c\phi;\Delta)$ tends continuously to $F_{\Delta^*}(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta)$ when $c\to 0+$, and is holomorphic with respect to $\mathbf{t}_{\Delta^*}$ around the origin.

Definition 3.8. For $\Delta^*\subset\Delta_+$ and $\mathbf{t}_{\Delta^*} = \{t_\alpha\}_{\alpha\in\Delta^*}$, we define $\mathcal{P}_{\Delta^*}(\mathbf{k}_{\Delta^*},\mathbf{y};\Delta)$ by

$$\begin{aligned} &F_{\Delta^*}(\mathbf{t}_{\Delta^*},\mathbf{y};\Delta) \\ &= \sum_{\mathbf{k}_{\Delta^*}\in\mathbb{N}_0^{|\Delta^*|}} \mathcal{P}_{\Delta^*}(\mathbf{k}_{\Delta^*},\mathbf{y};\Delta)\prod_{\alpha\in\Delta^*}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}. \end{aligned}$$

Theorem 3.9. *For $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+}$ with $k_\alpha\in\mathbb{Z}_{\ge2}$ $(\alpha\in\Delta^*)$, $k_\alpha = 0$ $(\alpha\in\Delta_+\setminus\Delta^*)$, we have*

$$\begin{aligned} &\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y};\Delta) \\ &= (-1)^{|\Delta_+|}\mathcal{P}_{\Delta^*}(\mathbf{k}_{\Delta^*},\mathbf{y};\Delta)\Bigl(\prod_{\alpha\in\Delta^*}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr) \end{aligned} \tag{3.9}$$

provided all the series on the left-hand side absolutely converge.

Assume that Δ is not simply-laced. Then we have the disjoint union $\Delta = \Delta_l \cup \Delta_s$, where Δ_l is the set of all long roots and Δ_s is the set of all short roots.

Note that if there is an odd k_i, then both hand sides vanish in (3.9). On the other hand, when all $k_i's$ are even, by applying Theorem 3.9 to $\Delta^* = \Delta_l$ or Δ_s, we obtain the following theorem immediately, which is a generalization of the explicit volume formula proved in [19, Theorem 4.6].

Theorem 3.10. *Let $\Delta_1 = \Delta_l$ (resp. Δ_s), $\Delta_2 = \Delta_s$ (resp. Δ_l), and $\Delta_{j+} = \Delta_j \cap \Delta_+$ ($j = 1, 2$). Then Δ_{j+} ($j = 1, 2$) is a root subset of Δ_+. For $\mathbf{s}_1 = \mathbf{k}_1 = (k_\alpha)_{\alpha \in \Delta_{1+}}$ with $k_\alpha = k \in 2\mathbb{N}$ (for all $\alpha \in \Delta_{1+}$) and $\nu \in P^\vee/Q^\vee$, we have*

$$\zeta_r(\mathbf{k}_1, \nu; \Delta_{1+}) = \frac{(-1)^{|\Delta_+|}}{|W|} \mathcal{P}_{\Delta_{1+}}(\mathbf{k}_1, \nu; \Delta) \Bigg(\prod_{\alpha \in \Delta_{1+}} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \Bigg). \tag{3.10}$$

Remark 3.11. Let $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+}$ with $k_\alpha = k \in 2\mathbb{N}$ ($\alpha \in \Delta_{1+}$) and $k_\alpha = 0$ ($\alpha \in \Delta_{2+}$). Then obviously $\zeta_r(\mathbf{k}_1, \nu; \Delta_{1+}) = \zeta_r(\mathbf{k}, \nu; \Delta)$. Our proof of Theorem 3.10 is actually based on the latter viewpoint.

4. Multiple zeta values and zeta-functions of root system of type C_r

Now we study MZVs from the viewpoint of zeta-functions of root systems of type C_r. For $\Delta = \Delta(C_r)$, we have the disjoint union $\Delta_+^\vee = (\Delta_{l+})^\vee \cup (\Delta_{s+})^\vee$, where $\Delta_{l+} = \Delta_{l+}(C_r) = \Delta_l(C_r) \cap \Delta_+(C_r)$, $\Delta_{s+} = \Delta_{s+}(C_r) = \Delta_s(C_r) \cap \Delta_+(C_r)$, and

$$(\Delta_{l+})^\vee = \{\alpha_r^\vee,\ \alpha_{r-1}^\vee + \alpha_r^\vee,\ \alpha_{r-2}^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee,\ \ldots, \alpha_1^\vee + \cdots + \alpha_r^\vee\}.$$

Since $P^\vee/Q^\vee = \{\mathbf{0}, \lambda_r^\vee\}$, Therefore, for $\mathbf{s}_l = (s_\alpha)_{\alpha \in \Delta_{l+}}$, we have

$$\zeta_r(\mathbf{s}_l, \mathbf{0}; \Delta_{l+}(C_r)) = \sum_{m_1, \ldots, m_r = 1}^\infty \prod_{i=1}^r \frac{1}{(\sum_{j=r-i+1}^{r-1} m_j + m_r)^{s_i}},$$

$$\zeta_r(\mathbf{s}_l, \lambda_r^\vee; \Delta_{l+}(C_r)) = \sum_{m_1, \ldots, m_r = 1}^\infty \prod_{i=1}^r \frac{(-1)^{m_r}}{(\sum_{j=r-i+1}^{r-1} m_j + m_r)^{s_i}},$$

where the first equation is exactly the Euler-Zagier sum $\zeta_r(s_1, \ldots, s_r)$ (see (1.1)). In order to apply Theorems 3.9 and 3.10 to MZVs, we rewrite the root system of type C_r in terms of standard orthonormal basis $\{e_1, \ldots, e_r\}$. We put $\alpha_i^\vee = e_i - e_{i+1}$ for $1 \le i \le r-1$ and $\alpha_r^\vee = e_r$. Then we have

$$(\Delta_{l+})^\vee = \{\alpha_r^\vee = e_r,\ \alpha_{r-1}^\vee + \alpha_r^\vee = e_{r-1},\ \alpha_{r-2}^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee = e_{r-2},\ \ldots, \alpha_1^\vee + \cdots + \alpha_r^\vee = e_1\}.$$

In this realization, we see that $W(C_r) = (\mathbb{Z}/2\mathbb{Z})^r \rtimes \mathfrak{S}_r$, where $\mathfrak{S}_r$ is the symmetric group of degree r which permutes bases, and the j-th $\mathbb{Z}/2\mathbb{Z}$ flips the sign of e_j. Since the sign flips act trivially on the variables $\mathbf{s}_l$, from Theorem 3.9 we obtain the following formulas. These are the "refined restricted sum formulas" for $\zeta_r(\mathbf{s})$, which we will discuss in Section 7.

Corollary 4.1. *Let $\Delta = \Delta(C_r)$. For $(2\mathbf{k})_l = (2k_\alpha)_{\alpha \in \Delta_{l+}} = (2k_1, \ldots, 2k_r) \in (2\mathbb{N})^r$ and $\mathbf{y} = \nu \in P^\vee/Q^\vee$,*

$$\sum_{\sigma \in \mathfrak{S}_r} \zeta_r(\sigma^{-1}(2\mathbf{k})_l, \nu; \Delta_{l+}) = \frac{(-1)^r}{2^r} \mathcal{P}_{\Delta_{l+}}((2\mathbf{k})_l, \nu; \Delta) \prod_{j=1}^r \frac{(2\pi i)^{2k_j}}{(2k_j)!} \in \mathbb{Q} \cdot \pi^{2\sum_{j=1}^r k_j}. \tag{4.1}$$

In particular when $\nu = \mathbf{0}$,

$$\sum_{\sigma \in \mathfrak{S}_r} \zeta_r(2k_{\sigma^{-1}(1)}, \ldots, 2k_{\sigma^{-1}(r)}) = \frac{(-1)^r}{2^r} \mathcal{P}_{\Delta_{l+}}((2\mathbf{k})_l, \mathbf{0}; \Delta) \prod_{j=1}^r \frac{(2\pi i)^{2k_j}}{(2k_j)!} \in \mathbb{Q} \cdot \pi^{2\sum_{j=1}^r k_j}. \tag{4.2}$$

Also Theorem 3.10 in the case of type C_r immediately gives the following.

Corollary 4.2. *Let* $\Delta=\Delta(C_r)$*. For* $(\mathbf{2k})_l=(2k,\ldots,2k)$ *with any* $k\in\mathbb{N}$*,*

$$\zeta_r(2k,2k,\ldots,2k)=\frac{(-1)^r}{2^r r!}\mathcal{P}_{\Delta_{l+}}((\mathbf{2k})_l,\mathbf{0};\Delta)\frac{(2\pi i)^{2kr}}{\{(2k)!\}^r}\in\mathbb{Q}\cdot\pi^{2kr}. \tag{4.3}$$

Remark 4.3. The fact that $\zeta_r(2k,\ldots,2k)\in\mathbb{Q}\cdot\pi^{2kr}$ was first proved by Zagier [38]. More generally, it is known that

$$\zeta_r(2k,\ldots,2k)=\mathcal{C}_r^{(k)}\frac{(2\pi i)^{2kr}}{(2kr)!}, \tag{4.4}$$

where

$$\mathcal{C}_0^{(k)}=1,\quad \mathcal{C}_n^{(k)}=\frac{1}{2n}\sum_{j=1}^{n}(-1)^j\binom{2nk}{2jk}B_{2jk}\mathcal{C}_{n-j}^{(k)}\quad(n\geq 1).$$

Formula (4.4) was first published in the lecture notes [1], [2] written in Japanese (Exercise 5, Section 1.1 of those lecture notes). See also Muneta [31]. We emphasize that (4.4) can be regarded as a kind of Witten's volume formula (4.3). Because (4.3) and (4.4) in the case $r=1$ are both Euler's well-known formula

$$\zeta(2k)=-B_{2k}\frac{(2\pi i)^{2k}}{2(2k)!}\qquad(k\in\mathbb{N}), \tag{4.5}$$

we can see that $\mathcal{P}_{\Delta_{l+}}((\mathbf{2k})_l,\mathbf{0};\Delta)$ and $\mathcal{C}_r^{(k)}$ are different types of generalizations of the ordinary Bernoulli number B_{2k}.

Example 4.4. Let $\Delta=\Delta(C_2)$ be the root system of type C_2. By Theorem 3.7, we have

$$\begin{aligned}(\mathfrak{D}_{\Delta_{s+}}F)(t_1,t_2,y_1,y_2;\Delta)&=1+\frac{t_1t_2e^{\{y_2\}t_1}}{(e^{t_1}-1)(t_1-t_2)}\\&\quad+\frac{t_1t_2e^{\{y_2\}t_2}}{(e^{t_2}-1)(-t_1+t_2)}+\frac{t_1t_2e^{(1-\{y_1-y_2\})t_1+\{y_1\}t_2}}{(e^{t_1}-1)(e^{t_2}-1)}\\&\quad-\frac{t_1t_2e^{(1-\{2y_1-y_2\})t_1}}{(e^{t_1}-1)(t_1+t_2)}-\frac{t_1t_2e^{\{2y_1-y_2\}t_2}}{(e^{t_2}-1)(t_1+t_2)}\\&=\sum_{k_1,k_2=1}^{\infty}\mathcal{P}_{\Delta_{l+}}(k_1,k_2,y_1,y_2;\Delta)\frac{t_1^{k_1}t_2^{k_2}}{k_1!k_2!}.\end{aligned}$$

Set $(y_1,y_2)=(0,0)$ and $\mathbf{k}=(0,k_1,k_2,0)$. Then $\zeta_2(0,k_1,k_2,0;y_1,y_2;\Delta)=\zeta_2(k_1,k_2)$ for $\Delta=\Delta(C_2)$. Hence it follows from (3.9) that

$$\begin{aligned}&(1+(-1)^{k_1})(1+(-1)^{k_2})\zeta_2(k_1,k_2)\\&\quad+(1+(-1)^{k_2})(1+(-1)^{k_1})\zeta_2(k_2,k_1)\\&=(-1)^4\mathcal{P}_{\Delta_{l+}}(k_1,k_2,0,0;\Delta)\frac{(2\pi i)^{k_1+k_2}}{k_1!k_2!}\end{aligned} \tag{4.6}$$

for $k_1,k_2\geq 2$.

For example, we can compute

$$\mathcal{P}_{\Delta_{l+}}(4,4,0,0;\Delta)=\frac{1}{6300}$$

from the above expansion. Hence we obtain

$$\zeta_2(4,4)=\frac{(-1)^4}{8}\frac{1}{6300}\frac{(2\pi i)^8}{(4!)^2}=\frac{\pi^8}{113400}.$$

Similarly we can compute $\zeta_2(2k,2k)$ for $k\in\mathbb{N}$, though in this case we can also compute $\zeta_2(2k,2k)$ by using the well-known harmonic product formula for double zeta values

$$\zeta(s)\zeta(t)=\zeta_2(s,t)+\zeta_2(t,s)+\zeta(s+t). \tag{4.7}$$

In the next section, we introduce a slight generalization of Corollary 4.2 which gives evaluation formulas of $\zeta_2(k,l)$ for odd $k+l$ in terms of $\zeta(s)$ (see Proposition 5.1).

Remark 4.5. In the general C_r case, considering the expansion of $(\mathfrak{D}_{\Delta_{s+}}F)(\mathbf{t}_{\Delta_{l+}},\mathbf{0};\Delta(C_r))$ similarly, we can systematically compute $\zeta_r(2k,\ldots,2k)$. Moreover, considering the case $\nu\neq\mathbf{0}$ for $\zeta_r(\mathbf{s},\nu;\Delta(C_r))$, we can give character analogues of Corollary 4.2 for multiple L-values, which were first proved by Yamasaki [37].

5. Some relations and parity results for double and triple zeta values

In Theorem 3.9, we considered the sum over W on the left-hand side of (3.9). Here, more generally, we consider the sum over a certain set of minimal coset representatives on the left-hand side of (3.9). In this case, it is not easy to execute its computation directly. Hence we use a more technical method which was already introduced in [18]. First we show the following result for double zeta values corresponding to a sub-root system of type C_2, where the number of the terms on the left-hand side is just the half of that on the left-hand side of (4.6).

Proposition 5.1. *For $p,q\in\mathbb{N}_{\geq 2}$,*

$$\begin{aligned}(1+(-1)^p)\,\zeta_2(p,q)&+(1+(-1)^q)\,\zeta_2(q,p)\\ &=2\sum_{j=0}^{[p/2]}\binom{p+q-2j-1}{q-1}\zeta(2j)\zeta(p+q-2j)\\ &\quad+2\sum_{j=0}^{[q/2]}\binom{p+q-2j-1}{p-1}\zeta(2j)\zeta(p+q-2j)-\zeta(p+q).\end{aligned}$$

Proof. The proof was essentially stated in [18, Theorem 3.1] which is a simpler form of a previous result for zeta-functions of type A_2 given by the third-named author [35, Theorem 4.5]. In fact, setting $(k,l,s)=(p,q,0)$ in [18, Theorem 3.1], we have

$$\begin{aligned}\zeta(p)\zeta(q)&+(-1)^p\zeta_2(p,q)+(-1)^q\zeta_2(q,p)\\ &=2\sum_{j=0}^{[p/2]}\binom{p+q-2j-1}{q-1}\zeta(2j)\zeta(p+q-2j)\\ &\quad+2\sum_{j=0}^{[q/2]}\binom{p+q-2j-1}{p-1}\zeta(2j)\zeta(p+q-2j).\end{aligned}$$

Combining this and (4.7), we have the assertion. □

In particular when p and q are of different parity, we see that $\zeta_2(p,q)\in\mathbb{Q}[\{\zeta(j+1)\,|\,j\in\mathbb{N}\}]$ which was first proved by Euler. For example, we have

$$\zeta_2(2,3)=3\zeta(2)\zeta(3)-\frac{11}{2}\zeta(5).$$

Next we consider triple zeta values. From the viewpoint of the root system of C_3 type, we have the following theorem. Note that, unlike the case of double zeta values, this result seems not to be led from the result on the case of type A_3 (cf. [28, Theorems 5.9 and 5.10]).

Theorem 5.2. *For $a,b,c\in\mathbb{N}_{\geq 2}$,*

$$\begin{aligned}&(1+(-1)^a)\zeta_3(a,b,c)+(1+(-1)^b)\{\zeta_3(b,a,c)+\zeta_3(b,c,a)\}+(-1)^b(1+(-1)^c)\zeta_3(c,b,a)\\ &=2\Bigg\{\sum_{\xi=0}^{[a/2]}\zeta(2\xi)\sum_{\omega=0}^{a-2\xi}\binom{\omega+b-1}{\omega}\binom{a+c-2\xi-\omega-1}{c-1}\zeta_2(b+\omega,a+c-2\xi-\omega)\end{aligned}$$

$$+\sum_{\xi=0}^{[b/2]}\zeta(2\xi)\sum_{\omega=0}^{a-1}\binom{\omega+b-2\xi}{\omega}\binom{a+c-\omega-2}{c-1}\zeta_2(b-2\xi+\omega+1,a+c-1-\omega)$$
$$+(-1)^b\sum_{\xi=0}^{[c/2]}\zeta(2\xi)\sum_{\omega=0}^{c-2\xi}\binom{\omega+b-1}{\omega}\binom{a+c-2\xi-\omega-1}{a-1}\zeta_2(b+\omega,a+c-2\xi-\omega)$$
$$+(-1)^b\sum_{\xi=0}^{[b/2]}\zeta(2\xi)\sum_{\omega=0}^{c-1}\binom{\omega+b-2\xi}{\omega}\binom{a+c-\omega-2}{a-1}\zeta_2(b-2\xi+\omega+1,a+c-1-\omega)\Bigg\}$$
$$-\zeta_2(a+b,c)-(1+(-1)^b)\zeta_2(b,a+c)-(-1)^b\zeta_2(b+c,a).$$

The proof of this theorem will be given in Section 10.

This theorem especially implies the following result which was proved by Borwein and Girgensohn (see [4]).

Corollary 5.3. *Let*
$$\mathfrak{X}=\mathbb{Q}\left[\{\zeta(j+1),\zeta_2(k,l+1)\}_{j,k,l\in\mathbb{N}}\right],$$
namely the $\mathbb{Q}$-algebra generated by Riemann zeta values and double zeta values at positive integers except singularities. Suppose $a,b,c\in\mathbb{N}_{\geq 2}$ satisfy that $a+b+c$ is even. Then $\zeta_3(a,b,c)\in\mathfrak{X}$.

Proof. We recall the harmonic product formula
$$\zeta_3(a,b,c)+\zeta_3(b,a,c)+\zeta_3(b,c,a)=\zeta(a)\zeta_2(b,c)-\zeta_2(b,c+a)-\zeta_2(a+b,c) \tag{5.1}$$
for $a,b,c\in\mathbb{N}_{\geq 2}$ (see [11]).

Let $a,b,c\in\mathbb{N}_{\geq 2}$ satisfying that $a+b+c$ is even. First we assume that a,b,c are all even. Then, combining Theorem 5.2 and (5.1), we see that $\zeta_3(c,b,a)\in\mathfrak{X}$.

Next we assume that a is even and b,c are odd. Then, by Theorem 5.2, we see that $\zeta_3(a,b,c)\in\mathfrak{X}$.

As for other cases, we can similarly obtain the assertions by using Theorem 5.2 and (5.1). Thus we complete the proof. □

Remark 5.4. The following property of the multiple zeta value is sometimes called the parity result:

The multiple zeta value $\zeta_r(k_1,k_2,\ldots,k_r)$ of depth r can be expressed as a rational linear combination of products of MZVs of lower depth than r, when its depth r and its weight $\sum_{j=1}^n k_j$ are of different parity.

The fact in case of depth 2 was proved by Euler, and that of depth 3 was proved by Borwein and Girgensohn (see [4]). Further they conjectured the above assertion in the case of an arbitrary depth. This conjecture was proved by the third-named author [34] and by Ihara, Kaneko and Zagier [10] independently. It should be stressed that our Corollary 5.3 gives an explicit expression of the parity result for the triple zeta value under the condition $a,b,c\in\mathbb{N}_{\geq 2}$.

Therefore it seems important to generalize Theorem 5.2 in order to give an explicit expression of the parity result of an arbitrary depth.

Example 5.5. Putting $(a,b,c)=(2,2,4)$ in Theorem 5.2, we have
$$\begin{aligned}&2\zeta_3(2,2,4)+2\{\zeta_3(2,2,4)+\zeta_3(2,4,2)\}+2\zeta_3(4,2,2)\\&=2\zeta(4)\zeta_2(2,2)+\zeta(2)\{8\zeta_2(4,2)+12\zeta_2(3,3)+16\zeta_2(2,4)+16\zeta_2(1,5)\}\\&\quad-16\zeta_2(6,2)-20\zeta_2(5,3)-25\zeta_2(4,4)-24\zeta_2(3,5)-17\zeta_2(2,6).\end{aligned}$$
Therefore, using (5.1), we obtain
$$\zeta_3(4,2,2)=\zeta(4)\zeta_2(2,2)+\zeta(2)\{4\zeta_2(4,2)+6\zeta_2(3,3)+7\zeta_2(2,4)+8\zeta_2(1,5)\}$$

$$- 8\zeta_2(6,2) - 10\zeta_2(5,3) - \frac{23}{2}\zeta_2(4,4) - 12\zeta_2(3,5) - \frac{15}{2}\zeta_2(2,6) \in \mathfrak{X}.$$

Note that this formula can be proved by combining known results for MZVs given by the double shuffle relations and harmonic product formulas (see, for example, [30, Section 5]).

Remark 5.6. If we replace (10.3) (in Section 10) by

$$\sum_{l\in\mathbb{N}}\sum_{m\in\mathbb{Z}^*}(-1)^{l+m}x^l y^m e^{i(l+m)\theta},$$

and argue along the same line as in the proof of Theorem 5.2, then we can obtain

$$\begin{aligned}&(1+(-1)^a)(1+(-1)^c)\{\zeta_3(a,b,c)+\zeta_3(a,c,b)+\zeta_3(c,a,b)\}\\&+\left(1+(-1)^b\right)(1+(-1)^c)\{\zeta_3(c,b,a)+\zeta_3(b,c,a)+\zeta_3(b,a,c)\}\\&\quad\in\mathbb{Q}[\{\zeta(j+1)\,|\,j\in\mathbb{N}\}]\end{aligned}$$

for $a,b,c\in\mathbb{N}_{\geq 2}$. In particular when a,b,c are both even, we have Zagier's formula (4.4) for the triple zeta value which can be regarded as a kind of Witten's volume formula (4.3) (see Section 4). Furthermore, when a is odd and both b and c are even, then

$$\zeta_3(c,b,a)+\zeta_3(b,c,a)+\zeta_3(b,a,c)\in\mathbb{Q}\left[\{\zeta(j+1)\,|\,j\in\mathbb{N}\}\right].$$

Note that this result can also be deduced by combining (5.1) and Proposition 5.1.

6. Multiple zeta values associated with the root system of type B_r

In this section we discuss the B_r-analogue of our theory developed in the preceding two sections.

As for the root system of type B_r, namely for $\Delta=\Delta(B_r)$, we see that

$$(\Delta_{s+})^\vee=\{\alpha_r^\vee,\ 2\alpha_{r-1}^\vee+\alpha_r^\vee, 2\alpha_{r-2}^\vee+2\alpha_{r-1}^\vee+\alpha_r^\vee,\ldots,2\alpha_1^\vee+\cdots+2\alpha_{r-1}^\vee+\alpha_r^\vee\}.$$

Therefore for $\mathbf{s}_s=(s_\alpha)_{\alpha\in\Delta_{s+}}$ we have

$$\zeta_r(\mathbf{s}_s,\mathbf{0};\Delta_{s+}(B_r))=\sum_{m_1,\ldots,m_r=1}^\infty\prod_{i=1}^r\frac{1}{(2\sum_{j=r-i+1}^{r-1}m_j+m_r)^{s_i}},\tag{6.1}$$

which is a partial sum of $\zeta_r(\mathbf{s})$. For example, we have

$$\zeta_2(\mathbf{s}_s,\mathbf{0};\Delta_{s+}(B_2))=\sum_{l,m=1}^\infty\frac{1}{m^{s_1}(2l+m)^{s_2}},\tag{6.2}$$

$$\zeta_3(\mathbf{s}_s,\mathbf{0};\Delta_{s+}(B_3))=\sum_{l,m,n=1}^\infty\frac{1}{n^{s_1}(2m+n)^{s_2}(2l+2m+n)^{s_3}},\tag{6.3}$$

where $s_j=s_{\alpha_j}$ corresponding to $\alpha_j\in\Delta_{s+}$.

From the viewpoint of zeta-functions of root systems, values of (6.1) at positive integers can be regarded as the objects dual to MZVs, in the sense that B_r and C_r are dual of each other. Hence we denote (6.1) by $\zeta_r^\sharp(s_1,\ldots,s_r)$.

Since $W(B_r)\simeq W(C_r)$, just like Corollary 4.1, from Theorem 3.9 we can obtain the following result, which gives the "refined restricted sum formulas" for $\zeta_r^\sharp(\mathbf{s})$.

Corollary 6.1. *Let $\Delta=\Delta(B_r)$. For $(2\mathbf{k})_s=(2k_\alpha)_{\alpha\in\Delta_{s+}}=(2k_1,\ldots,2k_r)\in(2\mathbb{N})^r$ and $\mathbf{y}=\nu\in P^\vee/Q^\vee$,*

$$\sum_{\sigma\in\mathfrak{S}_r}\zeta_r(\sigma^{-1}(2\mathbf{k})_s,\nu;\Delta_{l+})=\frac{(-1)^r}{2^r}\mathcal{P}_{\Delta_{s+}}((2\mathbf{k})_s,\nu;\Delta)\prod_{j=1}^r\frac{(2\pi i)^{2k_j}}{(2k_j)!}\in\mathbb{Q}\cdot\pi^{2\sum_{j=1}^r k_j}.\tag{6.4}$$

In particular when $\nu = \mathbf{0}$,

$$\sum_{\sigma\in\mathfrak{S}_r} \zeta_r^\sharp(2k_{\sigma^{-1}(1)},\ldots,2k_{\sigma^{-1}(r)}) = \frac{(-1)^r}{2^r}\mathcal{P}_{\Delta_{s+}}((\mathbf{2k})_s,\mathbf{0};\Delta)\prod_{j=1}^r \frac{(2\pi i)^{2k_j}}{(2k_j)!} \in \mathbb{Q}\cdot\pi^{2\sum_{j=1}^r k_j}. \quad (6.5)$$

From Theorem 3.10, we obtain an analogue of Corollary 4.2, which is a kind of Witten's volume formula and also a B_r-type analogue of Zagier's formula (4.4).

Corollary 6.2. *Let* $\Delta = \Delta(B_r)$. *For* $(\mathbf{2k})_s = (2k,\ldots,2k)$ *with any* $k \in \mathbb{N}$,

$$\zeta_r^\sharp(2k,\ldots,2k) = \frac{(-1)^r}{2^r r!}\mathcal{P}_{\Delta_{s+}}((\mathbf{2k})_s,\mathbf{0};\Delta)\prod_{j=1}^r \frac{(2\pi i)^{2k_j}}{(2k_j)!} \in \mathbb{Q}\cdot\pi^{2kr}.$$

Example 6.3.

$$\zeta_2^\sharp(2,2) = \sum_{m,n=1}^\infty \frac{1}{n^2(2m+n)^2} = \frac{1}{320}\pi^4,$$

$$\zeta_2^\sharp(4,4) = \sum_{m,n=1}^\infty \frac{1}{n^4(2m+n)^4} = \frac{23}{14515200}\pi^8,$$

$$\zeta_2^\sharp(6,6) = \sum_{m,n=1}^\infty \frac{1}{n^6(2m+n)^6} = \frac{1369}{871782912000}\pi^{12}.$$

These formulas can be obtained by calculating the generating function of type B_2 similarly to the case of type C_2 in Example 4.4 (see Section 4). Also we can obtain these formulas by Theorem 6.4 in the case $(p,q) = (2k,2k)$ for $k \in \mathbb{N}$. However, unlike the ordinary double zeta value, these cannot be easily deduced from (4.7).

Similarly, calculating the generating function of type B_3, we have explicit examples of Corollary 6.2:

$$\zeta_3^\sharp(2,2,2) = \sum_{l,m,n=1}^\infty \frac{1}{n^2(2m+n)^2(2l+2m+n)^2} = \frac{1}{40320}\pi^6,$$

$$\zeta_3^\sharp(4,4,4) = \sum_{l,m,n=1}^\infty \frac{1}{n^4(2m+n)^4(2l+2m+n)^4} = \frac{23}{697426329600}\pi^{12},$$

$$\zeta_3^\sharp(6,6,6) = \sum_{l,m,n=1}^\infty \frac{1}{n^6(2m+n)^6(2l+2m+n)^6} = \frac{1997}{170303140572364800000}\pi^{18}.$$

Also, similarly to Proposition 5.1, we can obtain the following result whose proof will be given in Section 10.

Theorem 6.4. *For* $p,q \in \mathbb{N}_{\geq 2}$,

$$\begin{aligned}&(1+(-1)^p)\zeta_2^\sharp(p,q) + (1+(-1)^q)\zeta_2^\sharp(q,p)\\ &= 2\sum_{j=0}^{[p/2]} \frac{1}{2^{p+q-2j}}\binom{p+q-1-2j}{q-1}\zeta(2j)\zeta(p+q-2j)\\ &+ 2\sum_{j=0}^{[q/2]} \frac{1}{2^{p+q-2j}}\binom{p+q-1-2j}{p-1}\zeta(2j)\zeta(p+q-2j) - \zeta(p+q).\end{aligned} \quad (6.6)$$

Theorem 6.4 in the case that p and q are of different parity implies the following.

Corollary 6.5. *Let* $p,q \in \mathbb{N}_{\geq 2}$. *Suppose* p *and* q *are of different parity, then*

$$\zeta_2^\sharp(p,q) \in \mathbb{Q}\left[\{\zeta(j+1)\,|\,j \in \mathbb{N}\}\right],$$

which is a parity result for $\zeta_2^\sharp$.

Remark 6.6. This parity result for $\zeta_2^\sharp(p,q)$ is important in a recent study of the dimension of the linear space spanned by double zeta values of level 2 given by Kaneko and Tasaka (see [12]).

For example, setting $(p,q)=(3,2)$ in (6.6), we have

$$\zeta_2^\sharp(2,3)=\sum_{m,n=1}^\infty \frac{1}{n^2(2m+n)^3}=-\frac{21}{32}\zeta(5)+\frac{3}{8}\zeta(2)\zeta(3).$$

It should be noted that this property can be given by combining the known facts for double zeta values and for their alternating series

$$\varphi_2(s_1,s_2)=\sum_{m,n=1}^\infty \frac{(-1)^m}{n^{s_1}(m+n)^{s_2}}.$$

Actually we see that

$$\zeta_2^\sharp(s_1,s_2)=\frac{1}{2}\left\{\zeta_2(s_1,s_2)+\varphi_2(s_1,s_2)\right\}.$$

When p and q are of different parity ($p,q\in\mathbb{N}$ and $q\geq 2$), Euler proved that

$$\zeta_2(p,q)\in\mathbb{Q}\left[\{\zeta(j+1)\,|\,j\in\mathbb{N}\}\right],$$

and Borwein et al. proved that

$$\varphi_2(p,q)\in\mathbb{Q}\left[\{\zeta(j+1)\,|\,j\in\mathbb{N}\}\right]$$

(see [3]), from which Corollary 6.5 follows. However (6.6) gives more explicit information on the parity result for $\zeta_2^\sharp(p,q)$.

Furthermore we can obtain the following result which can be regarded as an analogue of Theorem 5.2 for type B_3. This can be proved similarly to Theorem 5.2, hence we omit its proof here.

Theorem 6.7. *For* $a,b,c\in\mathbb{N}_{\geq 2}$,

$$\begin{aligned}
&(1+(-1)^a)\zeta_3^\sharp(a,b,c)+(1+(-1)^b)\{\zeta_3^\sharp(b,a,c)+\zeta_3^\sharp(b,c,a)\}+(-1)^b(1+(-1)^c)\zeta_3^\sharp(c,b,a)\\
&=2^{1-a-b-c}\Bigg\{\sum_{\xi=0}^{[a/2]}2^\xi\zeta(2\xi)\sum_{\omega=0}^{a-2\xi}\binom{\omega+b-1}{\omega}\binom{a+c-2\xi-\omega-1}{c-1}\zeta_2(b+\omega,a+c-2\xi-\omega)\\
&+\sum_{\xi=0}^{[b/2]}2^\xi\zeta(2\xi)\sum_{\omega=0}^{a-1}\binom{\omega+b-2\xi}{\omega}\binom{a+c-\omega-2}{c-1}\zeta_2(b-2\xi+\omega+1,a+c-1-\omega)\\
&+(-1)^b\sum_{\xi=0}^{[c/2]}2^\xi\zeta(2\xi)\sum_{\omega=0}^{c-2\xi}\binom{\omega+b-1}{\omega}\binom{a+c-2\xi-\omega-1}{a-1}\zeta_2(b+\omega,a+c-2\xi-\omega)\\
&+(-1)^b\sum_{\xi=0}^{[b/2]}2^\xi\zeta(2\xi)\sum_{\omega=0}^{c-1}\binom{\omega+b-2\xi}{\omega}\binom{a+c-\omega-2}{a-1}\zeta_2(b-2\xi+\omega+1,a+c-1-\omega)\Bigg\}\\
&-\zeta_2^\sharp(a+b,c)-(1+(-1)^b)\zeta_2^\sharp(b,a+c)-(-1)^b\zeta_2^\sharp(b+c,a).
\end{aligned}$$

Remark 6.8. In [24], we study zeta-functions of weight lattices of semisimple compact connected Lie groups. We can prove analogues of Theorem 3.9 for those zeta-functions by a method similar to the above. We will give the details in a forthcoming paper.

7. Certain restricted sum formulas for $\zeta_r(\mathbf{s})$ and for $\zeta_r^\sharp(\mathbf{s})$

In this section, we give certain restricted sum formulas for $\zeta_r(\mathbf{s})$ and for $\zeta_r^\sharp(\mathbf{s})$ of an arbitrary depth r which essentially include known results.

As we stated in Section 1, Gangl, Kaneko and Zagier [6] obtained the restricted sum formulas (1.5) for double zeta values. Recently Nakamura [32] gave certain analogues of (1.5).

More recently, Shen and Cai [33] gave the following restricted sum formulas for triple and fourth zeta values:

$$\sum_{\substack{a_1,a_2,a_3\in\mathbb{N}\\ a_1+a_2+a_3=N}} \zeta_3(2a_1,2a_2,2a_3)=\frac{5}{8}\zeta(2N)-\frac{1}{4}\zeta(2)\zeta(2N-2)\in\mathbb{Q}\cdot\pi^{2N}\quad (N\in\mathbb{Z}_{\geq 3}), \tag{7.1}$$

$$\sum_{\substack{a_1,a_2,a_3,a_4\in\mathbb{N}\\ a_1+a_2+a_3+a_4=N}} \zeta_4(2a_1,2a_2,2a_3,2a_4)=\frac{35}{64}\zeta(2N)-\frac{5}{16}\zeta(2)\zeta(2N-2)\in\mathbb{Q}\cdot\pi^{2N}\quad (N\in\mathbb{Z}_{\geq 4}). \tag{7.2}$$

Also Machide [25] gave certain restricted sum formulas for triple zeta values.

Now recall our Corollaries 4.1 and 6.1. In the above restricted sum formulas, the summations are taken over all tuples $(a_1,\ldots,a_r)$ satisfying $a_1+\cdots+a_r=N$. On the other hand, the summations in the formulas of Corollaries 4.1 and 6.1 are running over much smaller range, that is, just all the permutations of one fixed $(a_1,\ldots,a_r)$ with $a_1+\cdots+a_r=N$. Therefore our Corollaries give subdivisions, or refinements, of known restricted sum formulas.

Summing our formulas for all tuples $(a_1,\ldots,a_r)$ satisfying $a_1+\cdots+a_r=N$, we can obtain the r-ple generalization of (1.5), (7.1) and (7.2). Moreover we can show the following further generalization, which gives a new type of restricted sum formulas.

For $d\in\mathbb{N}$ and $N\in\mathbb{N}$, let

$$I_r(d,N)=\{(2da_1,\ldots,2da_r)\in(2d\mathbb{N})^r\,|\,a_1+\cdots+a_r=N\}.$$

Denote by P_r the set of all partitions of r, namely

$$P_r=\bigcup_{\nu=1}^{r}\{(j_1,\cdots,j_\nu)\in\mathbb{N}^\nu\,|\,j_1+\cdots+j_\nu=r\}.$$

For $J=(j_1,\cdots,j_\nu)\in P_r$, we set

$$\mathcal{A}_r(d,N,J)=\left\{((2dh_1)^{[j_1]},\ldots,(2dh_\nu)^{[j_\nu]})\in I_r(d,N)\,|\,h_1<\cdots<h_\nu\right\},$$

where $(2h)^{[j]}=(2h,\ldots,2h)\in(2\mathbb{N})^j$. Then we have the following restricted sum formulas of depth r.

Theorem 7.1. *For $d\in\mathbb{N}$ and $N\in\mathbb{N}$ with $N\geq r$,*

$$\sum_{\substack{a_1,\ldots,a_r\in\mathbb{N}\\ a_1+\cdots+a_r=N}} \zeta_r(2da_1,\ldots,2da_r)$$
$$=\frac{(-1)^r}{2^r}\sum_{J=(j_1,\cdots,j_\nu)\in P_r}\frac{1}{j_1!\cdots j_\nu!}\sum_{(2d\mathbf{k})_l\in\mathcal{A}_r(d,N,J)}\mathcal{P}_{\Delta_{l+}}((2d\mathbf{k})_l,\mathbf{0};\Delta(C_r))\prod_{\rho=1}^{r}\frac{(2\pi i)^{2dk_\rho}}{(2k_\rho)!}\in\mathbb{Q}\cdot\pi^{2dN}.$$

Remark 7.2. In the case $d=1$ and $r=2,3,4$, we essentially obtain (1.5), (7.1), (7.2). Also, in the case $N=r$, we obtain Zagier's formula stated in Corollary 4.2.

Proof of Theorem 7.1. Let $(2da_1,\ldots,2da_r)\in I_r(d,N)$. Denote a set of different elements in $\{a_1,\ldots,a_r\}$ by $\{h_1,\ldots,h_\nu\}$, and put $j_\mu=\sharp\{a_m\,|\,a_m=h_\mu\}$ $(1\leq\mu\leq\nu)$. We may assume

$h_1 < \cdots < h_\nu$. We can easily see that there exist $\sigma \in \mathfrak{S}_r$ and $((2dh_1)^{[j_1]}, \ldots, (2dh_\nu)^{[j_\nu]}) \in \mathcal{A}_r(d, N, J)$ with $J = (j_1, \cdots, j_\nu) \in P_r$ such that

$$(2da_1, \ldots, 2da_r) = ((2dh_1)^{[j_1]}, \ldots, (2dh_\nu)^{[j_\nu]})^\sigma,$$

where we use the notation

$$(k_1, \ldots, k_r)^\sigma = (k_{\sigma(1)}, \ldots, k_{\sigma(r)}).$$

On the other hand, the set $\{((2dh_1)^{[j_1]}, \ldots, (2dh_\nu)^{[j_\nu]})^\tau \mid \tau \in \mathfrak{S}_r\}$ contains $j_1! \cdots j_\nu!$-copies of each element. In fact, if we denote by $\mathfrak{S}(1, ..., j_1)$ the set of all permutations among $\{1, ..., j_1\}$, then

$$\mathfrak{X}(J) := \mathfrak{S}(1, \ldots, j_1) \times \mathfrak{S}(j_1 + 1, \ldots, j_1 + j_2) \times \cdots \times \mathfrak{S}(\sum_{\rho=1}^{\nu-1} j_\rho + 1, \ldots, \sum_{\rho=1}^{\nu} j_\rho) \subset \mathfrak{S}_r$$

forms the stabilizer subgroup of $((2dh_1)^{[j_1]}, \ldots, (2dh_\nu)^{[j_\nu]})$, and hence $\sharp\mathfrak{X}(J) = j_1! \cdots j_\nu!$. Therefore, using Corollary 4.1, we have

$$\begin{aligned}
&\sum_{\substack{a_1,\ldots,a_r \in \mathbb{N} \\ a_1+\cdots+a_r=N}} \zeta_r(2da_1, \ldots, 2da_r) = \sum_{(2da_1,\ldots,2da_r) \in I_r(d,N)} \zeta_r(2da_1, \ldots, 2da_r) \\
&= \sum_{J=(j_1,\cdots,j_\nu) \in P_r} \frac{1}{j_1! \cdots j_\nu!} \sum_{\substack{(2dk_1,\ldots,2dk_r) \\ \in \mathcal{A}_r(d,N,J)}} \sum_{\sigma \in \mathfrak{S}_r} \zeta_r(2dk_{\sigma(1)}, \ldots, 2dk_{\sigma(r)}) \\
&= \frac{(-1)^r}{2^r} \sum_{J=(j_1,\cdots,j_\nu) \in P_r} \frac{1}{j_1! \cdots j_\nu!} \sum_{(2d\mathbf{k})_l \in \mathcal{A}_r(d,N,J)} \mathcal{P}_{\Delta_{l+}(C_r)}((2d\mathbf{k})_l, \mathbf{0}; \Delta) \prod_{\rho=1}^{r} \frac{(2\pi i)^{2dk_\rho}}{(2dk_\rho)!}.
\end{aligned}$$

This completes the proof. □

Similarly, using Corollary 6.1, we obtain the following.

Theorem 7.3. *For $d \in \mathbb{N}$ and $N \in \mathbb{N}$ with $N \geq r$,*

$$\begin{aligned}
&\sum_{\substack{a_1,\ldots,a_r \in \mathbb{N} \\ a_1+\cdots+a_r=N}} \zeta_r^\sharp(2da_1, \ldots, 2da_r) \\
&= \frac{(-1)^r}{2^r} \sum_{J=(j_1,\cdots,j_\nu) \in P_r} \frac{1}{j_1! \cdots j_\nu!} \sum_{(2d\mathbf{k})_s \in \mathcal{A}_r(d,N,J)} \mathcal{P}_{\Delta_{s+}}((2d\mathbf{k})_s, \mathbf{0}; \Delta(B_r)) \prod_{\rho=1}^{r} \frac{(2\pi i)^{2dk_\rho}}{(2k_\rho)!} \in \mathbb{Q} \cdot \pi^{2dN}.
\end{aligned}$$

8. Analytically closed subclass

In this section we observe our theory from the analytic point of view.

First consider the case of type C_r. In Section 4 we have shown that the zeta-functions corresponding to the sub-root system of type C_r consisting of all long roots are exactly the family of Euler-Zagier sums. On the other hand, it is known that the Euler-Zagier r-fold sum can be expressed as an integral involving the Euler-Zagier $(r-1)$-fold sum in the integrand. In fact, it holds that

$$\zeta_r(s_1, \ldots, s_r) = \frac{1}{2\pi i} \int_{(\kappa)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z)\zeta(-z)dz \tag{8.1}$$

for $r \geq 2$, where $-\Re s_r < \kappa < -1$ and the path of integral is the vertical line from $\kappa - i\infty$ to $\kappa + i\infty$ (see [26, Section 12], [27, Section 3]). This formula is proved by applying the classical Mellin-Barnes integral formula ((8.2) below), so we may call (8.1) the Mellin-Barnes integral expression of $\zeta_r(s_1, \ldots, s_r)$.

Formula (8.1) implies that the family of Euler-Zagier sums is closed under the Mellin-Barnes integral operation. (Note that the Riemann zeta-function, also appearing in the integrand, is the Euler-Zagier sum with $r = 1$.) When some family of zeta-functions is closed in this sense, we call the family *analytically closed*. The aim of this section is to prove that the subclasses of type B_r and of type A_r discussed in our theory are both analytically closed.

Proposition 8.1. *The family of zeta-functions $\zeta_r(\mathbf{s}_s, \mathbf{0}; \Delta_{s+}(B_r))$ defined by (6.1) is analytically closed.*

Proof. Recall the Mellin-Barnes formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(\kappa)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{8.2}$$

where $s, \lambda \in \mathbb{C}$ with $\Re s > 0$, $\lambda \neq 0$, $|\arg \lambda| < \pi$, κ is real with $-\Re s < \kappa < 0$.

Dividing the factor $(2(m_1 + \cdots + m_{r-1}) + m_r)^{-s_r}$ as

$$(2(m_2 + \cdots + m_{r-1}) + m_r)^{-s_r}\left(1 + \frac{2m_1}{2(m_2 + \cdots + m_{r-1}) + m_r}\right)^{-s_r}$$

and applying (8.2) to the second factor with $\lambda = 2m_1/(2(m_2 + \cdots + m_{r-1}) + m_r)$, we obtain

$$\begin{aligned}
&\zeta_r((s_1, \ldots, s_r), \mathbf{0}; \Delta_{s+}(B_r)) \\
&= \frac{1}{2\pi i}\int_{(\kappa)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{i=1}^{r-1} \frac{1}{(2\sum_{j=r-i+1}^{r-1} m_j + m_r)^{s_i}} \\
&\quad \times (2(m_2 + \cdots + m_{r-1}) + m_r)^{-s_r}\left(\frac{2m_1}{2(m_2 + \cdots + m_{r-1}) + m_r}\right)^z dz \\
&= \frac{1}{2\pi i}\int_{(\kappa)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \sum_{m_1=1}^{\infty}(2m_1)^z \\
&\times \sum_{m_2,\ldots,m_r=1}^{\infty} \prod_{i=1}^{r-2} \frac{1}{(2\sum_{j=r-i+1}^{r-1} m_j + m_r)^{s_i}}(2(m_2 + \cdots + m_{r-1}) + m_r)^{-s_{r-1}-s_r-z} dz \\
&= \frac{1}{2\pi i}\int_{(\kappa)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} 2^z \zeta(z)\zeta_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z), \mathbf{0}; \Delta_{s+}(B_{r-1}))dz. \quad (8.3)
\end{aligned}$$

This implies the assertion. □

Next we consider the subclass of type A_r which we studied in [21], and prove that it is also analytically closed. This part may be regarded as a supplement of [21].

The explicit form of the zeta-function of the root system of type A_r is given by

$$\zeta_r(\mathbf{s}, \mathbf{0}; \Delta(A_r)) = \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{h=1}^{r}\prod_{j=h}^{r}\left(\sum_{k=h}^{r+h-j} m_k\right)^{-s_{hj}} \tag{8.4}$$

(where $\mathbf{s} = (s_{hj})_{h,j}$; see [21, formula (13)]). Let $a, b \in \mathbb{N}$, $c \in \mathbb{N}_0$ with $a + b + c = r$. The main result in [21] asserts that the shuffle product procedure can be completely described by the partial fraction decomposition of zeta-functions (8.4) at special values $\mathbf{s} = \mathbf{d} = (d_{hj})_{h,j}$, where d_{hj} for

$$\begin{cases} h = 1,\ 1 \leq j \leq c \\ h = 1,\ b + c + 1 \leq j \leq a + b + c \\ h = a + 1,\ a + c + 1 \leq j \leq a + b + c \end{cases} \tag{8.5}$$

are all positive integers, and all other d_{hj} are equal to 0. Let $\Delta_+^{(a,b,c)} = \Delta_+^{(a,b,c)}(A_r)$ be the set of all positive roots corresponding to s_{hj} with (h, j) in the list (8.5). Then this is a root set, and the above special values can be interpreted as special values of zeta-functions of $\Delta_+^{(a,b,c)}$.

Theorem 8.2. *The family of zeta-functions $\zeta_r(\mathbf{s}^{(a,b,c)}, \mathbf{0}; \Delta_+^{(a,b,c)}(A_r))$ is analytically closed, where $\mathbf{s}^{(a,b,c)} = (s_{hj})_{h,j}$ with (h, j) in the list (8.5).*

Proof. We prove that zeta-functions ζ_{r+1} belonging to the above family can be expressed as a Mellin-Barnes integral, or multiple integrals, involving ζ_r also belonging to the above family. Let $a, b \in \mathbb{N}$, $c \in \mathbb{N}_0$ with $a+b+c=r$. We show that all of the zeta-functions ζ_{r+1} associated with (i) $\Delta_+^{(a+1,b,c)}$, (ii) $\Delta_+^{(a,b+1,c)}$, (iii) $\Delta_+^{(a,b,c+1)}$ have integral expressions involving the zeta-function of $\Delta_+^{(a,b,c)}$.

From (8.4) we see that

$$\begin{aligned}\zeta_r(\mathbf{s}^{(a,b,c)}, \mathbf{0}; \Delta_+^{(a,b,c)}(A_r)) = &\sum_{m_1,\ldots,m_{a+b+c}=1}^{\infty} \prod_{j=1}^{c} (m_1+m_2+\cdots+m_{a+b+c+1-j})^{-s_{1j}} \\ &\times \prod_{j=b+c+1}^{a+b+c} (m_1+m_2+\cdots+m_{a+b+c+1-j})^{-s_{1j}} \\ &\times \prod_{j=a+c+1}^{a+b+c} (m_{a+1}+m_{a+2}+\cdots+m_{2a+b+c+1-j})^{-s_{a+1,j}}, \qquad (8.6)\end{aligned}$$

which is, by renaming the variables,

$$\begin{aligned}= &\sum_{m_1,\ldots,m_{a+b+c}=1}^{\infty} (m_1+\cdots+m_{a+b+1})^{-s_{11}} \cdots (m_1+\cdots+m_{a+b+c})^{-s_{1c}} \\ &\times m_1^{-s_{21}}(m_1+m_2)^{-s_{22}} \cdots (m_1+\cdots+m_a)^{-s_{2a}} \\ &\times m_{a+1}^{-s_{31}}(m_{a+1}+m_{a+2})^{-s_{32}} \cdots (m_{a+1}+\cdots+m_{a+b})^{-s_{3b}}. \qquad (8.7)\end{aligned}$$

Now we consider the above three cases (i), (ii) and (iii) separately.

The simplest case is (iii). When we replace c by $c+1$ in (8.7), the differences are that the summation is now with respect to $m_1, \ldots, m_{a+b+c+1}$, and a new factor $(m_1+\cdots+m_{a+b+c+1})^{-s_{1,c+1}}$ appears. Dividing this factor as

$$\begin{aligned}&(m_1+\cdots+m_{a+b+c+1})^{-s_{1,c+1}} \\ &= (m_1+\cdots+m_{a+b+c})^{-s_{1,c+1}} \left(1+\frac{m_{a+b+c+1}}{m_1+\cdots+m_{a+b+c}}\right)^{-s_{1,c+1}}\end{aligned}$$

and apply (8.2) as in the argument of (8.3), we find that the sum with respect to $m_{a+b+c+1}$ is separated, which produces a Riemann zeta factor, and hence the zeta-function of $\Delta_+^{(a,b,c+1)}$ can be expressed as an integral of Mellin-Barnes type, involving gamma factors, a Riemann zeta factor, and the zeta-function of $\Delta_+^{(a,b,c)}$.

Next consider the case (ii). When we replace b by $b+1$, (8.7) is changed to

$$\begin{aligned}= &\sum_{m_1,\ldots,m_{a+b+c+1}=1}^{\infty} (m_1+\cdots+m_{a+b+2})^{-s_{11}} \cdots (m_1+\cdots+m_{a+b+c+1})^{-s_{1c}} \\ &\times m_1^{-s_{21}}(m_1+m_2)^{-s_{22}} \cdots (m_1+\cdots+m_a)^{-s_{2a}} \\ &\times m_{a+1}^{-s_{31}}(m_{a+1}+m_{a+2})^{-s_{32}} \cdots (m_{a+1}+\cdots+m_{a+b})^{-s_{3b}} \\ &\times (m_{a+1}+\cdots+m_{a+b+1})^{-s_{3,b+1}}. \qquad (8.8)\end{aligned}$$

The last factor is

$$\begin{aligned}&= (m_{a+1}+\cdots+m_{a+b})^{-s_{3,b+1}} \left(1+\frac{m_{a+b+1}}{m_{a+1}+\cdots+m_{a+b}}\right)^{-s_{3,b+1}} \\ &= (m_{a+1}+\cdots+m_{a+b})^{-s_{3,b+1}} \\ &\qquad \times \frac{1}{2\pi i}\int_{(\kappa)} \frac{\Gamma(s_{3,b+1}+z)\Gamma(-z)}{\Gamma(s_{3,b+1})} \left(\frac{m_{a+b+1}}{m_{a+1}+\cdots+m_{a+b}}\right)^{z} dz. \qquad (8.9)\end{aligned}$$

The factors $(m_1+\cdots+m_{a+b+n})^{-s_{1,n-1}}$ $(2\le n\le c+1)$ also include the term m_{a+b+1}. We divide these factors as

$$(m_1+\cdots+m_{a+b}+m_{a+b+2}+\cdots+m_{a+b+n})^{-s_{1,n-1}} \times\left(1+\frac{m_{a+b+1}}{m_1+\cdots+m_{a+b}+m_{a+b+2}+\cdots+m_{a+b+n}}\right)^{-s_{1,n-1}}$$

and apply (8.2) to obtain

$$\begin{aligned}(m_1&+\cdots+m_{a+b+n})^{-s_{1,n-1}}\\ &=(m_1+\cdots+m_{a+b}+m_{a+b+2}+\cdots+m_{a+b+n})^{-s_{1,n-1}}\\ &\times\frac{1}{2\pi i}\int_{(\kappa_n)}\frac{\Gamma(s_{1,n-1}+z_n)\Gamma(-z_n)}{\Gamma(s_{1,n-1})}\left(\frac{m_{a+b+1}}{m_1+\cdots+m_{a+b}+m_{a+b+2}+\cdots+m_{a+b+n}}\right)^{z_n}dz_n\end{aligned} \tag{8.10}$$

for $2\le n\le c+1$. Substituting (8.9) and (8.10) into (8.8), we find that the sum with respect to m_{a+b+1} is separated and gives a Riemann zeta factor $\zeta(-z_2-\cdots-z_{c+1}-z)$. Since the remaining sum produces the zeta-function of $\Delta_+^{(a,b,c)}$, we obtain that the zeta-function of $\Delta_+^{(a,b+1,c)}$ can be expressed as a $(c+1)$-ple integral of Mellin-Barnes type involving $\zeta(-z_2-\cdots-z_{c+1}-z)$ and the zeta-function of $\Delta_+^{(a,b,c)}$.

The case (i) is similar; we omit the details, only noting that in this case the variable to be separated is m_{a+1}. The proof of Theorem 8.2 is now complete. □

9. Proof of fundamental formulas

In this section we prove fundamental formulas stated in Section 3.

Lemma 9.1. *For $B\subset\Delta_+$ and $\mathbf{V}\in\mathscr{V}$, we have*

$$\mathrm{L.h.}[\mathbf{V}\cap B]=\{v\in V\mid \langle v,\mu_\beta^{\mathbf{V}}\rangle=0\ \text{for all}\ \beta\in\mathbf{V}\setminus B\}. \tag{9.1}$$

Proof. Let v be an element of the right-hand side. We write $v=\sum_{\beta\in\mathbf{V}}c_\beta\beta$ and have $c_\beta=0$ for all $\beta\in\mathbf{V}\setminus B$ and hence

$$v=\sum_{\beta\in\mathbf{V}\cap B}c_\beta\beta\in\mathrm{L.h.}[\mathbf{V}\cap B]. \tag{9.2}$$

The converse is shown similarly. □

Proof of Theorem 3.7. For $\mathbf{t}=(t_\alpha)_{\alpha\in\Delta_+}\in\mathbf{T}$, $\mathbf{y}\in V$, $\mathbf{V}\in\mathscr{V}$, $B\subset\Delta_+$ and $q\in Q^\vee/L(\mathbf{V}^\vee)$, let

$$\begin{aligned}F(\mathbf{t},\mathbf{y};\mathbf{V},B,q)=(-1)^{|B\setminus\mathbf{V}|}&\Bigl(\prod_{\gamma\in\Delta_+\setminus(\mathbf{V}\cup B)}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}\setminus B}t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle}\Bigr)\\ &\times\Bigl(\prod_{\beta\in\mathbf{V}\setminus B}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigr),\end{aligned} \tag{9.3}$$

so that

$$F(\mathbf{t},\mathbf{y};\Delta)=\sum_{\mathbf{V}\in\mathscr{V}}\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}F(\mathbf{t},\mathbf{y};\mathbf{V},\emptyset,q). \tag{9.4}$$

Assume $\mathbf{y}\in V\setminus\mathfrak{H}_{\mathscr{R}}$, and let

$$F_j=F(\mathbf{t},\mathbf{y};\mathbf{V},A_j,q). \tag{9.5}$$

We calculate $\mathfrak{D}_{\nu_{j+1}}F_j$. First, since $\mathbf{y}\notin\mathfrak{H}_{\mathscr{R}}$, noting Remark 3.5 we find that

$$\partial_{\nu_{j+1}^\vee}F_j=\Bigl(\sum_{\beta\in\mathbf{V}\setminus A_j}t_\beta\langle\nu_{j+1}^\vee,\mu_\beta^{\mathbf{V}}\rangle\Bigr)F_j. \tag{9.6}$$

Consider the case $\nu_{j+1} \in \mathbf{V}$. Then $\langle \nu_{j+1}^\vee, \mu_\beta^{\mathbf{V}} \rangle = \delta_{\nu_{j+1},\beta}$ and

$$\sum_{\beta \in \mathbf{V} \setminus A_j} t_\beta \langle \nu_{j+1}^\vee, \mu_\beta^{\mathbf{V}} \rangle = t_{j+1}, \tag{9.7}$$

where we write $t_{\nu_{j+1}} = t_{j+1}$ for brevity. Hence we have $\partial_{\nu_{j+1}^\vee} F_j = t_{j+1} F_j$. Therefore we obtain

$$\begin{aligned} \mathfrak{D}_{\nu_{j+1}} F_j = (-1)^{|A_j \setminus \mathbf{V}|} \Bigl(\prod_{\gamma \in \Delta_+ \setminus (\mathbf{V} \cup A_j)} \frac{t_\gamma}{t_\gamma - \sum_{\beta \in \mathbf{V} \setminus (A_j \cup \{\nu_{j+1}\})} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \Bigr) \\ \times \Bigl(\prod_{\beta \in \mathbf{V} \setminus (A_j \cup \{\nu_{j+1}\})} \frac{t_\beta \exp(t_\beta \{\mathbf{y} + q\}_{\mathbf{V},\beta})}{e^{t_\beta} - 1} \Bigr) \end{aligned} \tag{9.8}$$

which is equal to F_{j+1} because $\Delta_+ \setminus (\mathbf{V} \cup (A_j \cup \{\nu_{j+1}\})) = \Delta_+ \setminus (\mathbf{V} \cup A_j)$ and $|(A_j \cup \{\nu_{j+1}\}) \setminus \mathbf{V}| = |A_j \setminus \mathbf{V}|$.

Next consider the case $\nu_{j+1} \notin \mathbf{V}$. If $\langle \nu_{j+1}^\vee, \mu_\beta^{\mathbf{V}} \rangle = 0$ for all $\beta \in \mathbf{V} \setminus A_j$, then

$$\partial_{\nu_{j+1}^\vee} F_j = \Bigl(\sum_{\beta \in \mathbf{V} \setminus A_j} t_\beta \langle \nu_{j+1}^\vee, \mu_\beta^{\mathbf{V}} \rangle \Bigr) F_j = 0 \tag{9.9}$$

and hence $\mathfrak{D}_{\nu_{j+1}} F_j = 0$. Otherwise, since

$$\frac{\partial}{\partial t_{j+1}} \Big|_{t_{j+1}=0} \Bigl(\frac{t_{j+1}}{t_{j+1} - \sum_{\beta \in \mathbf{V} \setminus A_j} t_\beta \langle \nu_{j+1}^\vee, \mu_\beta^{\mathbf{V}} \rangle} \Bigr) = -\frac{1}{\sum_{\beta \in \mathbf{V} \setminus A_j} t_\beta \langle \nu_{j+1}^\vee, \mu_\beta^{\mathbf{V}} \rangle} \tag{9.10}$$

we have

$$\begin{aligned} \mathfrak{D}_\nu F_j = (-1)^{|A_j \setminus \mathbf{V}|+1} \Bigl(\prod_{\gamma \in \Delta_+ \setminus (\mathbf{V} \cup A_j \cup \{\nu_{j+1}\})} \frac{t_\gamma}{t_\gamma - \sum_{\beta \in \mathbf{V} \setminus A_j} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \Bigr) \\ \times \Bigl(\prod_{\beta \in \mathbf{V} \setminus A_j} \frac{t_\beta \exp(t_\beta \{\mathbf{y} + q\}_{\mathbf{V},\beta})}{e^{t_\beta} - 1} \Bigr). \end{aligned} \tag{9.11}$$

By noting $\mathbf{V} \setminus (A_j \cup \{\nu_{j+1}\}) = \mathbf{V} \setminus A_j$ and $|(A_j \cup \{\nu_{j+1}\}) \setminus \mathbf{V}| = |A_j \setminus \mathbf{V}| + 1$ we find that the right-hand side is equal to F_{j+1}.

We see that the condition $\langle \nu_{j+1}, \mu_\beta^{\mathbf{V}} \rangle = 0$ for all $\beta \in \mathbf{V} \setminus A_j$ is equivalent to the condition $\nu_{j+1} \in \mathrm{L.h.}[\mathbf{V} \cap A_j]$. Therefore the above results can be summarized as

$$\mathfrak{D}_{\nu_{j+1}} F_j = \begin{cases} 0 & (\nu_{j+1} \in \mathrm{L.h.}[\mathbf{V} \cap A_j]), \\ F_{j+1} & (\nu_{j+1} \notin \mathrm{L.h.}[\mathbf{V} \cap A_j]). \end{cases} \tag{9.12}$$

Hence

$$\mathfrak{D}_A F_0 = \begin{cases} 0 & (\mathbf{V} \notin \mathscr{V}_A), \\ F_N & (\mathbf{V} \in \mathscr{V}_A). \end{cases} \tag{9.13}$$

Similarly to the above calculations, we see that $\mathfrak{D}_{A,2} F_0$ gives the same result as (9.13). Thus, since $F_0 = F(\mathbf{t}, \mathbf{y}; \mathbf{V}, \emptyset, q)$, from (9.4) we obtain (3.8).

The continuity follows from the limit

$$\lim_{c \to 0+} \{\mathbf{y} + q + c\phi\}_{\mathbf{V},\beta} = \{\mathbf{y} + q\}_{\mathbf{V},\beta} \tag{9.14}$$

(see the last part of the proof of [16, Theorem 4.1].) Finally, since $F(\mathbf{t}, \mathbf{y}; \Delta)$ is holomorphic with respect to $\mathbf{t}$ around the origin, so is $\bigl(\mathfrak{D}_A F\bigr)(\mathbf{t}_{\Delta^*}, \mathbf{y}; \Delta)$ with respect to $\mathbf{t}_{\Delta^*}$. The proof of Theorem 3.7 is thus complete. □

Proof of Theorem 3.9. First assume $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathscr{R}}$. Let $\mathbf{k}' = (k'_\alpha)_{\alpha\in\Delta_+}$ with $k'_\alpha = k_\alpha$ $(\alpha \in \Delta^*)$, $k'_\alpha = 2$ $(\alpha \in \Delta_+ \setminus \Delta^* = A)$. Then by Proposition 3.3, we have

$$\begin{aligned} S(\mathbf{k}', \mathbf{y}; \Delta) &= \sum_{\lambda\in P\setminus H_{\Delta^\vee}} e^{2\pi i\langle \mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee,\lambda\rangle^{k'_\alpha}} \\ &= \sum_{w\in W} \Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k'_\alpha} \Bigr) \zeta_r(w^{-1}\mathbf{k}', w^{-1}\mathbf{y}; \Delta) \\ &= (-1)^{|\Delta_+|} \mathcal{P}(\mathbf{k}', \mathbf{y}; \Delta) \Bigl(\prod_{\alpha\in\Delta_+} \frac{(2\pi i)^{k'_\alpha}}{k'_\alpha!} \Bigr). \end{aligned} \tag{9.15}$$

Applying $\prod_{\alpha\in A} \partial^2_{\alpha^\vee}$ to the above. From the first line we observe that each $\partial^2_{\alpha^\vee}$ produces the factor $(2\pi i\langle\alpha^\vee,\lambda\rangle)^2$. Hence the factor $\zeta_r(w^{-1}\mathbf{k}', w^{-1}\mathbf{y}; \Delta)$ on the second line is transformed into $(2\pi i)^{2|A|}\zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; \Delta)$. Therefore we have

$$\begin{aligned} &(2\pi i)^{2|A|} \sum_{w\in W} \Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \Bigr) \zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; \Delta) \\ &\qquad = (-1)^{|\Delta_+|} \Bigl(\prod_{\alpha\in A} \partial^2_{\alpha^\vee} \Bigr) \mathcal{P}(\mathbf{k}', \mathbf{y}; \Delta) \Bigl(\prod_{\alpha\in\Delta_+} \frac{(2\pi i)^{k'_\alpha}}{k'_\alpha!} \Bigr). \end{aligned} \tag{9.16}$$

Since

$$\Bigl(\prod_{\alpha\in\Delta_+} \frac{(2\pi i)^{k'_\alpha}}{k'_\alpha!} \Bigr) = \Bigl(\prod_{\alpha\in\Delta^*} \frac{(2\pi i)^{k'_\alpha}}{k'_\alpha!} \Bigr) \Bigl(\prod_{\alpha\in A} \frac{(2\pi i)^2}{2!} \Bigr),$$

we have

$$\sum_{w\in W} \Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \Bigr) \zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; \Delta) = (-1)^{|\Delta_+|} \Bigl(\prod_{\alpha\in A} \frac{1}{2}\partial^2_{\alpha^\vee} \Bigr) \mathcal{P}(\mathbf{k}', \mathbf{y}; \Delta) \Bigl(\prod_{\alpha\in\Delta^*} \frac{(2\pi i)^{k'_\alpha}}{k'_\alpha!} \Bigr). \tag{9.17}$$

From (3.1) it follows that

$$\Bigl(\prod_{\alpha\in A} \frac{1}{2} \frac{\partial^2}{\partial t_\alpha^2}\Big|_{t_\alpha=0} \partial^2_{\alpha^\vee} \Bigr) F(\mathbf{t}, \mathbf{y}; \Delta) = \sum_{\substack{\mathbf{m}=(m_\alpha)_{\alpha\in\Delta_+} \\ m_\alpha\in\mathbb{N}_0(\alpha\in\Delta^*) \\ m_\alpha=2(\alpha\in A)}} \Bigl(\prod_{\alpha\in A} \frac{1}{2}\partial^2_{\alpha^\vee} \Bigr) \mathcal{P}(\mathbf{m}, \mathbf{y}; \Delta) \prod_{\alpha\in\Delta^*} \frac{t_\alpha^{m_\alpha}}{m_\alpha!}. \tag{9.18}$$

By Theorem 3.7, we see that the left-hand side of (9.18) is equal to

$$F_{\Delta^*}(\mathbf{t}_{\Delta^*}, \mathbf{y}; \Delta) = \sum_{\mathbf{m}_{\Delta^*}\in\mathbb{N}_0^{|\Delta^*|}} \mathcal{P}_{\Delta^*}(\mathbf{m}_{\Delta^*}, \mathbf{y}; \Delta) \prod_{\alpha\in\Delta^*} \frac{t_\alpha^{m_\alpha}}{m_\alpha!}. \tag{9.19}$$

Comparing (9.18) with (9.19) we find that

$$\Bigl(\prod_{\alpha\in A} \frac{1}{2}\partial^2_{\alpha^\vee} \Bigr) \mathcal{P}(\mathbf{k}', \mathbf{y}; \Delta) = \mathcal{P}_{\Delta^*}(\mathbf{k}_{\Delta^*}, \mathbf{y}; \Delta).$$

Therefore (9.17) implies the desired result when $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathscr{R}}$. By the continuity with respect to $\mathbf{y}$, the result is also valid in the case when $\mathbf{y} \in \mathfrak{H}_{\mathscr{R}}$. □

Remark 9.2. It is possible to prove Theorem 3.9 by use of $\mathfrak{D}_A$ instead of $\mathfrak{D}_{A,2}$. In this method, we need to consider the case $k_\alpha = 1$ for some $\alpha \in A$ and such an argument is indeed valid. (See [19, Remark 3.2].)

10. Proofs of Theorems 5.2 and 6.4

In this final section we prove Theorems 5.2 and 6.4. The basic principle of the proofs of these theorems is similar to that of the argument developed in [18, Section 7]. We first state the following lemma.

Lemma 10.1. *For an arbitrary function $f : \mathbb{N}_0 \to \mathbb{C}$ and $d \in \mathbb{N}$, we have*

$$\sum_{k=0}^{d} \phi(d-k)\varepsilon_{d-k} \sum_{\nu=0}^{k} f(k-\nu)\frac{(i\pi)^\nu}{\nu!} = -\frac{i\pi}{2} f(d-1) + \sum_{\xi=0}^{[d/2]} \zeta(2\xi) f(d-2\xi), \tag{10.1}$$

where we denote the integer part of $x \in \mathbb{R}$ by $[x]$, $\varepsilon_j = (1+(-1)^j)/2$ $(j \in \mathbb{Z})$ and $\phi(s) = \sum_{m\geq 1}(-1)^m m^{-s} = \left(2^{1-s}-1\right)\zeta(s)$.

Proof. This can be immediately obtained by combining (2.6) and (2.7) (with the choice $g(x) = i\pi f(x-1)$) in [29, Lemma 2.1]. □

Proof of Theorem 5.2. From [18, (4.31) and (4.32)], we have

$$\sum_{n\in\mathbb{Z}^*} \frac{(-1)^n e^{in\theta}}{n^a} - 2\sum_{j=0}^{a} \phi(a-j)\varepsilon_{a-j}\frac{(i\theta)^j}{j!} = 0 \tag{10.2}$$

for $a \geq 2$ and $\theta \in [-\pi, \pi]$, where $\mathbb{Z}^* = \mathbb{Z} \smallsetminus \{0\}$. For $x, y \in \mathbb{R}$ with $|x| < 1$ and $|y| < 1$, multiply the above by

$$\sum_{l,m\in\mathbb{N}} (-1)^{l+m} x^l y^m e^{i(l+m)\theta}. \tag{10.3}$$

Separating the terms corresponding to $l + m + n = 0$, we obtain

$$\begin{aligned} &\sum_{l,m\in\mathbb{N}} \sum_{\substack{n\in\mathbb{Z}^* \\ l+m+n\neq 0}} \frac{(-1)^{l+m+n} x^l y^m e^{i(l+m+n)\theta}}{n^a} \\ &- 2\sum_{j=0}^{a} \phi(a-j)\varepsilon_{a-j} \sum_{l,m\in\mathbb{N}} (-1)^{l+m} x^l y^m e^{i(l+m)\theta} \frac{(i\theta)^j}{j!} \\ &= -(-1)^a \sum_{l,m\in\mathbb{N}} \frac{x^l y^m}{(l+m)^a} \end{aligned}$$

for $\theta \in [-\pi, \pi]$. The right-hand side of the above is constant with respect to θ. Therefore we can apply [18, Lemma 6.2] with $h = 1$, $a_1 = a$, $d = c \geq 2$,

$$C(N) = \sum_{\substack{l,m\in\mathbb{N}, n\in\mathbb{Z}^* \\ l+m+n=N}} \frac{x^l y^m}{n^a},$$

$$D(N; r; 1) = \begin{cases} \sum_{\substack{l,m\in\mathbb{N} \\ l+m=N}} x^l y^m & (N \geq 2, r = 0), \\ 0 & \text{(otherwise)} \end{cases}$$

in the notation of [18]. The result is

$$\begin{aligned} &\sum_{l,m\in\mathbb{N}} \sum_{\substack{n\in\mathbb{Z}^* \\ l+m+n\neq 0}} \frac{(-1)^{l+m+n} x^l y^m e^{i(l+m+n)\theta}}{n^a (l+m+n)^c} \\ &- 2\sum_{j=0}^{a} \phi(a-j)\varepsilon_{a-j} \sum_{\xi=0}^{j} \binom{j-\xi+c-1}{j-\xi} (-1)^{j-\xi} \sum_{l,m\in\mathbb{N}} \frac{(-1)^{l+m} x^l y^m e^{i(l+m)\theta}}{(l+m)^{c+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \end{aligned}$$

$$+2\sum_{j=0}^{c}\phi(c-j)\varepsilon_{c-j}\sum_{\xi=0}^{j}\binom{j-\xi+a-1}{a-1}(-1)^{a-1}\sum_{l,m\in\mathbb{N}}\frac{x^l y^m}{(l+m)^{a+j-\xi}}\frac{(i\theta)^\xi}{\xi!}=0.$$

Replace x by $-xe^{-i\theta}$ and separate the term corresponding to $m+n=0$ in the first member on the left-hand side, and apply [18, Lemma 6.2] again with $d=b\geq 2$. Then we can obtain

$$\begin{aligned}
&\sum_{l,m\in\mathbb{N}}\sum_{\substack{n\in\mathbb{Z}^*\\ m+n\neq 0\\ l+m+n\neq 0}}\frac{(-1)^{m+n}x^l y^m e^{i(m+n)\theta}}{n^a(m+n)^b(l+m+n)^c}\\
&=2\sum_{j=0}^{a}\phi(a-j)\varepsilon_{a-j}\sum_{\xi=0}^{j}\sum_{\omega=0}^{j-\xi}\binom{\omega+b-1}{\omega}(-1)^\omega\binom{j-\xi-\omega+c-1}{c-1}(-1)^{j-\xi-\omega}\\
&\qquad\times\sum_{l,m\in\mathbb{N}}\frac{(-1)^m x^l y^m e^{im\theta}}{m^{b+\omega}(l+m)^{c+j-\xi-\omega}}\frac{(i\theta)^\xi}{\xi!}\\
&-2\sum_{j=0}^{b}\phi(b-j)\varepsilon_{b-j}\sum_{\xi=0}^{j}\sum_{\omega=0}^{a-1}\binom{\omega+j-\xi}{\omega}(-1)^\omega\binom{a-1-\omega+c-1}{c-1}(-1)^{a-1-\omega}\\
&\qquad\times\sum_{l,m\in\mathbb{N}}\frac{x^l y^m}{m^{j-\xi+\omega+1}(l+m)^{a+c-1-\omega}}\frac{(i\theta)^\xi}{\xi!}\\
&-2\sum_{j=0}^{c}\phi(c-j)\varepsilon_{c-j}\sum_{\xi=0}^{j}\sum_{\omega=0}^{j-\xi}\binom{\omega+b-1}{\omega}(-1)^\omega\binom{j-\xi-\omega+a-1}{a-1}(-1)^{a-1}\\
&\qquad\times\sum_{l,m\in\mathbb{N}}\frac{(-1)^l x^l y^m e^{-il\theta}}{(-l)^{b+\omega}(l+m)^{a+j-\xi-\omega}}\frac{(i\theta)^\xi}{\xi!}\\
&+2\sum_{j=0}^{b}\phi(b-j)\varepsilon_{b-j}\sum_{\xi=0}^{j}\sum_{\omega=0}^{c-1}\binom{\omega+j-\xi}{\omega}(-1)^\omega\binom{a-1-\omega+c-1}{a-1}(-1)^{a-1}\\
&\qquad\times\sum_{l,m\in\mathbb{N}}\frac{x^l y^m}{(-l)^{j-\xi+\omega+1}(l+m)^{a+c-1-\omega}}\frac{(i\theta)^\xi}{\xi!}.
\end{aligned}\tag{10.4}$$

Since $a,b,c\geq 2$, we can let $x,y\to 1$ on the both sides because of absolute convergence. Then set $\theta=\pi$, and consider the left-hand side of the resulting formula first. The contribution of the terms corresponding to $m+2n=0$ is obviously $(-1)^a\zeta_2(a+b,c)$. The contribution of the terms corresponding to $l+m+2n=0$ is (with rewriting $-n$ by n)

$$(-1)^a\sum_{\substack{m,n\in\mathbb{N}\\ m\neq n, m<2n}}\frac{1}{n^{a+c}(m-n)^b},$$

which is, by separating into two parts according to $n<m<2n$ and $0<m<n$, equal to $(-1)^a(1+(-1)^b)\zeta_2(b,a+c)$. We can also see that the contribution of the terms corresponding to $l+2m+2n=0$ is

$$(-1)^a\sum_{\substack{m,n\in\mathbb{N}\\ n>m}}\frac{1}{n^a(m-n)^b(n-m)^c}=(-1)^{a+b}\zeta_2(b+c,a).$$

The remaining part of the left-hand side is

$$\sum_{l,m\in\mathbb{N}} \sum_{\substack{n\in\mathbb{Z}^* \\ m+n\neq 0 \\ m+2n\neq 0 \\ l+m+n\neq 0 \\ l+m+2n\neq 0 \\ l+2m+2n\neq 0}} \frac{1}{n^a(m+n)^b(l+m+n)^c}$$

$$= \zeta_3(a,b,c) + (-1)^a \sum_{l,m\in\mathbb{N}} \sum_{\substack{n\in\mathbb{N} \\ m\neq n \\ m\neq 2n \\ l+m\neq n \\ l+m\neq 2n \\ l+2m\neq 2n}} \frac{1}{n^a(m-n)^b(l+m-n)^c}.$$

On the above double sum, replace $j=m-n$ and $k=n-m$ correspondingly to $m>n$ and $m<n$, respectively. On the part corresponding to $m>n$, we further divide the sum into three parts according to $l+j<n$, $j<n<l+j$, $n<j$ and find that the contribution of this part is

$$(-1)^a\left\{\zeta_3(b,c,a)+\zeta_3(b,a,c)+\zeta_3(a,b,c)\right\}.$$

Similarly we treat the part $m<n$. Collecting the above results, we obtain that the left-hand side is

$$\begin{aligned}(-1)^a\Big\{&(1+(-1)^a)\zeta_3(a,b,c)+(1+(-1)^b)\left(\zeta_3(b,a,c)+\zeta_3(b,c,a)\right)\\ &+(-1)^b(1+(-1)^c)\zeta_3(c,b,a)+\zeta_2(a+b,c)\\ &+(1+(-1)^b)\zeta_2(b,a+c)+(-1)^b\zeta_2(b+c,a)\Big\}.\end{aligned}$$

On the other hand, applying Lemma 10.1, we can rewrite the right-hand side to

$$\begin{aligned}2(-1)^a\Bigg\{&\sum_{\xi=0}^{[a/2]}\zeta(2\xi)\sum_{\omega=0}^{a-2\xi}\binom{\omega+b-1}{\omega}\binom{a+c-2\xi-\omega-1}{c-1}\zeta_2(b+\omega,a+c-2\xi-\omega)\\ &+\sum_{\xi=0}^{[b/2]}\zeta(2\xi)\sum_{\omega=0}^{a-1}\binom{\omega+b-2\xi}{\omega}\binom{a+c-\omega-2}{c-1}\zeta_2(b-2\xi+\omega+1,a+c-1-\omega)\\ &+(-1)^b\sum_{\xi=0}^{[c/2]}\zeta(2\xi)\sum_{\omega=0}^{c-2\xi}\binom{\omega+b-1}{\omega}\binom{a+c-2\xi-\omega-1}{a-1}\zeta_2(b+\omega,a+c-2\xi-\omega)\\ &+(-1)^b\sum_{\xi=0}^{[b/2]}\zeta(2\xi)\sum_{\omega=0}^{c-1}\binom{\omega+b-2\xi}{\omega}\binom{a+c-\omega-2}{a-1}\zeta_2(b-2\xi+\omega+1,a+c-1-\omega)\Bigg\}.\end{aligned}$$

This completes the proof of Theorem 5.2. □

Finally we give the proof of Theorem 6.4.

Proof of Theorem 6.4. Let $p\in\mathbb{N}_{\geq 2}$ and $s\in\mathbb{R}_{>1}$. It follows from [23, Equation (4.7)] that

$$\begin{aligned}&\sum_{\substack{l\in\mathbb{Z}^*,m\in\mathbb{N}\\ l+m\neq 0}}\frac{(-1)^{l+m}x^m e^{i(l+m)\theta}}{l^p m^s}-2\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^s}\right\}\frac{(i\theta)^j}{j!}\\ &+(-1)^p\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p}}=0\end{aligned}$$

for $\theta \in [-\pi, \pi]$ and $x \in \mathbb{C}$ with $|x| \le 1$. Setting $x = -e^{i\theta}$ on the both sides and separating the term corresponding to $l + 2m = 0$ of the first term on the left-hand side, we have

$$\sum_{\substack{l\in\mathbb{Z}^*, m\in\mathbb{N} \\ l+m\neq 0 \\ l+2m\neq 0}} \frac{(-1)^l e^{i(l+2m)\theta}}{l^p m^s} - 2\sum_{j=0}^{p} \phi(p-j)\varepsilon_{p-j} \left\{ \sum_{m=1}^{\infty} \frac{e^{2im\theta}}{m^s} \right\} \frac{(i\theta)^j}{j!} + (-1)^p \sum_{m=1}^{\infty} \frac{(-1)^m e^{im\theta}}{m^{s+p}} = -\sum_{m=1}^{\infty} \frac{1}{(-2m)^p m^s}.$$

By [18, Lemma 6.2] with $d = q \ge 2$, we obtain

$$\begin{aligned} &\sum_{\substack{l\in\mathbb{Z}^*, m\in\mathbb{N} \\ l+m\neq 0 \\ l+2m\neq 0}} \frac{(-1)^l e^{i(l+2m)\theta}}{l^p m^s (l+2m)^q} \\ &= 2\sum_{j=0}^{p} \phi(p-j)\varepsilon_{p-j} \sum_{\xi=0}^{j} \binom{j-\xi+q-1}{j-\xi} \frac{(-1)^{j-\xi}}{2^{q+j-\xi}} \sum_{m=1}^{\infty} \frac{e^{2im\theta}}{m^{s+q+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ &- 2\sum_{j=0}^{q} \phi(q-j)\varepsilon_{q-j} \sum_{\xi=0}^{j} \binom{j-\xi+p-1}{j-\xi} \frac{(-1)^{p-1}}{2^{p+j-\xi}} \sum_{m=1}^{\infty} \frac{1}{m^{s+p+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ &- (-1)^p \sum_{m=1}^{\infty} \frac{(-1)^m e^{im\theta}}{m^{s+p+q}}. \end{aligned} \tag{10.5}$$

Let $\theta = \pi$ and using Lemma 10.1. Then the right-hand side of (10.5) is equal to

$$\begin{aligned} &2(-1)^p \sum_{\xi=0}^{[p/2]} \frac{1}{2^{p+q-2\xi}} \binom{p+q-1-2\xi}{q-1} \zeta(2\xi)\zeta(s+p+q-2\xi) \\ &+ 2(-1)^p \sum_{\xi=0}^{[q/2]} \frac{1}{2^{p+q-2\xi}} \binom{p+q-1-2\xi}{p-1} \zeta(2\xi)\zeta(s+p+q-2\xi) \\ &- (-1)^p \zeta(s+p+q). \end{aligned} \tag{10.6}$$

On the other hand, we can see that the left-hand side can be written in terms of the zeta-function of B_2. Recall that

$$\begin{aligned} \zeta_2(s_1, s_2, s_3, s_4; B_2) &= \zeta_2((s_1, s_2, s_3, s_4), \mathbf{0}; \Delta(B_2)) \\ &= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} (m_1+m_2)^{s_3} (2m_1+m_2)^{s_4}}. \end{aligned}$$

The contribution of the terms with $l > 0$ to the left-hand side is obviously $\zeta_2(s, p, 0, q; B_2)$. As for the terms with $l < 0$, we rewrite $-l$ by l, divide the sum into three parts according to the conditions $l < m$, $m < l < 2m$ and $l > 2m$, and evaluate each part in terms of the zeta-function of B_2. The conclusion is that the left-hand side is

$$\begin{aligned} &\zeta_2(s, p, 0, q; B_2) + (-1)^p \zeta_2(0, p, s, q; B_2) + (-1)^p \zeta_2(0, q, s, p; B_2) \\ &\quad + (-1)^{p+q} \zeta_2(s, q, 0, p; B_2). \end{aligned} \tag{10.7}$$

We combine (10.6) and (10.7) and multiply by $(-1)^p$. Then we can set $s = 0$ because (10.6) and (10.7) are absolutely convergent for $s > -1$. Noting $\zeta_2(0, p, 0, q; B_2) = \zeta_2^\sharp(p, q)$, we complete the proof of Theorem 6.4. □

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