Triangles on planar Jordan C^1-curves

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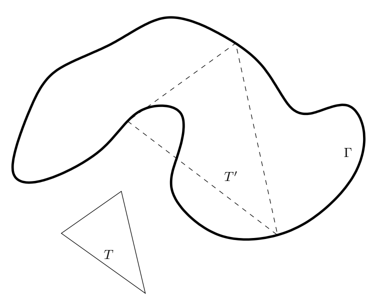
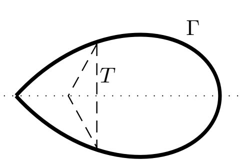
Abstract

We prove that a Jordan $\mathcal{C}^1$-curve in the plane contains any non-flat triangle up to translation and homothety with positive ratio. This is false if the curve is not C^1. The proof uses a bit configuration spaces, differential and algebraic topology as well as the Schoenflies theorem.

The aim of this note is to prove the main theorem below, in which the following standard definitions are used. A *Jordan $\mathcal{C}^1$-curve* Γ is a connected closed $\mathcal{C}^1$-submanifold of the plane. Equivalently, Γ is the image of an injective $\mathcal{C}^1$-immersion of the unit circle S^1 into the plane. A triangle is *flat* if it is contained in a straight line.

Main theorem. *Let Γ be a Jordan $\mathcal{C}^1$-curve and let T be a non-flat triangle in the plane. Then, by a translation and a homothety with positive ratio, T may be transformed into a triangle whose vertices lie on Γ.*





The main theorem is wrong if the curve Γ is not of class $\mathcal{C}^1$, as seen by the example of a half-lemniscate, with parametrization in polar coordinates given by $r(\theta) = \cos 2\theta$ ($|\theta| \leq \pi/4$). Let T be an isosceles triangle with a vertical basis AB and the third vertex C on the left of AB. If T is on Γ, then $C = 0$ by symmetry (since Γ and T are invariant under the reflection through the horizontal axis). If the angle at C is obtuse, this is impossible since Γ lies in in the polar domain $|\theta| \leq \pi/4$. Note that Γ is of class $\mathcal{C}^\infty$ except at 0, where the two tangents form a right angle.

However, a Jordan $\mathcal{C}^0$-curve contains any non-flat triangle up to similarity. This result was established in [8] with an elementary proof (see also [7, Section 11, Theorem 1.3]).

The proof of the main theorem is given in the last section while the previous ones are devoted to preliminary material. A version of the proof was written by Véronique Gonoyan in her master thesis (University of Geneva, 2003). Discussions with Anton Alekseev were useful.

1 The space of triangles

Identifying the plane with $\mathbb{C}$, the *space of triangles* (not reduced to a single point) Tri^0 is the smooth manifold

$$\mathrm{Tri}^0 = \{(z_0, z_1, z_2)) \in \mathbb{C}^3 - \Delta\}\,,$$

where $\Delta = \{(z,z,z)\}$ is the diagonal subset of $\mathbb{C}^3$. Let $\mathcal{G}_1 \approx \mathbb{C}$ be the group of translations of $\mathbb{C}$. The diagonal $\mathcal{G}_1$-action on $\mathbb{C}^3$ preserves Δ and is free and proper. Hence, $\mathrm{Tri}^1 = \mathrm{Tri}^0/\mathcal{G}_1$ inherits a structure of a smooth manifold and the correspondence $(z_0, z_1, z_2) \mapsto (z_1 - z_0, z_2 - z_0)$ induces a diffeomorphism

$$\mathrm{Tri}^1 \approx \mathbb{C}^2 - \{(0,0)\}\,.$$

Let $\mathcal{G}_2$ be the group of homothety of $\mathbb{C}$ with a positive ratio (isomorphic to the multiplicative group $\mathbb{R}_{>0}$). Again,

$$\mathrm{Tri} = \mathrm{Tri}^1/\mathcal{G}_2$$

is a smooth manifold and one has a diffeomorphism

$$\mathrm{Tri} \approx \left(\mathbb{C}^2 - \{(0,0)\}\right)\Big/\mathbb{R}_{>0} \approx S^3$$

from Tri to the standard sphere S^3.

Finally, the group $\mathcal{G}_3 \approx S^1$ of rotations of $\mathbb{C}$ acts on Tri^1 and Tri and one has diffeomorphisms

$$\mathrm{Tri}/\mathcal{G}_3 \approx S^3/S^1 \approx \mathbb{C}P^1 \approx \hat{\mathbb{C}}\,,$$

where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The following diagram

$$\begin{array}{ccccc} \mathrm{Tri}^1 & \twoheadrightarrow & \mathrm{Tri} & \twoheadrightarrow & \mathrm{Tri}/\mathcal{G}_3 \\ \downarrow{\scriptstyle\approx} & & & & \downarrow{\scriptstyle\approx} \\ \mathbb{C}^2 - \{(0,0)\} & & \xrightarrow{\ \beta\ } & & \hat{\mathbb{C}} \end{array}$$

is commutative, where β is the classical Hopf map

$$\beta(z_1, z_2) = \begin{cases} \dfrac{z_1}{z_2} & \text{if } z_2 \neq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Let Tri^0_{fl} be the subspace of Tri^0 formed by those triangles which are *flat*, namely contained in a line. Denote its images in Tri^1 (respectively: in Tri) by Tri^1_{fl} and (respectively: Tri_{fl}). The space Tri^1_{fl} in $\mathrm{Tri}^1 \approx \mathbb{C}^2 - \{(0,0)\}$ is formed by the couples (z_1, z_2) of complex numbers which are $\mathbb{R}$-linearly dependent. By the above definition of β, the image of Tri_{fl} in $\mathrm{Tri}/\mathcal{G}_3 \approx \hat{\mathbb{C}}$ is equal to $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. As β is a circle bundle, the surface $\mathcal{T} = \beta^{-1}(\hat{\mathbb{R}})$ is a circle bundle over $\hat{\mathbb{R}} \approx S^1$. Since τ separates S^3, it is orientable and thus diffeomorphic to a 2-torus. This proves the following proposition.

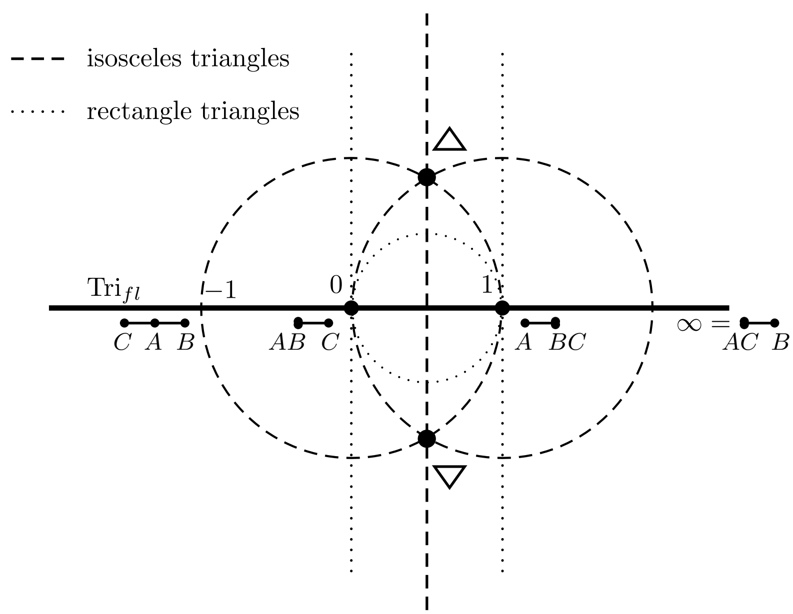
Proposition 1.1. *There is a diffeomorphism of manifold pairs*

$$(\mathrm{Tri}, \mathrm{Tri}_{fl}) \approx (S^3, \mathcal{T})$$

where $\mathcal{T}$ is a 2-torus in S^3, separating S^3 into two components. □

Remark 1.2. The reader may easily check the following facts about the images in $\mathrm{Tri}/\mathcal{G}_2 \approx \hat{\mathbb{C}}$ of the following subsets of Tri^0.

- the flats triangles (z_0, z_1, z_2) with two identical vertices have image equal to 1 (if $z_1 = z_2$), 0 (if $z_0 = z_1$) and ∞ (if $z_0 = z_2$).
- The equilateral triangles have image $e^{\pm i\pi/3}$.
- The isosceles triangles have image the circles of equation $|z| = 1$, $|z-1| = 1$ and the vertical line through 1/2 (union $\{\infty\}$).
- The rectangles triangles have image the circle $|z - 1/2| = 1/2$ and the two vertical lines (union $\{\infty\}$) through 0 and 1.



By a Moebius transformation h of $\bar{\mathbb{R}}^3$, one can move $\bar{\mathbb{C}}$ onto the unit sphere S^2, so that the equilateral triangles are the poles and Tri_{fl} is the equator. The locus of isosceles triangles is then formed by three meridians, dividing S^2 into six equal sectors (since h preserves the angles). Such pictures are used in statistical shape analysis (see e.g. [2, p. 37]).

2 The main map

2.1 Definitions

Let $c: S^1 \to \mathbb{C}$ be a $\mathcal{C}^1$-embedding (injective $\mathcal{C}^1$-immersion), parametrizing a Jordan curve Γ and let $\tilde{c}: \mathbb{R} \to \mathbb{C}$ be defined by $\tilde{c}(t) = c(e^{2i\pi t})$. That c is an immersion is equivalent to $\dot{\tilde{c}}(t) \neq 0$ for all $t \in \mathbb{R}$.

Let $V = (S^1)^3 - \Delta$ and define the map $F^0: V \to \mathrm{Tri}^0$ by

$$F^0(s_0, s_1, s_2) = (c(s_0), c(s_1), c(s_2))\,.$$

Taking the images in Tri^1 and Tri provides maps

$$F^1: V \to \mathrm{Tri}^1 \quad , \quad F^1(s_0, s_1, s_2) = (c(s_1) - c(s_0), c(s_2) - c(s_0))$$

and

$$F: V \to \mathrm{Tri}\,. \tag{2.1}$$

One can pre-compose the above maps with $\exp: \mathbb{R}^3 \to (S^1)^3$, the universal covering defined by $\exp(t_0, t_1, t_2) = (e^{2i\pi t_0}, e^{2i\pi t_1}, e^{2i\pi t_2})$. This provides maps $F_e^1: W \to \mathrm{Tri}^1$ and $F_e: W \to \mathrm{Tri}$, where $W = \mathbb{R}^3 - \tilde{\Delta}$ with

$$\tilde{\Delta} = \exp^{-1}(\Delta) = \{(t+p, t+q, t+r) \in \mathbb{R}^3 \mid t \in \mathbb{R} \text{ and } (p,q,r) \in \mathbb{Z}^3)\}\,.$$

All these maps are of class $\mathcal{C}^1$ and the maps F_e^0, F_e^1 and F_e are invariant under the $\mathbb{Z}^3$-action on W by translation. It will be useful to know what are the critical points of F. Recall that a point $x \in M$ is *critical* for a $\mathcal{C}^1$-map $f: M \to N$ between manifolds whenever the tangent map $T_x f: T_x M \to T_{f(x)} N$ is not surjective.

Proposition 2.2. *A point $(s_0, s_1, s_2) \in (S^1)^3 - \Delta$ is a critical point for the map F if and only if the tangents to Γ at the points $c(t_0)$, $c(t_1)$ and $c(t_2)$ are parallel or concurrent.*

Proof. It is equivalent to prove the statement for the map F_e, since exp is a covering. Since $\dim W = \dim \mathrm{Tri}$, a point $(t_0, t_1, t_2) \in W$ is critical for F_e if and only if $T_{(t_0,t_1,t_2)} F_e$ is not injective. Recall that, if U is an open subset of a real vector space X, the tangent space $T_u U$ at each point $u \in U$ is canonically identified with X. Under such identification, one has

$$T_{(t_0,t_1,t_2)} F_e^1(\lambda_0, \lambda_1, \lambda_2) = (\lambda_1 \dot{\tilde{c}}(t_1) - \lambda_0 \dot{\tilde{c}}(t_0), \lambda_2 \dot{\tilde{c}}(t_2) - \lambda_0 \dot{\tilde{c}}(t_0))\,.$$

The point $(t_0, t_1, t_2) \in W$ is critical for F if and only if

$$T_{(t_0,t_1,t_2)} F_e^1(\lambda_0, \lambda_1, \lambda_2) \in \ker T_{F_e(t_0,t_1,t_2)} \pi$$

for all $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$, where $\pi\colon \mathrm{Tri}^1 \to \mathrm{Tri}$ is the quotient map. But

$$\ker T_{(z_1,z_2)} \pi = \mathbb{R} \cdot (z_1, z_2)\,.$$

Indeed, $\mathbb{R}_{>0}$-action on Tri^1 corresponds to the "dilatation flow" $\Phi_t(z_1, z_2) = t(z_1, z_2)$ and $\frac{d}{dt}\Phi_t(z_1, z_2)_{|t=0} = (z_1, z_2)$. Therefore, $(t_0, t_1, t_2) \in W$ is critical for F if and only if there exists $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \lambda_1 \dot{\tilde{c}}(t_1) - \lambda_0 \dot{\tilde{c}}(t_0) & = & \lambda(\tilde{c}(t_1) - \tilde{c}(t_0)) \\ \lambda_2 \dot{\tilde{c}}(t_2) - \lambda_0 \dot{\tilde{c}}(t_0) & = & \lambda(\tilde{c}(t_2) - \tilde{c}(t_0))\,. \end{cases} \tag{2.3}$$

If $\lambda = 0$, the above system is equivalent to

$$\lambda_1 \dot{\tilde{c}}(t_1) = \lambda_0 \dot{\tilde{c}}(t_0) = \lambda_2 \dot{\tilde{c}}(t_2)$$

which is equivalent to the parallelism of the tangents to Γ at the points $c(t_0)$, $c(t_1)$ and $c(t_2)$. If $\lambda \neq 0$, then replacing λ_j by $\pm\lambda_j/\lambda$ in (2.3) makes the system equivalent to

$$\tilde{c}(t_1) + \mu_1 \dot{\tilde{c}}(t_1) = \tilde{c}(t_0) + \mu_0 \dot{\tilde{c}}(t_0) = \tilde{c}(t_2) + \mu_2 \dot{\tilde{c}}(t_2)$$

for some $(\mu_0, \mu_1, \mu_2) \in \mathbb{R}^3$. This is equivalent to the concurrency of the three tangents. □

2.2 Compactifications

We now define boundary compactifications $W \subset \hat{W}$ and $V \subset \hat{V}$ and extend the maps F_e and F to the manifolds with boundary $\hat{W}$ and $\hat{V}$. Let $D^2_{1/2} = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1/4\}$ be the disk of radius $1/2$, with boundary $S^1_{1/2}$. Let us parameterize an open tubular neighborhood of $\tilde{\Delta}$ in $\mathbb{R}^3$ by the embedding $h\colon \tilde{\Delta} \times S^1_{1/2} \times [0, 1) \to \mathbb{R}^3$ defined by

$$h(t + p, t + q, t + r, (u, v), \lambda) = (t + p, t + q + \lambda u, t + r + \lambda v)\,.$$

Define

$$\hat{W} = \Big(\big\{\tilde{\Delta} \times S^1_{1/2} \times [0, 1)\big\} \,\dot{\cup}\, W\Big) \Big/ \sim\,,$$

where "$\sim$" is the smallest equivalence relation such that

$$((t+p, t+q, t+r), (u, v), \lambda) \sim h(t+p, t+q, t+r, (u, v), \lambda) \;\text{ when }\; \lambda \neq 0\,. \tag{2.4}$$

As the map h is of class $\mathcal{C}^\infty$, we check that $\hat{W}$ is a $\mathcal{C}^\infty$-manifold with boundary $\partial\hat{W} = \tilde{\Delta} \times S^1_{1/2} \times \{0\}$. The inclusion $W \to \hat{W}$ is a diffeomorphism onto

$\hat{W} - \partial\hat{W}$. The inclusion $\tilde{\Delta} \times S^1_{1/2} \times [0,1) \to \hat{W}$ is an embedding onto an open collar of $\partial\hat{W}$.

The above construction is invariant under the action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ by translations, so $\mathbb{Z}^3$ acts freely and properly on $\hat{W}$. The quotient $\hat{V} = \hat{W}/\mathbb{Z}^3$ is a compact $\mathcal{C}^\infty$-manifold, with boundary $\partial\hat{V} \approx S^1 \times S^1$ and with a diffeomorphism from V onto $\hat{V} - \partial\hat{V}$.

Lemma 2.5. *Let $c: S^1 \to \mathbb{C}$ be a $\mathcal{C}^1$-embedding. Then, the map F of (2.1) extends to a continuous map of pairs*

$$\hat{F}: (\hat{V}, \partial\hat{V}) \to (\mathrm{Tri}, \mathrm{Tri}_{fl}) .$$

Proof. We define a $\mathbb{Z}^3$-invariant continuous extension $\hat{F}_e: (\hat{W}, \partial\hat{W}) \to (\mathrm{Tri}, \mathrm{Tri}_{fl})$. This provides $\hat{F}$ by passing to the quotient.

The map $\hat{F}_e$ restricts to F_e on W. Therefore, on $\tilde{\Delta} \times S^1_{1/2} \times (0,1)$, it must be equal to $F_e \circ h$. Let $\tilde{c}: \mathbb{R} \to \mathbb{C}$ defined by $\tilde{c}(t) = c(e^{2i\pi t})$. Note that $\tilde{c}(t+m) = c(t)$ for $m \in \mathbb{Z}$. Hence, for $Z = ((t+p, t+q, t+r), (u,v), \lambda)$ and $\lambda \neq 0$, one has

$$\begin{aligned} F^1_e \circ h(Z) &= (\tilde{c}(t+\lambda u) - \tilde{c}(t), \tilde{c}(t+\lambda v) - \tilde{c}(t)) \\ &= (\lambda u \dot{\tilde{c}}(t) + \circ(\lambda u), \lambda v \dot{\tilde{c}}(t) + \circ(\lambda v)) \qquad \text{since } c \text{ is } \mathcal{C}^1 \qquad (2.6) \\ &= (\lambda(u\dot{\tilde{c}}(t) + \tfrac{\circ(\lambda u)}{\lambda}), \lambda(v\dot{\tilde{c}}(t) + \tfrac{\circ(\lambda v)}{\lambda})) \end{aligned}$$

The last expression has same image in Tri as $(u\dot{\tilde{c}}(t) + \frac{\circ(\lambda u)}{\lambda}, v\dot{\tilde{c}}(t) + \frac{\circ(\lambda v)}{\lambda})$ and, in Tri^1, one has the following convergence

$$(u\dot{\tilde{c}}(t) + \tfrac{\circ(\lambda u)}{\lambda}, v\dot{\tilde{c}}(t) + \tfrac{\circ(\lambda v)}{\lambda}) \xrightarrow[(\lambda \to 0)]{} (u\dot{\tilde{c}}(t), v\dot{\tilde{c}}(t)) \in \mathrm{Tri}^1_{fl} . \qquad (2.7)$$

We are thus driven to define

$$\hat{F}_e \circ h(Z) = \begin{cases} F_e \circ h(Z) & \text{if } Z \in W \\ [(u\dot{\tilde{c}}(t), v\dot{\tilde{c}}(t))] & \text{if } \lambda = 0 \ , \end{cases}$$

where [ ] denotes the class in Tri. To check the continuity of $\hat{F}_e$, we have to take a converging sequence $Z_n \to Z_\infty$ in $\hat{W}$ and see that $\hat{F}_e \circ h(Z_n) \to \hat{F}_e \circ h(Z_\infty)$. Set

$$Z_n = ((t_n + p_n, t_n + q_n, t_n + r_n), (u_n, v_n), \lambda_n)$$

and

$$Z_\infty = ((t_\infty + p_\infty, t_\infty + q_\infty, t_\infty + r_\infty), (u_\infty, v_\infty), \lambda_\infty) .$$

Only the case $\lambda_\infty = 0$ requires a proof. As p_n, q_n and r_n are integers which play no role, one may assume that $(p_n, q_n, r_n) = (0,0,0)$. Since $\hat{F}_e | \partial\hat{W}$ is continuous, one may assume, restricting to a subsequence if necessary, that $\lambda_n \neq 0$ if $n < \infty$. Let us decompose $\tilde{c}$ in its real and imaginary part: $\tilde{c}(t) = \tilde{c}_{\mathrm{re}}(t) + i\,\tilde{c}_{\mathrm{im}}(t)$. By the mean value theorem, one has

$$\dot{\tilde{c}}_{\mathrm{re}}(t_n + \lambda_n u_n) - \dot{\tilde{c}}_{\mathrm{re}}(t_n) = \lambda_n u_n \dot{\tilde{c}}_{\mathrm{re}}(t_n + \mu_n)$$

with $|\mu_n| \le |\lambda_n u_n|$. We can write the analogous equation for $\dot{\hat{c}}_{\mathrm{re}}(t_n + \lambda_n v_n) - \dot{\hat{c}}_{\mathrm{re}}(t_n)$ and for the imaginary parts, and use them in the computations like in (2.6–7). This proves that $\hat{F}_e \circ h(Z_n) \to \hat{F}_e \circ h(Z_\infty)$. □

3 Bidegree

In this section $H_*(\,)$ denotes the singular homology with $\mathbb{Z}_2$ as coefficients. The various manifolds are topological manifolds and may be non-orientable. The following lemma will be useful.

Lemma 3.1. *Let X be a compact connected topological n-manifold with (possibly empty) boundary ∂X. Let K be a non-empty discrete set in $X - \partial X$. Then $H_n(X - K, \partial X) = 0$.*

Proof. We may suppose that $n > 1$, otherwise the easy proof is left to the reader. Since X is compact, K is finite. Let $\mathcal{D} \subset X - \partial X$ be a disjoint family of compact disks forming a tubular neighborhood of K. Then $Y = X - \operatorname{int}\mathcal{D}$ is a compact connected manifold with ∂Y being the disjoint union of ∂X and $\partial\mathcal{D}$. Since the inclusion $Y \subset X - K$ is a homotopy equivalence, one has $H_n(X - K, \partial X) \approx H_n(Y, \partial X)$. By Poincaré duality (see e.g. [3, Theorem] or [4, Corollary 29.11]), $H_n(Y, \partial X) \approx H^0(Y, \partial\mathcal{D})$. As $n > 1$, Y is connected. Therefore, since K is not empty, so is $\partial\mathcal{D}$ and thus $H^0(Y, \partial\mathcal{D}) = 0$. □

Let M be a closed connected manifold of dimension m, containing a closed submanifold N of codimension one which separates M into two connected manifolds $M_\pm$, with common boundary N. Let V be a tubular neighborhood of N in M. By homotopy and excision, one has the isomorphisms

$$H_n(M, N) \approx H_n(M, V) \approx H_n(M - \operatorname{int} V, \partial V)\,.$$

As N separates, V is of the form $N \times [-1, 1]$ with $N \times \{\pm 1\} \subset M_\pm$ and one has a homotopy equivalence of pairs

$$\begin{aligned}(M - \operatorname{int} V, \partial V) &= (M_- - \operatorname{int} V, N \times \{-1\}) \,\dot\cup\, (M_+ - \operatorname{int} V, N \times \{1\}) \\ &\simeq (M_-, N) \,\dot\cup\, (M_+, N)\end{aligned}$$

Hence,

$$H_n(M, N) \approx H_n(M_-, N) \oplus H_n(M_+, N) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2\,.$$

Let X be a compact connected manifold of dimension n and let $f\colon (X, \partial X) \to (M, N)$ be a continuous map of pairs. Consider the following commutative diagram

$$\begin{array}{ccc} H_n(X, \partial X) & \xrightarrow{f_*} & H_n(M, N) \\ \downarrow\approx & & \downarrow\approx \\ \mathbb{Z}_2 & \xrightarrow{\hat{f}_*} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}\quad.$$

The couple

$$\mathrm{bdg}(f) = \hat{f}_*(1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

is called the *bidegree* of f.

Proposition 3.2. *If* $\mathrm{bdg}(f) = (1,1)$*, then* f *is surjective.*

Proof. Suppose that f is not surjective. As X is compact, the set of points $u \in M$ with empty pre-image is open. Hence, there is such a point in $u \in M - N$, say $u \in M_+$. But $H_n(M_+ - \{u\}, N) = 0$ by Lemma 3.1. Therefore, $\mathrm{bdg}(f) \neq (1,1)$. □

The bidegree may be computed locally. A point $u \in M - N$ is a *topological regular value* for f if there is a neighborhood U of u in $M - N$ such that U is "*evenly covered*" by f. By this, we mean that $f^{-1}(U)$ is a disjoint union of subspaces U_j, indexed by a set $\mathcal{J}$, such that, for each $j \in \mathcal{J}$, the restriction of f to U_j is a homeomorphism from U_j to U. In particular, $f^{-1}(u)$ is a discrete closed subset of X indexed by $\mathcal{J}$, so $\mathcal{J}$ is finite since X is compact. For instance, a point u which is not in the range of f is a topological regular value of f (with $\mathcal{J}$ empty). For a topological regular value u of f, we define the *local degree* $d(f,u) \in \mathbb{Z}_2$ *of* f *at* u by

$$d(f,u) = \sharp f^{-1}(u) \mod 2\,.$$

Proposition 3.3. *Let* $f\colon (X, \partial X) \to (M, N)$ *as above. For any topological regular values* $u_\pm \in M\pm$ *of* f*, one has*

$$\mathrm{bdg}(f) = (d(f,u_-), d(f,u_+)).$$

Proof. We prove that the $H_n(M_+, N)$-component of $\mathrm{bdg}(f)$ is equal to $d(f,u_+)$. The argument for the other component is the same. The inclusions of pairs $i_\pm\colon (M_\pm, N) \to (M,N)$ and $j_\pm\colon (M,N) \to (M, M_\pm)$ induce homomorphisms

$$\begin{array}{ccc}
H_n(M_-,N) \oplus H_n(M_+,N) & \xrightarrow{H_*i_- + H_*i_+} & H_n(M,N) \\
& \searrow^{\approx} & \downarrow{\scriptstyle (H_*j_+, H_*j_-)} \\
& & H_n(M,M_+) \oplus H_n(M,M_-)
\end{array}$$

whose composition is an isomorphism by excision. The three groups above being isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, all the arrows are isomorphisms. Hence, the $H_n(M_+, N)$-component of $\mathrm{bdg}(f)$ is equal to $H_*j_- \circ \hat{f}(1)$. One may suppose that u_+ has a neighborhood D evenly covered by f which is homeomorphic to a compact n-

disk. Let $\mathcal{U} = f^{-1}(\{u_+\})$ and $\mathcal{D} = f^{-1}(D)$. One has the commutative diagram

$$\begin{array}{ccccc}
H_n(X, \partial X) & \xrightarrow{H_* f} & H_n(M, N) & \xrightarrow{H_* j_-} & H_n(M, M_-) \\
\big\downarrow & & & & \big\downarrow \approx \\
H_n(X, X - \mathcal{U}) & & \xrightarrow{\qquad H_* f \qquad} & & H_n(M, M - \{u_+\}) \\
\big\uparrow \approx & & & & \big\uparrow \approx \\
H_n(\mathcal{D}, \partial \mathcal{D}) & & \xrightarrow{\qquad H_* f_{\mathcal{D}} \qquad} & & H_n(D, \partial D)
\end{array}$$

where the bottom vertical arrows are isomorphisms by excision. The top left vertical arrows is injective, since, in the homology exact sequence of the triple $(X, X - \mathcal{U}, \partial X)$

$$H_n(X - \mathcal{U}, \partial X) \longrightarrow H_n(X, \partial X) \longrightarrow H_n(X, X - \mathcal{U}),$$

the left term vanishes by Lemma 3.1 (one may suppose that $\mathcal{U} \neq \emptyset$ since, otherwise the proof is trivial). The top right vertical arrows is injective by the same argument, so it is an isomorphism since its range is isomorphic to $\mathbb{Z}_2$. Writing $\mathcal{D}$ as a disjoint union of compact disks $\mathcal{D}_1, \dots, \mathcal{D}_k$, one has a commutative diagram

$$\begin{array}{ccccc}
H_n(\mathcal{D}, \partial \mathcal{D}) & \xleftarrow{\approx} & \bigoplus_{j=1}^k H_n(\mathcal{D}_j, \partial \mathcal{D}_j) & \xrightarrow{\approx} & \bigoplus_{j=1}^k \mathbb{Z}_2 \\
\big\downarrow H_* f_{\mathcal{D}} & & \big\downarrow \sum H_* f_{D_j} & & \big\downarrow + \\
H_n(D, \partial D) & \xleftrightarrow{=} & H_n(D, \partial D) & \xrightarrow{\approx} & \mathbb{Z}_2
\end{array}$$

which proves that the $H_n(M_+, N)$-component of $\mathrm{bdg}(f)$is equal to $d(f, u_+)$. □

Remark 3.4. Let $f, g\colon (X, \partial X) \to (M, N)$ be two maps as above which are homotopic (as maps of pairs). As $f_* = g_*$, one has $\mathrm{bdg}(f) = \mathrm{bdg}(g)$. □

4 Proof of the main theorem

Let $c\colon S^1 \to \mathbb{C}$ be a $\mathcal{C}^1$-embedding parametrizing Γ. Denote by $F_c : V \to \mathrm{Tri}$ the map F of (2.1) for the parametrization c and by $\hat{F}_c : (\hat{V}, \partial \hat{V}) \to (\mathrm{Tri}, \mathrm{Tri}_{fl})$ its continuous extension given by Lemma 2.5. It suffices to prove that $\mathrm{bdg}(\hat{F}_c) = (1, 1)$, which, using Proposition 3.2, implies that $\hat{F}_c$ is surjective. Indeed, as $\hat{F}_c(\partial \hat{V}) \subset \mathrm{Tri}_{fl}$, the class of a non-flat triangle will be in $F_c(V)$, which proves the main theorem.

Consider particular case where $\Gamma = S^1$, i.e. c is a $\mathcal{C}^1$-parametrisation of the unit circle. By Proposition 2.2, any non-flat triangle T on Γ is a smooth regular value of F_c and $F_c^{-1}(T)$ contains a single point. By the inverse function theorem, T is a topological regular value of $\hat{F}_c$. Using Proposition 3.3 for T and its conjugate $\bar{T}$, this proves that $\mathrm{bdg}(\hat{F}_c) = (1, 1)$.

For a general simple closed curve Γ of class $\mathcal{C}^1$, we shall prove that there is an isotopy of $\mathcal{C}^1$-embeddings $c_t\colon S^1 \to \mathbb{C}$ satisfying $c_0 = c$ and $c_1(S^1) = S^1$. The map $\hat{F}_{c_t}\colon (\hat{V}, \partial\hat{V}) \to (\mathrm{Tri}, \mathrm{Tri}_{fl})$ is then a homotopy of maps of pairs between $\hat{F}_c$ and $\hat{F}_{c_1}$ (the continuity of $\hat{F}_{c_t}$ may be checked with the arguments of the end of the proof of Lemma 2.5). Using Remark 3.4 and that $c_1(S^1) = S^1$, this will prove that $\mathrm{bdg}(\hat{F}_c) = \mathrm{bdg}(\hat{F}_{c_1}) = (1, 1)$.

The construction of the isotopy c_t proceeds as follows.

a) If $\gamma\colon S^1 \to \mathbb{C}$ is a $\mathcal{C}^1$-map which is close enough to c in the $\mathcal{C}^1$-metric, then γ is an injective immersions and $tc + (1-t)\gamma$ produces a $\mathbb{C}^1$-isotopy between c and γ. As such a γ may be chosen of class $\mathcal{C}^\infty$ [5, p. 49], we may thus suppose that c is of class $\mathcal{C}^\infty$.

b) By the Schoenflies theorem [6, Theorem 5.4], the embedding c extends to a $\mathcal{C}^\infty$-embedding $\Gamma\colon D \to \mathbb{C}$, were D is the unit disk. Such an embedding is isotopic to a diffeomorphism of D. Indeed, composing with a translation (isotopic to the identity), one may suppose that $\Gamma(0) = 0$. Then the map

$$\Gamma_t(z) = \begin{cases} \frac{1}{t}\Gamma(tz) & \text{if } t \neq 0 \\ D_0\Gamma(z) & \text{if } t = 0. \end{cases}$$

is a $\mathcal{C}^\infty$-isotopy between Γ and a $\mathbb{R}$-linear embedding. But $GL(2, \mathbb{R})$ has two connected components, both containing isometries.

Remark: the proof of Schoenflies theorem given in [6, Theorem 5.4] is not detailed for the smooth case. At least, the topological Schoenflies theorem implies that $c(S^1)$ bounds a disk. The embedding Γ may then be obtained by the Riemann mapping theorem with its extension to the boundary (see e.g. [1]).

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