THE THEORY OF HAHN MEROMORPHIC FUNCTIONS, A HOLOMORPHIC FREDHOLM THEOREM AND ITS APPLICATIONS

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Abstract. We introduce a class of functions near zero on the logarithmic cover of the complex plane that have convergent expansions into generalized power series. These power series are general enough to cover cases where non-integer powers of z and also terms containing $\log z$ can appear. We show that under natural assumptions some important theorems from complex analysis carry over to the class of these functions. In particular it is possible to define a field of functions that generalize meromorphic functions and one can formulate an analytic Fredholm theorem in this class. We show that this modified analytic Fredholm theorem can be used in spectral theory to prove convergent expansions of the resolvent for Bessel type operators.

1. Introduction

Asymptotic expansions of the form

$$f(z) \sim \sum_{k,m} z^{\alpha_k}(-\log(z))^{\beta_m}, \quad \text{as} \quad z \to 0$$

with non-integer α_k or β_m for functions f defined in some sector in the complex plane with base 0 appear quite frequently in mathematics and mathematical physics. Classical examples are solutions for differential equations (e.g. in Frobenius' method) or expansions of algebraic functions at singularities. More recently it was shown that low energy resolvent expansions in scattering problems are of this form (see e.g. [1], [3] for Schrödinger operators in $\mathbb{R}^n$, [4] for operators with constant leading coefficients in $\mathbb{R}^n$, and [2] for the case of the Laplace operator on a general manifold with a conical end).

The algebraic theory of generalized power series is well developed and can be found in the literature under the name Hahn series or Malcev-Neumann series. In this paper we are concerned with the analytic theory of such generalized power series, namely we will define a ring of functions, the Hahn holomorphic functions, that have convergent expansions into generalized power series, and we will show that this ring is actually a division ring. We show that the quotient field, the field of Hahn-meromorphic functions, has a nice description in terms of Hahn series and we generalize the notions of Hahn-holomorphic and Hahn-meromorphic functions to the operator valued case. The theory turns out to be very close to the case of analytic function theory. In particular one of our main theorems states that an analog of the analytic Fredholm theorem holds in the class of Hahn holomorphic functions.

This analytic Fredholm theorem has a straightforward application: is can be used to derive convergent resolvent expansions for Bessel type operators.

2. Hahn holomorphic functions

Let $(\Gamma, +)$ be a linearly ordered abelian group and let $(G, \cdot)$ be a group. Suppose $e : \Gamma \to G, \gamma \mapsto e_\gamma$ is a group homomorphism, in particular

$$e_0 = \mathbf{1} \in G, \qquad e_{\gamma_1+\gamma_2} = e_{\gamma_1} \cdot e_{\gamma_2} \quad \text{for all} \quad \gamma_1, \gamma_2 \in \Gamma.$$

The following definition and proposition are due to H. Hahn (see [6])

Definition 2.1. *Let $\mathcal{R}$ be a ring. A formal series*

$$\mathfrak{h} = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma, \qquad a_\gamma \in \mathcal{R}$$

is called a Hahn-series, *if the support of $\mathfrak{h}$,*

$$\operatorname{supp}(\mathfrak{h}) := \{g \in \Gamma \mid a_g \neq 0 \in \mathcal{R}\},$$

is a well-ordered subset of Γ. The set of Hahn-series will be denoted by $\mathcal{R}[[e_\Gamma]]$.

Proposition 2.2. *The set of Hahn series $\mathcal{R}[[e_\Gamma]]$ is a ring with multiplication*

$$\Big(\sum_{\alpha\in\Gamma} a_\alpha e_\alpha\Big)\Big(\sum_{\beta\in\Gamma} b_\beta e_\beta\Big) = \sum_{\gamma\in\Gamma} c_\gamma e_\gamma, \qquad c_\gamma := \sum_{\substack{(\alpha,\beta)\in\Gamma\times\Gamma\\ \alpha+\beta=\gamma}} a_\alpha b_\beta \tag{1}$$

and addition

$$\sum_{\alpha\in\Gamma} a_\alpha e_\alpha + \sum_{\beta\in\Gamma} b_\beta e_\beta = \sum_{\gamma\in\Gamma} (a_\gamma + b_\gamma) e_\gamma$$

If $\mathcal{R}$ is a field, then $\mathcal{R}[[e_\Gamma]]$ is a field.

If the support of $\mathfrak{h}$ is contained in $\Gamma^+ = \{\gamma \mid \gamma > 0\}$ then it is well known that $\mathbf{1} - \mathfrak{h}$ is invertible in $\mathcal{R}[[e_\Gamma]]$ and its inverse is given by the Neumann series

$$(\mathbf{1} - \mathfrak{h})^{-1} = \sum_{k=0}^{\infty} \mathfrak{h}^k.$$

This is due to the fact that for any well-ordered subset Σ of Γ^+ the semi-group generated by Σ is also well-ordered, see e.g. [7], Lemma 2.10. Here convergence of a sequence $(\mathfrak{p}_n) \subset \mathcal{R}[[e_\Gamma]]$ to $\mathfrak{p} \in \mathcal{R}[[e_\Gamma]]$ is understood in the sense that for every element $\alpha \in \Gamma$ there exists an $N > 0$ such that for all $n > N$ the coefficients of e_α in $\mathfrak{p}$ and $\mathfrak{p}_n$ are equal.

In the following let Z be the logarithmic covering surface of the complex plane without the origin. We will use polar coordinates (r, φ) as global coordinates to identify Z as a set with $\mathbb{R}_+ \times \mathbb{R}$. Adding a single point $\{0\}$ to Z we obtain a set Z_0 and a projection map $\pi : Z_0 \to \mathbb{C}$ by extending the covering map $Z \to \mathbb{C}\backslash\{0\}$ in sending $0 \in Z_0$ to $0 \in \mathbb{C}$. We endow Z with the covering topology and Z_0 with the topology generated by the open sets in Z together with the open discs $D_\epsilon := \{0\} \cup \{(r, \varphi) \mid 0 \le r < \epsilon\}$. This means a sequence $((r_n, \varphi_n))_n$ converges to zero if and only if $r_n \to 0$. The covering map is continuous with respect to this topology. For a point $z \in Z_0$ we denote by $|z|$ its r-coordinate and by $\arg(z)$ its φ coordinate. We will think of the positive real axis as embedded in Z as the subset $\{z \mid \arg(z) = 0\}$.

In the following $Y \subset Z$ will always denote an open subset containing an open interval $(0, \delta)$ for some $\delta > 0$ and such that $0 \notin Y$. The set Y_0 will denote $Y \cup \{0\}$. In the applications we have in mind the set Y is typically of the form $D_\delta^{[\sigma]}\backslash\{0\}$ where $D_\delta^{[\sigma]} = \{z \in Z_0 \mid 0 \le |z| < \delta,\ |\varphi| < \sigma\}$. For the discussion and the general theorems it is not necessary to restrict ourselves to this case.

In the remaining part of this article we assume that $G := \big(\mathrm{Hol}(Y \cap D_\epsilon), \cdot\big)^\times$ is a set of non-vanishing holomorphic functions and that the group homomorphism e satisfies the condition

$$\forall \gamma > 0 : e_\gamma \text{ is bounded on } Y \text{ and } \quad \lim_{z \to 0} |e_\gamma(z)| = 0. \tag{E1}$$

Definition 2.3. *Suppose that $\mathcal{R}$ is a vector space with norm $\|.\|$. A Hahn series $\mathfrak{f} = \sum_{\alpha\in\Gamma} a_\alpha e_\alpha$ is called* normally convergent *in $Y \cap D_\epsilon$ if its support is countable and*

$$\sum_{\alpha\in\Gamma} \|a_\alpha\| \, \|e_\alpha\|_{Y,\varepsilon} < \infty,$$

where $\|e_\alpha\|_{Y,\varepsilon} := \sup_{z\in Y\cap D_\epsilon} |e_\alpha(z)|$.

Since a normally convergent series converges absolutely and uniformly, the value of the function

$$f(z) = \sum_{\alpha\in\Gamma} a_\alpha e_\alpha(z), \qquad z \in Y \cap D_\varepsilon$$

does not depend on the order of summation and f is holomorphic in $z \neq 0$.

Definition 2.4. *Let $S \subset \Gamma_0^+ = \Gamma^+ \cup \{0\}$ be a subset of the non-negative group elements.*

- *The family* $\{e_\alpha\}_{\alpha \in S}$ *is called* weakly monotonous, *if there exists an* $r_S > 0$ *such that for every* $x \in (0, r_S)$ *there is a* radius $\rho(x)$ *with* $0 < \rho(x) \leq x$ *and with the property*

$$\alpha \in S \Rightarrow \|e_\alpha\|_{Y,\rho(x)} \leq |e_\alpha(x)|.$$

- *The set* S *is called* admissible for e *(or simply* admissible*), if* $\{e_\alpha\}_{\alpha \in S}$ *is weakly monotonous, and if for every* $B \subset S$ *also the family*

$$\{e_{\alpha - \min B}\}_{\substack{\alpha \in S \\ \alpha > \min B}}$$

is weakly monotonous.

Definition 2.5 (Hahn holomorphic functions)**.** *Suppose that* $\mathcal{R}$ *is a Banach algebra. A continuous function* $h : Y_0 \to \mathcal{R}$ *which is holomorphic in* Y*, is called* (Y, Γ)*-Hahn holomorphic (or simply* Hahn holomorphic*) if there is a Hahn-series*

$$\mathfrak{h} = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma, \qquad a_\gamma \in \mathcal{R},$$

with countable, admissible support, which converges normally on $Y \cap D_\delta$ *for some* $\delta > 0$*, and*

$$h(z) = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma(z), \quad z \in Y \cap D_\delta.$$

We will denote the Hahn series of a Hahn holomorphic function h by the corresponding "fraktur" letter $\mathfrak{h}$. Note that (E1) together with uniform convergence imply that $\operatorname{supp} \mathfrak{h} \subset \Gamma_0^+$ and $h(0) = a_0$. Of course any normally convergent Hahn series with admissible support gives rise to a Hahn holomorphic function.

A direct consequence of the support of Hahn holomorphic functions being admissible is

Lemma 2.6. *Let*

$$h(z) = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma(z), \quad z \in Y \cap D_{2r}.$$

be Hahn holomorphic with $\mathfrak{m} = \min \operatorname{supp}(\mathfrak{h})$*. Then*

$$e_{-\mathfrak{m}}(z) h(z) = \sum_{\gamma \geq \mathfrak{m}} a_\gamma e_{\gamma - \mathfrak{m}}(z)$$

is Hahn holomorphic.

Proof. Let ρ_1 be the radius for $\{e_\gamma\}$ such that for all $\gamma \in \operatorname{supp}(\mathfrak{h})$

$$\|e_\gamma\|_{\rho_1(r)} \leq |e_\gamma(r)|.$$

and similarily let ρ_2 the radius for $\{e_{\gamma-\mathfrak{m}}\}$. For $\rho(r)=\min\{\rho_1(r),\rho_2(r)\}$,

$$\|e_{\mathfrak{m}}\|_{\rho(r)}\sum_{\gamma\in\Gamma}\|a_\gamma\|\,\|e_{\gamma-\mathfrak{m}}\|_{\rho(r)}\leq|e_{\mathfrak{m}}(r)|\sum_{\gamma\in\Gamma}\|a_\gamma\||e_{\gamma-\mathfrak{m}}(r)|=\sum_{\gamma\in\Gamma}\|a_\gamma\||e_\gamma(r)|<\infty$$

Thus $\sum_{\gamma\in\Gamma}a_\gamma e_{\gamma-\mathfrak{m}}$ converges normally on $D_{\rho(r)}$. □

Proposition 2.7. *Let $f:Y\to\mathcal{R}$ be a Hahn holomorphic function represented by a Hahn series $\mathfrak{f}$ on $Y\cap D_\delta$. Suppose the zeros of f accumulate in $Y\cup\{0\}$. Then $f=0$ and $\mathfrak{f}=0$. In particular the Hahn series of a Hahn holomorphic function is completely determined by the germ of the function at zero.*

Proof. If the zero set of f has accumulation points in Y then the statement follows from the fact that f is holomorphic in this set. It remains to show that if $f\neq 0$ then 0 can not be an accumulation point of the zero set of f. Let $\mathfrak{f}$ be a Hahn series that represents the function on $Y\cap D_\epsilon$. Let $f\neq 0$, hence $\mathfrak{f}\neq 0$. Let $\mathfrak{m}=\min\operatorname{supp}\mathfrak{f}$. If there is no other element in the support of $\mathfrak{f}$ then $f(z)=a_{\mathfrak{m}}e_{\mathfrak{m}}(z)$ and the statement follows from the fact that $e_{\mathfrak{m}}$ has no zeros in Y. Otherwise, let $\mathfrak{m}_1$ be the smallest element in $\operatorname{supp}\mathfrak{f}$ which is larger than $\mathfrak{m}$. Then

$$f(z)=\sum_\alpha a_\alpha e_\alpha(z)=e_{\mathfrak{m}}(z)\Big(a_{\mathfrak{m}}+e_{\mathfrak{m}_1-\mathfrak{m}}(z)\sum_{\alpha\geq\mathfrak{m}_1}a_\alpha e_{\alpha-\mathfrak{m}_1}(z)\Big)=e_{\mathfrak{m}}(z)(a_{\mathfrak{m}}+h(z))$$

with a Hahn holomorphic function $h(z)$ such that $h(0)=0$. Since h is continuous and $e_{\mathfrak{m}}(z)\neq 0$ this shows $f(z)\neq 0$ in a neighborhood of 0. □

In the following suppose Y,Γ and the family of functions $(e_\gamma)_{\gamma\in\Gamma}$ is fixed and satisfies (E1).

We want to show that the space of Hahn holomorphic functions at 0 with values in a Banach algebra $\mathcal{R}$ is a ring. To that end we need

Lemma 2.8. *Let $A_1,A_2\subset\Gamma^+$ be admissible sets. Then the sets $A_1\cup A_2$, A_1+A_2 and $n\cdot A_1:=A_1+\ldots+A_1$ (n times), $\bigcup_{n=0}^\infty n\cdot A_1$ are admissible.*

Proof. First we show that $A_1\cup A_2$, A_1+A_2 and $n\cdot A_1$ are weakly monotonuos. Let ρ_i, $i=1,2$ be the radius for A_i and $\rho(x)=\min\{\rho_1(x),\rho_2(x)\}$. Then ρ is a radius for $A_1\cup A_2$ and as well for A_1+A_2, because for $\alpha_i\in A_i$,

$$\begin{aligned}\|e_{\alpha_1+\alpha_2}\|_{\rho(r)}&\leq\|e_{\alpha_1}\|_{\rho(r)}\|e_{\alpha_2}\|_{\rho(r)}\leq\|e_{\alpha_1}\|_{\rho_1(r)}\|e_{\alpha_2}\|_{\rho_2(r)}\\&\leq|e_{\alpha_1}(r)||e_{\alpha_2}(r)|=|e_{\alpha_1+\alpha_2}(r)|\end{aligned}$$

The same argument shows that ρ_1 is a radius for $n\cdot A_1$.

Now let $B\subset A:=A_1+A_2$. Then $B=B_1+B_2$ for some $B_i\subset A_i, i=1,2$ and $\min B=\min B_1+\min B_2$. Let $\alpha\in A$ with $\alpha=\alpha_1+\alpha_2,\alpha_i\in A_i$. Let $\rho_i(r)$ be the radius for $\{e_{\alpha_i - \min B_i}\}$ and $\rho = \min\{\rho_1, \rho_2\}$. The estimate

$$\|e_{\alpha-\min B}\|_{\rho(r)} = \|e_{\alpha_1-\min B_1+\alpha_2-\min B_2}\|_{\rho(r)} \le \|e_{\alpha_1-\min B_1}\|_{\rho_1(r)} \|e_{\alpha_2-\min B_2}\|_{\rho_2(r)}$$

shows that $A_1 + A_2$ is admissible. The other statements are proven similarily. □

Let $f(z) = \sum_\alpha a_\alpha e_\alpha$ and $g(z) = \sum_\beta b_\beta e_\beta$ be Hahn holomorphic functions on Y_f and Y_g respectively. First it is easy to see that $f + g$ is Hahn holomorphic on $Y = Y_f \cap Y_g$. Since $\mathfrak{f}$ and $\mathfrak{g}$ are Hahn-series with support contained in Γ_0^+, also $\mathrm{supp}(\mathfrak{f} \cdot \mathfrak{g}) \subset \Gamma_0^+$ for the multiplication as defined in (1). From Lemma 2.8 we obtain that the support of $\mathfrak{f} \cdot \mathfrak{g}$ is admissible. We claim that $h(z) = f(z) \cdot g(z)$ is represented by the product of Hahn-series $\mathfrak{h} = \mathfrak{f} \cdot \mathfrak{g}$ on $Y_f \cap Y_g$. Because f and g are normally convergent,

$$\begin{aligned}\sum_\gamma \Big\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big\| \, \|e_\gamma\| &\le \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} \|a_\alpha\| \|b_\beta\| \Big) \|e_\gamma\| \\ &\le \Big(\sum_\alpha \|a_\alpha\| \, \|e_\alpha\| \Big) \Big(\sum_\beta \|b_\beta\| \, \|e_\beta\| \Big)\end{aligned}$$

so that the series $\mathfrak{f} \cdot \mathfrak{g}$ is normally convergent in $Y_f \cap Y_g$. Thus the series $\mathfrak{f} \cdot \mathfrak{g}$ defines a Hahn holomorphic function on Y with values in $\mathcal{R}$ which equals $h(z)$.

Altogether we have found

Proposition 2.9. *Let $\mathcal{R}$ be a Banach algebra. The Hahn holomorphic functions with values in $\mathcal{R}$ on Y form a ring under usual addition and multiplication, and the map $\psi_{\mathcal{R}} : f \mapsto \mathfrak{f}$ is a ring isomorphism onto its image in $\mathcal{R}[[e_\gamma]]$.*

Corollary 2.10. *The ring of Hahn holomorphic functions on Y with values in an integral domain $\mathcal{R}$ is an integral domain.*

Proof. By looking at the coefficient c_γ with $\gamma = \min \mathrm{supp}\, \mathfrak{f}$ in (1), we observe that $\mathcal{R}[[e_\Gamma]]$ is an integral domain, if $\mathcal{R}$ is an integral domain. Because $\psi_{\mathcal{R}}$ is an isomorphism, the Hahn holomorphic functions must be an integral domain. □

Theorem 2.11. *Let $\mathcal{R}$ be a Banach algebra and suppose $f : Y_0 \to \mathcal{R}$ is Hahn holomorphic and $f(z)$ is invertible for all $z \in Y_0$. Then $f(z)^{-1}$ is also Hahn holomorphic on Y_0.*

Proof. Since $1/f$ is holomorphic in Y we only have to show that there is a Hahn series for $f(z)^{-1}$ that converges normally on some $Y_0 \cap D_\varepsilon$. Since $f(z)^{-1} = f(0)^{-1} \left(f(z) f(0)^{-1} \right)^{-1}$ we can assume without loss of generality that $f(0) = \mathrm{Id}$. Thus we can write $f(z) = \mathrm{Id} - h(z)$, where $\mathfrak{m} := \min \mathrm{supp}\, (\mathfrak{h}) > 0$.

By assumption the series $\mathfrak{h} := \sum_{\alpha\in\Gamma} a_\alpha e_\alpha$ defining $h(z)$ converges normally on the set $Y_0 \cap D_{\delta_0}$ for some $\delta_0 > 0$. The function $\tilde{h}$ defined by

$$\tilde{h}(t) = \sum_{\alpha\in\Gamma} \|a_\alpha\| \, \|e_\alpha\|_{Y_0,t} \leq \|e_\mathfrak{m}\|_{Y_0,t} \sum_{\alpha\geq\mathfrak{m}} \|a_\alpha\| \, \|e_{\alpha-\mathfrak{m}}\|_{Y_0,t}$$

converges to 0 for $t \to 0$ due to (E1) and Lemma 2.6. Therefore we can choose $\delta > 0$ so small that $\tilde{h} := \tilde{h}(\delta) < 1/2$. Because $|h(z)| \leq \tilde{h}$ for $z \in Y_0 \cap D_\delta$, the geometric series

$$f(z)^{-1} = \sum_{n=0}^{\infty} h(z)^n$$

then converges normally on $Y_0 \cap D_\delta$. But we also know that $\mathfrak{f}$ is invertible:

$$\mathfrak{f}^{-1} = \sum_{n=0}^{\infty} \mathfrak{h}^n =: \sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha, \quad \text{with} \quad \operatorname{supp}(\mathfrak{f}^{-1}) \subset \mathcal{S} := \bigcup_{n\geq 0} \operatorname{supp}(\mathfrak{h}^n).$$

¿From Lemma 2.8 we obtain that $\mathcal{S}$ is admissible. It remains to show that $\sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha(z)$ is normally convergent on $Y_0 \cap D_\delta$ and represents $f(z)^{-1}$.

Note that if $\sum_{n=0}^{N} \mathfrak{h}^n = \sum_{\alpha\in\mathcal{S}} c_\alpha(N) e_\alpha$ then

$$\sum_{\alpha\in\mathcal{S}} \|c_\alpha(N)\| \, \|e_\alpha\| \leq \sum_{n=0}^{N} \tilde{h}^n \quad \text{in} \quad Y_0 \cap D_\delta$$

as a simple consequence of the triangle inequality. For every fixed finite set $A \subset \mathcal{S}$ there exists an $N_A > 0$ such that for all $N \geq N_A$

$$\mathfrak{f}^{-1} - \sum_{n=0}^{N} \mathfrak{h}^n = \sum_{\alpha\in\mathcal{S}\setminus A} (b_\alpha - c_\alpha(N)) e_\alpha$$

has support away from A. In particular $c_\alpha(N) = b_\alpha$ for $\alpha \in A$ and $N \geq N_A$. Therefore for $N > N_A$

$$\sum_{\alpha\in A} \|b_\alpha\| \, \|e_\alpha\| \leq \sum_{\alpha\in\mathcal{S}} \|c_\alpha(N)\| \, \|e_\alpha\| \leq \sum_{n=0}^{N} \tilde{h}^n < \frac{1}{1-\tilde{h}},$$

and this proves convergence since this bound is independent of A.

In particular $\sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha(z)$ converges absolutely in $\mathcal{R}$, hence it converges and the value does not depend on the order of summation. After reordering,

$$\sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha(z) = \sum_{n=0}^{\infty} h(z)^n = f(z)^{-1}.$$

□

Because of Lemma 2.6, every complex valued Hahn holomorphic f can be inverted: Let $\mathfrak{m} := \min \operatorname{supp}(\mathfrak{f}) \geq 0$, then

$$f^{-1}(z) = a_{\mathfrak{m}}^{-1} e_{-\mathfrak{m}}(z) \sum_{n=0}^{\infty} \left(1 - a_{\mathfrak{m}}^{-1} e_{-\mathfrak{m}}(z) f(z)\right)^n$$

Theorem 2.12. *Suppose that $f : Y_0 \to \mathbb{C}$ is a Hahn holomorphic function with Hahn series $\mathfrak{f}$. Suppose that U is an open neighbourhood of $f(0)$ and $h : U \to \mathbb{C}$ is holomorphic. Then $h \circ f$ is Hahn holomorphic on its domain.*

Proof. Since holomorphicity away from zero is obvious it is enough to show that $h \circ f$ has a normally convergent expansion into a Hahn series. Replacing $f(z)$ by $f(z) - f(0)$ and $h(z)$ by $h(z - f(0))$ we can assume without loss of generality that $f(0) = 0$ and thus $\operatorname{supp}(\mathfrak{f}) \subset \Gamma^+$. Since h is holomorphic near $f(0)$ it has a uniformly and absolutely convergent expansion

$$h(z) = \sum_{k=0}^{\infty} a_k (z - f(0))^k.$$

Thus,

$$h \circ f(z) = \sum_{k=0}^{\infty} a_k (f(z))^k.$$

Note that $\sum_{k=0}^{\infty} a_k\, \mathfrak{f}^k$ is a Hahn series. A similar argument as in the proof of Theorem 2.11 shows that this Hahn series is normally convergent and represents $h \circ f(z)$. □

3. Hahn meromorphic functions

Definition 3.1. *A meromorphic function $h : Y \to \mathbb{C}$ is called* Hahn meromorphic *if h is represented by a Hahn series $\mathfrak{h}$ in $Y \cap D_\varepsilon$ for some $\varepsilon > 0$ and there exist Hahn holomorphic functions f, $g \neq 0$ on $Y_0 \cap D_\varepsilon$ such that $\mathfrak{h} \cdot \mathfrak{g} = \mathfrak{f}$.*

In this sense a Hahn meromorphic function can be written as a quotient $h = f/g$ of Hahn holomorphic functions in a neighborhood of 0.

Example 3.2. *The logarithm $\log z = \frac{z \log z}{z}$ is Hahn meromorphic for $\Gamma \subset \mathbb{Z} \times \mathbb{Z}$.*

Remark 3.3. *Since $\mathbb{C}$-valued Hahn holomorphic functions form an integral domain, the Hahn meromorphic functions form a field. More generally let $\mathcal{R}$ be a (commutative) integral domain. ¿From Corollary 2.10 we know that Hahn holomorphic functions with coefficients in $\mathcal{R}$ are a commutative integral domain, so that their quotient field is defined. Furthermore, the map $f \mapsto \mathfrak{f}$ induces an injective morphism from the quotient field of Hahn holomorphic functions to the* *quotient field $\mathcal{R}((e_\Gamma))$ of Hahn series $\mathcal{R}[[e_\Gamma]]$. Note that $\mathcal{R}((e_\Gamma)) = \mathcal{R}[[e_\Gamma]]$, if $\mathcal{R}$ is a field.*

An important difference with usual meromorphic functions is that Hahn meromorphic functions may have infinitely many negative exponents. For example the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{1-1/n}$$

is Hahn holomorphic and therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^{-1/n-1} = \frac{f(z)}{z^2}$$

is Hahn-meromorphic.

It follows from our analysis for Hahn holomorphic functions that every $\mathbb{C}$-valued Hahn meromorphic function h can be written as

$$h(z) = e_{\min \operatorname{supp} \mathfrak{h}}(z) f(z),$$

where f is Hahn holomorphic. Moreover, if $h \neq 0$ then $f(0) \neq 0$. In particular this implies that Hahn meromorphic functions which are bounded on $(0, \delta)$ are Hahn holomorphic in some neighborhood of 0.

We can also define Hahn meromorphic functions with values in a Banach algebra:

Definition 3.4. *Let $\mathcal{R}$ be a Banach algebra. A function $h : Y \to \mathcal{R}$ is called* Hahn meromorphic *if it is meromorphic on Y and there exists a $\delta > 0$ and a non-zero Hahn holomorphic function f on $Y_0 \cap D_\delta$ such that $f(z)h(z)$ is a Hahn holomorphic function on $Y_0 \cap D_\delta$ with values in $\mathcal{R}$.*

Remark 3.5. *Let $R > 0$ and $\sigma > 0$. If there exists one non-zero Hahn holomorphic function on $Y \cap D_R^{[\sigma]}$ one can use the Weierstrass product theorem together with theorem 2.12 to show that the set of complex valued Hahn meromorphic functions on $Y \cap D_R^{[\sigma]}$ can be identified with the quotient field of the division ring of Hahn holomorphic functions on $Y \cap D_R^{[\sigma]}$.*

4. A Hahn holomorphic Fredholm theorem

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{K}(\mathcal{H})$ the space of compact operators on $\mathcal{H}$.

Theorem 4.1. *Suppose $Y_0 \subset Z$ is connected and let $f : Y_0 \to \mathcal{K}(\mathcal{H})$ be Hahn holomorphic. Then either $(\mathrm{Id} - f(z)) \in \mathcal{B}(\mathcal{H})$ is invertible nowhere in Y_0 or its inverse $(\mathrm{Id} - f(z))^{-1}$ exists everywhere except at a discrete set of points in Y_0 and defines a Hahn meromorphic function. Moreover, all the negative coefficients in* *its Hahn series are finite rank operators and the residues of the poles away from* 0 *are finite rank operators, too.*

Proof. The proof generalizes that of Theorem VI.14 of [5]. Let A be a finite rank operator such that $\|f(0)-A\| < 1/2$ and let $\delta > 0$ be such that $\|f(x)-f(0)\| < 1/2$ for all $x \in U^{[\sigma]} := D_\delta^{[\sigma]} \cap Y$. Then $\|f(x) - A\| < 1$ and thus $(\mathrm{Id} - f(x) + A)^{-1}$ exists and is Hahn holomorphic by Theorem 2.11. Consequently $g(x) = A(\mathrm{Id} - f(x) + A)^{-1}$ is a Hahn holomorphic function on $U^{[\sigma]}$ with values in the Banach space $\mathcal{B}(\mathcal{H}, \mathrm{rg}(A))$. It is easy to see that

$$(\mathrm{Id} - f(x))^{-1} = (\mathrm{Id} - f(x) + A)^{-1}(\mathrm{Id} - g(x))^{-1} \tag{2}$$

where equality means here that the left hand side exists if and only of the right hand side exists. Let now P be the orthogonal projection onto $\mathrm{rg}(A)$ and let $G(x)$ be the endomorphisms of $\mathrm{rg}(A)$ defined by restricting $g(x)$ to $\mathrm{rg}(A)$, i.e. $G(x) = g(x) \circ P$.

Invertibility of $(\mathrm{Id} - g(x))$ in $\mathcal{B}(\mathcal{H})$ is equivalent to invertibility of $P(\mathrm{Id} - g(x))P$, and this is equivalent to $\det(\mathrm{Id}_{\mathrm{rg}(A)} - G(x)) \neq 0$. Moreover, a straightforward computation shows

$$(\mathrm{Id} - g(x))^{-1} = (P(\mathrm{Id} - g(x))P)^{-1} \big(P + g(x)(\mathrm{Id} - P)\big) + (\mathrm{Id} - P). \tag{3}$$

Now note that $G(x)$ is a Hahn holomorphic family of endomorphisms of $\mathrm{rg}(A)$. In particular $\det(\mathrm{Id} - G(x))$ is a Hahn holomorphic $\mathbb{C}$-valued function. As such, it is holomorphic in $U^{[\sigma]} \setminus \{0\}$, and together with Proposition 2.7 this shows that the set

$$S = \{z \in U^{[\sigma]} \mid \det(\mathrm{Id} - G(x)) = 0\}$$

is either discrete in $U^{[\sigma]}$ or $S = U^{[\sigma]}$.

If $\det(\mathrm{Id}-G(x)) \neq 0$, then after a choice of basis of $\mathrm{rg}(A)$ the inverse $(\mathrm{Id}-G(x))^{-1}$ can be computed with Cramer's rule, showing that with respect to this basis

$$\det(\mathrm{Id} - G(x))(\mathrm{Id} - G(x))^{-1} \in \mathrm{Mat}\big(\dim \mathrm{rg}(A), \mathbb{C}[[e_\Gamma]]\big)$$

is represented by a matrix with Hahn holomorphic entries. After the identification

$$\mathrm{Mat}\big(\dim \mathrm{rg}(A), \mathbb{C}[[e_\Gamma]]\big) = \mathrm{Mat}\big(\dim \mathrm{rg}(A), \mathbb{C}\big)[[e_\Gamma]]$$

we see that the function $(\mathrm{Id} - G(x))^{-1}$ is Hahn meromorphic with coefficients in $\mathrm{End}(\mathrm{rg}(A))$ if there is only a single point in $U^{[\sigma]}$ for which it exists. Consequently, due to (3) and (2), $(\mathrm{Id}-f(x))^{-1}$ is Hahn meromorphic with all negative coefficients being of finite rank, if there is only a single point in $U^{[\sigma]}$ for which $(\mathrm{Id} - f(x))$ is invertible.

So far we have proved the statement in $U^{[\sigma]}$. By the usual analytic Fredholm theorem, invertibility of $(\mathrm{Id} - f(x))$ at a single point in Y implies that the inverse exists as a meromorphic function on Y. Conversely, we have seen that invertibility of $(\mathrm{Id} - f(x))$ at a single point in $U^{[\sigma]}$ implies that $(\mathrm{Id} - f(x))^{-1}$ exists as a Hahn meromorphic function on $U^{[\sigma]}$. By the usual holomorphic Fredholm theorem it then exists as a Hahn meromorphic function on Y. □

5. z^α-Hahn holomorphic functions

The prominent class of Hahn holomorphic functions is defined by convergent power series with non-integer powers.

Let $\Gamma \subset \mathbb{R}$ be a subgroup with order inherited from the standard ordering of $\mathbb{R}$. As the group G we will take the group generated by the set of functions

$$e_\alpha(z) := z^\alpha, \qquad z \in D_r^{[\sigma]} \backslash \{0\}.$$

In this definition we choose the principal branch of the logarithm with $|\operatorname{Im} \log z| < \pi$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and as usual set $\log(re^{i\varphi}) = \log r + i\varphi$, $|\varphi| < \sigma$ and $z^\alpha := e^{\alpha \log z}$.

A z^α-Hahn holomorphic function f with values $\mathbb{C}$ then is a holomorphic function on $D_r^{[\sigma]} \backslash \{0\}$ such that the generalized power series

$$f(z) = \sum_\gamma a_\gamma z^\gamma, \qquad a_\gamma \in \mathbb{C}$$

is normally convergent in $Y \cap D_\delta^{[\sigma]}$ for some $\delta > 0$.

Note that every well-ordered subset of $W \subset \Gamma^+$ is admissible for e, because for every $\alpha \in W$,

$$|z^\alpha| = |z|^\alpha \leq |z|^{\min W}, \qquad z \in D_{1/2}^{[\sigma]}. \tag{4}$$

Example 5.1. *If $\Gamma = \mathbb{Z}$ and $e_k(z) = z^k$ then the set of Hahn series corresponds to the formal power series and the set of z-Hahn holomorphic functions can be identified with the set of functions that are holomorphic on the disc of radius $\delta > 0$ centered at the origin.*

Example 5.2. *The series*

$$z^\pi \sum_{k=0}^\infty \frac{z^{2k}}{(2k)!}$$

converges normally on D_r for any $r > 0$ and defines a z-Hahn holomorphic function for $\Gamma = \pi\mathbb{Z} + 2\mathbb{Z}$.

Example 5.3. *Puiseux series and Levi-Civita series as defined in e.g.* [8] *are special cases of Hahn series with certain $\Gamma \subset \mathbb{Q}$. In case they are normally convergent they define z-Hahn holomorphic functions.*

6. $z \log z$-Hahn holomorphic functions

In the following let $\mathbb{R}^2$ be equipped with the lexicographical order and let $\Gamma \subset \mathbb{R}^2$ be a subgroup with order inherited from that of $\mathbb{R}^2$. In the following $Y = D_{1/2}^{[\sigma]}$ for fixed $\sigma > 0$. The group G will be generated by

$$e_{(\alpha,\beta)}(z) := z^\alpha(-\log z)^{-\beta}, \qquad |z| < 1.$$

With the inclusion $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ this comprises the power functions z^α from the previous section. Note that

$$\lim_{z \to 0} e_{(\alpha,\beta)}(z) = 0 \iff \alpha > 0 \vee (\alpha = 0 \wedge \beta > 0)$$

which is equivalent to $(\alpha, \beta) > (0, 0)$ in the lexicographical ordering of $\mathbb{R}^2$.

The monotonicity (4) of power functions z^α, has to be replaced by the following "weak monotonicity" property.

Lemma 6.1. *Let $\mathcal{S} \subset \Gamma^+ = \{\gamma \in \Gamma \mid \gamma > 0\}$ be a set such that there exists an $N \in \mathbb{N}_0$ with*

$$-\beta \leq N\alpha \quad \textit{for all} \quad (\alpha, \beta) \in \mathcal{S}. \tag{$*$}$$

Then

a) There exists $r_N < 1$ such that for $(\alpha, \beta) \in \mathcal{S}$ and $|\theta| < \sigma$ the function

$$r \mapsto |re^{i\theta}|^\alpha |\log(re^{i\theta})|^{-\beta}$$

is monotonously increasing on $[0, r_N)$.

b) Given x with $0 < x < r_N$, there exists $\rho_N(x) \leq x$ such that for all z with $0 \leq |z| \leq \rho_N(x), |\arg z| < \sigma$ we have

$$(\alpha, \beta) \in \mathcal{S} \quad \Longrightarrow \quad |e_{(\alpha,\beta)}(z)| \leq e_{(\alpha,\beta)}(x)$$

Proof. The proof is elementary and will be omitted here. □

It is not difficult to see that if $\mathcal{S}$ satisfies $(*)$, then a similar inequality holds for the set $(S - A) \cap \Gamma^+$ where $A \subset \mathcal{S}$ and the constant N depends on A. Thus a set $\mathcal{S}$ with $(*)$ is admissible for e.

Now the assumptions from section 2 are all satisfied and we can consider Hahn holomorphic and meromorphic functions: A *$z \log z$-Hahn holomorphic function* with values in a Banach algebra $\mathcal{R}$ is defined by a normally convergent series

$$f(z) = \sum_{(\alpha,\beta) \in \Gamma} a_{(\alpha,\beta)} z^\alpha (-\log z)^{-\beta}, \qquad a_{(\alpha,\beta)} \in \mathcal{R}, \qquad z \in D_{1/2}^{[\sigma]},$$

such that $\mathrm{supp}(f)$ is contained in a set $\mathcal{S} \cup \{(0, 0)\}$ with $\mathcal{S}$ as in Lemma 6.1.

Note that the property $(*)$ is invariant under addition and multiplication of Hahn holomorphic functions, so that $z\log z$-Hahn holomorphic function indeed are a ring, and all results from section 2 apply.

Example 6.2. *The series*

$$\sum_{n=0}^{\infty} z^n(-\log z)^n = (1+z\log z)^{-1}$$

is a Hahn series in $\Gamma = \mathbb{Z}\times\mathbb{Z}$ *with support* $\{(n,-n) \mid n \in \mathbb{N}_0\}$. *It converges normally on the set* $\{z \in Z \mid |z\log z| < 1/2\}$ *and therefore defines a* $z\log z$*-Hahn holomorphic function on* $D_r^{[\sigma]}$ *for any* $\sigma, r > 0$.

Example 6.3. *The formal series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} z(-\log z)^n$$

is not *a Hahn series for* $\Gamma = \mathbb{Z}\times\mathbb{Z}$, *because the support*

$$\{(1,-n) \mid n \in \mathbb{N}_0\}$$

is not a well-ordered subset of Γ.

Example 6.4. *The series*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2} z^n(-\log z)^{(2n-1+\frac{1}{m})}$$

defines a $z\log z$*-Hahn holomorphic function neighborhood* $D_\epsilon^{[\sigma]}$ *for any* $\sigma > 0$ *and for small enough* $\epsilon = \epsilon(\sigma)$ *with* $\Gamma = \mathbb{Z}\times\mathbb{Q}$ *and support*

$$\{(n, 1-2n-1/m) \mid n, m \in \mathbb{N}\}.$$

7. Applications: Hahn meromorphic continuation of resolvent kernels

7.1. Suppose that $a > 0$. Then the differential operator

$$B_a := -\frac{\partial^2}{\partial x^2} + \frac{a^2 - \frac{1}{4}}{x^2}\mathrm{Id}$$

is essentially self-adjoint on the space $\{f \in C_c^\infty([1,\infty)) \mid f(1) = 0\}$, equipped with the inner product inherited from $L^2((1,\infty), dx)$. In the following we will denote the self-adjoint extension of B_a by the same symbol B_a.

Let $H_a^{(1)}$ and $H_a^{(2)}$ be the Hankel functions of order a of the first and second kinds respectively, and

$$\widetilde{H}_a^{(j)}(\lambda, x) := \sqrt{x}\frac{H_a^{(j)}(\lambda x)}{H_a^{(j)}(\lambda)}, \qquad j \in \{1, 2\}.$$

The following lemma summarizes some elementary properties of these functions.

Lemma 7.1.

a) The function $\psi_\lambda(x) := \widetilde{H}_a^{(1)}(\lambda, x)$ *is the unique solution of the boundary value problem*

$$(B_a - \lambda^2)\psi_\lambda = 0, \quad \psi_\lambda(1) = 1, \tag{5}$$

such that $\psi_\lambda \in L^2((1, \infty), dx)$ *for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Im}\lambda > 0$. *Similarly,* $\widetilde{H}_a^{(2)}$ *is the square integrable solution for* $\operatorname{Im}\lambda < 0$.

b) Furthermore, $\widetilde{H}_a(\lambda, x)$ *is a Hahn holomorphic function in* λ *for all* $a > 0$.

c) The cylinder function

$$G_a(\lambda, x) = \frac{i}{2}\big(H_a^{(2)}(\lambda)H_a^{(1)}(\lambda x) - H_a^{(1)}(\lambda)H_a^{(2)}(\lambda x)\big) \tag{6}$$

is the unique solution of the initial value problem

$$(B_a - \lambda^2)G_a(\lambda, \cdot) = 0, \qquad G_a(\lambda, 1) = 0, \quad \frac{d}{dx}G_a(\lambda, x)|_{x=1} = -\frac{2}{\pi}.$$

Moreover, $G_a(\lambda, x)$ *is holomorphic and even in* λ.

Proof. We will only sketch the arguments. All formulas can be found in [9] and the references there.

a) The Hankel functions $H_a^{(1)}$ are solutions of the Bessel equation. That $\widetilde{H}_a^{(1)}$ is a square integrable solution then follows from the asymptotics

$$H_a^{(1;2)}(z) \sim \sqrt{\frac{2}{\pi z}e^{\pm i(z - \frac{\pi}{2}a - \frac{\pi}{4})}} \quad , \qquad |z| \to \infty. \tag{7}$$

b) The Hankel function of the first kind is related with the Bessel function through

$$H_a^{(1)}(z) = \frac{i}{\sin a\pi}\big(e^{-ia\pi}J_a(z) - J_{-a}(z)\big). \tag{8}$$

There is an even, holomorphic function h_ν such that the first Bessel function has the representation

$$J_a(z) = \left(\frac{z}{2}\right)^a h_a(z),$$

which shows that J_a is a z-Hahn holomorphic function with support in $\mathbb{Z} + \nu\mathbb{Z}$. This shows that $\lambda \mapsto \widetilde{H}_a^{(j)}(\lambda, x)$ is a Hahn holomorphic function for $a \notin \mathbb{N}_0$, with support contained in $2\mathbb{N}_0+2\nu\mathbb{N}_0$. For $a = n \in \mathbb{N}_0$, one has to take the limit $a \to n$ in (8), which leads to logarithmic terms. In this way one obtains that $\lambda \mapsto \widetilde{H}_n^{(j)}(\lambda, x)$ is $z \log z$-Hahn holomorphic with support contained in $\mathbb{Z} \times \mathbb{Z}$.

c) That $G_a(\cdot, x)$ is holomorphic and even in λ follows for non-integer a from

$$G_a(\lambda, x) = \frac{1}{\sin a\pi}\big(h_a(\lambda)h_{-a}(\lambda x) - h_{-a}(\lambda)h_a(\lambda x)\big)$$

as above and is the result of a lengthy computation in the case when a is an integer. This can however also be derived more directly from the fact that this function is the unique solution to an initial value problem for an ordinary differential equations whose coefficients depend holomorphically on λ^2. □

The spectral resolution of B_a is well known and given by the Weber transform, which we recall now. Let $f \in C_0^\infty(1, \infty)$. For any $a \in \mathbb{R}$, define

$$\mathbb{W}_a(f)(\lambda) = \int_1^\infty \frac{G_a(\lambda, x)}{H_a^{(1)}(\lambda)H_a^{(2)}(\lambda)} f(x)\, x\, dx \quad \in C_0^\infty((0,\infty)).$$

It is well known that the Weber transform $\mathbb{W}_a$ extends continuously to a unitary map

$$\mathbb{W}_a : L^2([1,\infty), x\,dx) \to L^2([0,\infty), H_a^{(1)}(\lambda)H_a^{(2)}(\lambda)\lambda\, d\lambda) =: \mathfrak{L}.$$

Let $\eta : L^2((1,b), x\,dx) \to L^2((1,b), dx)$ by the isometry given by $\eta(f)(x) := \sqrt{x}f(x)$. The Weber transform diagonalizes the operator B_a in the sense that

$$\mathbb{W}_a\eta^{-1}B_a(\mathbb{W}_a\eta^{-1})^{-1}f(\lambda) = \lambda^2 f(\lambda)$$

and the domain of B_a can be described as

$$\big\{f \in L^2([1,\infty), dx) \mid (\lambda \mapsto \lambda^2(\mathbb{W}_a\eta^{-1}f)(\lambda)) \in \mathfrak{L}\big\}.$$

Thus $\mathbb{W}_a$ gives full control over the functional calculus of B_a.

The kernel of the resolvent $(B_a - \lambda^2)^{-1}$ can be constructed directly out of the fundamental system of the Sturm-Liouville equation (5) and this results in

$$r_{a,\lambda}(x,y) = \begin{cases} \frac{\pi}{2}\sqrt{x}G_a(\lambda,x)\widetilde{H}_a^{(1)}(\lambda y), & 1 \le x \le y \\ \frac{\pi}{2}\sqrt{y}G_a(\lambda,y)\widetilde{H}_a^{(1)}(\lambda x), & y < x < \infty \end{cases} \tag{9}$$

Lemma 7.2. *For any $r > 0$ and $\sigma > 0$ the resolvent $(B_a - \lambda^2)^{-1}$ extends, as a function of λ, to a Hahn meromorphic function on $D_r^{[\sigma]}$ with values in*

$$\mathcal{B}\big(L^2((1,\infty), e^{2rx}\,dx),\ L^2((1,\infty), e^{-2rx}\,dx)\big).$$

In a neighborhood of zero this function is Hahn holomorphic.

Proof. From this explicit description of its kernel and the asymptotics (7) one easily deduces that away from zero the integral kernel is meromorphic and defines the desired expression. It is therefore enough to show that $H_a^{(1)}(\lambda) r_{a,\lambda}(x,y)$ defines a Hahn holomorphic family of operators with values in

$$\mathcal{B}\big(L^2((1,\infty), e^{2rx}\,dx),\ L^2((1,\infty), e^{-2rx}\,dx)\big).$$

If a is an integer then $H_a^{(1)}(\lambda x)$ can be written as $(\log\lambda) F_1(\lambda, x) + F_2(\lambda, x)$, where both $F_1(\lambda, x)$ and $F_2(\lambda, x)$ are holomorphic in λ. If a is not an integer then, similarly, $H_a^{(1)}(\lambda x) = \lambda^a F_1(\lambda, x) + \lambda^{-a} F_2(\lambda, x)$ with F_1 and F_2 holomorphic in λ. This splitting can be used to show that the resolvent kernel has the form $g_1(\lambda)F_1(\lambda)+g_2(\lambda)F_2(\lambda)$, where g_1 and g_2 are Hahn-meromorphic function with values in $\mathbb{C}$ and $F_1(\lambda)$ and $F_2(\lambda)$ is a holomorphic family of operators. One can check directly, using the asymptotics (7), that in this case F_1 and F_2 are indeed holomorphic functions of λ with values in $\mathcal{B}\big(L^2((1,\infty), e^{2rx}\,dx),\ L^2((1,\infty), e^{-2rx}\,dx)\big)$. □

7.2. **The matrix valued case.** Let $\mathcal{H}$ be a finite dimensional Hilbert space and suppose A is a postive operator. On the Hilbert space $L^2((1,\infty), dx) \otimes \mathcal{H}$ define operator

$$B_A := -\frac{\partial^2}{\partial x^2} \otimes \mathrm{Id} + \frac{1}{x^2} \otimes (A^2 - \tfrac{1}{4}\mathrm{Id}).$$

on compactly supported functions on $(1,\infty)$ with values in M and Dirichlet boundary conditions at $x = 1$. The operator B_A is an unbounded selfadjoint operator on the Hilbert space $L^2((1,\infty), dx) \otimes \mathcal{H}$. Of course, $\mathcal{H} = \oplus_{k=1}^M \mathcal{H}_k$, where $\mathcal{H}_k$ are the eigenspaces of A, and $L^2((1,\infty), dx) \otimes \mathcal{H} = \oplus_{k=1}^M L^2((1,\infty), dx) \otimes \mathcal{H}_k$. On each of the spaces $L^2((1,\infty), dx) \otimes \mathcal{H}_k$ the operator B_A acts only on the first tensor factor as $B_{a_k} \otimes \mathrm{Id}$. Therefore, B_A can be written as a direct sum

$$B_A = \oplus_{k=1}^M B_{a_k} \otimes \mathrm{Id}$$

of self-adjoint operators. Of course then also the resolvent $(B_A - \lambda)^{-1}$ is a direct sum

$$(B_A - \lambda)^{-1} = \oplus_{k=1}^M (B_{a_k} - \lambda)^{-1} \otimes \mathrm{Id}.$$

Then for any $r > 0$ and $\sigma > 0$ the resolvent $(B_A - \lambda^2)^{-1}$ extends (as a function of λ) to a Hahn holomorphic function on $D_r^{[\sigma]}$ with values in

$$\mathcal{B}\big(L^2((1,\infty), e^{2rx}\,dx) \otimes \mathcal{H},\ L^2((1,\infty), e^{-2rx}\,dx) \otimes \mathcal{H}\big).$$

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