ON UNIFORM CONTINUOUS DEPENDENCE OF SOLUTION OF CAUCHY PROBLEM ON A PARAMETER

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ABSTRACT. Suppose that an n-dimensional Cauchy problem

$$\frac{dx}{dt} = f(t,x,\mu) \quad (t \in \mathcal{I},\ \mu \in \mathcal{M}), \quad x(t_0) = x^0$$

satisfies the conditions that guarantee existence, uniqueness and continuous dependence of solution $x(t,t_0,\mu)$ on parameter μ in an open set $\mathcal{M}$. We show that if one additionally requires that family $\{f(t,x,\cdot)\}_{(t,x)}$ is equicontinuous, then the dependence of solution $x(t,t_0,\mu)$ on parameter $\mu \in \mathcal{M}$ is uniformly continuous.

An analogous result for a linear $n \times n$-dimensional Cauchy problem

$$\frac{dX}{dt} = A(t,\mu)X + \Phi(t,\mu) \quad (t \in \mathcal{I},\ \mu \in \mathcal{M}), \quad X(t_0,\mu) = X^0(\mu)$$

is valid under the assumption that the integrals $\int_{\mathcal{I}} \|A(t,\mu_1) - A(t,\mu_2)\|\,dt$ and $\int_{\mathcal{I}} \|\Phi(t,\mu_1) - \Phi(t,\mu_2)\|\,dt$ can be made smaller than any given constant (uniformly with respect to $\mu_1, \mu_2 \in \mathcal{M}$) provided that $\|\mu_1 - \mu_2\|$ is sufficiently small.

1. Introduction

Let $\mathcal{I} \subset \mathbb{R}$ be an open interval, let $\mathcal{X} \subset \mathbb{R}^n, \mathcal{M} \subset \mathbb{R}^m$ be domains (i.e. open connected subsets). We set $\mathcal{D} \doteq \mathcal{I} \times \mathcal{X}$, $\mathcal{G} \doteq \mathcal{I} \times \mathcal{M}$, $\mathcal{O} \doteq \mathcal{I} \times \mathcal{X} \times \mathcal{M}$. Suppose that we are given $f : \mathcal{O} \to \mathbb{R}^n,\ t_0 \in \mathcal{I},\ x^0 \in \mathcal{X}$.

We consider Cauchy problem

$$\frac{dx}{dt} = f(t,x,\mu) \quad (t \in \mathcal{I},\ \mu \in \mathcal{M}), \quad x(t_0) = x^0. \tag{1.1}$$

The questions related to existence and uniqueness of solution of (1.1), its extension by continuity up to the boundary of $\mathcal{D}$ (to the maximal interval of existence), and its continuous dependence on a paramter $\mu \in \mathcal{M}$ are discussed, e.g. in [1, p.53–73] (see also [2, p. 19–28, 119]). The next theorem contains a number of basic results on (1.1) that can be found in [1].

Theorem 1.1. *Suppose that function f satisfies:*

1) f is measurable on $\mathcal{O}$;

2) f is continuous in $(x,\mu) \in \mathcal{X} \times \mathcal{M}$ for every fixed $t \in \mathcal{I}$;

3) there exists a Lebesgue locally summable function m on $\mathcal{I}$ such that

$$\|f(t,x,\mu)\| \le m(t) \qquad \big((x,\mu) \in \mathcal{X} \times \mathcal{M}\big);$$

4) for almost every $t \in \mathcal{I}$ and every $\mu \in \mathcal{M}$ function f satisfies the Lipschitz condition in x:

$$\|f(t,x',\mu) - f(t,x'',\mu)\| \le L\|x' - x''\| \ \ x', x'' \in \mathcal{X},$$

where Lipschitz constant L is independent of t and μ.

Then there exists a closed interval $[a,b] \subset \mathcal{I}$ such that for every $\mu \in \mathcal{M}$ problem (1.1) has the unique solution $x(t,\mu)$ on $[a,b]$ that is absolutely continuous in variable t, and depends continuously on $\mu \in \mathcal{M}$. ($\mathcal{I}$ is the maximal interval of existence of x.)

Key words and phrases. uniform continuity, equicontinuity.

Here and below $\|\cdot\|$ denotes a norm in $\mathbb{R}^k$. We use the same notation for norm of a matrix with entries in $\mathbb{R}$.

The study of ordinary differential equations in the space of Colombeau generalized functions (see [3]) requires that the solution x of (1.1) depends on *uniformly continuous* (cf. Theorem 1.1, where the dependence of x on μ is only *continuous*). We note that it is essential for our purposes that $\mathcal{M}$ is an open subset (for a closed and bounded $\mathcal{M}$ the uniform continuous dependence of x on μ would follow trivially from Cantor's theorem).

That under the assumptions of Theorem 1.1 the dependence of x on μ is not necessarily uniformly continuous is demonstrated by the following example. Consider Cauchy problem

$$\frac{dx}{dt} = \sin\frac{1}{\mu},\ x(0)=0,\ \ t\in[0,1],\ \mu\in\mathcal{M}\doteq(0,1).$$

The assumptions of Theorem 1.1 are satisfied: indeed, given $t>0$ and δ, denote

$$\mu_1=\frac{1}{\pi n},\ \mu_2=\frac{1}{\pi n+\frac{\pi}{2}}.$$

Then

$$|\mu_1-\mu_2|=\frac{1}{\pi n(2n+1)}<\delta$$

for a sufficiently large n, and

$$|x(t,\mu_1)-x(t,\mu_2)| = \left|t\sin\frac{1}{\mu_1}-t\sin\frac{1}{\mu_2}\right| = \ t.$$

The solution $x(t,\mu)=t\sin\frac{1}{\mu}$ $(t\in[0,1])$, however, is not uniformly continuous on $\mathcal{M}=(0,1)$.

The following question naturally arises: what additional assumptions are required (cf. Theorem 1.1) in order to ensure the uniform continuous dependence of solution $x(t,\mu)$ on parameter μ? Our answer to this question is proposed below.

2. Nonlinear Cauchy problem

Suppose that we are given a map $F:\mathcal{O}\to\mathbb{R}^{n\times p}$. We say that the family $\{F(t,x,\cdot)\}_{(t,x)\in\mathcal{D}}$ of maps $\mathcal{M}\to\mathbb{R}^{n\times p}$ is equicontinuous if

(2.2)

$$(\forall\varepsilon>0)\ (\exists\delta>0)\ \big(\forall(t,x)\in\mathcal{D},\ \forall\mu_1,\mu_2\in\mathcal{M}:\ \|\mu_1-\mu_2\|<\delta\big)\ \ \big(\|F(t,x,\mu_1)-F(t,x,\mu_2)\|<\varepsilon\big).$$

For example, if $F(t,x,\mu)=g(t,x)h(\mu)$, where $g:\mathcal{D}\to\mathbb{R}^{n\times m},\ h:\mathcal{M}\to\mathbb{R}^{m\times p}$, function g is continuous and bounded on $\mathcal{D}$, then $\{F(t,x,\cdot)\}_{(t,x)\in\mathcal{D}}$ is equicontinuous if and only if function h is uniformly continuous on $\mathcal{M}$.

Lemma 2.1. *Suppose that F satisfies Lipschitz condition in $\mu\in\mathcal{M}$ uniformly with respect to $(t,x)\in\mathcal{D}$, i.e. $\|F(t,x,\mu_1)-F(t,x,\mu_2)\|\le M\|\mu_1-\mu_2\|$ $((t,x)\in\mathcal{D})$, where M is independent of t and x. Then $\{F(t,x,\cdot)\}_{(t,x)\in\mathcal{D}}$ is equicontinuous.*

Proof. Let $\varepsilon>0$ be arbitrary, let $\mu_1,\ \mu_2\in\mathcal{M}$ be such that $\|\mu_1-\mu_2\|<\delta\doteq\frac{\varepsilon}{M}$. Then $\|F(t,x,\mu_1)-F(t,x,\mu_2)\|\le M\|\mu_1-\mu_2\|<\varepsilon$. □

Let us note that family $\{F(t,x,\cdot)\}_{(t,x)\in\mathcal{D}}$ can be equicontinuous even if function F does not satisfy the Lipschitz condition in variable μ. For instance, if in the example above we set $m=p=1, h(\mu)\doteq\mu\sin\frac{\pi}{\mu}, \mathcal{M}=(0,1)$, then h and, consequently, F do not satisfy the Lipschitz condition in variable μ on $\mathcal{M}$, although h is uniformly continuous on $(0,1)$, hence the corresponding function family is equicontinuous.

Theorem 2.2. *Suppose that function f satisfies conditions* 1) — 4) *of Theorem 1.1 and, additionally, condition* 5)*: family $\{f(t,x,\cdot)\}_{(t,x)\in\mathcal{D}}$ is equicontinuous on $\mathcal{M}$.*

Then the dependence of the solution $x(t,\mu)$ of Cauchy problem (1.1) *on $\mu\in\mathcal{M}$ is uniform continuous* (*uniformly with respect to variable $t\in[a,b]$*).

Proof. Let us fix an arbitrary $\varepsilon > 0$. Suppose that $\delta > 0$ in accordance with condition (2.2), and let $\mu_1, \mu_2 \in \mathcal{M}$ be such that $\|\mu_1 - \mu_2\| < \delta$.

Since $x(t,\mu_i) = x^0 + \int\limits_{t_0}^{t} f\big(s, x(s,\mu_i), \mu_i\big)\, ds$ $(i = 1,2)$ then, assuming first that $t > t_0$ and using conditions 4) and 5), we obtain an estmiate

$$\|x(t,\mu_1) - x(t,\mu_2)\| \le \int_{t_0}^{t} \big\| f\big(s, x(s,\mu_1),\mu_1\big) - f\big(s, x(s,\mu_2),\mu_2\big)\big\|\, ds \le$$
$$\le \int_{t_0}^{t} \big\| f\big(s, x(s,\mu_1),\mu_1\big) - f\big(s, x(s,\mu_2),\mu_1\big)\big\|\, ds + \int_{t_0}^{t} \big\| f\big(s, x(s,\mu_1),\mu_2\big) - f\big(s, x(s,\mu_2),\mu_2\big)\big\|\, ds <$$
$$< L \int_{t_0}^{t} \|x(t,\mu_1) - x(t,\mu_2)\|\, ds + \varepsilon(b-a).$$

Now, using Gronwall-Bellman inequality [2, p. 37] we obtain

$$\|x(t,\mu_1) - x(t,\mu_2)\| < \varepsilon(b-a)e^{L(b-a)}.$$

In the case $t < t_0$ the argument is analogous. The obtained inequality immediately yields the uniform continuous dependence of x on $\mu \in \mathcal{M}$ (uniformly with respect to $t \in [a,b]$). □

Using Lemma 2.1, we obtain

Corollary 2.3. *Suppose that function f satisfies conditions* 1)–4) *of Theorem 1.1 and a Lipschitz condition*

$$\big\| f(t,x,\mu_1) - f(t,x,\mu_2)\big\| \le M\|\mu_1 - \mu_2\| \quad \big((t,x) \in \mathcal{D},\ \ \mu_1,\mu_2 \in \mathcal{M}\big),$$

where M is independent of t and x. Then the assertion of Theorem 2.2 holds.

By the remark above the additional assumption 5) of Theorem 2.2 is weaker than the additional assumption of Corollary 2.3. Indeed, for the Cauchy problem

$$\frac{dx}{dt} = \mu \sin\frac{1}{\mu},\ x(0) = 0,\ \ t \in [0,1],\ \mu \in \mathcal{M} \doteq (0,1)$$

the assumptions of Theorem 2.2 are satisfied, while the assumptions of Corollary 2.3 are not.

3. Linear Cauchy problem

We now consider the linear variant of problem (1.1):

$$\frac{dx}{dt} = A(t,\mu)x + \varphi(t,\mu) \quad (t \in \mathcal{I},\ \mu \in \mathcal{M}), \quad x(t_0) = x^0. \tag{3.3}$$

The solution of (3.3) exists on the whole interval $\mathcal{I}$ and is possibly unbounded. Thus, in general one can not expect that family

$$\{f(t,x,\cdot)\}_{(t,x)\in\mathcal{D}} = \{A(t,\cdot)x + \varphi(t,\cdot)\}_{(t,x)\in\mathcal{D}}$$

is equicontinuous. Of course, we can restrict this family to a closed subinterval $[a,b] \subset \mathcal{I}$; then solution x of (3.3) is bounded on $[a,b]$ and we can apply Theorem 2.2 and Corollary 2.3. It is, however, desirable to have a sufficient condition that ensures the uniform continuous dependence of x on a parameter, when the argument t varies in the whole interval $\mathcal{I}$.

It will be convenient to consider a matrix-valued analogue of problem (3.3):

$$\frac{dX}{dt} = A(t,\mu)X + \Phi(t,\mu) \quad (t \in \mathcal{I},\ \mu \in \mathcal{M}), \quad X(t_0,\mu) = X^0(\mu), \tag{3.4}$$

where A, $\Phi : \mathcal{I} \times \mathcal{M} \to \mathbb{R}^{n\times n}$ are summable in t over $\mathcal{I}$ for every $\mu \in \mathcal{M}$, $X : \mathcal{I} \times \mathcal{M} \to \mathbb{R}^{n\times n}$ is absolutely continuous in t over $\mathcal{I}$ for every $\mu \in \mathcal{M}$, and $X^0 : \mathcal{M} \to \mathbb{R}^{n\times n}$ is continuous and bounded on $\mathcal{M}$.

Let $\mathfrak{X}$ be the set of absolutely continuous on $\mathcal{I}$ for all $\mu \in \mathcal{M}$ $n \times n$-matrices $X(t,\mu)$ endowed with metric

$$\rho\big(X(t,\mu_1),\, X(t,\mu_2)\big) \doteq \widehat{\rho}(\mu_1,\mu_2) \doteq \|X(t_0,\mu_1) - X(t_0,\mu_2)\| + \int_{\mathcal{I}} \|\dot{X}(t,\mu_1) - \dot{X}(t,\mu_2)\|\, dt.$$

This is a complete metric space. We denote by $\mathfrak{X}_{t_0} \subset \mathfrak{X}$ the subspace of non-degenerate $n \times n$-matrices $X(t,\mu)$ normed at point t_0 by the condition $X(t_0,\mu) = E$, where E is the identity matrix. The induced metric in $\mathfrak{X}_{t_0}$ is given by the formula

$$\rho\big(X(\cdot,\mu_1),\, X(\cdot,\mu_2)\big) \doteq \widehat{\rho}(\mu_1,\mu_2) \doteq \int_{\mathcal{I}} \|\dot{X}(t,\mu_1) - \dot{X}(t,\mu_2)\|\, dt.$$

Further, let $\mathfrak{A}$ be the set of summable on $\mathcal{I}$ for all $\mu \in \mathcal{M}$ $n \times n$ matrices $A(t,\mu)$ endowed with norm

$$\mathfrak{n}(A) = \ \big(\widehat{\mathfrak{n}}(\mu)\big) \ = \int_{\mathcal{I}} \|A(t,\mu)\|\, dt,$$

let $\mathfrak{A}_0$ be the space of bounded and continuous on $\mathcal{M}$ $n \times n$ matrices with norm $\|X^0\| \doteq \sup\limits_{\mu \in \mathcal{M}} \|X^0(\mu)\|$. Clearly, spaces $\mathfrak{A}$, $\mathfrak{A}_0$ are Banach. It follows from Theorems 1.1 and 2.2 that Cauchy problem (3.4) determines a continuous map $\mathfrak{F}(\mu) : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}_0 \to \mathfrak{X}$ that depends continuously on $\mu \in \mathcal{M}$. Similarly, Cauchy problem

$$\frac{dX}{dt} = A(t,\mu)X \qquad (t \in \mathcal{I},\ \mu \in \mathcal{M}), \quad X(t_0,\mu) = X^0(\mu) \tag{3.5}$$

can be viewed as a continuous (bijective) map $\mathfrak{F}_0(\mu) : \mathfrak{A} \times \mathfrak{A}_0 \to \mathfrak{X}_{t_0}$ that depends on $\mu \in \mathcal{M}$ continuously. Our goal is to establish the conditions under which these maps depend on μ *uniformly* continuously on $\mathcal{M}$.

We say that $B \in \mathcal{A}$ is *integrally uniformly continuous* on $\mathcal{M}$ if the following condition is satisfied:

$$(\forall \varepsilon > 0)\ \ (\exists \delta > 0)\ \ (\forall \mu_1, \mu_2 \in \mathcal{M} : \|\mu_1 - \mu_2\| < \delta) \quad \left(\int_{\mathcal{I}} \|B(t,\mu_1) - B(t,\mu_2)\|\, dt < \varepsilon\right).$$

Theorem 3.1. *Suppose that*

1) functions $\widehat{\mathfrak{n}}, \varphi, \eta : \mathcal{M} \to [0,+\infty)$, *where* $\varphi(\mu) = \int_{\mathcal{I}} \|\Phi(t,\mu)\|\, dt$, $\quad \eta(\mu) \doteq \|X^0(\mu)\|$, *are bounded on* $\mathcal{M}$*;*

2) function $X^0 \in \mathfrak{A}_0$ *is uniformly continuous, and functions* A, $\Phi \in \mathfrak{A}$ *are integrally uniformly continuous on* $\mathcal{M}$.

Then the solution $Y(t,\mu)$ *of problem* (3.4) *is uniformly continuous in* $\mu \in \mathcal{M}$ *(uniformly with respect to* $t \in \mathcal{I}$*), and maps* $\mathfrak{F}_0(\mu)(A,\, X^0)$ *and* $\mathfrak{F}(\mu)(A,\, \Phi,\, X^0)$ *are uniformly continuous on* $\mathcal{M}$ (*uniformly with respect to* $t \in \mathcal{I}$*).*

Proof. We choose a constant $K > 0$ such that the following inequalities are satisfied:

$$\widehat{\mathfrak{n}}(\mu) \le K, \quad \varphi(\mu) \le K, \quad \eta(\mu) \le K, \quad \xi \doteq \|E\| \le K. \tag{3.6}$$

Denote

$$\mathfrak{a}(\mu_1,\mu_2) \doteq \int_{\mathcal{I}} \|A(t,\mu_1) - A(t,\mu_2)\|\, dt, \quad \mathfrak{f}(\mu_1,\mu_2) \doteq \int_{\mathcal{I}} \|\Phi(t,\mu_1) - \Phi(t,\mu_2)\|\, dt,$$
$$\mathfrak{x}(\mu_1,\mu_2) \doteq \|X^0(\mu_1) - X^0(\mu_2)\|.$$

In the next four lemmas we obtain a number of estimates needed to complete the proof of the theorem. Some of these lemmas (e.g. Lemmas 3.3 and 3.5) are interesting in their own right.

Let $C(t,s,\mu) = X(t,\mu)X^{-1}(s,\mu)$ be the Cauchy matrix of the homogeneous system of differential equations (3.5) (let $X(t,\mu)$ be its fundamental matrix normed at point t_0).

Lemma 3.2. *The following estimates hold:*

$$\|X(t,\mu)\| \le \xi e^{\widehat{\mathfrak{n}}(\mu)} \le K e^{K} \quad (t\in\mathcal{I},\ \mu\in\mathcal{M}); \tag{3.7}$$

$$\|X^{-1}(t,\mu)\| \le \xi e^{\widehat{\mathfrak{n}}(\mu)} \le K e^{K} \quad (t\in\mathcal{I},\ \mu\in\mathcal{M}); \tag{3.8}$$

$$\|X(t,\mu_1)-X(t,\mu_2\| \le \xi^3 e^{2\widehat{\mathfrak{n}}(\mu_1)+\widehat{\mathfrak{n}}(\mu_2)} \int_{\mathcal{I}} \|A(s,\mu_1)-A(s,\mu_2)\|\,ds \le K^3 e^{3K}\mathfrak{a}(\mu_1,\mu_2) \tag{3.9}$$

$(t\in\mathcal{I},\ \mu_1,\mu_2\in\mathcal{M})$;

$$\|X^{-1}(t,\mu_1)-X^{-1}(t,\mu_2\| \le \xi^3 e^{2\widehat{\mathfrak{n}}(\mu_1)+\widehat{\mathfrak{n}}(\mu_2)} \int_{\mathcal{I}} \|A(s,\mu_1)-A(s,\mu_2)\|\,ds \le K^3 e^{3K}\mathfrak{a}(\mu_1,\mu_2) \tag{3.10}$$

$(t\in\mathcal{I},\ \mu_1,\mu_2\in\mathcal{M})$;

$$\|C(t,s,\mu)\| \le \xi^2 e^{2\widehat{\mathfrak{n}}(\mu)} \le K^2 e^{2K} \quad (t,s\in\mathcal{I},\ \mu\in\mathcal{M}), \tag{3.11}$$

$$\begin{aligned}&\|C(t,s,\mu_1)-C(t,s,\mu_2)\| \le \\ &\quad\le \xi^4 e^{2\widehat{\mathfrak{n}}(\mu_1)+\widehat{\mathfrak{n}}(\mu_2)}\big(e^{\widehat{\mathfrak{n}}(\mu_1)}+e^{\widehat{\mathfrak{n}}(\mu_2)}\big)\int_{\mathcal{I}} \|A(s,\mu_1)-A(s,\mu_2)\|\,ds \le 2K^4 e^{4K}\mathfrak{a}(\mu_1,\mu_2),\end{aligned} \tag{3.12}$$

where $t,s\in\mathcal{I},\ \mu_1,\mu_2\in\mathcal{M}$.

Proof of Lemma 3.2. From differential equations $\dot X(t,\mu_i)=A(t,\mu_i)X(t,\mu_i)$ $(i=1,2)$ we obtain the following differential equation (Cauchy problem)

$$\frac{d\big(X(t,\mu_1)-X(t,\mu_2)}{dt} = A(t,\mu_1)\big(X(t,\mu_1)-X(t,\mu_2)\big)+\big(A(t,\mu_1)-A(t,\mu_2)\big)X(t,\mu_2), \qquad \big(X(t_0,\mu_1)-X(t_0,\mu_2)\big)=0 \tag{3.13}$$

which we may consider as a non-homogeneous matrix differential equation with respect to the unknown function $Y \doteq X(t,\mu_1)-X(t,\mu_2)$, having $A(t,\mu_1)$ as its matrix, with the right-hand side $\big(A(t,\mu_1)-A(t,\mu_2)\big)X(t,\mu_2)$, and having zero initial value. Using Cauchy formula, we obtain

$$Y(t)=\int_{t_0}^{t} X(t,\mu_1)X^{-1}(s,\mu_1)\big(A(s,\mu_1)-A(s,\mu_2)\big)X(s,\mu_2)\,ds,$$

which implies that

$$\|Y\| \le \|X(t,\mu_1)\|\int_{t_0}^{t}\|X^{-1}(s,\mu_1)\|\cdot\|A(s,\mu_1)-A(s,\mu_2)\|\cdot\|X(s,\mu_2)\|\,ds. \tag{3.14}$$

Now, since $X(t,\mu_1)=E+\int_{t_0}^{t} A(s,\mu_1)X(s,\mu_1)\,ds$, we have $\|X(t,\mu_1)\|\le \xi+\int_{t_0}^{t}\|A(s,\mu_1)\|\cdot\|X(s,\mu_1)\|\,ds$. By Gronwall-Bellman inequality (see, e.g. [2, p. 37])

$$\|X(t,\mu_1)\| \le \xi\exp\left(\int_{t_0}^{t}\|A(s,\mu_1)\|\,ds\right) \le \xi\exp\left(\int_{\mathcal{I}}\|A(s,\mu_1)\|\,ds\right) \le \xi e^{\widehat{\mathfrak{n}}(\mu_1)} \quad (t\in\mathcal{I}).$$

Since

$$\frac{dX^{-1}(t,\mu_1)}{dt} = -X^{-1}(t,\mu_1)A(t,\mu_1),$$

we can apply the above argument to function $X^{-1}(t,\mu_1)$, thus arriving to estimate (3.8).

Using (3.7), (3.8), (3.14), we obtain estimate (3.9). A similar argument gives us (3.10).

Now, it follows from (3.7)–(3.8) that

$$\|C(t,s,\mu)\| \le \|X(t,\mu)\|\cdot\|X^{-1}(s,\mu)\| \le \xi^2 e^{2\widehat{\mathfrak{n}}(\mu)} \quad (t,s\in\mathcal{I}).$$

Furthermore, using estimates (3.9) and (3.10) we get

$$\|C(t,s,\mu_1)-C(t,s,\mu_2)\|=\|X(t,\mu_1)\big(X^{-1}(s,\mu_1)-X^{-1}(s,\mu_2)\big)-\big(X(t,\mu_1)-X(t,\mu_2)\big)X^{-1}(s,\mu_2)\|\leq$$
$$\leq \|X(t,\mu_1)\|\cdot\|X^{-1}(s,\mu_1)-X^{-1}(s,\mu_2)\|+\|X(t,\mu_1)-X(t,\mu_2)\|\cdot\|X^{-1}(s,\mu_2)\|\leq$$
$$\leq \xi^4 e^{2\widehat{\mathfrak{n}}(\mu_1)+\widehat{\mathfrak{n}}(\mu_2)}\big(e^{\widehat{\mathfrak{n}}(\mu_1)}+e^{\widehat{\mathfrak{n}}(\mu_2)}\big)\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds \quad (t,s\in\mathcal{I}).$$

The proof of Lemma 3.2 is complete. □

In the next lemma we esimate from above the distance $\rho\big(X(\cdot,\mu_1),\,X(\cdot,\mu_2)\big)=\widehat{\rho}(\mu_1,\mu_2)$.

Lemma 3.3.
$$\widehat{\rho}(\mu_1,\mu_2)\leq \xi e^{\widehat{\mathfrak{n}}(\mu_1)}\Big(\widehat{\mathfrak{n}}(\mu_1)\xi^2 e^{\widehat{\mathfrak{n}}(\mu_1)+\widehat{\mathfrak{n}}(\mu_2)}+1\Big)\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds\leq\big(K^4e^{3K}+Ke^K\big)\mathfrak{a}(\mu_1,\mu_2)$$
$(\mu_1,\mu_2\in\mathcal{M})$.

Proof of Lemma 3.3. Using estimates (3.13), (3.14) and (3.9) we obtain

$$\widehat{\rho}(\mu_1,\mu_2)=\int_{\mathcal{I}}\|\dot{X}(t,\mu_1)-\dot{X}(t,\mu_2)\|\,dt=$$
$$=\int_{\mathcal{I}}\|A(t,\mu_1)\big(X(t,\mu_1)-X(t,\mu_2)\big)+\big(A(t,\mu_1)-A(t,\mu_2)\big)X(t,\mu_2)\|\,dt\leq$$
$$\int_{\mathcal{I}}\|A(t,\mu_1)\|\,\|(X(t,\mu_1)-X(t,\mu_2)\|\,dt+\int_{\mathcal{I}}\|A(t,\mu_1)-A(t,\mu_2)\|\,\|X(t,\mu_2)\|\,dt\leq$$
$$\leq \xi e^{\widehat{\mathfrak{n}}(\mu_1)}\Big(\widehat{\mathfrak{n}}(\mu_1)\xi^2 e^{\widehat{\mathfrak{n}}(\mu_1)+\widehat{\mathfrak{n}}(\mu_2)}+1\Big)\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds$$

This completes the proof of Lemma 3.3. □

Next, we obtain estimates for the solution $Y=Y(t,\mu)$ of non-homogeneous Cauchy problem (3.4).

Lemma 3.4. *We have the following estimates:*

$$\|Y(t,\mu)\|\leq\|X^0(\mu)\|\,\xi e^{\mathfrak{n}(\mu)}+\xi^2e^{2\mathfrak{n}(\mu)}\cdot\int_{\mathcal{I}}\|\Phi(s,\mu)\|\,ds\leq K^2e^K+K^3e^{2K}\quad(t\in\mathcal{I},\,\mu\in\mathcal{M}). \tag{3.15}$$

$$\|Y(t,\mu_1)-Y(t,\mu_2)\|\leq\xi e^{\mathfrak{n}(\mu_2)}\|X^0(\mu_1)-X^0(\mu_2)\|+ \tag{3.16}$$
$$+\xi^3e^{2\mathfrak{n}(\mu_1)+\mathfrak{n}(\mu_2)}\left(\|X^0(\mu_1)\|+\xi e^{\mathfrak{n}(\mu_1)}+e^{\mathfrak{n}(\mu_2)}\int_{\mathcal{I}}\|\Phi(s,\mu_1)\|\,ds\right)\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds+$$
$$+\xi^2e^{2\mathfrak{n}(\mu_2)}\cdot\int_{\mathcal{I}}\|\Phi(s,\mu_1)-\Phi(s,\mu_2)\|\,ds\leq$$
$$\leq K_1\mathfrak{x}(\mu_1,\mu_2)+K_2\mathfrak{a}(\mu_1,\mu_2)+K_3\mathfrak{f}(\mu_1,\mu_2)\quad(t\in\mathcal{I},\,\mu_1,\mu_2\in\mathcal{M}),$$

where $K_1=Ke^K,\;\;K_2=K^4e^{3K}(1+K+e^K),\;\;K_3=K^2e^{2K}$

Proof of Lemma 3.4. Using Cauchy formula we obtain

$$Y(t,\mu)=X(t,\mu)X^0(\mu)+\int_{t_0}^{t}C(t,s,\mu)\Phi(s,\mu)\,ds\quad(t\in\mathcal{I},\,\mu\in\mathcal{M}). \tag{3.17}$$

Further, according to (3.7) and (3.11) we have the estimate

$$\|Y(t,\mu)\|\leq\|X(t,\mu)\|\,\|X^0(\mu)\|+\int_{t_0}^{t}\|C(t,s,\mu)\|\cdot\|\Phi(s,\mu)\|\,ds\leq$$
$$\leq\|X^0(\mu)\|\,\xi e^{\mathfrak{n}(\mu)}+\xi^2e^{2\mathfrak{n}(\mu)}\cdot\int_{\mathcal{I}}\|\Phi(s,\mu)\|\,ds\quad(t\in\mathcal{I},\,\mu\in\mathcal{M}).$$

Now, in virtue of (3.7), (3.11), (3.9) and (3.12) we have

$$\|Y(t,\mu_1)-Y(t,\mu_2)\| \le \|X(t,\mu_1)-X(t,\mu_2)\|\,\|X^0(\mu_1)\| + \|X(t,\mu_2)\|\,\|X^0(\mu_1)-X^0(\mu_2)\|+$$
$$+\max_{(t,s)\in=\mathcal{I}^2}\|C(t,s,\mu_1)-C(t,s,\mu_2)\|\cdot\int_{\mathcal{I}}\|\Phi(s,\mu_1)\|\,ds+$$
$$+\max_{(t,s)\in=\mathcal{I}^2}\|C(t,s,\mu_2)\|\cdot\int_{\mathcal{I}}\|\Phi(s,\mu_1)-\Phi(s,\mu_2)\|\,ds\le$$
$$\le\|X^0(\mu_1)\|\,\|E\|^3e^{2\mathfrak{n}(\mu_1)+\mathfrak{n}(\mu_2)}\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds+\|X^0(\mu_1)-X^0(\mu_2)\|\cdot\xi e^{\mathfrak{n}(\mu_2)}+$$
$$+\xi^4e^{2\mathfrak{n}(\mu_1)+\mathfrak{n}(\mu_2)}\big(e^{\mathfrak{n}(\mu_1)}+e^{\mathfrak{n}(\mu_2)}\big)\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds\cdot\int_{\mathcal{I}}\|\Phi(s,\mu_1)\|\,ds+$$
$$+\xi^2e^{2\mathfrak{n}(\mu_2)}\cdot\int_{\mathcal{I}}\|\Phi(s,\mu_1)-\Phi(s,\mu_2)\|\,ds=\xi e^{\mathfrak{n}(\mu_2)}\,\|X^0(\mu_1)-X^0(\mu_2)\|+$$
$$+\xi^3e^{2\mathfrak{n}(\mu_1)+\mathfrak{n}(\mu_2)}\left(\|X^0(\mu_1)\|+\xi e^{\mathfrak{n}(\mu_1)}+e^{\mathfrak{n}(\mu_2)}\int_{\mathcal{I}}\|\Phi(s,\mu_1)\|\,ds\right)\int_{\mathcal{I}}\|A(s,\mu_1)-A(s,\mu_2)\|\,ds+$$
$$+\xi^2e^{2\mathfrak{n}(\mu_2)}\cdot\int_{\mathcal{I}}\|\Phi(s,\mu_1)-\Phi(s,\mu_2)\|\,ds\quad(t\in\mathcal{I},\ \mu_1,\mu_2\in\mathcal{M}),$$

which concludes the proof of Lemma 3.4. □

Now, we estimate from above the distance $\rho\big(Y(\cdot,\mu_1),\,Y(\cdot,\mu_2)\big)=\widehat{\rho}(\mu_1,\mu_2)$.

Lemma 3.5.

$$\widehat{\rho}(\mu_1,\mu_2)\le \tag{3.18}$$
$$\le\widetilde{K}_1\,\|X^0(\mu_1)-X^0(\mu_2)\|+\widetilde{K}_2\int_{\mathcal{I}}\|A(t,\mu_1)-A(t,\mu_2)\|\,dt+\widetilde{K}_3\int_{\mathcal{I}}\|\Phi(t,\mu_1)-\Phi(t,\mu_2)\|\,dt,$$

where $\widetilde{K}_1=1+K^2e^K,\ \ \widetilde{K}_2=K^2e^K(1+Ke^K+K^3e^{2K}+2K^3e^{3K}),\ \ \widetilde{K}_3=1+K^3e^{2K}$.

Proof of Lemma 3.5. According to (3.4) we have

$$\widehat{\rho}(\mu_1,\mu_2)=\|X^0(\mu_1)-X^0(\mu_2)\|+\int_{\mathcal{I}}\|\dot{Y}(t,\mu_1)-\dot{Y}(t,\mu_2)\|\,dt\le\mathfrak{x}(\mu_1,\mu_2)+$$
$$+\int_{\mathcal{I}}\|A(t,\mu_1)\|\,\|Y(t,\mu_1)-Y(t,\mu_2)\|\,dt+\int_{\mathcal{I}}\|A(t,\mu_1)-A(t,\mu_2)\|\,\|Y(t,\mu_2)\|\,dt+\mathfrak{f}(\mu_1,\mu_2)\le$$
$$\le\mathfrak{x}\,(\mu_1,\mu_2)+K\Big(Ke^K\mathfrak{x}\,(\mu_1,\mu_2)+K^4e^{3K}(1+2e^K)\mathfrak{a}(\mu_1,\mu_2)+K^2e^{2K}\mathfrak{f}\,(\mu_1,\mu_2)\Big)+$$
$$+K^2\big(e^K+Ke^{2K}\big)\mathfrak{a}\,(\mu_1,\mu_2)+\mathfrak{f}(\mu_1,\mu_2)=\widetilde{K}_1\mathfrak{x}\,(\mu_1,\mu_2)+\widetilde{K}_2\mathfrak{a}(\mu_1,\mu_2)+\widetilde{K}_3\mathfrak{f}(\mu_1,\mu_2),$$

as needed. □

Now, we are ready to complete the proof of Theorem 3.1. Let $\varepsilon>0$ be arbitrary. By our assumptions there is $\delta>0$ such that if $\|\mu_1-\mu_2\|<\delta$, then

$$\mathfrak{x}(\mu_1,\mu_2)<\frac{\varepsilon}{3K_1},\quad\mathfrak{a}(\mu_1,\mu_2)<\frac{\varepsilon}{3K_2},\quad\mathfrak{f}(\mu_1,\mu_2)<\frac{\varepsilon}{3K_3},$$
$$\mathfrak{x}(\mu_1,\mu_2)<\frac{\varepsilon}{3\widetilde{K}_1},\quad\mathfrak{a}(\mu_1,\mu_2)<\frac{\varepsilon}{3\widetilde{K}_2},\quad\mathfrak{f}(\mu_1,\mu_2)<\frac{\varepsilon}{3\widetilde{K}_3}.$$

This esimate, combined with Lemma 3.4 (estimates (3.16)) and 3.5), implies that

$$\|Y(t,\mu_1)-Y(t,\mu)\|<\varepsilon\quad(t\in\mathcal{I}),\qquad\rho\big(Y(\cdot,\mu_1),\,Y(\cdot,\mu_2)\big)<\varepsilon.$$

The proof of Theorem 3.1 is complete. □

Theorem 3.6. *The assertion of Theorem 3.1 remains true if we replace in its formulation* $\widehat{\mathfrak{n}}$, φ, $\mathfrak{a}$, $\mathfrak{f}$ *with, respectively,*

$$\widehat{\mathfrak{n}}_q(\mu) \doteq \left(\int_{\mathcal{I}} \|A(\mu)\|^q \, dt\right)^{\frac{1}{q}}, \quad \varphi_q(\mu) \doteq \left(\int_{\mathcal{I}} \|\Phi(\mu)\|^q \, dt\right)^{\frac{1}{q}},$$

$$\mathfrak{a}_p(\mu_1, \mu_2) \doteq \left(\int_{\mathcal{I}} \|A(t,\mu_1) - A(t,\mu_2)\|^p \, dt\right)^{\frac{1}{p}}, \quad \mathfrak{f}_p(\mu_1, \mu_2) \doteq \left(\int_{\mathcal{I}} \|\Phi(t,\mu_1) - \Phi(t,\mu_2)\|^p \, dt\right)^{\frac{1}{p}}$$

$\left(1 < p < +\infty,\ \frac{1}{p} + \frac{1}{q} = 1\right)$,

Proof. In the proof of Theorem 3.1, in the estimates of the integrals of products one has to apply Hölder inequality. □

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