The Approximate Sum Capacity of the Symmetric Gaussian K-User Interference Channel

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Abstract—**Interference alignment has emerged as a powerful tool in the analysis of multi-user networks. Despite considerable recent progress, the capacity region of the Gaussian K-user interference channel is still unknown in general, in part due to the challenges associated with alignment on the signal scale using lattice codes. This paper develops a new framework for lattice interference alignment, based on the compute-and-forward approach. Within this framework, each receiver decodes by first recovering two or more linear combinations of the transmitted codewords with integer-valued coefficients and then solving these equations for its desired codeword. For the special case of symmetric channel gains, this framework is used to derive the approximate sum capacity of the Gaussian interference channel, up to an explicitly defined outage set of the channel gains. The key contributions are the capacity lower bounds for the weak through strong interference regimes, where each receiver should jointly decode its own codeword along with part of the interfering codewords. As part of the analysis, it is shown that decoding K linear combinations of the codewords can approach the sum capacity of the K-user Gaussian multiple-access channel up to a gap of no more than $\frac{K}{2}\log K$ bits.**

I. INTRODUCTION

Handling interference efficiently is a major challenge in multi-user wireless communication. Recently, it has become clear that this challenge can sometimes be overcome via *interference alignment* [1], [2]. For instance, consider the K-user Gaussian interference channel, where K transmitter-receiver pairs wish to communicate simultaneously. Through the use of clever encoding strategies, it is possible to align the transmitted signals so that each receiver only observes its desired signal along with a single effective interferer. As a result, each user can achieve roughly half the rate that would be available were there no interference whatsoever, i.e., $K/2$ degrees-of-freedom (DoF) are available. However, many schemes, such as the Cadambe-Jafar framework [2] and ergodic interference alignment [3], require a large number of independent channel realizations to achieve near-perfect alignment. In certain settings, this level of channel diversity may not be attainable; ideally, we would like to achieve alignment over a single channel realization.

The capacity region of the (static) Gaussian K-user interference channel [4] is unknown in general, although significant

progress has been made recently, in part due to the discovery of interference alignment and the shift from exact capacity results to capacity approximations [5]–[7]. It has been shown by Motahari *et al.* that $K/2$ degrees-of-freedom are achievable for almost all channel realizations [8] but it is an open question as to whether this result translates to real gains outside of the very high signal-to-noise ratio (SNR) regime. One promising direction is the use of lattice codes [9]–[11], as they can enable alignment on the signal scale. By taking advantage of the fact that the sum of lattice codewords is itself a lattice codeword, a receiver can treat several users as one effective user, thereby reducing the number of effective interferers. A compelling example of this approach is the derivation of the approximate capacity of the many-to-one interference channel by Bresler, Parekh, and Tse [7]. For fully connected channels, much less is known, owing to the difficulty of choosing lattices that simultaneously align at several receivers.

In some cases, focusing on the special case of symmetric channel gains has yielded important insights. For instance, in the two-user case, Etkin, Tse, and Wang [5] used the symmetric interference channel to develop the notion of generalized degrees-of-freedom. This in turn revealed five operating regimes, based on relative interference strength:

- *Noisy:* Each receiver treats interference as noise, which is optimal for sufficiently weak interference [12]–[14].
- *Weak and Moderately Weak:* Each transmitter sends a public and a private codeword following the scheme of Han and Kobayashi [15]. Each receiver jointly decodes both public codewords and its desired private codeword while treating the interfering private codeword as noise.
- *Strong:* Each receiver jointly decodes both users' codewords. This regime and its capacity was discovered by Sato [16] as well as Han and Kobayashi [15].
- *Very Strong:* Each receiver decodes and subtracts the interference before recovering its desired codeword. This regime and its capacity was discovered by Carleial [17].

Using these regimes as a guideline, they were able to approximate the capacity region to within half a bit.

In this paper, we focus on the special case of the symmetric Gaussian K-user interference channel. Each receiver observes

$$\mathbf{y}_k = \mathbf{x}_k + g \sum_{m \neq k} \mathbf{x}_m + \mathbf{z}_k$$

where $\mathbf{x}_k$ is the codeword sent by the kth transmitter, g is the cross-channel gain, and $\mathbf{z}_k$ is additive white Gaussian noise. If each transmitter uses the same lattice codebook, then each receiver observes an effective two-user multiple-access channel (MAC). The first effective transmitter sends the desired codeword and the second sends the sum of interfering

codewords, which is itself a codeword. A natural approach is to first decode and cancel this sum of codewords $\sum_{m \neq k} \mathbf{x}_m$, leaving only the desired signal plus noise. In the very strong interference regime, this approach is optimal [18] but it does not suffice in general. In our setting, joint decoding must cope with the fact that all users employ the same lattice codebook, and distinct pairs of lattice codewords may result in the same sum.

Our main contribution is the derivation of new achievable rate regions for the weak, moderately weak, and strong interference regimes, for which no capacity approaching schemes were known in the literature. This is enabled using a new technique, based on compute-and-forward [19], that allows each receiver to jointly decode its desired codeword and the sum of the interfering codewords. We also propose a generalization of the two-user Han-Kobayashi scheme [15] in which each user transmits a superposition of one private lattice codeword and one public lattice codeword. Each receiver decodes its private and public lattice codewords as well as linear combinations of all interfering public lattice codewords while treating the interfering private lattice codewords as noise. Using these new tools, we can mimic the achievable scheme of [5], and obtain an approximation of the sum capacity in all regimes. In the weak interference regime, the approximation is valid for all channel gains, whereas in the strong and moderately weak interference regimes, it is valid for all channel gains except for an outage set. The measure of this outage set can be made as small as desired, but this comes at the expense of increasing the gap between the inner and outer bounds on the capacity. Our capacity approximation closely follows the basic shape of the generalized degrees-of-freedom (shown in Figure 3), which was derived for this channel by Jafar and Vishwanath [20]. Their approach is inspired by the deterministic model [6], and relies upon coding across an infinite number of signal levels. As a result, their coding scheme cannot be directly translated into a finite SNR result, as existing techniques for decoding the sum of codewords incur a penalty for each additional codeword layer [19].

In its original incarnation, compute-and-forward makes it possible for relays in a network to decode integer combinations of the transmitted codewords and send them towards a destination [19]. The effective SNR at each relay is determined by how closely the integer coefficients match the channel coefficients. If together the relays recover a full-rank set of linear combinations, the destination can recover all of the messages. In our setting, each receiver decodes *two linear combinations of the transmitted codewords*,

$$a_{11}\mathbf{x}_k + a_{12} \sum_{m \neq k} \mathbf{x}_m \qquad a_{21}\mathbf{x}_k + a_{22} \sum_{m \neq k} \mathbf{x}_m \ ,$$

where $a_{11}, a_{12}, a_{21},$ and a_{22} are integer-valued coefficients. If the vectors $\mathbf{a}_1 = [a_{11} \ a_{12}]^T$ and $\mathbf{a}_2 = [a_{21} \ a_{22}]^T$ are linearly independent, then each receiver can solve for its desired codeword $\mathbf{x}_k$. Through this approach, we can derive closed-form lower bounds on the performance of joint decoding. A direct analysis of joint decoding, although possible [21], presents additional difficulties that in turn lead to looser bounds.

As part of the derivation of our sum capacity bounds, we develop a new decoding framework, *the compute-and-forward transform*, that may be of independent interest. Consider a Gaussian K-user MAC where the encoders employ lattice codes. Under our framework, the receiver attempts to recover K linearly independent equations of the transmitted codewords and then solve them for its desired messages. This transforms the MAC into an effective multiple-input multiple-output (MIMO) channel with an integer-valued channel matrix. A striking phenomenon we discover is that while the computation rate for each of these K equations is very sensitive to the exact channel gains, the sum of their computation rates is equal to the sum capacity of the MAC up to a constant gap, independent of the channel gains and the SNR (see Figure 1). To give this an operational meaning, the computation rate for each equation is associated with a single transmitter's codeword. The receiver successively decodes the equations, ordered by decreasing computation rate. As we will show, each decoded equation can be used to remove its associated codeword from the receiver's observation in a way that does not alter the effective SNR for the remaining equations.
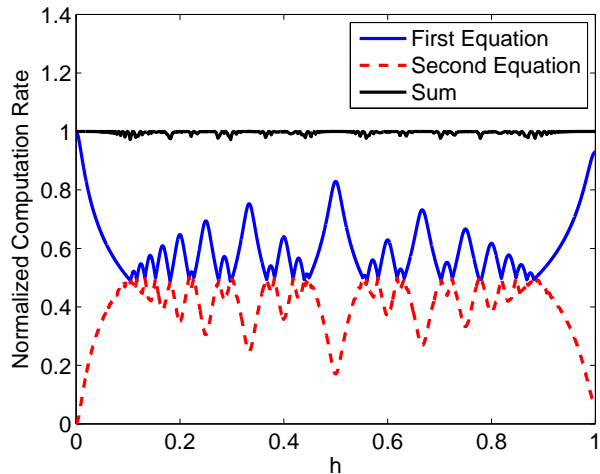

Fig. 1. Computation rates for the best two linearly independent equations vs. h for the channel $\mathbf{y} = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}$ at SNR=40dB. The sum of these computation rates is nearly equal to the multiple-access sum capacity. All rates are normalized by this sum capacity $1/2 \log(1 + (1 + h^2)\mathsf{SNR})$.

A. Related Work

Interference alignment has generated a great deal of excitement, due to the promise of higher throughputs in wireless networks [1], [2] as well as other applications, including coding for distributed storage [22]. See the recent monograph by Jafar for a comprehensive survey [23]. Of particular note is a series of recent papers that delineate the degrees-of-freedom limits of linear beamforming strategies for alignment over a finite number of channel realizations [24], [25]. Beamforming strategies can only approach perfect alignment asymptotically, whereas lattice-based schemes can achieve $K/2$ degrees-of-freedom over a single channel realization [8]. However, lattice-based alignment at finite SNR has to date been limited to special cases, such as symmetric [18], [21], [26], integer [27],

and many-to-one interference channels [7], [28]. Capacity approximations are also available for one-to-many [7] and cyclic interference channels [29], although these coding schemes do not employ alignment.

The results in this paper are connected to the recent work of Niesen and Maddah-Ali [30], which proposes a finer-grain deterministic model for interference networks that is rich enough to capture the phenomenon of alignment. Via this model, they derive the approximate capacity region of the two-user Gaussian X-channel. As in our work, they characterize the performance of joint decoding over a MAC where all users employ the same linear code. While both approaches yield "constant-gap" approximations, our framework naturally yields achievable rate expressions that are easy to plot and often much closer to the upper bound than the constant-gap analysis suggests.

Bandemer and El Gamal have recently proposed a class of three-user deterministic channels where the interfering signals are passed through a function on their way to the receiver, which, in a certain sense, models interference alignment [31]. They develop a new rate region based on interference decoding for this model. In a recent paper, Wu, Shamai, and Verdú have derived a general formula for the degrees-of-freedom of the K-user Gaussian interference channel via Rényi's information dimension [32].

Nested lattice codes have been thoroughly studied as a framework for efficient source and channel coding with side information [9], [10], [33]. Recently, it has become clear that the inherent linear structure of lattices can enable many interesting new schemes, including distributed dirty paper coding [34], distributed source coding of linear functions [35]–[37], distributed antenna systems [38]–[40], and physical-layer network coding [19], [41]–[44], to name a few. The origins of these schemes can be traced to the work of Körner and Marton [45], who showed that linear binning is optimal for the distributed compression of the parity of a doubly symmetric binary source.

The remainder of the paper is organized as follows. Section II gives a formal problem statement as well as the approximate sum capacity of the Gaussian K-user interference channel. Next, Section III provides a brief review of nested lattice codes and the compute-and-forward strategy. Section IV shows how the Gaussian multiple-access sum capacity can be approached within a constant gap using compute-and-forward combined with a type of successive interference cancellation. It also shows how an effective multiple-access channel emerges in the context of lattice interference alignment. The upper bounds needed to establish our approximate sum capacity result are reviewed in Section V. Afterwards, Section VI develops our two achievable schemes and develops closed-form expressions for each interference regime. Finally, Section VII derives the degrees-of-freedom attained by our strategy.

II. SYMMETRIC GAUSSIAN K-USER INTERFERENCE CHANNEL

A. Problem Statement

We begin with some notational conventions. We will denote column vectors with boldface lowercase letters and matrices with boldface uppercase letters. For instance, $\mathbf{a} \in \mathbb{Z}^K$ and $\mathbf{A} \in \mathbb{Z}^{K \times K}$. Let $\|\mathbf{a}\| = \sqrt{\sum_{k=1}^{K} a_k^2}$ denote the ℓ_2-norm of the vector $\mathbf{a}$. Also, let $\mathbf{0}$ denote the zero vector and $\mathbf{I}_{K \times K}$ denote the identity matrix of size K. We use $\lfloor \cdot \rceil$ to denote rounding to the nearest integer, $\lfloor \cdot \rfloor$ to denote the floor operation and $\lceil \cdot \rceil$ for the ceiling operation. In general, the letters a and b are used in this paper whenever the variables they describe are integer valued. All logarithms are to base 2. We also occasionally use the notation $\log^+(x) \triangleq \max(0, \log(x))$.
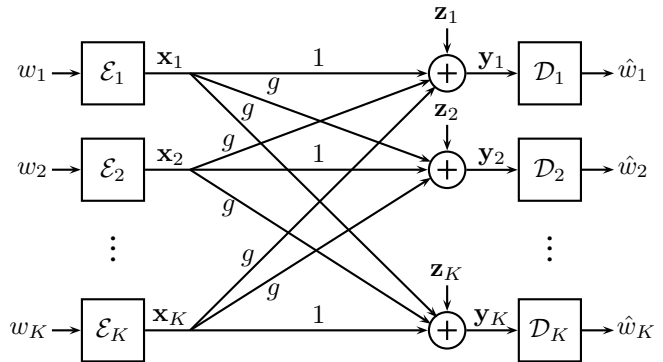

Fig. 2. Block diagram of a symmetric Gaussian K-user interference channel.

There are K transmitter-receiver pairs that wish to simultaneously communicate across a shared channel over n time slots, where the channel gains are constant over all n channel uses. We assume a real-valued channel model throughout.

Definition 1 (Messages): Each transmitter has a *message* w_k drawn independently and uniformly over $\{1, 2, \ldots, 2^{nR_{\text{SYM}}}\}$.

Definition 2 (Encoders): Each transmitter is equipped with an *encoder*, $\mathcal{E}_k : \{1, 2, \ldots, 2^{nR_{\text{SYM}}}\} \to \mathbb{R}^n$, that maps its message into a length-n channel input $\mathbf{x}_k = \mathcal{E}_k(w_k)$ that satisfies the power constraint,

$$\|\mathbf{x}_k\|^2 \leq n\mathsf{SNR}$$

where $\mathsf{SNR} > 0$ is the signal-to-noise ratio.

Definition 3 (Channel Model): The channel output at each receiver is a noisy linear combination of its desired signal and the sum of the interfering terms, of the form

$$\mathbf{y}_k = \mathbf{x}_k + g \sum_{m \neq k} \mathbf{x}_m + \mathbf{z}_k \ , \qquad (1)$$

where $g > 0$ parametrizes the interference strength and $\mathbf{z}_k$ is an i.i.d. Gaussian vector with mean 0 and variance 1. We define the *interference-to-noise ratio* to be

$$\mathsf{INR} \triangleq g^2 \mathsf{SNR}$$

and the *interference level* to be

$$\alpha \triangleq \frac{\log(\mathsf{INR})}{\log(\mathsf{SNR})} \ .$$

Remark 1: Note that our definition of INR ignores the fact that there are $K - 1$ interferers observed at each receiver. This is for two reasons. First, this definition parallels that of the two-user case [5], which will make it easier to compare

the two rate regions. Second, the receivers will often be able to treat the interference as stemming from a single effective transmitter, via interference alignment.

Definition 4 (Decoders): Each receiver is equipped with a decoder, $\mathcal{D}_k : \mathbb{R}^n \to \{1, 2, \ldots, 2^{nR_{\mathrm{SYM}}}\}$, that produces an estimate $\hat{w}_k = \mathcal{D}_k(\mathbf{y}_k)$ of its desired message w_k.

Definition 5 (Symmetric Capacity): A symmetric rate R_{SYM} is *achievable* if, for any $\epsilon > 0$ and n large enough, there exist encoders and decoders that can attain probability of error at most ϵ,

$$\Pr\left(\{\hat{w}_1 \neq w_1\} \cup \cdots \cup \{\hat{w}_K \neq w_K\}\right) < \epsilon .$$

The *symmetric capacity* C_{SYM} is the supremum of all achievable symmetric rates.

Remark 2: Due to the symmetry of the channel, the symmetric capacity is equal to the sum capacity, normalized by the number of users. To see this, assume that the users employ different rates and that a rate tuple $(R_1, R_2, \ldots, R_K)$ is achievable. Since each transmitter-receiver pair sees the same effective channel, we can simply exchange the encoders and decoders to achieve the rate tuple $(R_{\pi(1)}, R_{\pi(2)}, \ldots, R_{\pi(K)})$ for any permutation π. By time-sharing across all permutations, we find that each user can achieve $\frac{1}{K}\sum_{k=1}^{K} R_k$, corresponding to a symmetric rate. Thus, the sum of any achievable rate tuple is upper bounded by KC_{SYM}.

Definition 6 (Generalized Degrees-of-Freedom): The *generalized degrees-of-freedom* (GDoF) specifies the fraction of the point-to-point Gaussian capacity that can be attained per user for a given interference level $\alpha \geq 0$ as SNR tends to infinity,

$$d(\alpha) = \lim_{\mathrm{SNR}\to\infty} \frac{C_{\mathrm{SYM}}}{\frac{1}{2}\log(1+\mathrm{SNR})} .$$

B. Approximate Sum Capacity

As shown by Jafar and Vishwanath [20, Theorem 3.1], the GDoF of the symmetric K-user interference channel is identical to that of the two-user channel, except for a singularity at $\alpha = 1$,

$$d(\alpha) = \begin{cases} 1 - \alpha & 0 \leq \alpha < \frac{1}{2} \ \ (\text{noisy}) \\ \alpha & \frac{1}{2} \leq \alpha < \frac{2}{3} \ \ (\text{weak}) \\ 1 - \frac{\alpha}{2} & \frac{2}{3} \leq \alpha < 1 \ \ (\text{moderately weak}) \\ \frac{1}{K} & \alpha = 1 \\ \frac{\alpha}{2} & 1 < \alpha < 2 \ \ (\text{strong}) \\ 1 & \alpha \geq 2 \ \ (\text{very strong}). \end{cases}$$

See Figure 3 for a plot. Notice that since SNR is taken to infinity, the GDoF characterization treats all channel gains g that do not scale with SNR as a single point at $\alpha = 1$. A finer view of this regime is possible at high SNR by simply setting g to be some fixed value and then taking SNR to infinity, corresponding to the standard notion of degrees-of-freedom. Surprisingly, this degrees-of-freedom characterization is everywhere discontinuous with respect to g [46]. This presents an obstacle towards a clean capacity approximation at finite SNR.
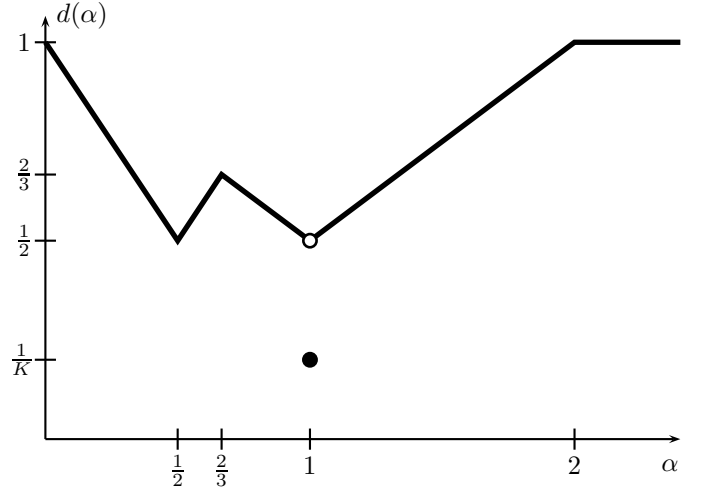

Fig. 3. Generalized degrees-of-freedom for the symmetric Gaussian K-user interference channel.

To overcome this difficulty, our approximations allow for the possibility of an *outage set*, which is explicitly characterized. Specifically, in the regime around $\alpha = 1$, our capacity results take the following shape: for any constant $c > 0$, the capacity is approximated within at most $c + 9 + \log K$ bits over the entire range of SNR, and all channel gains g, except for a set of measure $\mu(c)$ which vanishes rapidly with c. This type of capacity approximation has also been used by Niesen and Maddah-Ali for the two-user Gaussian X channel [30] and seems to arise from the capacity region itself, not just the lower bound. That is, it appears that the capacity may in fact simultaneously vary rapidly with the fine scale of the channel gains (e.g., the distance to an appropriately scaled integer) and slowly on the coarse scale (e.g., relative interference strength). In the high SNR limit, this behavior shows up as a discontinuity on the rationals but, at reasonable SNRs, our achievable scheme shows that this variation is in fact fairly smooth. The theorem below captures our capacity approximations in a simple form.

Theorem 1: The symmetric capacity of the symmetric Gaussian K-user interference channel can be lower and upper bounded as follows:

- *Noisy Interference Regime, $0 \leq \alpha < \frac{1}{2}$,*

$$\frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{1+\mathrm{INR}}\right) - \frac{1}{2}\log(K-1)$$
$$\leq C_{\mathrm{SYM}} < \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{1+\mathrm{INR}}\right) + 1$$

- *Weak Interference Regime, $\frac{1}{2} \leq \alpha < \frac{2}{3}$,*

$$\frac{1}{2}\log^+(\mathrm{INR}) - \frac{7}{2} - \log(K) \leq C_{\mathrm{SYM}} \leq \frac{1}{2}\log^+(\mathrm{INR}) + 1$$

for all channel gains.

- *Moderately Weak Interference Regime, $\frac{2}{3} \leq \alpha < 1$,*

$$\frac{1}{2}\log^+\left(\frac{\mathrm{SNR}}{\sqrt{\mathrm{INR}}}\right) - c - 8 - \log(K)$$
$$\leq C_{\mathrm{SYM}} \leq \frac{1}{2}\log^+\left(\frac{\mathrm{SNR}}{\sqrt{\mathrm{INR}}}\right) + 1$$
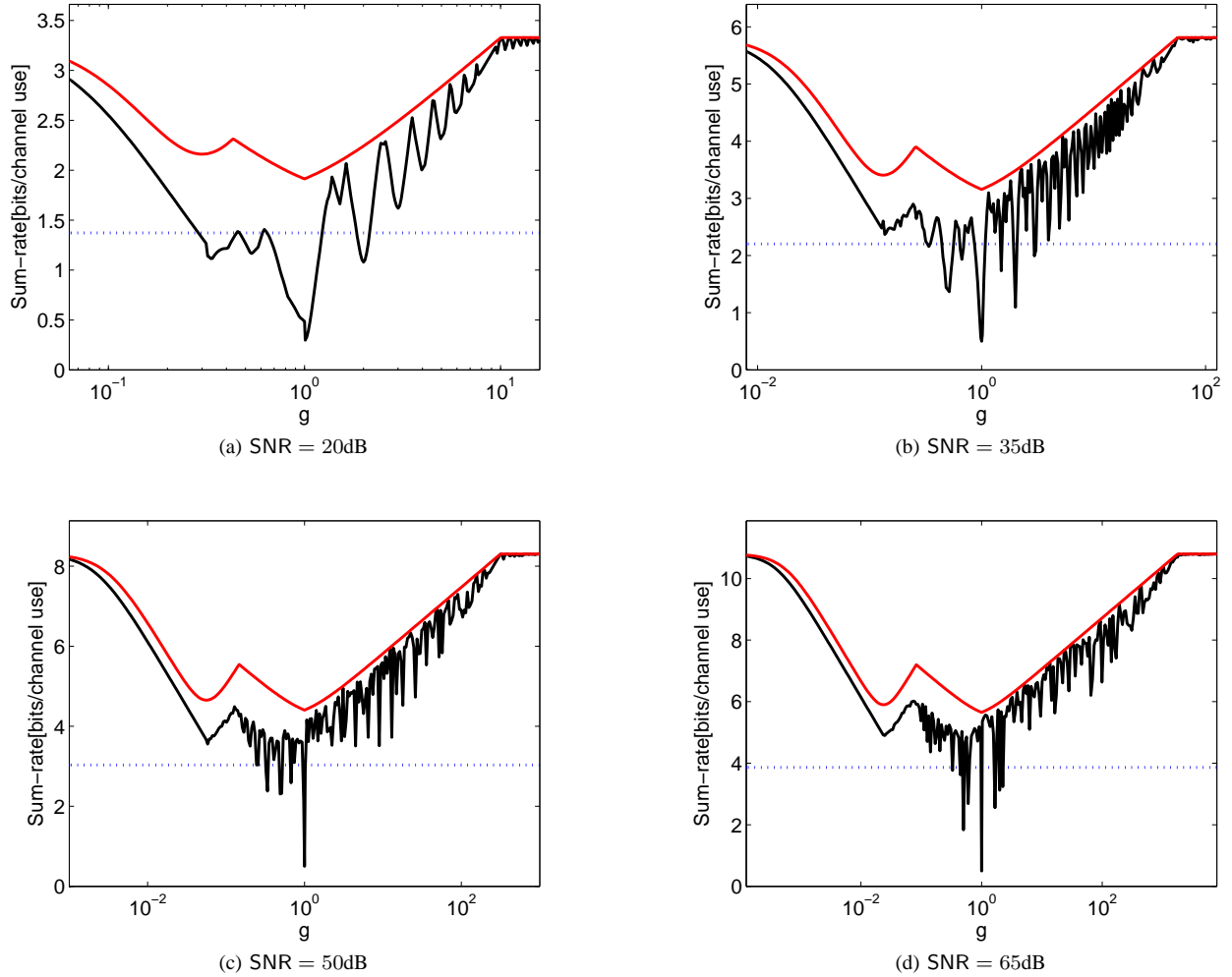
Fig. 4. Upper and lower bounds on the sum capacity of a 3-user symmetric Gaussian interference channel with respect to the cross-gain g. The upper bound (red line) is given by (42) and the lower bound (black line) is the maximum of the achievable rates from Theorem 9 and Corollary 3, which were computed numerically, and Theorem 10. The lower bounds from Theorem 1 are not plotted in this figure. For reference, we have also plotted the rate achievable via time-division (dotted blue line).

for all channel gains except for an outage set of measure $\mu < 2^{-c}$ for any $c > 0$.

- *Strong Interference Regime*, $1 \le \alpha < 2$,

$$\frac{1}{4} \log^+(\mathsf{INR}) - \frac{c}{2} - 3 \le C_{\text{SYM}} \le \frac{1}{4} \log^+(\mathsf{INR}) + 1$$

for all channel gains except for an outage set whose measure is a fraction of 2^{-c} of the interval $1 < |g| < \sqrt{\mathsf{SNR}}$, for any $c > 0$.

- *Very Strong Interference Regime*, $\alpha \ge 2$,

$$\frac{1}{2} \log(1 + \mathsf{SNR}) - 1 \le C_{\text{SYM}} \le \frac{1}{2} \log(1 + \mathsf{SNR})$$

III. PRELIMINARIES

In this section we give some basic definitions and results that will be extensively used in the sequel.

A. K-user Gaussian MAC

Consider the K-user Gaussian MAC

$$\mathbf{y} = \sum_{k=1}^{K} h_k \mathbf{x}_k + \mathbf{z}, \qquad (2)$$

where the vector $\mathbf{h} = [h_1 \cdots h_K]^T \in \mathbb{R}^K$ represents the channel gains, $\mathbf{x}_k \in \mathbb{R}^n$, $k = 1, \ldots, K$, are the channel inputs, $\mathbf{z} \in \mathbb{R}^n$ is additive white Gaussian noise (AWGN) with zero mean and unit variance and $\mathbf{y} \in \mathbb{R}^n$ is the channel output. Without loss of generality, we assume all K users are subject to the same power constraint[1]

$$\|\mathbf{x}_k\|^2 \le n\mathsf{SNR}, \ k = 1, \ldots, K. \qquad (3)$$

The capacity region of the channel (2) is known (see e.g., [47, Theorem 15.3.6]) to be the set of all rate tuples $(R_1, \ldots, R_K)$

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[1] As otherwise the different powers can be absorbed into the channel gains.

satisfying

$$\sum_{k \in \mathcal{S}} R_k < \frac{1}{2} \log \left(1 + \mathsf{SNR} \sum_{k \in \mathcal{S}} |h_k|^2 \right) \qquad (4)$$

for all subsets $\mathcal{S} \subseteq \{1, \ldots, K\}$. The achievability part of the capacity theorem is established using i.i.d. Gaussian codebooks for all users. Motivated by lattice interference alignment, we are interested in establishing the achievability of certain rate tuples under the constraint that the codebooks employed by the K users form a chain of nested lattice codes.

B. Nested Lattice Codes

We employ the nested lattice framework originally proposed in [10]. A lattice Λ is a discrete subgroup of $\mathbb{R}^n$ which is closed under reflection and real addition. Formally, for any $\mathbf{t}_1, \mathbf{t}_2 \in \Lambda$, we have that $-\mathbf{t}_1, -\mathbf{t}_2 \in \Lambda$ and $\mathbf{t}_1 + \mathbf{t}_2 \in \Lambda$. Note that by definition the zero vector $\mathbf{0}$ is always a member of the lattice. Any lattice Λ in $\mathbb{R}^n$ is spanned by some $n \times n$ matrix $\mathbf{G}$ such that

$$\Lambda = \{\mathbf{t} = \mathbf{G}\mathbf{q} : \mathbf{q} \in \mathbb{Z}^n\}.$$

We say that a lattice is full-rank if its spanning matrix $\mathbf{G}$ is full-rank.

We denote the nearest neighbor quantizer associated with the lattice Λ by

$$Q_\Lambda(\mathbf{x}) = \arg \min_{\mathbf{t} \in \Lambda} \|\mathbf{x} - \mathbf{t}\|. \qquad (5)$$

The Voronoi region of Λ, denoted by $\mathcal{V}$, is the set of all points in $\mathbb{R}^n$ which are quantized to the zero vector, where ties in (5) are broken in a systematic manner. The modulo operation returns the quantization error w.r.t. the lattice,

$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}),$$

and satisfies the distributive law,

$$\big[a[\mathbf{x}] \bmod \Lambda + b[\mathbf{y}] \bmod \Lambda \big] \bmod \Lambda = [a\mathbf{x} + b\mathbf{y}] \bmod \Lambda,$$

for all $a, b \in \mathbb{Z}$.

A lattice Λ is said to be nested in Λ_1 if $\Lambda \subseteq \Lambda_1$. The coding schemes presented in this paper utilize a chain of $K+1$ nested lattices satisfying

$$\Lambda \subseteq \Lambda_K \subseteq \cdots \subseteq \Lambda_1. \qquad (6)$$

From these lattices, we construct K codebooks, one for each user. Specifically, user k is allocated the codebook $\mathcal{L}_k = \Lambda_{\theta(k)} \cap \mathcal{V}$, where $\mathcal{V}$ is the Voronoi region of Λ and the function $\theta(k) : \{1, \ldots, K\} \to \{1, \ldots, K\}$ maps between users and lattices. The rate of each codebook $\mathcal{L}_k$ is

$$R_k = \frac{1}{n} \log \big| \Lambda_{\theta(k)} \cap \mathcal{V} \big|.$$

User k encodes its message into a lattice point from its codebook, $\mathbf{t}_k \in \mathcal{L}_k$. Each user also has a random[2] dither vector $\mathbf{d}_k$ which is generated independently and uniformly over $\mathcal{V}$.

[2]It can be shown that these random dithers can be replaced with deterministic ones, meaning that no common randomness is required.

These dithers are made available to the decoder. The signal transmitted by user k is

$$\mathbf{x}_k = [\mathbf{t}_k - \mathbf{d}_k] \bmod \Lambda.$$

Remark 3: The nested lattice construction from [10] employs Construction A. To create each fine lattice, this procedure first embeds codewords drawn from a linear code into the unit cube, and then applies the generator matrix for the coarse lattice Λ. As shown in [10], this ensemble of nested lattice codes can approach the capacity of a point-to-point Gaussian channel. If the integers $\mathbb{Z}^n$ are selected as the coarse lattice, the resulting nested lattice code is equivalent to a linear code coupled with a pulse amplitude modulation (PAM) constellation. Furthermore, the $\bmod \Lambda$ operation simplifies to the quantization error from rounding to the integers. It can be shown that the cost of this simplification is only the shaping gain, which corresponds to at most $1/2 \log(2\pi e/12) \simeq 0.255$ bits per channel use.
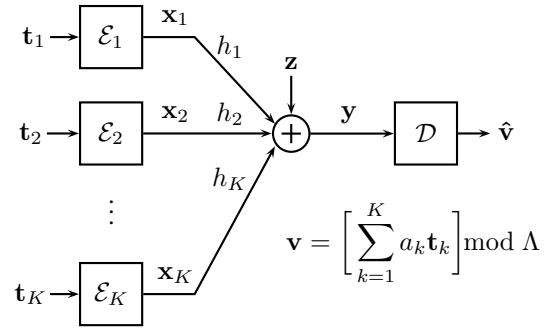
C. Compute-and-Forward

Fig. 5. Compute-and-forward on a Gaussian multiple-access channel. The transmitters send lattice points $\mathbf{t}_k$ and the receiver decodes an integer combination of them, modulo the coarse lattice Λ. The rate is determined by how closely the equation coefficients a_k match the channel coefficients h_k.

Our objective is to communicate over the MAC using the compute-and-forward scheme from [19]. To this end, the receiver first decodes a set of K lattice equations with linearly independent coefficient vectors. Afterwards, it solves this set of equations for the transmitted lattice points. Assume the receiver is interested in decoding the lattice equation

$$\mathbf{v} = \left[\sum_{k=1}^{K} a_k \mathbf{t}_k \right] \bmod \Lambda$$

with coefficient vector $\mathbf{a} = [a_1 \ \cdots \ a_K]^T \in \mathbb{Z}^K$. Following the scheme of [19], the receiver scales the observation $\mathbf{y}$ by a factor β, removes the dithers, and reduces modulo Λ to get

$$\mathbf{s} = \left[\beta \mathbf{y} + \sum_{k=1}^{K} a_k \mathbf{d}_k \right] \bmod \Lambda$$

$$= \left[\sum_{k=1}^{K} a_k \mathbf{x}_k + \sum_{k=1}^{K} a_k \mathbf{d}_k + \sum_{k=1}^{K} (\beta h_k - a_k) \mathbf{x}_k + \beta \mathbf{z} \right] \bmod \Lambda$$

$$= [\mathbf{v} + \mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \beta)] \bmod \Lambda, \qquad (7)$$

where

$$\mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \beta) = \sum_{k=1}^{K} (\beta h_k - a_k)\mathbf{x}_k + \beta \mathbf{z} \qquad (8)$$

is effective noise. From [19], we have that $\mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \beta)$ is statistically independent of $\mathbf{v}$ and its effective variance, defined as

$$\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, \beta) \triangleq \frac{1}{n} \mathbb{E}\|\mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \beta)\|^2 \qquad (9)$$

is

$$\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, \beta) = \|\beta\mathbf{h} - \mathbf{a}\|^2 \cdot \text{SNR} + \beta^2. \qquad (10)$$

Let $k^* = \min_{k:a_k \neq 0} \theta(k)$ be the index of the densest lattice participating in the lattice equation $\mathbf{v}$. The receiver produces an estimate for $\mathbf{v}$ by applying to $\mathbf{s}$ the lattice quantizer associated with Λ_{k^*},

$$\hat{\mathbf{v}} = [Q_{\Lambda_{k^*}}(\mathbf{s})] \bmod \Lambda. \qquad (11)$$

Let $\mathcal{V}_{k^*}$ be the Voronoi region of Λ_{k^*}, and define the error probability

$$\Pr(\hat{\mathbf{v}} \neq \mathbf{v}) \leq \Pr(\mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}, \beta) \notin \mathcal{V}_{k^*}). \qquad (12)$$

The next theorem summarizes and reformulates relevant results from Sections IV.C, IV.D, and V.A of [19].

Theorem 2: For any $\epsilon > 0$ and n large enough there exists a chain of n-dimensional nested lattices $\Lambda \subseteq \Lambda_K \subseteq \cdots \subseteq \Lambda_1$ forming the set of codebooks $\mathcal{L}_1, \ldots, \mathcal{L}_K$ having rates $R_1, \ldots, R_K$ and satisfying the power constraint (3) such that:

(a) For all channel vectors $\mathbf{h} \in \mathbb{R}^K$ and coefficient vectors $\mathbf{a} \in \mathbb{Z}^K$, the average error probability in decoding the lattice equation $\mathbf{v} = \left[\sum_{k=1}^{K} a_k \mathbf{t}_k\right] \bmod \Lambda$ of transmitted lattice points $\mathbf{t}_k \in \mathcal{L}_k$ can be made smaller than ϵ so long as the message rates do not exceed the computation rate,

$$R_k < R_{\text{comp}}(\mathbf{h}, \mathbf{a}, \beta) \triangleq \frac{1}{2} \log\left(\frac{\text{SNR}}{\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)}\right), \qquad (13)$$

for all k such that $a_k \neq 0$ and some $\beta \in \mathbb{R}$.

(b) The codebooks $\mathcal{L}_1, \ldots, \mathcal{L}_K$ are isomorphic to some set of linear codebooks $\mathcal{C}_1, \ldots, \mathcal{C}_K$ over the finite field $\mathbb{Z}_p$, where p is a sufficiently large prime number.

(c) For the same p, the equation $[p \cdot \mathbf{t}] \bmod \Lambda = \mathbf{0}$ holds $\forall \mathbf{t} \in \Lambda_k$, $k = 1, \ldots, K$.

Corollary 1: Given K lattice equations $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_K]$ with coefficient vectors $\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_K]^T$, the lattice points $\mathbf{t}_1, \ldots, \mathbf{t}_K$ can be recovered if $[\mathbf{A}] \bmod p$ is full-rank over $\mathbb{Z}_p$.

Remark 4: Note that it is also possible to map both the messages and the lattice equations into an appropriately chosen finite field. That is, the messages can be written as vectors with elements that take values in a prime-sized finite field, and the equations are linear combinations of the messages over the same finite field. See [19] for more details.

It follows from Theorem 2(a) that in order to maximize the computation rate $R_{\text{comp}}(\mathbf{h}, \mathbf{a}, \beta)$ for a given coefficient vector, one has to minimize $\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)$ over β. It is seen from (10) that the expression for $\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)$ is equal to the mean

squared error (MSE) for linear estimation of $\tilde{X} = \sum_{k=1}^{K} a_k X_k$ from $Y = \sum_{k=1}^{K} h_k X_k + Z$ where $\{X_k\}_{k=1}^{K}$ are i.i.d. random variables with zero mean and variance SNR and Z is statistically independent of $\{X_k\}_{k=1}^{K}$ with zero mean and unit variance. Hence the minimizing value of β is the linear minimum mean squared error (MMSE) estimation coefficient of $\tilde{X}$ from Y. This value of β was found in [19, Theorem 2] and the MSE is given by

$$\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}) \triangleq \min_{\beta \in \mathbb{R}} \sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)$$

$$= \text{SNR}\left(\|\mathbf{a}\|^2 - \frac{\text{SNR}(\mathbf{h}^T\mathbf{a})^2}{1 + \text{SNR}\|\mathbf{h}\|^2}\right)$$

$$= \text{SNR}\,\mathbf{a}^T\left(\mathbf{I}_{K\times K} - \frac{\text{SNR}\,\mathbf{h}\mathbf{h}^T}{1 + \text{SNR}\|\mathbf{h}\|^2}\right)\mathbf{a}$$

$$= \mathbf{a}^T\left(\text{SNR}^{-1}\mathbf{I}_{K\times K} + \mathbf{h}\mathbf{h}^T\right)^{-1}\mathbf{a} \qquad (14)$$

$$= \left\|\left(\text{SNR}^{-1}\mathbf{I}_{K\times K} + \mathbf{h}\mathbf{h}^T\right)^{-1/2}\mathbf{a}\right\|^2, \qquad (15)$$

where (14) can be verified using Woodbury's matrix identity (i.e., the Matrix Inversion Lemma) [48, Thm 18.2.8]. Accordingly, we define

$$R_{\text{comp}}(\mathbf{h}, \mathbf{a}) \triangleq \max_{\beta \in \mathbb{R}} R_{\text{comp}}(\mathbf{h}, \mathbf{a}, \beta)$$

$$= \frac{1}{2} \log\left(\frac{\text{SNR}}{\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a})}\right). \qquad (16)$$

The following definition identifies the K linearly independent coefficient vectors which yield the highest computation rates.

Definition 7: We say that an ordered set of linearly independent integer coefficient vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_K\}$ with corresponding computation rates $R_{\text{comp},k} \triangleq R_{\text{comp}}(\mathbf{h}, \mathbf{a}_k)$ is *optimal* if $R_{\text{comp},1} \geq \cdots \geq R_{\text{comp},K}$ and for any $k = 1, \ldots, K$ and any set of integer coefficient vectors $\{\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k\}$ of rank k

$$\min_{\ell=1,\ldots,k} R_{\text{comp}}(\mathbf{h}, \tilde{\mathbf{a}}_\ell) \leq R_{\text{comp},k},$$

or equivalently

$$\max_{\ell=1,\ldots,k} \sigma_{\text{eff}}^2(\mathbf{h}, \tilde{\mathbf{a}}_\ell) \geq \sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}_k).$$

Note that this set is not unique. For example, if $\{\mathbf{a}_1, \ldots, \mathbf{a}_K\}$ is an optimal set of coefficient vectors, so is the set $\{-\mathbf{a}_1, \ldots, -\mathbf{a}_K\}$.

Remark 5: Several recent papers have proposed families of constellations and codes that are well-suited for low-complexity implementations of compute-and-forward [39], [44], [49]–[52]. These codes could serve as building blocks for a practical implementation of our alignment scheme.

D. Numerical Evaluations

The optimal coefficient vectors and computation rates from Definition 7 play an important role in the achievable rate regions derived in this paper. The problem of determining the optimal coefficient vectors is that of finding the set of K linearly

independent *integer-valued* vectors that minimizes the effective noise (15). This problem is equivalent to finding the shortest K linearly independent lattice vectors in the lattice[3] $\Lambda(\mathbf{F})$ spanned by the matrix $\mathbf{F} = \left(\mathsf{SNR}^{-1}\mathbf{I}_{K \times K} + \mathbf{h}\mathbf{h}^T\right)^{-1/2}$.

It is shown in [19, Lemma 1] that only integer vectors $\mathbf{a} \in \mathbb{Z}^K$ that satisfy the condition

$$\|\mathbf{a}\|^2 < 1 + \|\mathbf{h}\|^2\mathsf{SNR} \qquad (17)$$

yield positive rates. Therefore, in our considerations it suffices to enumerate all integer vectors (other than the zero vector) that satisfy (17), and then exhaustively search over these vectors in order to find the optimal set. At moderate values of SNR this task is computationally reasonable. Nevertheless, it is sometimes simpler to find a set of short linearly independent lattice vectors in $\Lambda(\mathbf{F})$, which is not necessarily optimal, in order to obtain lower bounds on the set of optimal computation rates. A simple low-complexity algorithm for computing a short lattice basis (which forms a set of K linearly independent lattice vectors) is the LLL algorithm [53].[4] In producing the figures for this paper we have employed the LLL algorithm, meaning that the plotted achievable rates in Figure 4 are in fact lower bounds on the rates given by Theorems 9 and 11.

We note that a similar procedure for finding the optimal coefficient vectors was also described in [44], where the optimal coefficient vectors are termed dominated solutions.

IV. MULTIPLE-ACCESS VIA COMPUTE-AND-FORWARD

This section introduces a new coding technique for reliable communication over the K-user Gaussian multiple-access channel. This technique is based on the receiver decoding K linearly independent equations of the transmitted codewords, and then solving them for obtaining the messages transmitted by each user. We begin this section with a high-level overview of the scheme, which is illustrated in Figures 6 and 7.

Each user k maps its message to a lattice point $\mathbf{t}_k$ in its codebook $\mathcal{L}_k$ and transmits a dithered version of it. The K lattice codebooks utilized by the different users form a chain of nested lattices as in (6). Assume for now that the users are ordered with descending rates $R_1 \geq R_2 \geq \cdots \geq R_K$, i.e., $\theta(k) = k$ for $k = 1, \ldots, K$. The receiver, which sees a noisy real-valued linear combination of the transmitted codewords, begins by decoding the integer-valued linear combination $\mathbf{v}_1 = \left[\sum a_{1m}\mathbf{t}_m\right] \mod \Lambda$ which yields the highest computation rate $R_{\text{comp},1}$. Using the compute-and-forward framework, this is possible if $R_1 < R_{\text{comp},1}$. Then, it proceeds to decode the equation $\mathbf{v}_2 = \left[\sum a_{2m}\mathbf{t}_m\right] \mod \Lambda$ which yields the second highest computation rate $R_{\text{comp},2}$. In general, $\mathbf{t}_1$ participates in this equation and the condition for correct decoding of $\mathbf{v}_2$ is therefore $R_1 < R_{\text{comp},2}$. Nevertheless, this condition can be relaxed using the first equation $\mathbf{v}_1$ that was already decoded. Specifically, after appropriate scaling of the channel's output and dithers removal, the receiver has a noisy observation

$$\mathbf{s}_2 = [\mathbf{v}_2 + \mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}_2)] \mod \Lambda$$

of the desired equation $\mathbf{v}_2$. If $\mathbf{t}_1$ participates in $\mathbf{v}_1$, it is possible to cancel out $\mathbf{t}_1$ from the second equation by adding a scaled version of $\mathbf{v}_1$ to $\mathbf{s}_2$. Namely, the receiver adds $r_{21}\mathbf{v}_1$ to $\mathbf{s}_2$, where r_{21} is an integer chosen such that $[(a_{11} + r_{21}a_{21})] \mod p = 0$, which assures that $[(a_{11} + r_{21}a_{21})\mathbf{t}_1] \mod \Lambda = \mathbf{0}$ for any $\mathbf{t}_1 \in \mathcal{L}_1$. After reducing $\mod \Lambda$ this yields

$$\begin{aligned} \mathbf{s}_2^{\text{SI}} &= [\mathbf{v}_2 + r_{21}\mathbf{v}_1 + \mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}_2)] \mod \Lambda \\ &= [\tilde{\mathbf{v}}_2 + \mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}_2)] \mod \Lambda, \end{aligned}$$

where $\mathbf{t}_1$ does not participate in $\tilde{\mathbf{v}}_2$. Since the effective noise $\mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}_2)$ is unchanged by this process, the receiver can decode $\tilde{\mathbf{v}}_2$ as long as $R_2 < R_{\text{comp},2}$. Now, the receiver can obtain $\mathbf{v}_2$ by subtracting $r_{21}\mathbf{v}_1$ from $\tilde{\mathbf{v}}_2$ and reducing $\mod \Lambda$.[5] The receiver decodes the remaining equations in a similar manner, i.e., before decoding the kth equation $\mathbf{v}_k$ with computation rate $R_{\text{comp},k}$ the receiver adds to

$$\mathbf{s}_k = [\mathbf{v}_k + \mathbf{z}_{\text{eff}}(\mathbf{h}, \mathbf{a}_k)] \mod \Lambda$$

an integer-valued linear combination $\left[\sum_{\ell=1}^{k-1} r_{k\ell}\mathbf{v}_\ell\right] \mod \Lambda$ of the lattice equations that were already decoded. The coefficients in the linear combination are chosen such that the effect of $\mathbf{t}_1, \ldots, \mathbf{t}_{k-1}$ is canceled out from $\mathbf{v}_k$. Assuming that such coefficients $\{r_{k1}, \ldots, r_{k,k-1}\}$ exist, the receiver can decode $\tilde{\mathbf{v}}_k = \left[\mathbf{v}_k + \sum_{\ell=1}^{k-1} r_{k\ell}\mathbf{v}_\ell\right] \mod \Lambda$ as long as $R_k < R_{\text{comp},k}$.

Lemma 2, stated in the sequel, establishes that for any set of K linearly independent coefficient vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_K\}$ there indeed always exist integer-valued coefficients $\{r_{ij}\}$ such that in each kth decoding step the receiver can cancel out $k - 1$ lattice points from the desired equation $\mathbf{v}_k$, using the previously decoded equations $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$. The procedure for finding these coefficients is reminiscent of the Gaussian elimination procedure of a full-rank matrix. One of the basic operations in Gaussian elimination is row switching. In our considerations, this would correspond to using an equation that was not decoded yet for eliminating lattice points from another equation. Since our successive cancelation procedure only uses decoded equations, this is not possible. Therefore, a major difference between our procedure for finding a good set of coefficients $\{r_{ij}\}$ and Gaussian elimination is that row switching is not permitted. This incurs a constraint on the order in which we cancel out users from equations. Nevertheless, there always exists at least one order of successive cancelation that is possible. In other words, we can always cancel out the effect of $k - 1$ users from $\mathbf{v}_k$ using the decoded equation $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$, but we cannot always control which of the K users to cancel. As a result, there always exists at least one permutation vector π such that all K equations can be decoded as long as

$$R_{\pi(k)} < R_{\text{comp},k}, \quad k = 1, \ldots, K. \qquad (18)$$

It follows that a sum-rate of $\sum_{k=1}^{K} R_{\text{comp},k}$ is achievable over the K-user MAC with our scheme, where all users are using

---

[3]Notice that this K-dimensional lattice is induced by the channel matrix, not the n-dimensional coding scheme.

[4]Pseudocode for the LLL algorithm can be found, e.g., in [54].

[5]The operation of extracting $\mathbf{v}_2$ from $\tilde{\mathbf{v}}_2$ is in fact not necessary as the receiver is only interested in decoding *any* set of K linearly independent equations. We describe this step only to simplify the exposition of the scheme.

nested lattice codebooks. As we shall see, this sum rate is within a constant gap, smaller than $K/2 \log(K)$ bits, from the sum capacity of the MAC, for any channel gains and SNR.

A. The Compute-and-Forward Transform

We first introduce a transformation of a MAC to a multiple-input multiple-output (MIMO) mod-Λ channel, where the $K \times K$ channel matrix is integer-valued. This transformation, dubbed the *compute-and-forward transform*, will play an important role in our decoding scheme for the interference channel.

Definition 8: Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_K\}$ be a set of optimal integer coefficient vectors (see Definition 7), $\beta_1, \ldots, \beta_K$ the corresponding optimal scaling factors, and $R_{\text{comp},1} \geq \cdots \geq R_{\text{comp},K}$ the corresponding optimal computation rates. We define the *compute-and-forward transform* of the MAC with nested lattice codes as

$$\mathbf{S} = \begin{pmatrix} \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_K \end{pmatrix} = \begin{pmatrix} \left[\beta_1 \mathbf{y} + \sum_{k=1}^{K} a_{1k} \mathbf{d}_k \right] \bmod \Lambda \\ \vdots \\ \left[\beta_K \mathbf{y} + \sum_{k=1}^{K} a_{Kk} \mathbf{d}_k \right] \bmod \Lambda \end{pmatrix}$$

$$= \left[\mathbf{A} \begin{pmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_K \end{pmatrix} + \mathbf{Z}_{\text{eff}} \right] \bmod \Lambda, \qquad (19)$$

where we have written the channel output $\mathbf{y}$, dithers $\mathbf{d}_k$, and lattice codewords $\mathbf{t}_k$ as length-n row vectors. We also denote $\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_K]^T$ and $\mathbf{Z}_{\text{eff}} = [\mathbf{z}_{\text{eff},1}^T \cdots \mathbf{z}_{\text{eff},K}^T]^T$.

Remark 6: The transform is not unique as the set of optimal integer coefficient vectors is not unique. Nevertheless, the set of optimal computation rates is unique. As we shall see, the set of optimal computation rates dictates the rates attained over the transformed channel. Therefore, we use the term *the* compute-and-forward transform of the channel, with the understanding that although there may be multiple options for the transform, they are all equivalent.

The kth output $\mathbf{s}_k$ of the transformed channel corresponds to a lattice equation plus effective noise. Due to Theorem 2, each such lattice equation can be reliably decoded as long as all lattice points participating in it belong to codes $\mathcal{L}_k$ of rates smaller than $R_{\text{comp},k}$. We now lower bound the sum of K optimal computation rates, and in the sequel we show that this sum can be translated to a valid MAC sum rate.

Theorem 3: The sum of optimal computation rates is lower bounded by

$$\sum_{k=1}^{K} R_{\text{comp},k} \geq \frac{1}{2} \log \left(1 + \|\mathbf{h}\|^2 \mathsf{SNR} \right) - \frac{K}{2} \log(K) . \quad (20)$$

We will need the following definition for the proof. Note that the lattice referred to below is over the K dimensions induced by the channel vector $\mathbf{h} \in \mathbb{R}^K$, rather than the n dimensions used for coding.

Definition 9 (Successive minima): Let $\Lambda(\mathbf{F})$ be a full-rank lattice in $\mathbb{R}^K$ spanned by the matrix $\mathbf{F} \in \mathbb{R}^{K \times K}$. For $k = 1, \ldots, K$, we define the kth successive minimum as

$$\lambda_k(\mathbf{F}) \triangleq \inf \left\{ r : \dim \left(\text{span} \left(\Lambda(\mathbf{F}) \bigcap \mathcal{B}(\mathbf{0}, r) \right) \right) \geq k \right\}$$

where $\mathcal{B}(\mathbf{0}, r) = \left\{ \mathbf{x} \in \mathbb{R}^K : \|\mathbf{x}\| \leq r \right\}$ is the closed ball of radius r around $\mathbf{0}$. In words, the kth successive minimum of a lattice is the minimal radius of a ball centered around $\mathbf{0}$ that contains k linearly independent lattice points.

The product of successive minima can be upper bounded using the following well-known theorem due to Minkowski [55, Theorem 1.5].

Theorem 4 (Minkowski): For any lattice $\Lambda(\mathbf{F})$ which is spanned by a full-rank $K \times K$ matrix $\mathbf{F}$

$$\prod_{k=1}^{K} \lambda_k^2(\mathbf{F}) \leq K^K \left| \det(\mathbf{F}) \right|^2 . \qquad (21)$$

We are now ready to prove Theorem 3.

Proof of Theorem 3: Let $\Lambda(\mathbf{F})$ be a lattice spanned by the matrix $\mathbf{F} = \left(\mathsf{SNR}^{-1} \mathbf{I}_{K \times K} + \mathbf{h}\mathbf{h}^T \right)^{-1/2}$, and let $\lambda_1(\mathbf{F}), \ldots, \lambda_K(\mathbf{F})$ be its K successive minima. Let $\mathbf{a}_1, \ldots, \mathbf{a}_K$ denote the optimal coefficient vectors. By Definition 7, Definition 9 and (15) we have $\|\mathbf{F}\,\mathbf{a}_k\| = \lambda_k(\mathbf{F})$ for $k = 1, \ldots, K$. The sum of optimal computation rates is

$$\sum_{k=1}^{K} R_{\text{comp},k} = \sum_{k=1}^{K} R_{\text{comp}}(\mathbf{h}, \mathbf{a}_k)$$

$$= \sum_{k=1}^{K} \frac{1}{2} \log \left(\frac{\mathsf{SNR}}{\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}_k)} \right)$$

$$= \frac{K}{2} \log(\mathsf{SNR}) - \frac{1}{2} \log \left(\prod_{k=1}^{K} \|\mathbf{F}\,\mathbf{a}_k\|^2 \right)$$

$$= \frac{K}{2} \log(\mathsf{SNR}) - \frac{1}{2} \log \left(\prod_{k=1}^{K} \lambda_k^2(\mathbf{F}) \right).$$

Applying Theorem 4 to the product $\prod_{k=1}^{K} \lambda_k^2(\mathbf{F})$ yields

$$\sum_{k=1}^{K} R_{\text{comp},k} \geq \frac{K}{2} \log(\mathsf{SNR}) - \frac{1}{2} \log \left(K^K \left| \det(\mathbf{F}) \right|^2 \right). \quad (22)$$

Using Sylvester's determinant identity (see e.g., [48])

$$\det(\mathbf{I}_{K \times K} + \mathsf{SNR}\,\mathbf{h}\mathbf{h}^T) = \det(1 + \|\mathbf{h}\|^2 \mathsf{SNR}),$$

we have that

$$\left| \det(\mathbf{F}) \right|^2 = \frac{\mathsf{SNR}^K}{1 + \|\mathbf{h}\|^2 \mathsf{SNR}} . \qquad (23)$$

Substituting (23) into (22) proves the theorem. ∎

Remark 7: It is possible to avoid the loss of the constant factor $K/2 \log K$ in (20) using successive compute-and-forward, as described in [56]. However, in this case the operational interpretation of the sum of computation rates becomes more complicated than that described in the sequel. We will elaborate on this issue in future work.

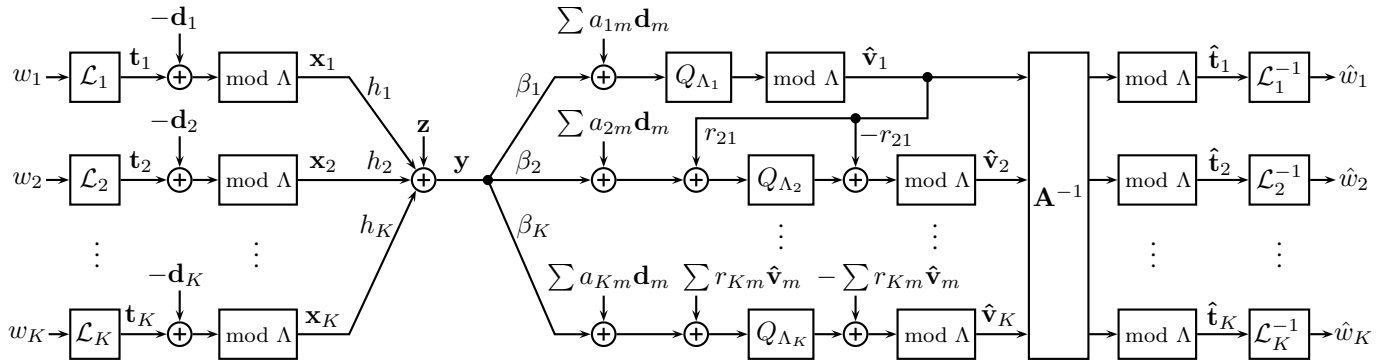Next, we give an operational meaning to the K optimal computation rates.

Fig. 6. System diagram of the nested lattice encoding and decoding operations employed as part of the compute-and-forward transform. Each message w_k is mapped to a lattice codeword $\mathbf{t}_k$ according to codebook $\mathcal{L}_k$, dithered, and transmitted as $\mathbf{x}_k$. The multiple-access channel scales codeword k by h_k and outputs the sum plus Gaussian noise $\mathbf{z}$. The decoder attempts to recover K linearly independent equations with coefficients $\mathbf{A} = \{a_{km}\}$. For the figure, we have assumed that $R_1 \geq R_2 \geq \cdots \geq R_K$ and that $R_k < R_{\mathrm{comp}}(\mathbf{h}, \mathbf{a}_k, \beta_k)$. To decode the first equation $\mathbf{v}_1 = [\sum a_{1m}\mathbf{t}_m] \bmod \Lambda$, the receiver scales $\mathbf{y}$ by β_1, removes the dithers, quantizes using Q_{Λ_1}, and takes $\bmod \Lambda$. For the second equation $\mathbf{v}_2 = [\sum a_{2m}\mathbf{t}_m] \bmod \Lambda$, the decoder scales by β_2, removes the dithers, and then eliminates the lattice point $\mathbf{t}_1$ using the first equation $\hat{\mathbf{v}}_1$ so that the rate of the remaining lattice points is at most R_2. It then quantizes using Q_{Λ_2}, adds back in $\hat{\mathbf{v}}_1$, and takes $\bmod \Lambda$. Decoding proceeds in this fashion, using a form of successive interference cancellation to keep the rates of the lattice points below the computation rates. Afterwards, the receiver solves for the original lattice points by multiplying by $\mathbf{A}^{-1}$ and taking $\bmod \Lambda$. Finally, it maps these estimates $\hat{\mathbf{t}}_k$ of the transmitted lattice points back to the corresponding messages.
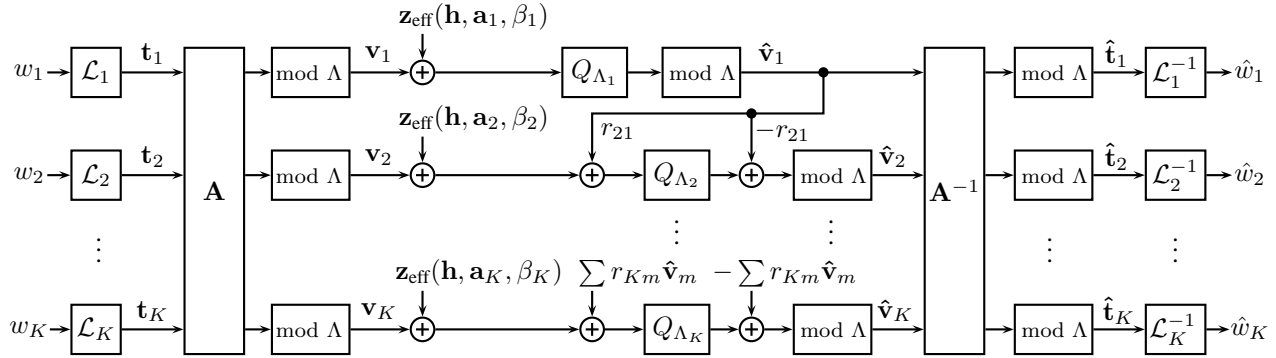

Fig. 7. Effective MIMO channel induced by the compute-and-forward transform of a Gaussian multiple-access channel. The channel output $\mathbf{y} = \sum h_k \mathbf{x}_k + \mathbf{z}$ is converted into K linearly independent lattice equations $\mathbf{v}_k = [\sum a_{km}\mathbf{t}_m] \bmod \Lambda$ plus effective noise $\mathbf{z}_{\mathrm{eff}}(\mathbf{h}, \mathbf{a}_k, \beta_k) = \beta_k \mathbf{z} + \sum(\beta_k h_k - a_k)\mathbf{x}_k$. As in Figure 6, these lattice equations can be decoded using a version of successive cancellation.

B. Multiple-Access Sum Capacity to within a Constant Gap

We now show that the compute-and-forward transform can be used for achieving several rate tuples within a constant gap from the boundary of the capacity region of the K-user MAC. The main technique used for establishing this result is successive cancelation. Namely, each decoded lattice equation is used to cancel out the effect of one user from the lattice equations that have yet to be decoded. We first illustrate the coding scheme by an example, and then formalize our result in Theorem 5.

Example 1: Consider the two-user MAC

$$\mathbf{y} = \sqrt{5}\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z},$$

at $\mathsf{SNR} = 15\mathrm{dB}$. It can be shown using (15) and (16) that the compute-and-forward transform of this channel is

$$\begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} = \left[\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{z}_{\mathrm{eff},1} \\ \mathbf{z}_{\mathrm{eff},2} \end{pmatrix} \right] \bmod \Lambda$$

with $R_{\mathrm{comp},1} \simeq 2.409$ bits and $R_{\mathrm{comp},2} \simeq 1.372$ bits. Note that $(R_{\mathrm{comp},1} + R_{\mathrm{comp},2})/(1/2 \log(1 + \|\mathbf{h}\|^2 \mathsf{SNR})) \simeq 0.998$. We use

a chain of three nested lattices $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$ that satisfy the conditions of Theorem 2 in order to construct the codebooks $\mathcal{L}_1 = \Lambda_1 \cap \mathcal{V}$ with rate R_1 arbitrarily close to $R_{\mathrm{comp},1}$ for user 1 and $\mathcal{L}_2 = \Lambda_2 \cap \mathcal{V}$ with rate R_2 arbitrarily close to $R_{\mathrm{comp},2}$ for user 2.

From Theorem 2(a), we know that $\mathbf{v}_1 = [2\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda$ can be decoded from $\mathbf{s}_1$ since R_1 and R_2 are smaller than $R_{\mathrm{comp},1}$. However, Theorem 2 does not guarantee that the second lattice equation $\mathbf{v}_2 = [3\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda$ can be decoded from $\mathbf{s}_2$ since the first user employs a codebook with a rate $R_1 \approx R_{\mathrm{comp},1}$ which is higher than the second computation rate $R_{\mathrm{comp},2}$. To circumvent this issue, we use the decoded equation $\hat{\mathbf{v}}_1$ as side information in order to cancel out the lattice point $\mathbf{t}_1 \in \Lambda_1 \cap \mathcal{V}$ from $\mathbf{s}_2$. Note that Theorem 2(c) guarantees that $[p \cdot \mathbf{t}_k] \bmod \Lambda = \mathbf{0}$, $k = 1, 2$ for some sufficiently large prime number p. Let $2^{-1} \in \mathbb{Z}$ be an integer

that satisfies $[2^{-1} \cdot 2] \bmod p = 1$. The receiver computes

$$
\begin{aligned}
\mathbf{s}_2^{\text{SI}} &= \left[\mathbf{s}_2 - 3 \cdot 2^{-1} \hat{\mathbf{v}}_1 \right] \bmod \Lambda \\
&\overset{(a)}{=} \left[(3 - 3 \cdot 2^{-1} \cdot 2) \mathbf{t}_1 + (1 - 3 \cdot 2^{-1}) \mathbf{t}_2 + \mathbf{z}_{\text{eff},2} \right] \bmod \Lambda \\
&\overset{(b)}{=} \left[[1 - 3 \cdot 2^{-1}] \bmod p \cdot \mathbf{t}_2 + \mathbf{z}_{\text{eff},2} \right] \bmod \Lambda \\
&= \left[\tilde{a}_{12} \cdot \mathbf{t}_2 + \mathbf{z}_{\text{eff},2} \right] \bmod \Lambda, \quad\quad (24)
\end{aligned}
$$

where $\tilde{a}_{12} = [1 - 3 \cdot 2^{-1}] \bmod p$. Step (a) in (24) follows from the distributive law. Step (b) follows since $3 - 3 \cdot 2^{-1} \cdot 2 = M \cdot p$ for some $M \in \mathbb{Z}$. Thus,

$$
\begin{aligned}
\left[(3 - 3 \cdot 2^{-1} \cdot 2) \mathbf{t}_1 \right] \bmod \Lambda &= [M \cdot p \cdot \mathbf{t}_1] \bmod \Lambda \\
&= [M \cdot [p \cdot \mathbf{t}_1] \bmod \Lambda] \bmod \Lambda \\
&= \mathbf{0},
\end{aligned}
$$

where the last equality is justified by Theorem 2(c).

Now only $\mathbf{t}_2$ participates in the lattice equation $\tilde{\mathbf{v}}_2 = [\tilde{a}_{12} \mathbf{t}_2] \bmod \Lambda$ and, since R_2 is smaller than $R_{\text{comp},2}$, Theorem 2 guarantees that it can be decoded from $\mathbf{s}_2^{\text{SI}}$. This is accomplished by quantizing onto Λ_2 and reducing modulo Λ,

$$
\hat{\tilde{\mathbf{v}}}_2 = [Q_{\Lambda_2}(\mathbf{s}_2^{\text{SI}})] \bmod \Lambda.
$$

After decoding both lattice equations $\mathbf{v}_1$ and $\tilde{\mathbf{v}}_2$ the receiver can solve for the transmitted lattice points $\mathbf{t}_1$ and $\mathbf{t}_2$, as the two equations are full-rank over $\mathbb{Z}_p$. We have therefore shown that the rate region $R_1 < R_{\text{comp},1}$ and $R_2 < R_{\text{comp},2}$ is achievable. In a similar manner, we can show that the rate region $R_1 < R_{\text{comp},2}$ and $R_2 < R_{\text{comp},1}$ is achievable with this scheme.

In order to formally characterize the achievable rate region, we will need the following definition which identifies the orders for which successive cancelation can be performed.

Definition 10: For a full-rank $K \times K$ matrix $\mathbf{A}$ with integer-valued entries we define the *pseudo-triangularization* process, which transforms the matrix $\mathbf{A}$ to a matrix $\tilde{\mathbf{A}}$ which is upper triangular up to column permutation $\pi = [\pi(1)\ \pi(2)\ \cdots\ \pi(K)]$. This is accomplished by left-multiplying $\mathbf{A}$ by a lower triangular matrix $\mathbf{L}$ with unit diagonal, such that $\tilde{\mathbf{A}} = \mathbf{L}\mathbf{A}$ is upper triangular up to column permutation π. Although the matrix $\mathbf{A}$ is integer valued, the matrices $\mathbf{L}$ and $\tilde{\mathbf{A}}$ need not necessarily be integer valued. Note that the pseudo-triangularization process is reminiscent of Gaussian elimination except that row switching and row multiplication are prohibited.

Example 2: The 2×2 matrix

$$
\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}
$$

from Example 1 can be pseudo-triangularized with two different permutation vectors

$$
\begin{aligned}
\tilde{\mathbf{A}} &= \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix} \cdot \mathbf{A} \\
&= \begin{pmatrix} 2 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \pi = [1\ 2],
\end{aligned}
$$

or

$$
\begin{aligned}
\tilde{\mathbf{A}} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \mathbf{A} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi = [2\ 1].
\end{aligned}
$$

Remark 8: Any full-rank matrix can be triangularized using the Gaussian elimination process, and therefore any full-rank matrix can be pseudo-triangularized with at least one permutation vector π. In particular, since for any MAC the integer-valued matrix $\mathbf{A}$ from the compute-and-forward transform is full-rank, it can always be pseudo-triangularized with at least one permutation vector π. There are full-rank matrices that can be pseudo-triangularized with several different permutation vectors, such as $\mathbf{A}$ from Example 2. However, there are also full-rank matrices $\mathbf{A}$ that can be pseudo-triangularized with only one permutation vector π. An example of such a matrix is the identity matrix $\mathbf{I}_{K \times K}$.

The next theorem gives an achievable rate region for the MAC under the compute-and-forward transform. The proof is given in Appendix A and follows along the same lines as Example 1 .

Theorem 5: Consider the MAC (2). For any $\epsilon > 0$ and n large enough, there exists a chain of n-dimensional nested lattices $\Lambda \subseteq \Lambda_K \subseteq \cdots \subseteq \Lambda_1$ forming the set of codebooks $\mathcal{L}_1, \ldots, \mathcal{L}_K$ with rates $R_1, \ldots, R_K$ such that for all $\mathbf{h} \in \mathbb{R}^K$, if:

1) each user k encodes its message using the codebook $\mathcal{L}_k$,
2) the integer-valued matrix from the compute-and-forward transform of the MAC (2) can be pseudo-triangularized with the permutation vector π, and the optimal computation rates are $R_{\text{comp},1} \geq \cdots \geq R_{\text{comp},K}$,
3) all rates $R_1, \ldots, R_K$ satisfy

$$
R_k < R_{\text{comp},\pi^{-1}(k)}, \quad \text{for } k = 1, \ldots, K \quad (25)
$$

where π^{-1} is the inverse permutation vector of π, then all messages can be decoded with error probability smaller than ϵ.

Combining Theorems 3 and 5 gives the following theorem.

Theorem 6: The sum rate achieved by the compute-and-forward transform has a gap of no more than $K/2 \log K$ bits from the sum capacity of the MAC.

Proof: Let $R_{\text{comp},1} \geq \cdots \geq R_{\text{comp},K}$ be the optimal computation rates in the compute-and-forward transform of the MAC (2). The integer-valued matrix from the compute-and-forward transform can be pseudo-triangularized with at least one permutation vector π. By Theorem 5, the rate tuple

$$
R_k = R_{\text{comp},\pi^{-1}(k)} - \delta, \quad \text{for } k = 1, \ldots, K \quad (26)
$$

is achievable for any $\delta > 0$. For this rate tuple we have

$$
\begin{aligned}
\sum_{k=1}^{K} R_k &= \sum_{k=1}^{K} \left(R_{\text{comp},\pi^{-1}(k)} - \delta \right) \\
&= \sum_{k=1}^{K} R_{\text{comp},k} - K\delta \\
&\geq \frac{1}{2} \log \left(1 + \|\mathbf{h}\|^2 \mathsf{SNR} \right) - \frac{K}{2} \log(K) - K\delta, \quad (27)
\end{aligned}
$$

where (27) follows from Theorem 3. Since this is true for any $\delta > 0$, the result follows. ∎

C. Effective Multiple-Access Channel

A channel that often arises in the context of lattice interference alignment is a K-user Gaussian multiple-access channel (MAC) with integer-valued ratios between some of the channel coefficients. Specifically, the output of such a channel can be written as

$$\mathbf{y} = \sum_{\ell=1}^{L} g_\ell \left(\sum_{m \in \mathcal{K}_\ell} b_m \mathbf{x}_m \right) + \mathbf{z}, \qquad (28)$$

where $\mathcal{K}_1, \ldots, \mathcal{K}_L$ are disjoint subsets of $\{1, \ldots, K\}$. We assume that the $b_m \in \mathbb{Z}$ are non-zero integers, which opens up the possibility of lattice alignment.

The channel (28) may describe the signal seen by a receiver in an interference network, perhaps after appropriate precoding at the transmitters. In such networks, each receiver is only interested in the messages from some of the users while the others act as interferers. Hence, it is beneficial to align several interfering users into one effective interferer, by taking advantage of the fact that the sum of lattice codewords is itself a lattice codeword.

Definition 11 (Effective users): For the MAC specified by (28), we define L *effective users*

$$\mathbf{x}_{\text{eff},\ell} \triangleq \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{x}_m, \ \ell = 1, \ldots, L.$$

Definition 12 (Effective MAC): The K-user MAC (28) induces the *effective L-user MAC*

$$\mathbf{y} = \sum_{\ell=1}^{L} g_\ell \mathbf{x}_{\text{eff},\ell} + \mathbf{z}, \qquad (29)$$

with the vector of effective channel coefficients $\mathbf{g} = [g_1 \ \cdots \ g_L]^T \in \mathbb{R}^L$. The effective channel is further characterized by the effective users' weights

$$b_{\text{eff},\ell}^2 \triangleq \sum_{m \in \mathcal{K}_\ell} b_m^2$$

for $\ell = 1, \ldots, L$, and the effective (diagonal) weight matrix

$$\mathbf{B} \triangleq \text{diag}(b_{\text{eff},1}^2, \ldots, b_{\text{eff},L}^2). \qquad (30)$$

Definition 13 (Effective lattice points): Let $\mathbf{t}_m$ be the lattice point transmitted by user m. We define the *effective lattice point* corresponding to effective user ℓ as

$$\mathbf{t}_{\text{eff},\ell} = \left[\sum_{m \in \mathcal{K}_\ell} b_m \mathbf{t}_m \right] \text{mod} \ \Lambda.$$

Let $\theta_{\text{eff}}(\ell) = \min_{m \in \mathcal{K}_\ell} \theta(m)$ (where $\theta(\cdot)$ is the mapping between users and fine lattices defined in Section III) be the index of the densest lattice contributing to $\mathbf{t}_{\text{eff},\ell}$. Since all lattices are nested, it follows that $\mathbf{t}_{\text{eff},\ell} \in \Lambda_{\theta_{\text{eff}}(\ell)}$.

Example 3 (Symmetric K-user interference channel):
Consider the symmetric K-user interference channel (1). The channel seen by the kth receiver is of the form of (28)

with $g_1 = 1$, $g_2 = g$, $\mathcal{K}_1 = \{k\}$, $\mathcal{K}_2 = \{1, \ldots, K\} \setminus k$ and $b_m = 1 \ \forall m = 1, \ldots, K$. If each of the K users transmits a single codeword drawn from a common nested lattice code, the channel becomes an effective two-user MAC as (29)

$$\mathbf{y}_k = \mathbf{x}_{\text{eff},k1} + g\mathbf{x}_{\text{eff},k2} + \mathbf{z}_k,$$

where the effective users are $\mathbf{x}_{\text{eff},k1} = \mathbf{x}_k$ and $\mathbf{x}_{\text{eff},k2} = \sum_{m \neq k} \mathbf{x}_m$, and the effective users' weights are $b_{\text{eff},1}^2 = 1$ and $b_{\text{eff},2}^2 = K - 1$. The effective lattice points are $\mathbf{t}_{\text{eff},k1} = \mathbf{t}_k$ and $\mathbf{t}_{\text{eff},k2} = [\sum_{m \neq k} \mathbf{t}_m] \text{mod} \ \Lambda$.

Our achievable schemes for the symmetric K-user interference channel, developed in Section VI, are based on transforming the K-user MAC seen by each receiver into an effective MAC with less effective users. We will develop two schemes: One transforms the channel into an effective two-user MAC as in the example above. The other, which mimics the Han-Kobayashi approach, transforms the channel into an effective three-user MAC.

When lattice interference alignment schemes are designed properly, the message intended for the receiver is mapped into a unique effective user, while multiple interfering users are folded into a smaller number of effective users. In this case, it suffices for the receiver to decode only the L effective lattice points corresponding to the effective users, rather than the K lattice points transmitted by all users. In our considerations, the effective lattice points are recovered by first decoding L lattice equations of the type

$$\mathbf{v} = \left[\sum_{\ell=1}^{L} a_\ell \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{t}_m \right] \text{mod} \ \Lambda$$

$$= \left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_{\text{eff},\ell} \right] \text{mod} \ \Lambda \qquad (31)$$

with linearly independent coefficient vectors, and then solving for $\mathbf{t}_{\text{eff},1}, \ldots, \mathbf{t}_{\text{eff},L}$.

As in Section III, in order to decode a lattice equation $\mathbf{v}$, the receiver first scales its observation by a factor β, removes the dithers and reduces modulo Λ which yields

$$\mathbf{s} = \left[\beta\mathbf{y} + \sum_{\ell=1}^{L} a_\ell \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{d}_m \right] \text{mod} \ \Lambda$$

$$= \left[\sum_{l=1}^{L} a_\ell \mathbf{x}_{\text{eff},\ell} + \sum_{\ell=1}^{L} a_\ell \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{d}_m \right.$$

$$\left. + \sum_{\ell=1}^{L} (\beta g_\ell - a_\ell)\mathbf{x}_{\text{eff},\ell} + \beta\mathbf{z} \right] \text{mod} \ \Lambda$$

$$= [\mathbf{v} + \mathbf{z}_{\text{eff}}(\mathbf{g}, \mathbf{a}, \beta, \{b_m\})] \text{mod} \ \Lambda, \qquad (32)$$

where

$$\mathbf{z}_{\text{eff}}(\mathbf{g}, \mathbf{a}, \beta, \{b_m\}) = \beta\mathbf{y} - \sum_{\ell=1}^{L} a_\ell \mathbf{x}_{\text{eff},\ell}$$

$$= \sum_{\ell=1}^{L} (\beta g_\ell - a_\ell) \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{x}_m + \beta\mathbf{z} \qquad (33)$$

is effective noise which is statistically independent of $\mathbf{v}$. Its effective variance is

$$\sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B}) = \mathsf{SNR} \sum_{\ell=1}^{L} (\beta g_\ell - a_\ell)^2 b_{\text{eff},\ell}^2 + \beta^2, \quad (34)$$

where $\mathbf{B}$ is defined in (30). Let $\ell^* = \min_{\ell:a_\ell \neq 0} \theta_{\text{eff}}(\ell)$ be the index of the densest lattice participating in the lattice equation $\mathbf{v}$. Since all lattices are nested, then $\mathbf{v} \in \Lambda_{\ell^*}$. The receiver produces an estimate for $\mathbf{v}$ by applying to $\mathbf{s}$ the lattice quantizer associated with Λ_{ℓ^*},

$$\hat{\mathbf{v}} = [Q_{\Lambda_{\ell^*}}(\mathbf{s})] \bmod \Lambda. \quad (35)$$

It follows from Theorem 2 that there exists a chain of $K + 1$ nested lattices which allows to decode $\mathbf{v}$ with a vanishing error probability so long as

$$R_m < R_{\text{comp}}(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B}) = \frac{1}{2} \log \left(\frac{\mathsf{SNR}}{\sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B})} \right), \quad (36)$$

for all $m \in \bigcup_{\ell:a_\ell \neq 0} \mathcal{K}_\ell$.

The expression for $\sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B})$ is equal to the MSE for linear estimation of $\tilde{X} = \sum_{\ell=1}^{L} a_\ell X_\ell$ from $Y = \sum_{\ell=1}^{L} g_\ell X_\ell + Z$ where $\{X_\ell\}_{\ell=1}^{L}$ are statistically independent random variables with zero mean and variances $b_{\text{eff},\ell}^2 \mathsf{SNR}$ respectively and Z is statistically independent of $\{X_\ell\}_{\ell=1}^{L}$ with zero mean and unit variance. Hence, the minimizing value of β is the linear MMSE estimation coefficient of $\tilde{X}$ from Y. A straightforward calculation shows that the minimizing value of β is

$$\beta = \frac{\mathbb{E}(\tilde{X} Y)}{\text{Var}(Y)} = \frac{\mathsf{SNR} \, \mathbf{g}^T \mathbf{B} \mathbf{a}}{1 + \mathsf{SNR} \, \mathbf{g}^T \mathbf{B} \mathbf{g}}$$

and the MSE it achieves is

$$\sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \mathbf{B}) \triangleq \min_{\beta \in \mathbb{R}} \sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B})$$

$$= \mathsf{SNR} \, \mathbf{a}^T \left(\mathbf{B} - \frac{\mathsf{SNR} \, \mathbf{B} \mathbf{g} \mathbf{g}^T \mathbf{B}}{1 + \mathsf{SNR} \cdot \mathbf{g}^T \mathbf{B} \mathbf{g}} \right) \mathbf{a} \quad (37)$$

$$= \mathbf{a}^T \left(\mathsf{SNR}^{-1} \mathbf{B}^{-1} + \mathbf{g} \mathbf{g}^T \right)^{-1} \mathbf{a} \quad (38)$$

$$= \left\| \left(\mathsf{SNR}^{-1} \mathbf{B}^{-1} + \mathbf{g} \mathbf{g}^T \right)^{-1/2} \mathbf{a} \right\|^2,$$

where again (38) can be verified using Woodbury's matrix identity [48, Thm 18.2.8]. Accordingly, we define

$$R_{\text{comp}}(\mathbf{g}, \mathbf{a}, \mathbf{B}) \triangleq \frac{1}{2} \log \left(\frac{\mathsf{SNR}}{\sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \mathbf{B})} \right). \quad (39)$$

As in Section III, we define the set of optimal L coefficient vectors for the equivalent channel (29) as the L linearly independent vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_L\}$ that yield the highest computation rates $R_{\text{comp},1} = R_{\text{comp}}(\mathbf{g}, \mathbf{a}_1, \mathbf{B}) \geq \cdots \geq R_{\text{comp},L} = R_{\text{comp}}(\mathbf{g}, \mathbf{a}_L, \mathbf{B})$ (see Definition 7). The compute-and-forward transform of the induced L-user MAC is

$$\mathbf{S} = \begin{pmatrix} \left[\beta_1 \mathbf{y} + \sum_{\ell=1}^{L} a_{1\ell} \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{d}_m \right] \bmod \Lambda \\ \vdots \\ \left[\beta_L \mathbf{y} + \sum_{\ell=1}^{L} a_{L\ell} \sum_{m \in \mathcal{K}_\ell} b_m \mathbf{d}_m \right] \bmod \Lambda \end{pmatrix}$$

$$= \left[\mathbf{A} \begin{pmatrix} \mathbf{t}_{\text{eff},1} \\ \vdots \\ \mathbf{t}_{\text{eff},L} \end{pmatrix} + \mathbf{Z}_{\text{eff}} \right] \bmod \Lambda, \quad (40)$$

where $\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_L]^T$ and $\mathbf{Z}_{\text{eff}} = [\mathbf{z}_{\text{eff},1}^T \cdots \mathbf{z}_{\text{eff},L}^T]^T$.

The next two theorems are simple extensions of Theorems 3 and 5. Their proofs are given in Appendix B.

Theorem 7: The sum of optimal computation rates for the effective L-user MAC (29) is lower bounded by

$$\sum_{\ell=1}^{L} R_{\text{comp},\ell} \geq \frac{1}{2} \log \left(\frac{1 + \mathsf{SNR} \sum_{\ell=1}^{L} g_\ell^2 b_{\text{eff},\ell}^2}{\det(\mathbf{B})} \right) - \frac{L}{2} \log(L).$$

Theorem 8: Consider the effective L-user MAC (29), induced from the K-user MAC (28), characterized by the effective channel vector $\mathbf{g}$ and the effective weight matrix $\mathbf{B}$. For any $\epsilon > 0$ and n large enough there exists a chain of n-dimensional nested lattices $\Lambda \subseteq \Lambda_L \subseteq \cdots \subseteq \Lambda_1$ forming the set of codebooks $\mathcal{L}_1, \ldots, \mathcal{L}_L$ with rates $R_1, \ldots, R_L$ such that for all $\mathbf{g} \in \mathbb{R}^L$ and $\mathbf{B}$, if:

1) each user $m \in \mathcal{K}_\ell$ encodes its message using the codebook $\mathcal{L}_\ell$ or a codebook nested in $\mathcal{L}_\ell$,
2) the integer-valued matrix from the compute-and-forward transform of the effective MAC (29) can be pseudo-triangularized with the permutation vector π, and the optimal computation rates are $R_{\text{comp},1} \geq \cdots \geq R_{\text{comp},L}$,
3) all rates $R_1, \ldots, R_L$ satisfy

$$R_\ell < R_{\text{comp},\pi^{-1}(\ell)}, \text{ for } \ell = 1, \ldots, L \quad (41)$$

where π^{-1} is the inverse permutation vector of π,

then all effective lattice points $\mathbf{t}_{\text{eff},\ell}$ can be decoded with error probability smaller than ϵ.

Corollary 2 (Achievable symmetric rate): Consider the effective L-user MAC (29), induced from the K-user MAC (28), characterized by channel coefficients $\mathbf{g}$ and the effective weight matrix $\mathbf{B}$. There exists a pair of n-dimensional nested lattices $\Lambda \subseteq \Lambda_1$ forming the codebook $\mathcal{L}$ of rate R such that for all $\mathbf{g} \in \mathbb{R}^L$ and $\mathbf{B}$, if

1) all users encode their messages using $\mathcal{L}$ (or codebooks nested in $\mathcal{L}$),
2) The Lth optimal computation rate in the compute-and-forward transform of (29) is $R_{\text{comp},L}$,
3) $R < R_{\text{comp},L}$,

then, for n large enough, all effective lattice points $\mathbf{t}_{\text{eff},\ell}$ can be decoded with an arbitrarily small error probability.

Remark 9: Corollary 2 is easily obtained from Theorem 8. However, it can also be established without incorporating the compute-and-forward transform machinery. Indeed, if all users transmit from the same lattice codebook with rate smaller than $R_{\text{comp},L}$, by Theorem 2, each of the L equations with optimal coefficient vectors can be decoded (without using successive decoding as in the compute-and-forward transform approach). Then, the decoded equations can be solved for the effective lattice points.

In Section VI we introduce two achievable schemes for the K-user Gaussian interference channel. One of them is a simple transmission scheme where all users transmit from the same nested lattice code. The result of Corollary 2 suffices to establish the rates achieved by this scheme. In the second achievable scheme, which mimics the Han-Kobabyshi scheme

for the two-user interference channel, each user transmits a superposition of codewords taken from two nested lattice codebooks. In this case Corollary 2 does not suffice and Theorem 8, which uses the compute-and-forward transform machinery, is needed.

In Section VI, we leverage the above results to lower bound the capacity of the symmetric Gaussian K-user interference channel.

V. Symmetric Capacity Upper Bounds

In this section we give upper bounds on the symmetric capacity of the symmetric K-user Gaussian interference channel. To We follow the same arguments given in [20] for showing that the symmetric capacity of the symmetric K-user interference channel is upper bounded by that of the symmetric two-user interference channel. Namely, eliminating all but two users, say users 1 and 2, the symmetric capacity is upper bounded by the results of [5]. This is simply because removing interferers cannot decrease the symmetric rates for users 1 and 2. Thus, the upper bounds from [5] hold for the symmetric rates of user 1 and 2 in the K-user symmetric interference channel. Repeating the same argument for each pair of users we see that the upper bounds on C_{SYM} developed in [5] for $K = 2$ continue to hold for all $K > 2$ as well. Therefore, the symmetric capacity of the symmetric K-user Gaussian interference channel is upper bounded as [5]

$$C_{\text{SYM}} \leq \begin{cases} \frac{1}{2}\log\left(1 + \text{INR} + \frac{\text{SNR}}{1+\text{INR}}\right) & 0 \leq \alpha < \frac{2}{3} \\ \frac{1}{4}\log\left(1 + \text{SNR}\right) + \frac{1}{4}\log\left(1 + \frac{\text{SNR}}{1+\text{INR}}\right) & \frac{2}{3} \leq \alpha < 1 \\ \frac{1}{4}\log\left(1 + \text{SNR} + \text{INR}\right) & 1 \leq \alpha < 2 \\ \frac{1}{2}\log\left(1 + \text{SNR}\right) & 2 \leq \alpha. \end{cases} \tag{42}$$

Since we are only after an approximate capacity characterization, we further upper bound C_{SYM} for $\text{SNR} > 1$ as

$$C_{\text{SYM}} \leq \begin{cases} \frac{1}{2}\log\left(2 + \frac{2\text{SNR}}{1+\text{INR}}\right) & 0 \leq \alpha < \frac{1}{2} \\ \frac{1}{2}\log\left(1 + 2\text{INR}\right) & \frac{1}{2} \leq \alpha < \frac{2}{3} \\ \frac{1}{4}\log\left(1 + \frac{4\text{SNR}^2}{1+\text{INR}}\right) & \frac{2}{3} \leq \alpha < 1 \\ \frac{1}{4}\log\left(1 + 2\text{INR}\right) & 1 \leq \alpha < 2 \\ \frac{1}{2}\log\left(1 + \text{SNR}\right) & 2 \leq \alpha. \end{cases}$$

Thus, for all values of SNR we have

$$C_{\text{SYM}} \leq \begin{cases} \frac{1}{2}\log\left(1 + \frac{\text{SNR}}{1+\text{INR}}\right) + 1 & 0 \leq \alpha < \frac{1}{2} \\ \frac{1}{2}\log^+\left(\text{INR}\right) + 1 & \frac{1}{2} \leq \alpha < \frac{2}{3} \\ \frac{1}{2}\log^+\left(\frac{\text{SNR}}{\sqrt{\text{INR}}}\right) + 1 & \frac{2}{3} \leq \alpha < 1 \\ \frac{1}{4}\log^+\left(\text{INR}\right) + 1 & 1 \leq \alpha < 2 \\ \frac{1}{2}\log\left(1 + \text{SNR}\right) & 2 \leq \alpha. \end{cases} \tag{43}$$

VI. Achievable Schemes

This section introduces two simple achievable schemes for reliable communication over the symmetric K-user interference channel which are based on nested lattice codes. These schemes are then shown to approximately achieve C_{SYM}, the

symmetric capacity of the channel, for all channel gains g, except for an outage set of bounded measure. This outage set is explicitly characterized.

We begin by describing the two schemes and deriving their achievable symmetric rates. These rates are given in terms of the optimal computation rates corresponding to a certain effective multiple access channel, i.e., the rates are given as a solution to an optimization problem. This optimization problem, which amounts to finding the optimal coefficient vectors, can be efficiently solved numerically, as described in Section III-D. Figure 4 shows our achievable rates for the three-user symmetric interference channel as a function of the interference level g, for several values of SNR. It is evident that the obtained rates significantly improve over time-sharing even for moderate values of SNR.

In order to establish the approximate optimality of these schemes, we derive explicit lower bounds on the rates they achieve which depend only on the SNR and INR. As in the two-user case, the symmetric capacity exhibits a different behavior for different regimes of interference strength, characterized by the parameter α.

We now present the two achievable schemes. The first achieves the approximate symmetric capacity in the noisy, strong and very strong interference regimes, while the second achieves the approximate symmetric capacity in the weak and moderately weak interference regimes.

First scheme - A single-layer lattice code: A pair of nested lattices $\Lambda \subseteq \Lambda_1$ is utilized to construct the codebook $\mathcal{L} = \Lambda_1 \cap \Lambda$ of rate R_{SYM}. All users encode their messages using this codebook. Since all interferers arrive at the kth transmitter with the same gain, they will be aligned into one effective lattice point. Thus, the K-user MAC seen by the kth receiver becomes an effective two-user MAC of the form defined in Section IV-C (see Example 3)

$$\mathbf{y}_k = \mathbf{x}_{\text{eff},k1} + g\mathbf{x}_{\text{eff},k2} + \mathbf{z}_k, \tag{44}$$

where $\mathbf{x}_{\text{eff},k1} = \mathbf{x}_k$, $\mathbf{x}_{\text{eff},k2} = \sum_{m \neq k} \mathbf{x}_m$ are the effective users, $b_{\text{eff},1}^2 = 1$, $b_{\text{eff},2}^2 = K - 1$ are the effective users' weights and $\mathbf{g} = \begin{bmatrix} 1 & g \end{bmatrix}^T$ is the vector of channel gains.

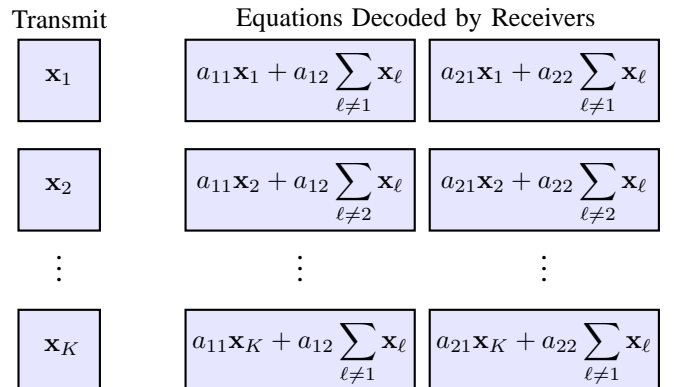| Transmit	Equations Decoded by Receivers	
$\mathbf{x}_1$	$a_{11}\mathbf{x}_1 + a_{12}\sum_{\ell \neq 1}\mathbf{x}_\ell$	$a_{21}\mathbf{x}_1 + a_{22}\sum_{\ell \neq 1}\mathbf{x}_\ell$
$\mathbf{x}_2$	$a_{11}\mathbf{x}_2 + a_{12}\sum_{\ell \neq 2}\mathbf{x}_\ell$	$a_{21}\mathbf{x}_2 + a_{22}\sum_{\ell \neq 2}\mathbf{x}_\ell$
$\vdots$	$\vdots$	$\vdots$
$\mathbf{x}_K$	$a_{11}\mathbf{x}_K + a_{12}\sum_{\ell \neq 1}\mathbf{x}_\ell$	$a_{21}\mathbf{x}_K + a_{22}\sum_{\ell \neq 1}\mathbf{x}_\ell$

Fig. 8. Illustration of the single-layer lattice scheme. Each transmitter sends a codeword drawn from a common lattice. Each receiver decodes two equations of the codewords, which it can then solve for its desired message.

The next theorem gives an achievable rate region for the

K-user interference channel when each receiver jointly decodes both the effective user $\mathbf{x}_{\text{eff},k1}$ which carries the desired information, and the effective user $\mathbf{x}_{\text{eff},k2}$ which carries the sum of interfering codewords. The theorem relies on decoding two independent linear combinations of the effective lattice points. See Figure 8 for an illustration. This is in contrast to the successive decoding technique used in [18], where first the interference is decoded and removed, and only then the desired lattice point is decoded.

Theorem 9: Let $R_{\text{comp},1} \geq R_{\text{comp},2}$ be the optimal computation rates for the effective MAC (44) induced by the symmetric K-user interference channel (1). Any symmetric rate $R_{\text{SYM}} < R_{\text{comp},2}$ is achievable for the symmetric K-user interference channel (1).

Proof: Corollary 2 implies that for any symmetric rate $R_{\text{SYM}} < R_{\text{comp},2}$ there exists a pair of nested lattices $\Lambda \subseteq \Lambda_1$ such that both effective lattice points can be decoded at each receiver. Since the first effective user $\mathbf{x}_{\text{eff},k1}$ carries all the desired information for the kth receiver, it follows that any $R_{\text{SYM}} < R_{\text{comp},2}$ is achievable. ∎

The next theorem gives an achievable rate region for the K-user interference channel when each receiver decodes only its desired codeword, while treating all other interfering codewords as noise. This theorem can be trivially proved using i.i.d. Gaussian codebooks. Nevertheless, we prove the theorem using nested lattice codebooks for completeness.

Theorem 10: Any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2}\log\left(1 + \frac{\text{SNR}}{1 + (K-1)g^2\text{SNR}}\right)$$

is achievable for the symmetric K-user interference channel (1).

Proof: Decoding $\mathbf{x}_k$ at the kth receiver of the symmetric K-user interference channel (1), while treating all other users as noise, is equivalent to decoding the equation with coefficient vector $\mathbf{a} = [1\ 0]^T$ in the effective two-user MAC (44). Therefore, any symmetric rate satisfying $R_{\text{SYM}} < R_{\text{comp}}(\mathbf{g}, [1\ 0]^T, \mathbf{B})$ is achievable. The effective variance for decoding this equation is found using (37) to be

$$\sigma_{\text{eff}}^2(\mathbf{g}, [1\ 0]^T, \mathbf{B}) = \text{SNR}\left(1 + \frac{\text{SNR}}{1 + (K-1)g^2\text{SNR}}\right)^{-1},$$

which, using (39), implies that

$$R_{\text{comp}}(\mathbf{g}, [1\ 0]^T, \mathbf{B}) = \frac{1}{2}\log\left(1 + \frac{\text{SNR}}{1 + (K-1)g^2\text{SNR}}\right).$$

∎

For the two-user case, it is known that in the weak and moderately weak interference regimes each receiver should decode only part of the message transmitted by the other user [5]. A natural extension of this Han-Kobayashi [15] approach to the K-user case is for each receiver to decode linear combinations that only include parts of the interfering messages. This is enabled by using a superposition of two lattice codewords at each transmitter, as we describe next. See Figure 9 for an illustration.

Second scheme - Lattice Han-Kobayashi: This scheme employs a chain of nested lattices $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$ to construct two codebooks $\mathcal{L}_1$ and $\mathcal{L}_2$ with rates R_1 and R_2, respectively. Each user k splits its message w_k into two messages, a public message w_{k1} that is mapped into codebook $\mathcal{L}_1$ and a private message w_{k2} that is mapped into codebook $\mathcal{L}_2$. It is convenient to treat each user k as two virtual users with codewords $\mathbf{x}_{k1}$ and $\mathbf{x}_{k2}$ that carry messages w_{k1} and w_{k2}, respectively. User k transmits a superposition of its virtual users' codewords,

$$\mathbf{x}_k = \sqrt{1-\gamma^2}\mathbf{x}_{k1} + \gamma\mathbf{x}_{k2},$$

for $\gamma \in [0,1)$. The signal seen by the kth receiver is

$$\begin{aligned}
\mathbf{y}_k = {} & \sqrt{1-\gamma^2}\mathbf{x}_{k1} + \gamma\mathbf{x}_{k2} \\
& + g\sqrt{1-\gamma^2}\sum_{m\neq k}\mathbf{x}_{m1} + g\gamma\sum_{m\neq k}\mathbf{x}_{m2} + \mathbf{z}_k,
\end{aligned} \quad (45)$$

which induces the effective four-user MAC

$$\begin{aligned}
\mathbf{y}_k = {} & \sqrt{1-\gamma^2}\mathbf{x}_{\text{eff},k1} + \gamma\mathbf{x}_{\text{eff},k2} \\
& + g\sqrt{1-\gamma^2}\mathbf{x}_{\text{eff},k3} + g\gamma\mathbf{x}_{\text{eff},k4} + \mathbf{z}_k,
\end{aligned} \quad (46)$$

with effective users $\mathbf{x}_{\text{eff},k1} = \mathbf{x}_{k1}$, $\mathbf{x}_{\text{eff},k2} = \mathbf{x}_{k2}$, $\mathbf{x}_{\text{eff},k3} = \sum_{m\neq k}\mathbf{x}_{m1}$ and $\mathbf{x}_{\text{eff},k4} = \sum_{m\neq k}\mathbf{x}_{m2}$. The effective users' weights are $b_{\text{eff},1}^2 = 1$, $b_{\text{eff},2}^2 = 1$, $b_{\text{eff},3}^2 = K-1$ and $b_{\text{eff},4}^2 = K-1$, and

$$\mathbf{g} = \left[\sqrt{1-\gamma^2}\ \ \gamma\ \ g\sqrt{1-\gamma^2}\ \ g\gamma\right]^T$$

is the vector holding the channel gains.

The receiver aims to decode the effective codewords $\mathbf{x}_{\text{eff},k1}$, $\mathbf{x}_{\text{eff},k2}$ and $\mathbf{x}_{\text{eff},k3}$ while treating the fourth effective codeword $\mathbf{x}_{\text{eff},k4}$ as noise. The next lemma will be useful for the derivation of rates achieved by this scheme. Its proof is given in Appendix C.

Lemma 1: Consider the effective L-user MAC (29), where the decoder is only interested in the first $L-1$ effective lattice points $\mathbf{t}_{\text{eff},1},\ldots,\mathbf{t}_{\text{eff},L-1}$ and let $\kappa = 1/\sqrt{1 + \text{SNR}g_L^2 b_{\text{eff},L}^2}$. Any rate tuple achievable for decoding $\mathbf{t}_{\text{eff},1},\ldots,\mathbf{t}_{\text{eff},L-1}$ over the effective $(L-1)$-user MAC

$$\sum_{\ell=1}^{L-1}\kappa g_\ell\mathbf{x}_{\text{eff},\ell} + \mathbf{z} \quad (47)$$

is also achievable for decoding the desired $L-1$ lattice points over (29).

The next theorem gives the achievable rate region for the lattice Han-Kobayashi scheme.

Theorem 11: Let $\kappa(\gamma) = 1/\sqrt{1 + \text{SNR}g^2\gamma^2(K-1)}$ and consider the effective MAC

$$\begin{aligned}
\mathbf{y}_k = {} & \kappa(\gamma)\sqrt{1-\gamma^2}\mathbf{x}_{\text{eff},k1} + \kappa(\gamma)\gamma\mathbf{x}_{\text{eff},k2} \\
& + \kappa(\gamma)g\sqrt{1-\gamma^2}\mathbf{x}_{\text{eff},k3} + \mathbf{z}_k,
\end{aligned} \quad (48)$$

with effective channel vector

$$\mathbf{g} = \left[\kappa(\gamma)\sqrt{1-\gamma^2}\ \ \kappa(\gamma)\gamma\ \ \kappa(\gamma)g\sqrt{1-\gamma^2}\right]^T,$$

and effective users' weights $b_{\text{eff},1}^2 = 1$, $b_{\text{eff},2}^2 = 1$, and $b_{\text{eff},3}^2 = K-1$. Let $\{\mathbf{a}_1(\gamma), \mathbf{a}_2(\gamma), \mathbf{a}_3(\gamma)\}$ and $R_{\text{comp},1}(\gamma) \geq$

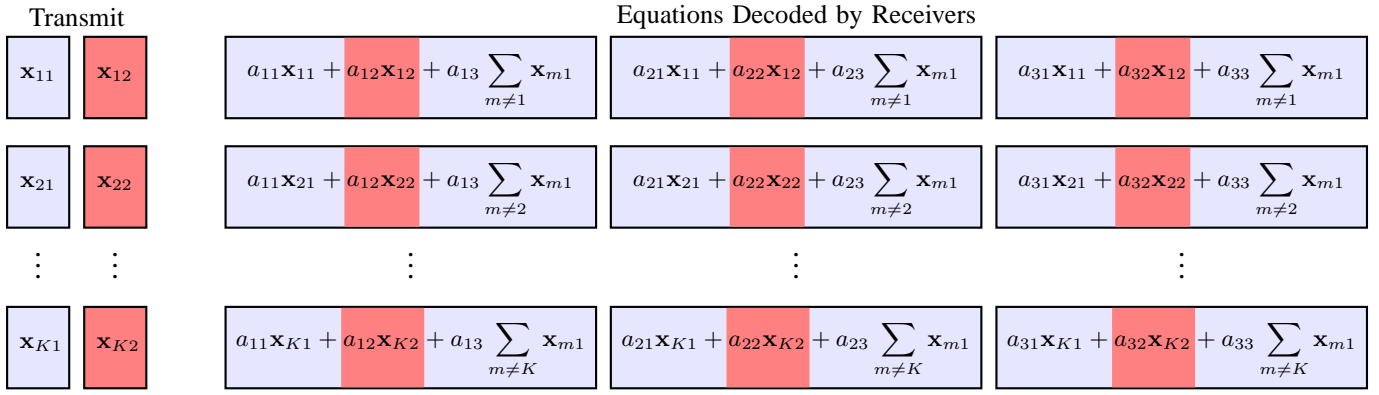Transmit   Equations Decoded by Receivers

Fig. 9. Illustration of the lattice Han-Kobayashi scheme. Each transmitter sends a public (blue) and a private (red) lattice codeword. Each receiver decodes three equations of the public codewords as well as its desired private codeword while treating the other private codewords as noise. From these equations, the receivers can infer their desired public and private messages.

$R_{\text{comp},2}(\gamma) \geq R_{\text{comp},3}(\gamma)$ be the optimal coefficient vectors and computation rates. Any symmetric rate satisfying

$$R_{\text{SYM}} < \max_{\gamma \in [0,1)} R_{\text{comp},2}(\gamma) + R_{\text{comp},3}(\gamma)$$

is achievable for the symmetric K-user interference channel (1).

Proof: The receiver is only interested in the effective lattice points $\mathbf{t}_{\text{eff},k1}$, $\mathbf{t}_{\text{eff},k2}$. Nevertheless, we require that it decodes the three effective lattice points $\mathbf{t}_{\text{eff},k1}$, $\mathbf{t}_{\text{eff},k2}$ and $\mathbf{t}_{\text{eff},k3}$. Due to Lemma 1, any rate tuple that is achievable over the effective channel (48) is also achievable for decoding $\mathbf{t}_{\text{eff},k1}$, $\mathbf{t}_{\text{eff},k2}$ and $\mathbf{t}_{\text{eff},k3}$ from the original effective channel (46) induced by the lattice Han-Kobayashi scheme.

Note that $\mathbf{t}_{\text{eff},k1}$ and $\mathbf{t}_{\text{eff},k3}$ are points from the same codebook $\mathcal{L}_1$ with rate R_1, and $\mathbf{t}_{\text{eff},k2}$ is a codeword from $\mathcal{L}_2$ with rate R_2.

Consider a compute-and-forward transform coefficient matrix $\mathbf{A}(\gamma) = [\mathbf{a}_1(\gamma) \ \mathbf{a}_2(\gamma) \ \mathbf{a}_3(\gamma)]^T$ for (48). For any such full-rank, integer-valued matrix, there exists at least one order of pseudo-triangularization. Therefore, there exists a pseudo-triangularization of $\mathbf{A}(\gamma)$ with at least one permutation vector π.

Consider first the case where $\pi(3) = 2$, i.e., the effective lattice point $\mathbf{t}_{\text{eff},2}$ is the last to be removed in the successive cancelation decoding procedure of the compute-and-forward transform. According to Theorem 8, for any $R_1 < R_{\text{comp},2}(\gamma)$ and $R_2 < R_{\text{comp},3}(\gamma)$ there exists a chain $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$ such that $\mathbf{t}_{\text{eff},k1}$, $\mathbf{t}_{\text{eff},k2}$ and $\mathbf{t}_{\text{eff},k3}$ can be decoded from the effective channel (48) via the compute-and-forward transform.

Otherwise, $\pi(1) = 2$ or $\pi(2) = 2$, which means that the effective lattice point $\mathbf{t}_{\text{eff},2}$ is either removed first or second from the proceeding equations in the successive cancelation decoding procedure of the compute-and-forward transform. According to Theorem 8 for any $R_1 < R_{\text{comp},3}(\gamma)$ and $R_2 < R_{\text{comp},2}(\gamma)$ there exists a chain $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$ such that $\mathbf{t}_{\text{eff},k1}$, $\mathbf{t}_{\text{eff},k2}$ and $\mathbf{t}_{\text{eff},k3}$ can be decoded from the effective channel (48) via the compute-and-forward transform.

Since $R_{\text{SYM}} = R_1 + R_2$, and γ can be chosen such as to maximize R_{SYM}, the theorem is proved. ■

The problem of optimizing the power allocation γ between the private and public codewords, played a major role in the approximation of the two-user interference channel capacity [5]. Here, we follow this approach and choose γ such that, at each unintended receiver, the received power of each private codeword is equal to that of the additive noise. Specifically, in the sequel we set $\gamma^2 = 1/(g^2\text{SNR})$. While this choice of γ may be sub-optimal, it suffices to develop our capacity approximations in closed form. The achievable symmetric rate for $\gamma^2 = 1/(g^2\text{SNR})$ is given in the following corollary to Theorem 11.

Corollary 3: Assume $g^2\text{SNR} > 1$ and consider the effective MAC

$$\mathbf{y}_k = \sqrt{\frac{g^2\text{SNR} - 1}{K \cdot g^2\text{SNR}}} \mathbf{x}_{\text{eff},k1} + \sqrt{\frac{1}{K \cdot g^2\text{SNR}}} \mathbf{x}_{\text{eff},k2}$$
$$+ g\sqrt{\frac{g^2\text{SNR} - 1}{K \cdot g^2\text{SNR}}} \mathbf{x}_{\text{eff},k3} + \mathbf{z}_k, \qquad (49)$$

with effective channel vector

$$\mathbf{g} = \left[\sqrt{\frac{g^2\text{SNR} - 1}{K \cdot g^2\text{SNR}}} \ \ \sqrt{\frac{1}{K \cdot g^2\text{SNR}}} \ \ g\sqrt{\frac{g^2\text{SNR} - 1}{K \cdot g^2\text{SNR}}} \right]^T, \qquad (50)$$

and effective users' weights $b_{\text{eff},1}^2 = 1$, $b_{\text{eff},2}^2 = 1$, and $b_{\text{eff},3}^2 = K - 1$. Let $\{\mathbf{a}_1^{\text{HK}}, \mathbf{a}_2^{\text{HK}}, \mathbf{a}_3^{\text{HK}}\}$ and $R_{\text{comp},1}^{\text{HK}} \geq R_{\text{comp},2}^{\text{HK}} \geq R_{\text{comp},3}^{\text{HK}}$ be the optimal coefficient vectors and computation rates for this effective MAC. Any symmetric rate

$$R_{\text{SYM}} < R_{\text{comp},2}^{\text{HK}} + R_{\text{comp},3}^{\text{HK}}$$

is achievable for the symmetric K-user interference channel (1).

Computing the achievable rates given by Theorem 9 and Corollary 3 requires finding the optimal computation rates for the effective MACs (44) and (49), which involves solving an integer least-squares optimization problem (see Section III-D). In the remainder of this section, we derive lower bounds on these achievable rates that depend only on the values of SNR and INR and can therefore be directly compared to the upper

bounds (43). To simplify the exposition, we assume $g > 0$ in the sequel, although all results easily follow for $g < 0$ as well.

A. Very Strong Interference Regime

The sum capacity in this regime was characterized exactly by Sridharan *et al.* [18] using a lattice encoding scheme very similar to the one used in Theorem 9. The key difference is that in [18] each receiver decodes successively: it first decodes the sum of interfering codewords and then subtracts it in order to get a clean view of the desired signal. Recall that in our scheme, each receiver decodes two linear combinations of its signal and the interference.

We now proceed to lower bound the achievable rate of Theorem 9 for the case $\alpha \geq 2$, i.e., $g^2 \geq \mathsf{SNR}$. We do this, rather than directly using the results of [18], for two reasons. The first is that the results of [18] only apply for

$$g^2 \geq \frac{(\mathsf{SNR} + 1)^2}{\mathsf{SNR}}, \tag{51}$$

whereas we need the result to be valid for any $g^2 \geq \mathsf{SNR}$. The second is to show that our lattice encoding and decoding framework suffices to achieve the approximate capacity in all regimes.

Using the single-layer scheme presented above, the channel seen by each receiver is converted to an effective two-user MAC (44). Applying Theorem 7 to this effective MAC, we find that the sum of the optimal computation rates is lower bounded by

$$R_{\text{comp},1} + R_{\text{comp},2} \geq \frac{1}{2} \log \left(\frac{1 + \mathsf{SNR}(1 + g^2(K-1))}{K-1} \right) - 1. \tag{52}$$

Let $R_{\text{comp}}(\mathbf{g}, [0\ 1]^T, \mathbf{B})$ be the computation rate for decoding the lattice equation with coefficient vector $\mathbf{a} = [0\ 1]^T$ over the effective MAC (44) with $\mathbf{g} = [1\ g]$ and $\mathbf{B} = \text{diag}(1, K-1)$. The effective variance for the equation $\mathbf{a} = [0\ 1]^T$, which is calculated using (37), is given in (53) at the top of the next page. Substituting $\sigma_{\text{eff}}^2(\mathbf{g}, [0\ 1]^T, \mathbf{B})$ into (36) gives

$$R_{\text{comp}}(\mathbf{g}, [0\ 1]^T, \mathbf{B}) = \frac{1}{2} \log \left(\frac{1 + \mathsf{SNR}\left(1 + g^2(K-1)\right)}{(K-1)(1 + \mathsf{SNR})} \right). \tag{54}$$

For $\mathsf{SNR} > 1$ we can lower bound (54) as

$$R_{\text{comp}}(\mathbf{g}, [0\ 1]^T, \mathbf{B}) > \frac{1}{2} \log \left(\frac{g^2(K-1)\mathsf{SNR}}{(K-1)(1 + \mathsf{SNR})} \right)$$
$$> \frac{1}{2} \log(g^2) - \frac{1}{2}. \tag{55}$$

We would like to find a lower bound on $R_{\text{comp},2}$. Consider two cases. If the coefficient vector $\mathbf{a} = [0\ 1]^T$ is optimal, i.e., $R_{\text{comp},1} = R_{\text{comp}}(\mathbf{g}, [0\ 1]^T, \mathbf{B})$, we have

$$R_{\text{comp},2} = R_{\text{comp},1} + R_{\text{comp},2} - R_{\text{comp}}(\mathbf{g}, [0\ 1]^T, \mathbf{B})$$
$$\geq \frac{1}{2} \log(1 + \mathsf{SNR}) - 1,$$

where we have used (52) and (54) for the last inequality. If the coefficient vector $\mathbf{a} = [0\ 1]^T$ is not optimal, we must have

$$R_{\text{comp},2} \geq R_{\text{comp}}(\mathbf{g}, [0\ 1]^T, \mathbf{B})$$
$$> \frac{1}{2} \log(g^2) - \frac{1}{2},$$

where the last inequality follows from (55). Taking the minimum of the two bounds above we obtain

$$R_{\text{comp},2} \geq \min \left(\frac{1}{2} \log(1 + \mathsf{SNR}) - 1, \frac{1}{2} \log(g^2) - \frac{1}{2} \right)$$
$$= \frac{1}{2} \log(1 + \mathsf{SNR}) - 1, \tag{56}$$

for any $\mathsf{SNR} > 1$ and $g^2 \geq \mathsf{SNR}$. Since for any $\mathsf{SNR} \leq 1$ the bound (56) is negative, the restriction $\mathsf{SNR} > 1$ is redundant. Applying Theorem 9, we conclude that for $g^2 \geq \mathsf{SNR}$ any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2} \log(1 + \mathsf{SNR}) - 1 \tag{57}$$

is achievable, and is within 1 bit of the outer bound (43) in this regime.

B. Strong Interference Regime

The strong interference regime corresponds to $1 \leq \alpha < 2$, or equivalently $1 \leq g^2 < \mathsf{SNR}$. As in the previous subsection, we lower bound $R_{\text{comp},2}$ in order to obtain a closed-form expression for the achievable symmetric rate. In contrast to the very strong interference regime, where the lower bound on $R_{\text{comp},2}$ is valid for any $g^2 \geq \mathsf{SNR}$, here we must exclude certain channel gains in order to get a constant gap from the outer bound (43). That is, the lower bounds we derive for the strong interference regime are only valid for a predefined subset of the interval $g^2 \in [1, \mathsf{SNR})$. As we increase the measure of this subset, our approximation gap worsens. This somewhat strange behavior is to be expected from the existing literature. The results of [8] and [46] show that for the K-user interference channel the DoF are everywhere discontinuous. The notion of DoF corresponds to $\alpha \approx 1$. Since the strong interference regime contains values of α near 1, we cannot expect to achieve rates which are a constant gap from the upper bounds of [5] for all values of g. Instead, we show that these upper bounds can be approached up to a constant gap for all $1 \leq g^2 < \mathsf{SNR}$ except for some outage set whose measure can be controlled at the price of increasing the gap. We will see a similar phenomenon when we analyze the moderately weak interference regime.

Substituting $\mathbf{g} = [1\ g]$ and $\mathbf{B} = \text{diag}(1, K-1)$ into (34), the optimal computation rate for the effective MAC (44) can be written as

$$R_{\text{comp},1} = \frac{1}{2} \log(\mathsf{SNR}) - \frac{1}{2} \log(\sigma_g^2) \tag{58}$$

$$\sigma_g^2 = \min_{\beta, a_1, a_2} \left(\left((\beta - a_1)^2 + (\beta g - a_2)^2 (K-1) \right) \mathsf{SNR} + \beta^2 \right),$$

where σ_g^2 is the effective noise variance and the minimization is over $\beta \in \mathbb{R}$, and $\mathbf{a} = [a_1\ a_2] \in \mathbb{Z}^2 \setminus \mathbf{0}$. From (52) combined

$$\sigma_{\text{eff}}^2(\mathbf{g}, [0\ 1]^T, \mathbf{B}) = \text{SNR} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & K-1 \end{bmatrix} - \frac{\text{SNR} \begin{bmatrix} 1 & 0 \\ 0 & K-1 \end{bmatrix} \begin{bmatrix} 1 \\ g \end{bmatrix} \begin{bmatrix} 1 & g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K-1 \end{bmatrix}}{1 + \text{SNR} \begin{bmatrix} 1 & g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K-1 \end{bmatrix} \begin{bmatrix} 1 \\ g \end{bmatrix}} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \text{SNR} \cdot \frac{(K-1)(1+\text{SNR})}{1 + \text{SNR} + (K-1)g^2\text{SNR}} \tag{53}$$

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with (58), we have

$$R_{\text{comp},2} \geq \frac{1}{2} \log \left(\frac{1 + \text{SNR}(1 + g^2(K-1))}{K-1} \right) - 1 - R_{\text{comp},1}$$

$$> \frac{1}{2} \log \left(g^2\text{SNR} \right) - 1 - R_{\text{comp},1}$$

$$= \frac{1}{2} \log \left(g^2 \right) + \frac{1}{2} \log \left(\sigma_g^2 \right) - 1,$$

and combining with Theorem 9 we see that any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2} \log \left(g^2 \right) + \frac{1}{2} \log \left(\sigma_g^2 \right) - 1, \tag{59}$$

is achievable over the K-user interference channel. Thus, in order to obtain a lower bound on C_{SYM} it suffices to lower bound σ_g^2. The effective noise σ_g^2 can be bounded as

$$\sigma_g^2 \geq \min_{\beta, a_1, a_2} \left(\left((\beta - a_1)^2 + (\beta g - a_2)^2 \right) \text{SNR} + \beta^2 \right). \tag{60}$$

We first hold β constant and minimize over a_1, a_2. If $|\beta| \geq 1/(2g)$, the optimal choices for the integers a_1, a_2 are

$$a_1 = \lfloor \beta \rceil, \ a_2 = \lfloor \beta g \rceil. \tag{61}$$

If $|\beta| < 1/(2g)$, rounding the gains will set both a_1 and a_2 to zero, which is not allowed. Since $g \geq 1$, the optimal choice is

$$a_1 = 0, \ a_2 = \text{sign}(\beta). \tag{62}$$

Now, we are left with the problem of minimizing (60) over β. Instead of performing this minimization problem, we give a lower bound on its solution. We do that by splitting the real line into three intervals, and lower bound σ_g^2 for all values of β within each one. Then, we take the minimum over these three bounds.

Interval 1 : $0 < |\beta| \leq 1/(2g)$
In this interval it is optimal to set $a_2 = \text{sign}(\beta)$. Moreover, $|\beta g| \leq 1/2$, and therefore $|\beta g - a_2| > 1/2$. Combining this with (60) gives

$$\sigma_g^2 \geq \frac{\text{SNR}}{4}. \tag{63}$$

Interval 2 : $1/(2g) < |\beta| \leq 1/2$
Here, it is optimal to set $a_1 = \lfloor \beta \rceil = 0$. Substituting $a_1 = 0$ in (60) gives

$$\sigma_g^2 \geq \beta^2\text{SNR} > \frac{\text{SNR}}{4g^2} > \frac{1}{4|g|}\text{SNR}^{1/2}, \tag{64}$$

where the last inequality follows since $g^2 < \text{SNR}$ in the strong interference regime.

Interval 3 : $1/2 < |\beta|$

Since $|\beta| > 1/2$, we can always decompose β into a sum of a *nonzero* integer q and a real number $\varphi \in [-1/2, 1/2)$ giving

$$\sigma_g^2 \geq \min_{\varphi, q, a_1, a_2} \left((\varphi + q - a_1)^2\text{SNR} \right.$$

$$+ (qg - a_2 + \varphi g)^2\text{SNR} + (\varphi + q)^2 \bigg)$$

$$\geq \min_{\varphi, q, a_2} \left(\left(\varphi^2 + (qg - a_2 + \varphi g)^2 \right)\text{SNR} + \frac{q^2}{4} \right). \tag{65}$$

The minimization of (65) with respect to φ (where the constraint $\varphi \in [-1/2, 1/2)$ is ignored) can be obtained by differentiation. The minimizing value of φ is

$$\varphi^* = -\frac{g}{1+g^2}(qg - a_2).$$

Substituting φ^* into (65) gives

$$\sigma_g^2 \geq \min_{q, a_2} \left(\frac{1}{1+g^2}(qg - a_2)^2\text{SNR} + \frac{q^2}{4} \right), \tag{66}$$

which, using the fact that $g^2 \geq 1$, can be further bounded by

$$\sigma_g^2 \geq \frac{1}{4} \min_{q, a_2} \max \left(\frac{1}{g^2}(qg - a_2)^2\text{SNR}, q^2 \right). \tag{67}$$

We would like to obtain a lower bound on σ_g^2 which is valid for all $g \notin \mathcal{S}$, where $\mathcal{S}$ is an outage set with bounded measure. Assume first that $g \in [b, b+1)$ for some integer $1 \leq b < \sqrt{\text{SNR}}$. Let

$$q_{\text{max},b} \triangleq \frac{1}{\sqrt{b+1/2}}\text{SNR}^{1/4-\delta/2}, \tag{68}$$

for some $\delta > 0$ to be defined later, and note that $q_{\text{max},b}$ is not necessarily an integer. Define

$$\Phi_b \triangleq \sqrt{b+1/2}\ \text{SNR}^{-1/4-\delta/2} \tag{69}$$

and let $\mathcal{S}_b$ be the set of all values of $g \in [b, b+1)$ such that the inequality

$$|qg - a_2| < \Phi_b \tag{70}$$

has at least one solution with integers q and a_2, where q is in the range $0 < q \leq q_{\text{max},b}$. Let $\bar{\mathcal{S}}_b = [b, b+1)\backslash\mathcal{S}_b$. By (67), (68) and (69), we have that for all $g \in \bar{\mathcal{S}}_b$

$$\sigma_g^2 \geq \frac{1}{4} \min \left(\min_{0 < q \leq \lfloor q_{\text{max},b} \rfloor, a_2} \max \left(\frac{1}{g^2}(qg - a_2)^2\text{SNR}, q^2 \right), \right.$$

$$\left. \min_{\lceil q_{\text{max},b} \rceil \geq q, a_2} \max \left(\frac{1}{g^2}(qg - a_2)^2\text{SNR}, q^2 \right) \right)$$

$$\geq \frac{1}{4} \min \left(\frac{1}{g^2}\Phi_b^2\text{SNR}, q_{\text{max},b}^2 \right)$$

$$= \frac{1}{4} \min \left(\frac{b+1/2}{g^2}\text{SNR}^{1/2-\delta}, \frac{1}{b+1/2}\text{SNR}^{1/2-\delta} \right). \tag{71}$$
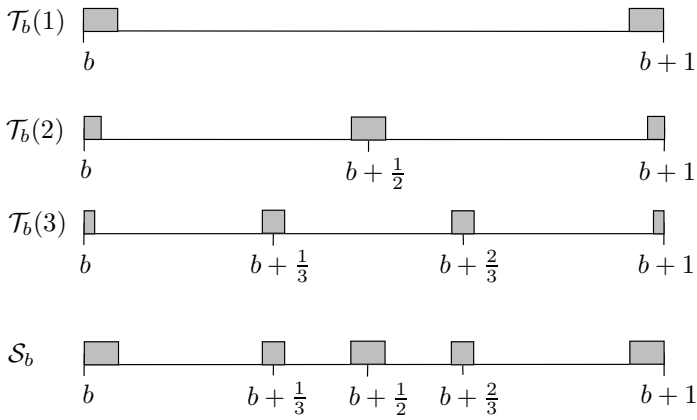
Fig. 10. An illustration of the sets $\mathcal{T}_b(1)$, $\mathcal{T}_b(2)$, $\mathcal{T}_b(3)$ and their union $\mathcal{S}_b$. In this illustration $q_{\max,b} = 3$ and $\Phi_b = 1/16$.

Since $b \geq 1$, we have that

$$\frac{g}{2} < b + \frac{1}{2} < 2g.$$

Thus, (76) can be further bounded by

$$\sigma_g^2 \geq \frac{1}{8\sqrt{g^2}} \mathsf{SNR}^{1/2-\delta}. \tag{72}$$

We now turn to upper bound the Lebesgue measure of the set $\mathcal{S}_b$. Let $\mathcal{I} = [-1, 1)$ and define the set

$$\mathcal{T}_b(q) = \left[\left\{ b, b + \frac{1}{q}, \cdots, b + \frac{q-1}{q} \right\} + \frac{\Phi_b}{q} \mathcal{I} \right] \mod [b, b+1),$$

where the sum of the two sets is a Minkowski sum. Writing (70) as

$$\left| g - \frac{a_2}{q} \right| < \frac{\Phi_b}{q}, \tag{73}$$

we see that for a given q and $g \in [b, b+1)$ the inequality admits a solution if and only if $g \in \mathcal{T}_b(q)$. It follows that

$$\mathcal{S}_b = \bigcup_{q=1}^{\lfloor q_{\max,b} \rfloor} T_b(q). \tag{74}$$

See Figure 10 for an illustration of the sets $\mathcal{T}_b(q)$ and $\mathcal{S}_b$. Thus, the Lebesgue measure of $\mathcal{S}_b$ can be upper bounded by

$$\begin{aligned}
\mu(\mathcal{S}_b) &= \mathrm{Vol}\,(\mathcal{S}_b) \\
&\leq \sum_{q=1}^{\lfloor q_{\max,b} \rfloor} \mathrm{Vol}\,(\mathcal{T}_b(q)) \\
&\leq \sum_{q=1}^{\lfloor q_{\max,b} \rfloor} q \cdot 2\frac{\Phi_b}{q} \\
&\leq 2q_{\max,b}\Phi_b \\
&= 2\mathsf{SNR}^{-\delta}.
\end{aligned} \tag{75}$$

Setting $\delta = (c+1)/\log(\mathsf{SNR})$ and substituting into (72) and (75) gives

$$\sigma_g^2 \geq \frac{2^{-c}}{16|g|} \mathsf{SNR}^{1/2} \tag{76}$$

for all $g \in [b, b+1)$ up to an outage set $\mathcal{S}_b$ of measure not greater than 2^{-c}.

Combining the three bounds (63), (64) and (76) gives

$$\begin{aligned}
\sigma_g^2 &\geq \min\left(\frac{\mathsf{SNR}}{4}, \frac{1}{4|g|}\mathsf{SNR}^{1/2}, \frac{2^{-c}}{16|g|}\mathsf{SNR}^{1/2} \right) \\
&\geq \frac{2^{-c}}{16|g|}\mathsf{SNR}^{1/2}
\end{aligned} \tag{77}$$

for all $g \in [b, b+1)$ up to an outage set $\mathcal{S}_b$ of measure not greater than 2^{-c}.

Combining (59) and (77) we see that for all $g \in [b, b+1)$ up to an outage set $\mathcal{S}_b$ of measure not greater than 2^{-c} any symmetric rate satisfying

$$\begin{aligned}
R_{\mathrm{SYM}} &< \frac{1}{4}\log(g^2\mathsf{SNR}) - \frac{c}{2} - 3 \\
&= \frac{1}{4}\log(\mathsf{INR}) - \frac{c}{2} - 3
\end{aligned} \tag{78}$$

is achievable. We conclude that the symmetric rate (78) is achievable for all channel gains in the strong interference regime except for an outage set whose measure is a fraction of 2^{-c} of the interval $1 \leq |g| < \sqrt{\mathsf{SNR}}$, for any $c > 0$.

Remark 10: We note that for any $c > 0$ the set of channel coefficients that fall in the outage set $\mathcal{S}$ can be easily determined.

C. Moderately Weak Interference Regime

The moderately weak interference regime is characterized by $2/3 \leq \alpha < 1$, or equivalently, $\mathsf{SNR}^{-1/3} \leq g^2 < 1$. As in the strong interference regime, we show the achievability of symmetric rates which are a constant gap from the upper bound for a certain fraction of the channel gains. As opposed to the very strong and strong interference regimes, where a single-layered lattice scheme suffices to achieve the approximate capacity, here we will need the second scheme, which employs two layers of lattice codes at each transmitter.

As mentioned in the description of the second scheme, we set the power of the private lattice codewords so that they are perceived at noise level at the unintended receivers. The achievable rate for this choice is given by Corollary 3 and we now turn to lower bounding this achievable rate in closed form. Let g_1, g_2 and g_3 be the channel gains in the effective three-user MAC (49) from Corollary 3, and recall that, for this effective channel, the effective weight matrix is $\mathbf{B} = \mathrm{diag}(1, 1, K-1)$. We begin by applying Theorem 7 to the effective channel (49) to get the following lower bound on the sum of optimal computation rates,

$$\begin{aligned}
\sum_{i=1}^{3} & R_{\mathrm{comp},i}^{\mathrm{HK}} \\
&\geq \frac{1}{2}\log\left(\frac{1 + \mathsf{SNR}(g_1^2 + g_2^2 + (K-1)g_3^2)}{K-1} \right) - \frac{3}{2}\log(3) \\
&> \frac{1}{2}\log\left(\mathsf{SNR}(g_1^2 + g_2^2) \right) - \frac{1}{2}\log\left(3^3(K-1) \right) \\
&= \frac{1}{2}\log(\mathsf{SNR}) - \frac{1}{2}\log\left(27K(K-1) \right).
\end{aligned} \tag{79}$$

The optimal computation rate can be written as

$$R_{\text{comp},1}^{\text{HK}} = \frac{1}{2}\log(\text{SNR}) - \frac{1}{2}\log(\sigma_{\text{HK}}^2) \qquad (80)$$

where σ_{HK}^2 is given in (81) at the top of the next page. The minimization in (81) is performed over all $\beta \in \mathbb{R}$ and $\mathbf{a}_1^{\text{HK}} = [a_1\ a_2\ a_3] \in \mathbb{Z}^3 \setminus \mathbf{0}$. Applying Corollary 3, we see that any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2}\log(\sigma_{\text{HK}}^2) - \frac{1}{2}\log\left(27K(K-1)\right) \qquad (82)$$

is achievable for the K-user interference channel. Therefore, it suffices to lower bound the effective variance σ_{HK}^2. Substituting

$$\beta = \sqrt{\frac{Kg^2\text{SNR}}{g^2\text{SNR} - 1}}\tilde{\beta}$$

in (81), which is allowed since β can take any value in $\mathbb{R}$, gives

$$\sigma_{\text{HK}}^2 = \min_{\tilde{\beta},a_1,a_2,a_3} \left(\tilde{\beta}^2 \cdot \frac{Kg^2\text{SNR}}{g^2\text{SNR} - 1} + (\tilde{\beta} - a_1)^2\text{SNR} \right.$$
$$\left. + \left(\frac{\tilde{\beta}}{\sqrt{g^2\text{SNR} - 1}} - a_2 \right)^2 \text{SNR} + (K-1)(\tilde{\beta}g - a_3)^2\text{SNR} \right). \qquad (83)$$

In the sequel, we assume[6] $\text{SNR} > 4$. With this assumption, $\sqrt{g^2\text{SNR} - 1} > 1$ for all $g^2 \geq \text{SNR}^{-1/3}$, i.e., for all values of g in the moderately weak interference regime. We will also use the fact that the inequality $\sqrt{g^2\text{SNR} - 1} > 1$ continues to hold for all $g^2 \geq \text{SNR}^{-1/2}$, i.e., for all values of g in the weak interference regime. Since $\sqrt{g^2\text{SNR} - 1} > 1$ implies $g^2\text{SNR} > 2$, we also have $g^2\text{SNR}/(g^2\text{SNR} - 1) > 1$ and hence (83) can be lower bounded as

$$\sigma_{\text{HK}}^2 \geq \min_{\tilde{\beta},a_1,a_2,a_3} \left(K\tilde{\beta}^2 + (\tilde{\beta} - a_1)^2\text{SNR} \right.$$
$$\left. + \left(\frac{\tilde{\beta}}{\sqrt{g^2\text{SNR} - 1}} - a_2 \right)^2 \text{SNR} + (K-1)(\tilde{\beta}g - a_3)^2\text{SNR} \right). \qquad (84)$$

We first hold $\tilde{\beta}$ constant, and minimize over a_1, a_2, a_3. If $|\tilde{\beta}| \geq 1/2$, the optimal choices for the integers a_1, a_2, a_3 are

$$a_1 = \lfloor\tilde{\beta}\rceil, \ \ a_2 = \left\lfloor \tilde{\beta}/\sqrt{g^2\text{SNR} - 1} \right\rceil, \ \ a_3 = \lfloor\tilde{\beta}g\rceil. \qquad (85)$$

If $|\tilde{\beta}| < 1/2$ all three integers a_1, a_2, a_3 from (85) are zero and thus are invalid. Therefore, for these values of $\tilde{\beta}$ one of the integers must take the value 1 or −1. Since for $\text{SNR} > 4$ and $\text{SNR}^{-1/2} \leq g^2 < 1$ we have

$$\max\left(|\tilde{\beta}|, |\tilde{\beta}/\sqrt{g^2\text{SNR} - 1}|, |\tilde{\beta}g| \right) = |\tilde{\beta}|,$$

the optimal choices of a_1, a_2, a_3 for values of $|\tilde{\beta}| < 1/2$ are

$$a_1 = \text{sign}(\tilde{\beta}), \ \ a_2 = 0, \ \ a_3 = 0 \ . \qquad (86)$$

---

[6]This assumption is valid, since for $\text{SNR} \leq 4$ the symmetric capacity is upper bounded by $1/2\log(1 + 4) = 1.161$bits. Our capacity approximations in this subsection, and also in the next subsection, exhibit a constant gap greater than $7/2$ bits, and therefore hold for $\text{SNR} < 4$.

Now, the problem of lower bounding σ_{HK}^2 reduces to minimizing (84) over $\tilde{\beta}$. Rather than solving this cumbersome minimization problem, we split the real line into four intervals, and lower bound σ_{HK}^2 for all values of $\tilde{\beta}$ within each one. Then, we take the minimum over these four lower bounds. In a similar manner to the previous subsection, we define $\delta = (2c + 8)/\log(\text{SNR})$, where $c > 0$ is some constant. The lower bounds below are derived in Appendix D-A.

Interval 1 : $0 < |\tilde{\beta}| \leq 1/2$

$$\sigma_{\text{HK}}^2 \geq \frac{\text{SNR}}{4}. \qquad (87)$$

Interval 2 : $1/2 < |\tilde{\beta}| \leq \sqrt{|g|}\text{SNR}^{1/4-\delta/2}/2$ [7]
For all values of $\text{SNR}^{-1/3} < |g| \leq 1$ except for an outage set with measure not greater than 2^{-c} we have

$$\sigma_{\text{HK}}^2 > \frac{2^{-2c}}{4 \cdot 2^8}\frac{\text{SNR}^{1/2}}{\sqrt{g^2}}. \qquad (88)$$

Interval 3 : $\sqrt{|g|}\text{SNR}^{1/4-\delta/2}/2 < |\tilde{\beta}| \leq \text{SNR}^{1/4}/\sqrt{8|g|}$

$$\sigma_{\text{HK}}^2 \geq \frac{2^{-2c}}{4 \cdot 2^8}\frac{\text{SNR}^{1/2}}{\sqrt{g^2}}. \qquad (89)$$

Interval 4 : $\text{SNR}^{1/4}/\sqrt{8|g|} < |\tilde{\beta}|$

$$\sigma_{\text{HK}}^2 \geq \frac{1}{8}\frac{\text{SNR}^{1/2}}{\sqrt{g^2}}. \qquad (90)$$

Combining the four lower bounds (87), (88), (89) and (90) we have

$$\sigma_{\text{HK}}^2 \geq \min\left(\frac{1}{4}\text{SNR}, \frac{2^{-2c}}{2^{10}}\frac{\text{SNR}^{1/2}}{\sqrt{g^2}}, \frac{1}{8}\frac{\text{SNR}^{1/2}}{\sqrt{g^2}} \right)$$
$$= \frac{2^{-2c}}{2^{10}}\frac{\text{SNR}^{1/2}}{\sqrt{g^2}}$$

for all $\text{SNR}^{-1/3} \leq g^2 < 1$ up to an outage set of measure not greater than 2^{-c}. Thus, substituting our lower bound for σ_{HK}^2 into (82), we find that any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2}\log\left(\frac{\text{SNR}^{1/2}}{\sqrt{g^2}} \right) - c - 5 - \frac{1}{2}\log(27) - \frac{1}{2}\log(K^2)$$
$$< \frac{1}{2}\log\left(\frac{\text{SNR}^{1/2}}{\sqrt{g^2}} \right) - c - 8 - \log(K)$$
$$= \frac{1}{2}\log\left(\frac{\text{SNR}}{\sqrt{\text{INR}}} \right) - c - 8 - \log(K) \qquad (91)$$

is achievable over the symmetric K-user interference channel for all $\text{SNR}^{-1/3} \leq g^2 < 1$ up to an outage set of measure not greater than 2^{-c}.

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[7]If $\sqrt{|g|}\text{SNR}^{1/4-\delta/2}/2 < 1/2$ this interval is empty, and we skip to interval 3.

$$\sigma_{\text{HK}}^2 = \min_{\beta, \mathbf{a}_1^{\text{HK}}} \left(\beta^2 + \left(\left(\beta \sqrt{\frac{g^2 \text{SNR} - 1}{K \cdot g^2 \text{SNR}}} - a_1 \right)^2 + \left(\beta \sqrt{\frac{1}{K \cdot g^2 \text{SNR}}} - a_2 \right)^2 + (K-1) \left(\beta g \sqrt{\frac{g^2 \text{SNR} - 1}{K \cdot g^2 \text{SNR}}} - a_3 \right)^2 \right) \text{SNR} \right) \tag{81}$$

D. Weak Interference Regime

This regime is characterized by $1/2 \leq \alpha < 2/3$, or equivalently, $\text{SNR}^{-1/2} \leq g^2 < \text{SNR}^{-1/3}$. As in the moderately weak interference regime, we develop a closed-form lower bound on the achievable symmetric rate of Corollary 3. A key difference is that the bound derived here is valid for all channel gains, rather than up to an outage set.

We first note that equations (82) and (84) continue to hold in this regime as well as in the moderately weak interference regime, and the optimal choices of a_1, a_2, a_3 are also as in (85) and (86). As before, we divide the real line into four intervals, give lower bounds on σ_{HK}^2 which hold for all values of $\tilde{\beta}$ in each one, and conclude that σ_{HK}^2 is lower bounded by the minimum of these four bounds. The lower bounds below are derived in Appendix D-B

Interval 1 : $0 < |\tilde{\beta}| \leq 1/2$

$$\sigma_{\text{HK}}^2 \geq \frac{\text{SNR}}{4}. \tag{92}$$

Interval 2 : $1/2 < |\tilde{\beta}| \leq 1/(2|g|)$

$$\sigma_{\text{HK}}^2 \geq \frac{g^2 \text{SNR}}{4}. \tag{93}$$

Interval 3 : $1/(2|g|) < |\tilde{\beta}| \leq \sqrt{g^2 \text{SNR}/8}$

$$\sigma_{\text{HK}}^2 \geq \frac{1}{4g^4}. \tag{94}$$

Interval 4 : $\sqrt{g^2 \text{SNR}/8} < |\tilde{\beta}|$

$$\sigma_{\text{HK}}^2 > \frac{g^2 \text{SNR}}{4}. \tag{95}$$

Combining the four lower bounds (92), (93), (94) and (95) we have

$$\sigma_{\text{HK}}^2 \geq \frac{1}{4} \min \left(\text{SNR}, g^2 \text{SNR}, g^{-4} \right)$$
$$= \frac{g^2 \text{SNR}}{4}, \tag{96}$$

where (96) is true since $\text{SNR}^{-1/2} \leq g^2 < \text{SNR}^{-1/3}$. It follows by substituting (96) into (82) that any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2} \log \left(\frac{g^2 \text{SNR}}{4} \right) - \frac{1}{2} \log(27) - \frac{1}{2} \log(K^2)$$
$$< \frac{1}{2} \log \left(g^2 \text{SNR} \right) - \frac{7}{2} - \log(K)$$
$$= \frac{1}{2} \log (\text{INR}) - \frac{7}{2} - \log(K) \tag{97}$$

is achievable for the symmetric K-user interference channel with $\text{SNR}^{-1/2} \leq g^2 < \text{SNR}^{-1/3}$.

E. Noisy Interference Regime

The noisy interference regime is characterized by $\alpha < 1/2$, or equivalently $g^2 < \text{SNR}^{-1/2}$. In this regime, each receiver decodes its desired codeword while treating all interfering codewords as noise. Lattice codes are not necessary in this regime in order to approximate the symmetric capacity: random i.i.d. Gaussian codebooks suffice. Nevertheless, the same performance can be achieved with lattice codes as shown in Theorem 10 which states that any symmetric rate

$$R_{\text{SYM}} < \frac{1}{2} \log \left(1 + \frac{\text{SNR}}{1 + (K-1)g^2 \text{SNR}} \right)$$

is achievable. It follows that any symmetric rate satisfying

$$R_{\text{SYM}} < \frac{1}{2} \log \left(1 + \frac{\text{SNR}}{1 + g^2 \text{SNR}} \right) - \frac{1}{2} \log(K-1)$$
$$= \frac{1}{2} \log \left(1 + \frac{\text{SNR}}{1 + \text{INR}} \right) - \frac{1}{2} \log(K-1) \tag{98}$$

is achievable.

VII. DEGREES-OF-FREEDOM

We have shown that the compute-and-forward transform can approximate the capacity of the symmetric K-user interference channel up to a constant gap for all channel gains outside a small outage set. Ideally, we would like to use a similar approach to approximate the capacity of the general (non-symmetric) interference channel. In contrast to the symmetric case, where all interferers are automatically aligned (if they all use the same lattice codebook), the interferers will be observed through different channel gains. A linear combination of lattice codewords is a codeword only if all of the coefficients are integers. Thus, in order to induce alignment, all of the interfering gains should be steered towards integers, which is an overconstrained problem.

Recently, Motahari *et al.* [8] introduced a precoding scheme which achieves $K/2$ DoF for almost every channel realization of a K-user Gaussian interference channel. Their technique relies on symbol-level alignment. Namely, each user encodes its message into several layers. Each symbol in a layer belongs to the one-dimensional lattice $\mathbb{Z}$. Each user superimposes its layers in a clever way, such that at each unintended receiver symbols from different layers are received aligned with those transmitted by other interfering users. The resulting effective channel at each receiver is an effective multiple access channel of the type introduced in Section IV-C. When the numbers of layers is large, roughly half of the effective users at each receiver carry valuable information while the other half are effective interferers. Through the use of techniques drawn from Diophantine approximation, they showed that each of the L effective users achieves essentially $1/L$ DoF. Thus, each receiver obtains $1/2$ useful DoF.

In this section, we show that the same asymptotic results can be achieved using the compute-and-forward transform. Specifically, we show that nested lattice codes allows each of the L effective users in the effective MAC (29) introduced in Section IV-C to achieve $1/L$ degree of freedom for almost all channel realizations. Therefore, the one-dimensional lattice used in [8] can be replaced with n-dimensional lattice codes while maintaining the same asymptotic performance. The benefit of using nested lattice codes combined with the precoding scheme of [8] is that one can obtain an achievable rate region at any SNR rather than only a DoF characterization. We begin by analyzing the DoF offered by the K optimal computation rates in the K-user MAC. Then, we translate the results to the effective L-user MAC.

Theorem 3 in Section IV guarantees that the sum of the optimal computation rates is close to the sum capacity of the MAC. However, the theorem does not tell us how the sum rate is divided between the K rates. We now show that, in a DoF sense, the sum is equally split between all K rates for almost every channel realization. Recall the definition for DoF:

$$d_{\text{comp},k} = \lim_{\text{SNR}\to\infty} \frac{R_{\text{comp},k}(\text{SNR})}{\frac{1}{2}\log(1+\text{SNR})} \ . \tag{99}$$

First, we upper bound $d_{\text{comp},1}$, the DoF provided by the best lattice equation.

Theorem 12: Let $f_1,\ldots,f_K$ be functions from $\mathbb{R}^m$ to $\mathbb{R}$ satisfying

1) f_k for $k = 1,\ldots,K$ is analytic in $\mathbb{R}^m$,
2) $1, f_1,\ldots,f_K$ are linearly independent over $\mathbb{R}$,

and define the manifold

$$\mathcal{M} = \left\{ \begin{bmatrix} f_1(\tilde{\mathbf{h}}) & \cdots & f_K(\tilde{\mathbf{h}}) \end{bmatrix} \ : \ \tilde{\mathbf{h}} \in \mathbb{R}^m \right\}. \tag{100}$$

For almost every $\mathbf{h} \in \mathcal{M}$ the DoF offered by the best lattice equation is upper bounded by

$$d_{\text{comp},1} \leq \frac{1}{K} \ . \tag{101}$$

The proof is given in Appendix E, and is based on showing that restricting the scaling coefficient β from (7) to the form $\beta = q/h_1$ for $q \in \mathbb{Z}$ incurs no loss from a DoF point of view. This way, the first coefficient of $\beta\mathbf{h}$ is an integer. Then, a result from the field of Diophantine approximation which is due to Kleinbock and Margulis [57] is used in order to lower bound the error in approximating the remaining $K-1$ channel gains with integers.

Remark 11: The manifold $\mathcal{M}$ from Theorem 12, which appears in all of our DoF results in this paper, is the same manifold used in [8] to prove the achievability of $K/2$ DoF for the K-user interference channel. Thus, all achievable DoF results from [8] are also achievable using the compute-and-forward transform.

As a special case of Theorem 12 we may choose the manifold $\mathcal{M}$ as $\mathbb{R}^K$ which implies the following corollary.

Corollary 4: For almost every $\mathbf{h} \in \mathbb{R}^K$ the DoF offered by the best lattice equation is upper bounded by

$$d_{\text{comp},1} \leq \frac{1}{K}.$$

Remark 12: Niesen and Whiting [58] studied the DoF offered by the best lattice equation and showed that

$$d_{\text{comp},1} \leq \left\{ \begin{array}{ll} 1/2 & K = 2 \\ 2/(K+1) & K > 2 \end{array} \right. \tag{102}$$

for almost every $\mathbf{h} \in \mathbb{R}^K$. Our bound therefore agrees with that of [58] for $K = 2$ and improves it for $K > 2$.

The next corollary shows that all K optimal computation rates offer $1/K$ DoF for almost every $\mathbf{h}$ satisfying mild conditions.

Corollary 5: Let $\mathcal{M}$ be a manifold satisfying the conditions of Theorem 12. For almost every $\mathbf{h} \in \mathcal{M}$ the DoF provided by each of the K optimal computation rates is $d_{\text{comp},k} = 1/K$.

Proof: Theorem 3 implies that $\sum_{k=1}^K d_{\text{comp},k} \geq 1$. Using the fact that $d_{\text{comp},k}$ is monotonically decreasing in k and that $d_{\text{comp},1} \leq 1/K$ for almost every $\mathbf{h} \in \mathcal{M}$, the corollary follows. ∎

The corollary above implies that in the limit of very high SNR not only is the sum of computation rates close to the sum capacity of the MAC, but each computation rate scales like the symmetric capacity of the MAC for almost all channel gains. Note that our analysis (as well as that of [58]) is within the context of the achievable computation rates stemming from Theorem 2.

The next corollary follows from Corollary 5 and Theorem 5.

Corollary 6: Let $\mathcal{M}$ be a manifold satisfying the conditions of Theorem 12. The DoF attained by each user in the K-user MAC under the compute-and-forward transform is $1/K$ for almost every $\mathbf{h} \in \mathcal{M}$. In particular, the DoF attained by each user in the K-user MAC under the compute-and-forward transform is $1/K$ for almost every $\mathbf{h} \in \mathbb{R}^K$

The next theorem shows that for almost every effective L-user multiple access channel of the form introduced in Section IV-C each of the effective users achieves $1/L$ degree of freedom. The proof is given in Appendix F.

Theorem 13: Let $f_1,\ldots,f_L$ be functions from $\mathbb{R}^m$ to $\mathbb{R}$ satisfying

1) f_ℓ for $\ell = 1,\ldots,L$ is analytic in $\mathbb{R}^m$,
2) $1, f_1,\ldots,f_L$ are linearly independent over $\mathbb{R}$,

and define the manifold

$$\mathcal{M} = \left\{ \begin{bmatrix} f_1(\tilde{\mathbf{g}}) & \cdots & f_L(\tilde{\mathbf{g}}) \end{bmatrix} \ : \ \tilde{\mathbf{g}} \in \mathbb{R}^m \right\}.$$

For almost every $\mathbf{g} \in \mathcal{M}$ the DoF offered by each of the L optimal computation rates for the effective MAC (29) is

$$d_{\text{comp},\ell} = \lim_{\text{SNR}\to\infty} \frac{R_{\text{comp},\ell}(\text{SNR})}{\frac{1}{2}\log(1+\text{SNR})} = \frac{1}{L}. \tag{103}$$

VIII. DISCUSSION

In this paper, we have developed a new decoding framework for lattice-based interference alignment. We used this framework as a building block for two lattice-based interference alignment schemes for the symmetric real Gaussian K-user interference channel. These schemes perform well starting

from the moderate SNR regime, and are within a constant gap from the upper bounds on the capacity for all channel gains outside of some outage set whose measure can be controlled.

A natural question for future research is how to extend the results above to the general Gaussian K-user interference channel. The main problem is that, in the general case, the interfering lattice codewords are not naturally aligned, as their gains are not integer-valued. Therefore, in order to successfully apply lattice interference alignment, some form of precoding, aimed towards forcing the cross channel gains to be integers, is required. Unfortunately, simple power-backoff strategies do not suffice, even in the three-user case.

One option for overcoming this problem is to use many layers at each transmitter, as in [8], and create partial alignment between interfering layers. While this achieves the optimal DoF, it performs poorly at reasonable values of SNR, as there will be a rate loss for each additional layer. As a result the rate region obtained by combining the compute-and-forward transform with the precoding scheme of [8] is inferior to that obtained by time-sharing, for values of SNR of practical interest. Another option is to precode not only using power-backoff, but also over time, which may partially compensate for the lack of sufficient free parameters. An example for such a precoding scheme is the power-time code introduced in [21].

A positive feature of the compute-and-forward framework is that it does not require perfect alignment of the lattice points participating in the equations. Namely, the effect of not perfectly equalizing the channel gains to integers is an enhanced effective noise. For the general interference channel, this suggests that it may suffice to find precoding schemes that only approximately force the cross-channel gains to integers.

Another issue left for future work is extending the capacity approximation for the real symmetric Gaussian K-user interference channel to the complex symmetric case. The achievable rate region, as given by Theorems 9 and Corollary 3, naturally extends to the complex case using the results of [19] for the complex case. An open question is whether the lower bounds on these achievable rate regions, which depend only on SNR and INR, also extend to the complex case. The answer seems positive since the bounds mainly rely on volume considerations that seem to remain valid in the complex case.

APPENDIX A
PROOF OF THEOREM 5

We begin with two lemmas which will be useful for the proof of Theorem 5.

Lemma 2: Let $\mathbf{A}$ be a $K \times K$ matrix with integer entries of magnitudes bounded from above by some constant $a_{\max}$. If there exists a lower triangular matrix $\mathbf{L}$ with unit diagonal such that $\tilde{\mathbf{A}} = \mathbf{LA}$ is upper triangular up to column permutation π, then for a prime $p > K(K!)^2(Ka_{\max})^{2K}a_{\max}$ there also exists a lower triangular matrix $\mathbf{L}^{(p)}$ with elements from $\{0, 1, \ldots, p-1\}$ and unit diagonal such that $\tilde{\mathbf{A}}^{(p)} = \left[\mathbf{L}^{(p)}\mathbf{A}\right] \bmod p$ is upper triangular up to column permutation π.

Proof: Assume that there exists a lower triangular matrix $\mathbf{L}$ with unit diagonal such that $\tilde{\mathbf{A}} = \mathbf{LA}$ is upper triangular

up to column permutation π. We begin by showing that all elements in the ith $(i > 1)$ row of $\mathbf{L}$ can be written as rational numbers with the same denominator $1 \leq q_i \leq K!(Ka_{\max}^2)^K$. To see this note that if $\tilde{\mathbf{A}}$ is triangular up to column permutation vector π, then its ith row contains at least $i - 1$ zeros, namely $\tilde{a}_{ij} = 0$ for $j = \pi(1), \ldots, \pi(i-1)$. Since $\mathbf{L}$ is lower triangular, the following equations must hold

$$\tilde{a}_{ij} = \sum_{m=1}^{i} \ell_{im} a_{mj} = 0, \text{ for } j = \pi(1), \ldots, \pi(i-1). \quad (104)$$

By definition $\ell_{ii} = 1$, therefore (104) can be written as

$$\sum_{m=1}^{i-1} \ell_{im} a_{mj} = -a_{ij}, \text{for } j = \pi(1), \ldots, \pi(i-1). \quad (105)$$

Define the vectors $\ell^{(i)} = [\ell_{i1} \cdots \ell_{i,i-1}]^T$, $\mathbf{a}^{(i,\pi)} = -[a_{i,\pi(1)} \cdots a_{i,\pi(i-1)}]^T$ and the matrix

$$\mathbf{A}^{(i,\pi)} = \begin{pmatrix} a_{1\pi(1)} & \cdots & a_{i-1\pi(1)} \\ \vdots & \ddots & \vdots \\ a_{1\pi(i-1)} & \cdots & a_{i-1\pi(i-1)} \end{pmatrix}.$$

We have,

$$\mathbf{A}^{(i,\pi)}\ell^{(i)} = \mathbf{a}^{(i,\pi)}. \quad (106)$$

From the fact that $\mathbf{A}$ can be pseudo-triangularized with permutation vector π we know that the system of equations (106) has a solution. Assume that

$$\text{rank}\left(\mathbf{A}^{(i,\pi)}\right) = u \leq K.$$

It follows that there are u linearly independent columns in $\mathbf{A}^{(i,\pi)}$. Let $\mathcal{U} \subseteq \{1, \ldots, K\}$ be a set of indices corresponding to u such linearly independent columns, and $\bar{\mathcal{U}}$ be its complement. Let $\mathbf{A}_{\mathcal{U}}^{(i,\pi)} \in \mathbb{Z}^{K \times u}$ be the matrix obtained by taking the columns of $\mathbf{A}^{(i,\pi)}$ with indices in $\mathcal{U}$. Since (106) has a solution, we have $\mathbf{a}^{(i,\pi)} \in \text{span}\left(\mathbf{A}_{\mathcal{U}}^{(i,\pi)}\right)$. Thus, we can set $\ell^{(i)}(k) = 0$ for all $k \in \bar{\mathcal{U}}$, and (106) will still have a solution. Letting $\ell_{\mathcal{U}}^{(i)} \in \mathbb{R}^{u \times 1}$ be the vector obtained by taking from $\ell^{(i)}$ only the entries with indices in $\mathcal{U}$, it follows that

$$\mathbf{A}_{\mathcal{U}}^{(i,\pi)}\ell_{\mathcal{U}}^{(i)} = \mathbf{a}^{(i,\pi)} \quad (107)$$

has a solution. Now, multiplying both sides of (107) by $\left(\mathbf{A}_{\mathcal{U}}^{(i,\pi)}\right)^T$ gives

$$\mathbf{A}'^{(i,\pi)}\ell_{\mathcal{U}}^{(i)} = \mathbf{a}'^{(i,\pi)}, \quad (108)$$

where $\mathbf{A}'^{(i,\pi)} = \left(\mathbf{A}_{\mathcal{U}}^{(i,\pi)}\right)^T \mathbf{A}_{\mathcal{U}}^{(i,\pi)} \in \mathbb{Z}^{u \times u}$ is a full-rank matrix and $\mathbf{a}'^{(i,\pi)} = \left(\mathbf{A}_{\mathcal{U}}^{(i,\pi)}\right)^T \mathbf{a}^{(i,\pi)} \in \mathbb{Z}^{u \times 1}$. Note that all entries of $\mathbf{A}'^{(i,\pi)}$ as well as all entries of $\mathbf{a}'^{(i,\pi)}$ have magnitude bounded from above by $\tilde{a}_{\max} \triangleq u a_{\max}^2$. We have

$$\ell_{\mathcal{U}}^{(i)} = \left(\mathbf{A}'^{(i,\pi)}\right)^{-1} \mathbf{a}'^{(i,\pi)}.$$

Letting $q_i = |\det(\mathbf{A}'^{(i,\pi)})|$ and applying Cramer's rule for matrix inversion (see e.g. [48]) we see that all elements of $\ell_{\mathcal{U}}^{(i)}$ can be expressed as rational numbers with denominator q_i.

Recall the Leibnitz formula (see e.g., [48]) for the determinant of an $n \times n$ matrix $\mathbf{G}$

$$\det(\mathbf{G}) = \sum_{\sigma \in \mathcal{S}_n} \text{sign}(\sigma) \prod_{i=1}^{n} G_{i,\sigma_i}, \qquad (109)$$

where $\mathcal{S}_n$ is the set of all permutations of $\{1, \ldots, n\}$. It follows that $\det\left(\mathbf{A}'^{(i,\pi)}\right)$ must be an integer and in addition $1 \leq |\det\left(\mathbf{A}'^{(i,\pi)}\right)| \leq u!(\tilde{a}_{\max})^u$. Thus, $1 \leq q_i \leq u!(\tilde{a}_{\max})^u$. Moreover, Cramer's rule also implies that the numerator of each element in $\ell_{\mathcal{U}}^{(i)}$ is an integer not greater than $u!(\tilde{a}_{\max})^u$ in magnitude. Since $u \leq K$, and since each element of $\ell^{(i)}$ is either zero or corresponds to an element in $\ell_{\mathcal{U}}^{(i)}$, each element ℓ_{ij}, $j \leq i$ of $\mathbf{L}$ can be written as a rational number $\ell_{ij} = m_{ij}/q_i$ with $1 \leq q_i \leq K!(Ka_{\max}^2)^K$ and $|m_{ij}| \leq K!(Ka_{\max}^2)^K$ for $i = 1, \ldots, K$.

Now, define the matrix $\tilde{\mathbf{L}} = \text{diag}(q_1, \ldots, q_K)\mathbf{L}$ and note that $\tilde{\mathbf{L}} \in \mathbb{Z}^{K \times K}$ due to the above. Let $\tilde{\mathbf{A}}'^{(p)} = [\tilde{\mathbf{L}}\mathbf{A}] \mod p$. Since multiplying a row in a matrix by a constant leaves its zero entries unchanged, the entries of the matrix

$$\begin{aligned}
\tilde{\mathbf{A}}'^{(p)} &= [\tilde{\mathbf{L}}\mathbf{A}] \mod p \\
&= [\text{diag}(q_1, \ldots, q_K)\mathbf{L}\mathbf{A}] \mod p \\
&= [\text{diag}(q_1, \ldots, q_K)\tilde{\mathbf{A}}] \mod p, \qquad (110)
\end{aligned}$$

are zero whenever the entries of $\tilde{\mathbf{A}}$ are equal to zero. Moreover, since all elements of $\mathbf{L}$ are bounded in magnitude by $K!(Ka_{\max}^2)^K$ and all elements of $\mathbf{A}$ are bounded in magnitude by $a_{\max}$, all elements of $\tilde{\mathbf{A}} = \mathbf{L}\mathbf{A}$ are bounded in magnitude by $KK!(Ka_{\max}^2)^K a_{\max}$. Combining with the fact that $1 \leq q_i \leq K!(Ka_{\max}^2)^K$, we have $|\tilde{a}_{ij}'^{(p)}| \leq K(K!)^2(Ka_{\max})^{2K}a_{\max}$ for all $i = 1, \ldots, K$, $j = 1, \ldots, K$. Therefore, for a prime number $p > K(K!)^2(Ka_{\max})^{2K}a_{\max}$ the modulo reduction in (110) does not change any of the non-zero entries of $\text{diag}(q_1, \ldots, q_K)\tilde{\mathbf{A}}$ to zero.

Recall that if $\mathbf{A}$ can be pseudo-triangularized with a matrix $\mathbf{L}$ and permutation vector π then $\tilde{a}_{i,\pi(i)} \neq 0$, and hence also $\tilde{a}_{i,\pi(i)}'^{(p)} \neq 0$ for $i = 1, \ldots, K$. We have therefore shown that for p large enough there exists a lower-triangular matrix $\tilde{\mathbf{L}}^{(p)} = [\text{diag}(q_1, \ldots, q_K)\mathbf{L}] \mod p$ with elements from $\{0, 1, \ldots, p-1\}$ such that $\tilde{\mathbf{A}}'^{(p)} = [\tilde{\mathbf{L}}^{(p)}\mathbf{A}] \mod p$ is upper-triangular up to column permutation π. In order to complete the proof, it is left to transform $\tilde{\mathbf{L}}^{(p)}$ to a lower-triangular matrix with elements from $\{0, 1, \ldots, p-1\}$ and *unit diagonal*. Let $(q_i)^{-1}$ be an integer that satisfies $[(q_i)^{-1}q_i] \mod p = 1$. Such an integer always exists since q_i is an integer different than zero, and p is prime. It is easy to verify that the matrix $\mathbf{L}^{(p)} = [\text{diag}\left((q_1)^{-1}, \ldots, (q_K)^{-1}\right)\tilde{\mathbf{L}}^{(p)}] \mod p$ is a lower-triangular matrix with elements from $\{0, 1, \ldots, p-1\}$ and unit diagonal, and $\tilde{\mathbf{A}}^{(p)} = [\mathbf{L}^{(p)}\mathbf{A}] \mod p$ is upper triangular up to column permutation π. ∎

Lemma 3: Let $\mathbf{t}_1, \ldots, \mathbf{t}_k$ be lattice points from a chain of nested lattices satisfying the conditions of Theorem 2. Let $\mathbf{v} = [\sum_{k=1}^{K} a_k \mathbf{t}_k] \mod \Lambda$ and $\mathbf{u} = [\sum_{k=1}^{K} b_k \mathbf{t}_k] \mod \Lambda$ be integer-valued lattice equations of these points. Then

$$[\mathbf{v} + \mathbf{u}] \mod \Lambda = \left[\sum_{k=1}^{K}\left((a_k + b_k) \mod p\right)\mathbf{t}_k\right] \mod \Lambda.$$

Proof: Due to the distributive property of the modulo operation we have

$$\begin{aligned}
[\mathbf{v} + \mathbf{u}] \mod \Lambda &= \left[\sum_{k=1}^{K}(a_k + b_k)\mathbf{t}_k\right] \mod \Lambda. \\
&= \left[\sum_{k=1}^{K}[(a_k + b_k) \mod p + M_k \cdot p]\mathbf{t}_k\right] \mod \Lambda \\
&= \left[\sum_{k=1}^{K}\left((a_k + b_k) \mod p\right)\mathbf{t}_k\right. \\
&\quad \left. + \sum_{k=1}^{K} M_k \cdot [p \cdot \mathbf{t}_k] \mod \Lambda\right] \mod \Lambda \qquad (111)
\end{aligned}$$

where $\{M_k\}_{k=1}^{K}$ are some integers. Utilizing the fact that $[p \cdot \mathbf{t}_k] \mod \Lambda = \mathbf{0}$ for all lattice points in the chain, which follows from Theorem 2(c), the lemma is established. ∎

We are now ready to prove Theorem 5.

Proof of Theorem 5: Let $\mathbf{T} = [\mathbf{t}_1^T \cdots \mathbf{t}_K^T]^T$ and $\mathbf{V} = [\mathbf{v}_1^T \cdots \mathbf{v}_K^T]^T = [\mathbf{A}\mathbf{T}] \mod \Lambda$. The compute-and-forward transform of the MAC (2) can be written as

$$\begin{aligned}
\mathbf{S} &= \left[\mathbf{A}\begin{pmatrix}\mathbf{t}_1 \\ \vdots \\ \mathbf{t}_K\end{pmatrix} + \mathbf{Z}_{\text{eff}}\right] \mod \Lambda \\
&= [\mathbf{A}\mathbf{T} + \mathbf{Z}_{\text{eff}}] \mod \Lambda \\
&= [\mathbf{V} + \mathbf{Z}_{\text{eff}}] \mod \Lambda.
\end{aligned}$$

Assume there exists a pseudo-triangularization of $\mathbf{A}$ with permutation vector π, i.e., there exists a lower triangular matrix $\mathbf{L}$ with unit diagonal such that $\tilde{\mathbf{A}} = \mathbf{L}\mathbf{A}$ is upper triangular up to column permutation π. Lemma 2 implies that there exists a lower triangular matrix $\mathbf{L}^{(p)}$ with elements from $\{0, 1, \ldots, p-1\}$ and unit diagonal such that $\tilde{\mathbf{A}}^{(p)} = [\mathbf{L}^{(p)}\mathbf{A}] \mod p$ is upper triangular up to column permutation π. Since $\mathbf{L}^{(p)}$ has a unit diagonal it can be written as $\mathbf{L}^{(p)} = \mathbf{I} + \mathbf{R}$ where $\mathbf{I}$ is the identity matrix and $\mathbf{R}$ has non-zero entries only below the main diagonal.

Assume the receiver had access to the side information $\mathbf{v}_1, \ldots, \mathbf{v}_{K-1}$. As the entries of $\mathbf{R}$ are non-zero only below the main diagonal, the receiver can compute $\mathbf{R} \cdot \mathbf{V}$, add it to $\mathbf{S}$ and reduce modulo Λ, giving rise to

$$\begin{aligned}
\mathbf{S}^{\text{SI}} &= [\mathbf{S} + \mathbf{R} \cdot \mathbf{V}] \mod \Lambda \\
&= [\mathbf{A}\mathbf{T} + \mathbf{R}\mathbf{A}\mathbf{T} + \mathbf{Z}_{\text{eff}}] \mod \Lambda \\
&= [(\mathbf{I} + \mathbf{R})\mathbf{A}\mathbf{T} + \mathbf{Z}_{\text{eff}}] \mod \Lambda \\
&= \left[\mathbf{L}^{(p)}\mathbf{A}\mathbf{T} + \mathbf{Z}_{\text{eff}}\right] \mod \Lambda \\
&= \left[[\mathbf{L}^{(p)}\mathbf{A}] \mod p \cdot \mathbf{T} + \mathbf{Z}_{\text{eff}}\right] \mod \Lambda \qquad (112) \\
&= \left[\tilde{\mathbf{A}}^{(p)}\begin{pmatrix}\mathbf{t}_1 \\ \vdots \\ \mathbf{t}_K\end{pmatrix} + \mathbf{Z}_{\text{eff}}\right] \mod \Lambda
\end{aligned}$$

where (112) follows from Lemma 3. Let $\tilde{\mathbf{V}} = [\tilde{\mathbf{A}}^{(p)}\mathbf{T}] \mod \Lambda$ and recall that $\tilde{\mathbf{A}}^{(p)}$ is upper-triangular up to column permutation π, thus $\tilde{a}_{j,\pi(k)}^{(p)} = 0$ for all $j = \pi(k)+1, \ldots, K$. It follows

that for any $k < K$ the lattice point $\mathbf{t}_{\pi(k)}$ does not participate in any of the lattice equations $\tilde{\mathbf{v}}_{k+1}, \ldots, \tilde{\mathbf{v}}_K$

Assume the mapping function between users and lattices is chosen as $\theta(k) = \pi^{-1}(k)$, i.e., each user k employs the codebook $\mathcal{L}_k = \Lambda_{\pi^{-1}(k)} \cap \Lambda$. In this case, the densest lattice participating in lattice equation $\tilde{\mathbf{v}}_k$ is Λ_k. The decoder uses $\mathbf{s}_k^{SI}$ in order to produce an estimate

$$\hat{\tilde{\mathbf{v}}}_k = \left[Q_{\Lambda_k}(\mathbf{s}_k^{SI}) \right] \bmod \Lambda \qquad (113)$$

for each one of the lattice equations $\tilde{\mathbf{v}}_k$. It follows from Theorem 2 that there exists a chain of nested lattices $\Lambda \subseteq \Lambda_K \subseteq \cdots \subseteq \Lambda_1$ forming the set of codebooks $\mathcal{L}_1, \ldots, \mathcal{L}_K$ with rates $R_1, \ldots, R_K$ such that all equations $\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_K$ can be decoded with a vanishing error probability as long as the rates of all users satisfy the constraints of (25).

We have shown that if the receiver has access to $\mathbf{v}_1, \ldots, \mathbf{v}_{K-1}$ it can decode the set of equations $\tilde{\mathbf{V}}$. We now show a sequential decoding procedure which guarantees that the receiver has the right amount of side information at each step. First, note that

$$\mathbf{s}_k^{SI} = \left[\mathbf{s}_k + \sum_{m=1}^{k-1} r_{km} \mathbf{v}_m \right] \bmod \Lambda, \qquad (114)$$

thus the necessary side information for decoding $\tilde{\mathbf{v}}_k$ is only $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. In particular, $\mathbf{s}_1^{SI} = \mathbf{s}_1$ and hence $\mathbf{v}_1$ can be decoded with a vanishing error probability with no side information. After decoding $\mathbf{v}_1$ the receiver has it as side information, and can therefore compute $\mathbf{s}_2^{SI}$ and decode $\tilde{\mathbf{v}}_2$. As $\tilde{\mathbf{v}}_2 = [r_{21}\mathbf{v}_1 + \mathbf{v}_2] \bmod \Lambda$ and the receiver knows $\mathbf{v}_1$, it can use it in order to recover $\mathbf{v}_2$. Now, the receiver has $\mathbf{v}_1$ and $\mathbf{v}_2$ as side information and can use it to compute $\mathbf{s}_3^{SI}$. The process continues sequentially until all equations $\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_K$ are decoded.

When the process ends we are left with K noiseless equations

$$\begin{pmatrix} \tilde{\mathbf{v}}_1 \\ \vdots \\ \tilde{\mathbf{v}}_K \end{pmatrix} = \left[\tilde{\mathbf{A}}^{(p)} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_K \end{pmatrix} \right] \bmod \Lambda. \qquad (115)$$

Since $\tilde{\mathbf{A}}^{(p)}$ is upper-triangular up to column permutation, and in particular full-rank modulo p, the original lattice points $\mathbf{t}_1, \ldots, \mathbf{t}_K$ each user transmitted can be recovered. ∎

APPENDIX B
PROOF OF THEOREMS 7 AND 8

Proof of Theorem 7: The proof is identical to that of Theorem 3 with $\mathbf{F} = \left(\mathsf{SNR}^{-1}\mathbf{B}^{-1} + \mathbf{gg}^T \right)^{-1/2}$. ∎

Proof of Theorem 8: Let

$$\mathbf{S} = \left[\mathbf{A} \begin{pmatrix} \mathbf{t}_{\mathrm{eff},1} \\ \vdots \\ \mathbf{t}_{\mathrm{eff},L} \end{pmatrix} + \mathbf{Z}_{\mathrm{eff}} \right] \bmod \Lambda$$

be the compute-and-forward transform of the effective L-user MAC, and assume that $\mathbf{A}$ can be pseudo-triangularized with

permutation vector π. Repeating the proof of Theorem 5 it easy is to see that for any set of rates

$$R_\ell < R_{\mathrm{comp},\pi^{-1}(\ell)}, \ \ell = 1, \ldots, L$$

there exists a chain of nested lattices $\Lambda \subseteq \Lambda_L \subseteq \cdots \subseteq \Lambda_1$ inducing the codebooks $\mathcal{L}_\ell = \Lambda_{\pi^{-1}(\ell)} \cap \mathcal{V}$ with rates R_ℓ, such that if $\mathbf{t}_{\mathrm{eff},\ell} \in \mathcal{L}_\ell$ for all $\ell = 1, \ldots, L$, all effective lattice points can be decoded from $\mathbf{S}$.

If each of the users $m \in \mathcal{K}_\ell$ that comprise effective user ℓ uses the lattice codebook $\mathcal{L}_\ell$ (or any codebook nested in $\mathcal{L}_\ell$), then $\mathbf{t}_{\mathrm{eff},\ell} \in \mathcal{L}_\ell$ and all effective lattice points can be decoded. ∎

APPENDIX C
PROOF OF LEMMA 1

In order to decode the desired effective lattice points, it suffices to decode $L-1$ linearly independent lattice equations of them, in which $\mathbf{t}_{\mathrm{eff},L}$ does not participate. Let $\bar{\mathbf{a}} = [\bar{a}_1 \ \cdots \ \bar{a}_{L-1} \ 0]$ be some coefficient vector for such an equation. The effective rate for computing the lattice equation $\bar{\mathbf{v}} = [\sum_{\ell=1}^{L-1} \bar{a}_\ell \mathbf{t}_{\mathrm{eff},\ell}] \bmod \Lambda$ with the coefficient vector $\bar{\mathbf{a}}$ over the channel (29) is

$$R_{\mathrm{comp}}(\mathbf{g}, \bar{\mathbf{a}}, \mathbf{B}) = \frac{1}{2} \log \left(\frac{\mathsf{SNR}}{\sigma_{\mathrm{eff}}^2(\mathbf{g}, \bar{\mathbf{a}}, \mathbf{B})} \right), \qquad (116)$$

where

$$\begin{aligned} \sigma_{\mathrm{eff}}^2(\mathbf{g}, \bar{\mathbf{a}}, \mathbf{B}) &= \min_{\bar{\beta} \in \mathbb{R}} \mathsf{SNR} \sum_{\ell=1}^{L-1} (\bar{\beta}g_\ell - \bar{a}_\ell)^2 b_{\mathrm{eff},\ell}^2 \\ &\quad + \bar{\beta}^2 (1 + \mathsf{SNR}g_L^2 b_{\mathrm{eff},L}^2) \\ &= \min_{\beta \in \mathbb{R}} \mathsf{SNR} \sum_{\ell=1}^{L-1} (\beta\kappa g_\ell - \bar{a}_\ell)^2 b_{\mathrm{eff},\ell}^2 + \beta^2, \quad (117) \end{aligned}$$

where (117) follows by substituting $\bar{\beta} = \beta\kappa$. The effective variance and computation rate for computing an equation with coefficient vector $\bar{\mathbf{a}} = [\bar{a}_1 \ \cdots \ \bar{a}_{L-1} \ 0]$ over the effective channel (29) are therefore the same as those of computing an equation with $\mathbf{a} = [\bar{a}_1 \ \cdots \ \bar{a}_{L-1}]$ over the effective channel (47). Thus, for purposes of computing equations with the effective lattice points $\mathbf{t}_{\mathrm{eff},1}, \ldots, \mathbf{t}_{\mathrm{eff},L-1}$ the two channels are equivalent. Since this is all we need in order to decode $\mathbf{t}_{\mathrm{eff},1}, \ldots, \mathbf{t}_{\mathrm{eff},L-1}$, the lemma follows. ∎

APPENDIX D
DERIVATION OF THE UPPER BOUNDS ON σ_{HK}^2 WITHIN THE DIFFERENT INTERVALS

A. Moderately weak interference regime

We upper bound σ_{HK}^2 for all values of $\tilde{\beta}$ within each of the four intervals. Recall that in the moderately weak interference regime $\mathsf{SNR}^{-1/3} \leq g^2 \leq 1$. Define $\delta = (2c+8)/\log(\mathsf{SNR})$, where $c > 0$ is some constant.

Interval $1 : 0 < |\tilde{\beta}| \leq 1/2$

In this interval the choice $a_1 = \mathrm{sign}(\tilde{\beta})$ is optimal due to (86). Therefore, for all $|\tilde{\beta}| \leq 1/2$ we have $(\tilde{\beta} - a_1)^2 \geq 1/4$. Thus,

$$\sigma_{\mathrm{HK}}^2 \geq \frac{\mathsf{SNR}}{4}.$$

Interval $2 : 1/2 < |\tilde{\beta}| \leq \sqrt{|g|}\mathsf{SNR}^{1/4-\delta/2}/2$

Since $|\tilde{\beta}| > 1/2$ we can express it as $\tilde{\beta} = q + \varphi$ with $q \in \mathbb{Z} \setminus 0$ and $\varphi \in [-1/2, 1/2]$. We can further lower bound σ_{HK}^2 as

$$\sigma_{\mathrm{HK}}^2 > \min_{\varphi,q,a_1,a_3} \left(\left((\varphi + q - a_1)^2 + (qg - a_3 + \varphi g)^2\right)\mathsf{SNR} \right)$$
$$= \min_{\varphi,q,a_3} \left(\left(\varphi^2 + (qg - a_3 + \varphi g)^2\right)\mathsf{SNR} \right). \quad (118)$$

The minimizing value of φ (ignoring the constraint $\varphi \in [-1/2, 1/2]$) is found by derivation to be

$$\varphi^* = -\frac{g}{1+g^2}(qg - a_3).$$

Substituting φ^* into (118) gives

$$\sigma_{\mathrm{HK}}^2 > \min_{q,a_3} \left(\frac{1}{1+g^2}(qg - a_3)^2\mathsf{SNR} \right)$$
$$\geq \frac{1}{2}\min_{q,a_3} \left((qg - a_3)^2\mathsf{SNR} \right). \quad (119)$$

For $b = 1, 2, \ldots, \lceil 1/3 \log(\mathsf{SNR}) \rceil$ we define the sets

$$\mathcal{G}_b = \left\{ g : g \in \left[2^{-b}, 2^{-b+1}\right) \right\}, \quad (120)$$

and the quantities

$$q_{\mathrm{max},b} \triangleq \sqrt{2^{-b+1}}\mathsf{SNR}^{1/4-\delta/2},$$
$$\Phi_b \triangleq \frac{1}{\sqrt{2^{-b+1}}}\mathsf{SNR}^{-1/4-\delta/2}.$$

Let $\mathcal{S}_b$ be the set of all values of $g \in \mathcal{G}_b$ such that the inequality

$$|qg - a_3| < \Phi_b \quad (121)$$

has at least one solution with $0 < |q| \leq q_{\mathrm{max},b}$ and $a_3 \in \mathbb{Z}$. Note that since $q = \tilde{\beta} + \varphi$ and we assume in this interval that $1/2 < |\tilde{\beta}| \leq \sqrt{|g|}\mathsf{SNR}^{1/4-\delta/2}/2$, we have

$$|q| < \sqrt{|g|}\mathsf{SNR}^{1/4-\delta/2}.$$

Thus, for all $g \in \mathcal{G}_b$ and $\tilde{\beta}$ in the considered interval, we have

$$|q| < q_{\mathrm{max},b}.$$

Let $\bar{\mathcal{S}}_b = \mathcal{G}_b \backslash \mathcal{S}_b$. Using (119), we have that for all $g \in \bar{\mathcal{S}}_b$ and $\tilde{\beta}$ in the considered interval

$$\sigma_{\mathrm{HK}}^2 \geq \frac{1}{2}\Phi_b^2\mathsf{SNR}$$
$$\geq \frac{1}{2}\frac{\mathsf{SNR}^{1/2-\delta}}{2^{-b+1}}$$
$$\geq \frac{1}{4}\frac{\mathsf{SNR}^{1/2-\delta}}{\sqrt{g^2}}. \quad (122)$$

The condition (121), which defines the set $\mathcal{S}_b$, can be written equivalently as

$$|q \cdot 2^b g - 2^b a_3| < 2^b \Phi_b. \quad (123)$$

Define $\tilde{g} = 2^b \cdot g$, and note that for all $g \in \mathcal{G}_b$ we have $\tilde{g} \in [1, 2)$. With this notation, (123) becomes

$$\left| \tilde{g} - \frac{2^b a_3}{q} \right| < 2^b \frac{\Phi_b}{q}. \quad (124)$$

Define

$$\mathcal{T}_b(q) = \left[\left\{ \frac{0}{q}, \frac{1 \cdot 2^b}{q}, \frac{2 \cdot 2^b}{q}, \ldots, \frac{\lfloor \frac{2q-1}{2^b} \rfloor \cdot 2^b}{q} \right\} \right.$$
$$\left. + 2^b \frac{\Phi_b}{q}\mathcal{I} \right] \bmod [0, 2), \quad (125)$$

where $\mathcal{I} = [-1, 1)$ and the sum in (125) is a Minkowski sum. It is easy to verify that

$$\mathcal{S}_b \subseteq 2^{-b} \bigcup_{q=1}^{\lfloor q_{\mathrm{max}} \rfloor} \mathcal{T}_b(q). \quad (126)$$

Since $\lfloor \frac{2q-1}{2^b} \rfloor = 0$ for all $0 < q < 2^{b-1}$, for all values of q in this range we have

$$\mathcal{T}_b(q) = \left[2^b \frac{\Phi_b}{q}\mathcal{I} \right] \bmod [0, 2)$$
$$\subseteq \left[2^b \Phi_b \mathcal{I} \right] \bmod [0, 2)$$
$$= \mathcal{T}_b(1). \quad (127)$$

Therefore,

$$\mathcal{S}_b \subseteq 2^{-b} \left(\left(\bigcup_{q=1}^{2^{b-1}-1} \mathcal{T}_b(q) \right) \cup \left(\bigcup_{q=2^{b-1}}^{\lfloor q_{\mathrm{max}} \rfloor} \mathcal{T}_b(q) \right) \right)$$
$$= 2^{-b} \left(\mathcal{T}_b(1) \cup \left(\bigcup_{q=2^{b-1}}^{\lfloor q_{\mathrm{max}} \rfloor} \mathcal{T}_b(q) \right) \right). \quad (128)$$

The Lebesgue measure of $\mathcal{S}_b$ is bounded by

$$\mu(\mathcal{S}_b) = \mathrm{Vol}\,(\mathcal{S}_b)$$
$$\leq 2^{-b} \left(\mathrm{Vol}\,(\mathcal{T}_b(1)) + \sum_{q=2^{b-1}}^{\lfloor q_{\mathrm{max},b} \rfloor} \mathrm{Vol}\,(\mathcal{T}_b(q)) \right)$$
$$\leq 2^{-b} \left(2 \cdot 2^b \Phi_b + \sum_{q=2^{b-1}}^{\lfloor q_{\mathrm{max},b} \rfloor} \left\lceil \frac{2q}{2^b} \right\rceil \cdot 2 \cdot 2^b \frac{\Phi_b}{q} \right)$$
$$\leq 2\Phi_b + 2\Phi_b \sum_{q=2^{b-1}}^{\lfloor q_{\mathrm{max},b} \rfloor} 2\frac{2q}{2^b}\frac{1}{q}$$
$$\leq 2\Phi_b + 8 \cdot 2^{-b}\Phi_b q_{\mathrm{max},b}$$
$$= 2\Phi_b + 8 \cdot 2^{-b}\mathsf{SNR}^{-\delta}$$
$$= \sqrt{2} \cdot 2^{b/2}\mathsf{SNR}^{-1/4-\delta/2} + 8 \cdot 2^{-b}\mathsf{SNR}^{-\delta}. \quad (129)$$

We can now upper bound the measure of the outage set

$$\mathcal{S} = \bigcup_{b=1}^{\lceil 1/3 \log(\mathsf{SNR}) \rceil} \mathcal{S}_b,$$

of all values of $\mathsf{SNR}^{-1/3} \leq g < 1$ for which (122) does not

necessarily hold, as

$$\mu(\mathcal{S}) = \sum_{b=1}^{\lceil 1/3 \log(\mathsf{SNR}) \rceil} \mu(\mathcal{S}_b)$$

$$< \sqrt{2}\mathsf{SNR}^{-1/4-\delta/2} \sum_{b=1}^{1/3 \log(\mathsf{SNR})+1} (\sqrt{2})^b$$

$$+ 8\mathsf{SNR}^{-\delta} \sum_{b=1}^{1/3 \log(\mathsf{SNR})+1} 2^{-b}.$$

Using the identity

$$\sum_{b=1}^{B} \rho^b = \frac{\rho}{\rho-1}(\rho^B - 1),$$

which is valid for all $\rho \neq 1$, and the fact that $\sum_{b=1}^{\infty} 2^{-b} < 1$, we have

$$\mu(\mathcal{S}) < \sqrt{2}\mathsf{SNR}^{-1/4-\delta/2} \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{2}\mathsf{SNR}^{1/6} + 8\mathsf{SNR}^{-\delta}$$

$$< 7\mathsf{SNR}^{-\delta/2} + 8\mathsf{SNR}^{-\delta}$$

$$< 16\mathsf{SNR}^{-\delta/2}. \qquad (130)$$

Substituting $\delta = (2c+8)/\log(\mathsf{SNR})$ into (122) and (130) we see that in the interval $1/2 < |\tilde{\beta}| \leq \sqrt{|g|}\mathsf{SNR}^{1/4-\delta/2}/2$ for all values of $\mathsf{SNR}^{-1/3} < |g| \leq 1$ except for an outage set with measure not greater than 2^{-c} we have

$$\sigma_{\mathrm{HK}}^2 > \frac{2^{-2c}}{4 \cdot 2^8} \frac{\mathsf{SNR}^{1/2}}{\sqrt{g^2}}.$$

Interval 3 : $\sqrt{|g|}\mathsf{SNR}^{1/4-\delta/2}/2 < |\tilde{\beta}| \leq \mathsf{SNR}^{1/4}/\sqrt{8|g|}$
Since $\mathsf{SNR}^{-1/3} \leq g^2 < 1$ and we assumed $\mathsf{SNR} > 4$, we have

$$g^2\mathsf{SNR} - 1 > \frac{g^2\mathsf{SNR}}{2}. \qquad (131)$$

Note that (131) continues to hold for all $g^2 > \mathsf{SNR}^{-1/2}$. This will be useful in the weak interference regime. For all values of $|\tilde{\beta}|$ in this interval

$$\left| \frac{\tilde{\beta}}{\sqrt{g^2\mathsf{SNR}-1}} \right| \leq \frac{\mathsf{SNR}^{1/4}/\sqrt{8|g|}}{\sqrt{g^2\mathsf{SNR}-1}}$$

$$< \frac{\mathsf{SNR}^{1/4}}{\sqrt{8|g| \cdot \frac{g^2\mathsf{SNR}}{2}}}$$

$$\leq \frac{1}{2}|g|^{-3/2}\mathsf{SNR}^{-1/4}$$

$$\leq \frac{1}{2},$$

and hence, using (85), the optimal value of a_2 is

$$a_2 = \left\lfloor \tilde{\beta}/\sqrt{g^2\mathsf{SNR}-1} \right\rceil = 0.$$

Therefore, using the fact that $\delta = (2c+8)/\log(\mathsf{SNR})$, we can upper bound (84) as

$$\sigma_{\mathrm{HK}}^2 \geq \frac{\tilde{\beta}^2\mathsf{SNR}}{g^2\mathsf{SNR}-1} \geq \frac{2^{-2c}}{4 \cdot 2^8} \frac{\mathsf{SNR}^{1/2}}{\sqrt{g^2}}.$$

Interval 4 : $\mathsf{SNR}^{1/4}/\sqrt{8|g|} < |\tilde{\beta}|$
In this interval

$$\sigma_{\mathrm{HK}}^2 \geq K\tilde{\beta}^2 \geq \frac{1}{8} \frac{\mathsf{SNR}^{1/2}}{\sqrt{g^2}}.$$

B. Weak Interference Regime

We upper bound σ_{HK}^2 for all values of $\tilde{\beta}$ within each of the four intervals. Recall that in this regime $\mathsf{SNR}^{-1/2} \leq g^2 < \mathsf{SNR}^{-1/3}$.

Interval 1 : $0 < |\tilde{\beta}| \leq 1/2$
As $a_1 \neq 0$, in this interval $(\tilde{\beta} - a_1)^2 > 1/4$. Thus,

$$\sigma_{\mathrm{HK}}^2 \geq \frac{\mathsf{SNR}}{4}.$$

Interval 2 : $1/2 < |\tilde{\beta}| \leq 1/(2|g|)$
In this interval $a_3 = \lfloor \tilde{\beta}g \rceil = 0$. Thus,

$$\sigma_{\mathrm{HK}}^2 \geq (\tilde{\beta}g)^2\mathsf{SNR} \geq \frac{g^2\mathsf{SNR}}{4}.$$

Interval 3 : $1/(2|g|) < |\tilde{\beta}| \leq \sqrt{g^2\mathsf{SNR}/8}$
Under our regular assumption that $\mathsf{SNR} > 4$, for all values of $|\tilde{\beta}|$ in this interval we have

$$\left| \frac{\tilde{\beta}}{\sqrt{g^2\mathsf{SNR}-1}} \right| \leq \frac{\sqrt{g^2\mathsf{SNR}/8}}{\sqrt{g^2\mathsf{SNR}-1}}$$

$$< \frac{\sqrt{g^2\mathsf{SNR}/8}}{\sqrt{g^2\mathsf{SNR}/2}}$$

$$\leq \frac{1}{2},$$

where the second inequality follows from (131). Thus, the optimal choice for a_2 is

$$a_2 = \left\lfloor \tilde{\beta}/\sqrt{g^2\mathsf{SNR}-1} \right\rceil = 0. \qquad (132)$$

Therefore, (84) can be lower bounded by

$$\sigma_{\mathrm{HK}}^2 \geq \frac{\tilde{\beta}^2}{g^2\mathsf{SNR}}\mathsf{SNR} \geq \frac{1}{4g^4}.$$

Interval 4 : $\sqrt{g^2\mathsf{SNR}/8} < |\tilde{\beta}|$
In this interval

$$\sigma_{\mathrm{HK}}^2 \geq K\tilde{\beta}^2 > \frac{g^2\mathsf{SNR}}{4}.$$

APPENDIX E
PROOF OF THEOREM 12

For the proof we will need a key result from the field of Diophantine approximation which is due to Kleinbock and Margulis. The following theorem is a special case of [57, Theorem A].

Theorem 14: Let $f_1, f_2, \cdots, f_K$ be real analytic functions in $\tilde{\mathbf{h}} \in \mathcal{U}$, $\mathcal{U}$ a domain in $\mathbb{R}^m$, which together with 1 are linearly independent over $\mathbb{R}$, and define the manifold

$$\mathcal{M} = \left\{ \begin{bmatrix} f_1(\tilde{\mathbf{h}}) & \cdots & f_K(\tilde{\mathbf{h}}) \end{bmatrix} : \tilde{\mathbf{h}} \in \mathcal{U} \right\}.$$

For almost every $\mathbf{h} \in \mathcal{M}$ and any $\delta > 0$ the inequality

$$\max_{\ell=1,\ldots,K} |qh_\ell - a_\ell| \leq |q|^{-\frac{1}{K}-\delta} \qquad (133)$$

has at most finitely many solutions $(q, \mathbf{a}) \in \mathbb{Z} \times \mathbb{Z}^K$.

For the proof of Theorem 12 we will need a corollary of Theorem 14.

Corollary 7: Let $f_1, f_2, \ldots, f_K$ be functions from $\mathbb{R}^m$ to $\mathbb{R}$ satisfying the following conditions:

1) f_i for $i = 1, \ldots, K$ is analytic in $\mathbb{R}^m$,
2) $1, f_1, \ldots, f_K$ are linearly independent over $\mathbb{R}$.

Let $\mathcal{D} = \left\{ \tilde{\mathbf{h}} \in \mathbb{R}^m \ : \ f_1(\tilde{\mathbf{h}}) = 0 \right\}$ and $\mathcal{D}(\epsilon) = \mathcal{D} + \mathcal{B}(\mathbf{0}, \epsilon)$, where the sum is a Minkowski sum and $\mathcal{B}(\mathbf{0}, \epsilon)$ is an m-dimensional closed ball with some radius $\epsilon > 0$. Define the set $\mathcal{U}(\epsilon) = \mathbb{R}^m \setminus \mathcal{D}(\epsilon)$, the set of functions $\tilde{f}_k(\tilde{\mathbf{h}}) = f_k(\tilde{\mathbf{h}})/f_1(\tilde{\mathbf{h}})$ from $\mathcal{U}(\epsilon)$ to $\mathbb{R}$ for $k = 2, \ldots, K$, and the manifold

$$\bar{\mathcal{M}}(\epsilon) = \left\{ \begin{bmatrix} \tilde{f}_2(\tilde{\mathbf{h}}) & \cdots & \tilde{f}_K(\tilde{\mathbf{h}}) \end{bmatrix} \ : \ \tilde{\mathbf{h}} \in \mathcal{U}(\epsilon) \right\}. \qquad (134)$$

For all $\epsilon > 0$, almost every $\bar{\mathbf{h}} \in \mathcal{M}'(\epsilon)$, and any $\delta > 0$ the inequality

$$\max_{\ell=1,\ldots,K-1} |q\bar{h}_\ell - a_\ell| \leq |q|^{-\frac{1}{K-1}-\delta} \qquad (135)$$

has at most finitely many solutions $(q, \mathbf{a}) \in \mathbb{Z} \times \mathbb{Z}^{K-1}$.

Proof of Corollary 7: We would like to apply Theorem 14 for the set of functions $\tilde{f}_2, \ldots, \tilde{f}_K$ from $\mathcal{U}(\epsilon)$ to $\mathbb{R}$. To that end we have to show that for all $\epsilon > 0$ the functions $\tilde{f}_2, \ldots, \tilde{f}_K$ are analytic in $\mathcal{U}(\epsilon)$ and together with 1 are linearly independent over $\mathbb{R}$.

The reciprocal of an analytic function that is nowhere zero is analytic. Thus, for any $\epsilon > 0$ the function $1/f_1(\tilde{\mathbf{h}})$ is analytic in $\mathcal{U}(\epsilon)$. In addition the product of two analytic functions is analytic. Therefore, for any $\epsilon > 0$ the functions $\tilde{f}_k = f_k(\tilde{\mathbf{h}}) \cdot (1/f_1(\tilde{\mathbf{h}}))$ are analytic in $\mathcal{U}(\epsilon)$ for $k = 2, \ldots, K$.

We show that the functions $1, \tilde{f}_2, \ldots, \tilde{f}_K$ from $\mathcal{U}(\epsilon)$ to $\mathbb{R}$ are linearly independent for all $\epsilon > 0$ by contradiction. Assume they are linearly dependent. Thus, there a exists a measurable set $\mathcal{S} \in \mathcal{U}(\epsilon)$ and a set of coefficients $\{t_1(\epsilon), \ldots, t_K(\epsilon)\} \in \mathbb{R}$ not all zero such that $\forall \tilde{\mathbf{h}} \in \mathcal{S}$

$$t_1(\epsilon) \cdot 1 + t_2(\epsilon) \cdot \frac{f_2(\tilde{\mathbf{h}})}{f_1(\tilde{\mathbf{h}})} + \cdots + t_K(\epsilon) \cdot \frac{f_K(\tilde{\mathbf{h}})}{f_1(\tilde{\mathbf{h}})} = 0.$$

This implies that $\forall \tilde{\mathbf{h}} \in \mathcal{S}$

$$0 \cdot 1 + t_1(\epsilon) \cdot f_1(\tilde{\mathbf{h}}) + t_2(\epsilon) \cdot f_2(\tilde{\mathbf{h}}) + \cdots + t_K(\epsilon) \cdot f_K(\tilde{\mathbf{h}}) = 0,$$

in contradiction to the assumption that the functions $1, f_1, \ldots, f_K$ from $\mathbb{R}^m$ to $\mathbb{R}$ are linearly independent over $\mathbb{R}$.

We can therefore apply Theorem 14 to the set of functions $\tilde{f}_2, \ldots, \tilde{f}_K$ from $\mathcal{U}(\epsilon)$ to $\mathbb{R}$ for all $\epsilon > 0$, and the corollary follows. ∎

We are now ready to prove Theorem 12. Define the sets $\mathcal{D}$, $\mathcal{D}(\epsilon)$ and $\mathcal{U}(\epsilon)$ as in Corollary 7, and the manifold

$$\tilde{\mathcal{M}}(\epsilon) = \left\{ \begin{bmatrix} f_1(\tilde{\mathbf{h}}) & \cdots & f_K(\tilde{\mathbf{h}}) \end{bmatrix} \ : \ \tilde{\mathbf{h}} \in \mathcal{U}(\epsilon) \right\}.$$

We begin by showing that for any $\epsilon > 0$ the number of DoF offered by the first computation rate is upper bounded by

$$d_{\mathrm{comp},1} \leq \frac{1}{K} \qquad (136)$$

for almost every $\mathbf{h} \in \tilde{\mathcal{M}}(\epsilon)$. Then we take ϵ to zero in order to show that this holds for almost every

$$\mathbf{h} \in \mathcal{M} = \left\{ \begin{bmatrix} f_1(\tilde{\mathbf{h}}) & \cdots & f_K(\tilde{\mathbf{h}}) \end{bmatrix} \ : \ \tilde{\mathbf{h}} \in \mathbb{R}^m \right\}.$$

Consider the manifold $\tilde{\mathcal{M}}(\epsilon)$ for some $\epsilon > 0$. Note that $h_1 = f_1(\tilde{\mathbf{h}}) \neq 0$ for any $\mathbf{h} \in \tilde{\mathcal{M}}(\epsilon)$, and we can therefore define $\bar{\mathbf{h}} = \mathbf{h}/h_1$. We have $\bar{h}_1 = 1$ and $[\bar{h}_2 \cdots \bar{h}_K] \in \bar{\mathcal{M}}(\epsilon)$, where $\bar{\mathcal{M}}(\epsilon)$ is the manifold from (134) in Corollary 7.

The channel (2) is equivalent to the channel

$$\bar{\mathbf{y}} = \frac{1}{h_1}\mathbf{y} = \mathbf{x}_1 + \sum_{k \neq 1} \bar{h}_k \mathbf{x}_k + \frac{1}{h_1}\mathbf{z}. \qquad (137)$$

Let $\mathbf{a}$ be a vector of integer coefficients, and β be the scaling factor used by the receiver in order to compute the lattice equation $\mathbf{v} = \left[\sum_{k=1}^K a_k \mathbf{t}_k\right] \mod \Lambda$, see Section III. The effective noise for computing the equation $\mathbf{v}$ with coefficient vector $\mathbf{a}$ is

$$\mathbf{z}_{\mathrm{eff}}(\mathbf{h}, \mathbf{a}, \beta) = (\beta - a_1)\mathbf{x}_1 + \sum_{k \neq 1}(\beta \bar{h}_k - a_k)\mathbf{x}_k + \frac{\beta}{h_1}\mathbf{z},$$

and its effective variance is given by

$$\sigma_{\mathrm{eff}}^2(\mathbf{h}, \mathbf{a}, \beta) = (\beta - a_1)^2 \mathsf{SNR}$$
$$+ \sum_{k \neq 1}(\beta \bar{h}_k - a_k)^2 \mathsf{SNR} + \frac{\beta^2}{|h_1|^2}. \qquad (138)$$

Recall that

$$R_{\mathrm{comp},1} = \max_{\mathbf{a},\beta} \frac{1}{2} \log\left(\frac{\mathsf{SNR}}{\sigma_{\mathrm{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)}\right)$$
$$= \frac{1}{2}\log(\mathsf{SNR}) - \frac{1}{2}\log\left(\min_{\mathbf{a},\beta} \sigma_{\mathrm{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)\right). \qquad (139)$$

Thus, in order to obtain an upper bound on $R_{\mathrm{comp},1}$ we need to lower bound $\sigma_{\mathrm{eff}}^2(\mathbf{h}, \mathbf{a}, \beta)$ for all values of $\beta \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{Z}^K \setminus \mathbf{0}$. Let

$$\bar{h}^* = \max_{k=1,\ldots,K} \bar{h}_k,$$

and

$$k^* = \arg\max_{k=1,\ldots,K} \bar{h}_k.$$

Note that if $|\beta| < 1/(2\bar{h}^*)$ the minimizing corresponding choice of integers $a_1, \ldots, a_K$ in (138) is $a_{k^*} = \mathrm{sign}(\beta)$, and $a_k = 0$ for all $k \neq k^*$. This in turn, implies that for $|\beta| < 1/(2\bar{h}^*)$ we have

$$\sigma_{\mathrm{eff}}^2(\mathbf{h}, \mathbf{a}, \beta) > (\beta \bar{h}^* - \mathrm{sign}(\beta))^2 \mathsf{SNR} > \frac{\mathsf{SNR}}{4}, \qquad (140)$$

which means $d_{\mathrm{comp},1} = 0$. Thus, in order to obtain a positive number of DoF, $|\beta|$ must be greater than $1/(2\bar{h}^*)$.

If $1/(2\bar{h}^*) \leq |\beta| \leq 1/2$, then the minimizing value of a_1 in (138) is $a_1 = 0$. This implies that for all values of $1/(2\bar{h}^*) \leq |\beta| \leq 1/2$ we have

$$\sigma_{\mathrm{eff}}^2(\mathbf{h}, \mathbf{a}, \beta) > \beta^2 \mathsf{SNR} > \frac{\mathsf{SNR}}{4(\bar{h}^*)^2}, \qquad (141)$$

which also means $d_{\text{comp},1} = 0$. Thus, in order to obtain a positive number of DoF, $|\beta|$ must be greater than $1/2$.

Hence, in order to lower bound (138) in the limit of very high SNR, it suffices to limit the optimization space of β to $|\beta| > 1/2$. For such values, β can be written in the form $\beta = q + \varphi$ where $\varphi \in [-1/2, 1/2)$, and $q \in \mathbb{Z} \setminus 0$ is the nearest integer to β.

For any $|\varphi| < 1/2$, $q \in \mathbb{Z} \setminus 0$ and $\mathbf{a} \in \mathbb{Z}^K \setminus \mathbf{0}$ we have

$$
\begin{aligned}
&\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, q, \varphi) \\
&= (\varphi + q - a_1)^2 \mathsf{SNR} \\
&\quad + \max_{k \neq 1} \left(q\bar{h}_k - a_k \varphi \bar{h}_k\right)^2 \mathsf{SNR} + \frac{(q/2)^2}{|h_1|^2} \\
&\geq \varphi^2 \mathsf{SNR} + \max_{k \neq 1} \left(q\bar{h}_k - a_k + \varphi \bar{h}_k\right)^2 \mathsf{SNR} + \frac{(q/2)^2}{|h_1|^2} \\
&= \max_{k \neq 1} \left(\left(\varphi^2 + (q\bar{h}_k - a_k + \varphi \bar{h}_k)^2\right) \mathsf{SNR} + \frac{q^2}{|2h_1|^2} \right)
\end{aligned}
\tag{142}
$$

We further bound (142) by substituting the minimizing value of φ for each $k \neq 1$. It follows by simple differentiation that for each $k \neq 1$ the minimum is obtained at

$$
\varphi^*(k) = \frac{-\bar{h}_k}{1 + \bar{h}_k^2}(q\bar{h}_k - a_k).
$$

Substituting $\varphi^*(k)$ into (142) yields

$$
\begin{aligned}
\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, q, \varphi) &\geq \max_{k \neq 1} \left(\frac{1}{1 + \bar{h}_k^2}(q\bar{h}_k - a_k)^2 \mathsf{SNR} + \frac{q^2}{|2h_1|^2} \right) \\
&> \frac{1}{1 + \max_{k \neq 1} \bar{h}_k^2} \cdot \max_{k \neq 1} |q\bar{h}_k - a_k|^2 \mathsf{SNR} + \frac{q^2}{|2h_1|^2} \\
&= c_0(\mathbf{h}) \cdot \max_{k \neq 1} |q\bar{h}_k - a_k|^2 \mathsf{SNR} + \frac{q^2}{|2h_1|^2},
\end{aligned}
\tag{143}
$$

where $c_0(\mathbf{h}) > 0$ is some constant independent of the SNR.

Consider the limit of $\mathsf{SNR} \to \infty$, and assume $|q|$ is upper bounded by some finite integer $q_0 > 0$. Then, for almost every $\mathbf{h} \in \tilde{\mathcal{M}}(\epsilon)$ there exists a constant $c_1(\mathbf{h}, q_0) > 0$, independent of the SNR, for which

$$
\max_{k \neq 1} |q\bar{h}_k - a_k| > c_1(\mathbf{h}, q_0)
\tag{144}
$$

for all $0 < |q| \leq q_0$ and $\mathbf{a} \in \mathbb{Z}^{K-1}$. Note that $\mathbf{h}$ does not satisfy (144) only if all elements of $\bar{\mathbf{h}}$ are rational. Substituting (144) into (143) gives $\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, q, \varphi) > c_2(\mathbf{h}, q_0)\mathsf{SNR}$ which means that the DoF is zero. Therefore, in order to get a positive DoF, q must tend to infinity when the SNR tends to infinity.

Any positive integer $|q|$ can be expressed as $|q| = \mathsf{SNR}^\gamma$ for some $\gamma > 0$. From Corollary 7 we know that for any $\epsilon > 0$, for almost every $\bar{\mathbf{h}} \in \bar{\mathcal{M}}(\epsilon)$ and any $\delta > 0$, for q large enough we have

$$
\max_{k \neq 1} |q\bar{h}_k - a_k| > |q|^{-\frac{1}{K-1} - \delta} = \mathsf{SNR}^{-\frac{\gamma}{K-1} - \gamma\delta}.
\tag{145}
$$

Thus, for $|q|$ large enough and almost every $\mathbf{h} \in \tilde{\mathcal{M}}(\epsilon)$ we have

$$
\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, q, \varphi) \geq \max\left(c_0^2(\mathbf{h})\mathsf{SNR}^{1 - \frac{2\gamma}{K-1} - 2\gamma\delta}, c_3(\mathbf{h})\mathsf{SNR}^{2\gamma} \right),
\tag{146}
$$

where $c_3(\mathbf{h}) > 0$ is a constant independent of the SNR.

Minimizing (146) with respect to γ gives

$$
\gamma = \frac{K - 1}{2(K + \delta K - \delta)}.
$$

Hence, for all $q \in \mathbb{Z}$, $\varphi \in [-1/2, 1/2)$, $\mathbf{a} \in \mathbb{Z}^K \setminus \mathbf{0}$ and almost every $\mathbf{h} \in \tilde{\mathcal{M}}(\epsilon)$

$$
\sigma_{\text{eff}}^2(\mathbf{h}, \mathbf{a}, q, \varphi) > c_4(\mathbf{h})\mathsf{SNR}^{\frac{K-1}{K + \delta(K-1)}},
\tag{147}
$$

where $c_4(\mathbf{h}) > 0$ is also a constant independent of the SNR. Substituting into (139) gives

$$
R_{\text{comp},1} < \frac{1 + \delta(K-1)}{K + \delta(K-1)} \cdot \frac{1}{2} \log(\mathsf{SNR}) - \frac{1}{2} \log(c_4(\mathbf{h}))
\tag{148}
$$

for any $\delta > 0$. Taking $\delta \to 0$, it follows that the number of DoF the best equation offers is upper bounded by

$$
\lim_{\mathsf{SNR} \to \infty} \frac{R_{\text{comp},1}}{\frac{1}{2} \log(1 + \mathsf{SNR})} \leq \frac{1}{K},
\tag{149}
$$

for almost every $\mathbf{h} \in \tilde{\mathcal{M}}(\epsilon)$. Since this holds for all $\epsilon > 0$ we can now take ϵ to zero (note that the bound does not depend on ϵ). The set $\mathcal{D}$ has measure zero since f_1 is analytic on $\mathbb{R}^m$ and is not identically zero (otherwise, the set of functions $1, f_1, \ldots, f_K$ is not linearly independent). Note that the measure of $\mathcal{D}(\epsilon)$ goes to zero as $\epsilon \to 0$, and furthermore $\mathcal{D} = \cap_{\epsilon > 0} \mathcal{D}(\epsilon)$. Therefore, the claim holds for almost every $\mathbf{h} \in \mathcal{M}$.

APPENDIX F
PROOF OF THEOREM 13

Consider the reference L-user MAC

$$
\mathbf{y}_{\text{ref}} = \sum_{\ell=1}^{L} g_\ell \mathbf{x}_\ell + \mathbf{z},
\tag{150}
$$

where $\mathbf{z}$ is AWGN with zero mean and unit variance and all users are subject to the power constraint $\|\mathbf{x}_\ell\|^2 \leq n\mathsf{SNR}$. Applying Corollary 5 to this channel implies that for almost every $\mathbf{g} \in \mathcal{M}$ the DoF that each optimal computation rate offers is $1/L$. Let $R_{\text{comp}}^{\text{ref}}(\mathbf{g}, \mathbf{a})$ be the computation rate corresponding to the coefficient vector $\mathbf{a}$ over the reference MAC (150). We now show the computation rate of the same coefficient vector $R_{\text{comp}}(\mathbf{g}, \mathbf{a}, \mathbf{B})$ over the effective MAC (29) is within a constant number of bits from $R_{\text{comp}}^{\text{ref}}(\mathbf{g}, \mathbf{a})$.

For the reference channel (150) the effective noise variance for a given $\mathbf{a}$ and β is

$$
\sigma_{\text{ref}}^2(\mathbf{g}, \mathbf{a}, \beta) = \mathsf{SNR}\|\beta\mathbf{g} - \mathbf{a}\|^2 + \beta^2,
$$

while for the effective L-user MAC (29) the effective variance for the same $\mathbf{a}$ and β is

$$
\sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B}) = \mathsf{SNR} \sum_{\ell=1}^{L} (\beta g_\ell - a_\ell)^2 b_{\text{eff},\ell}^2 + \beta^2.
$$

Letting $b^* = \max_{\ell=1,\ldots,L} b_{\text{eff},\ell}^2$ and noting that $b^* \geq 1$ gives

$$
\sigma_{\text{ref}}^2(\mathbf{g}, \mathbf{a}, \beta) \leq \sigma_{\text{eff}}^2(\mathbf{g}, \mathbf{a}, \beta, \mathbf{B}) \leq b^* \sigma_{\text{ref}}^2(\mathbf{g}, \mathbf{a}, \beta).
$$

Since the above inequalities are valid for any value of β, in particular they hold true for the optimal value of β and it follows that

$$R_{\text{comp}}^{\text{ref}}(\mathbf{g}, \mathbf{a}) - \frac{1}{2} \log(b^*) \leq R_{\text{comp}}(\mathbf{g}, \mathbf{a}, \mathbf{B}) \leq R_{\text{comp}}^{\text{ref}}(\mathbf{g}, \mathbf{a}).$$

As b^* is independent of the SNR it follows that the DoF of each computation rate over the reference and effective MACs (150) and (29) offer are equal, and, in particular, this is the case for the optimal computation rates, thus the theorem follows.

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