Duality methods for a class of quasilinear systems

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Abstract

Duality methods are used to generate solutions to nonlinear Hodge systems and to reveal, via the Hodge–Bäcklund transformation, underlying symmetries among a variety of models in the physics literature. *MSC2010*: 58A14, 58A15, 35J47, 35J62, 35M10.

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1 Introduction

After well over a half-century, the equations of Hodge and Kodaira remain a fruitful approach to the theory of conservative fields, which they endow with the rich topological structure of de Rahm cohomology. See, *e.g.*, Ch. 7 of [24], or [31], for introductions. A solution to the Hodge–Kodaira equations is a k-form ω which is *closed* ($d\omega = 0$) and *co-closed* ($\delta\omega = 0$) under the exterior derivative d and its formal adjoint δ.

Most of the interesting classical fields are quasilinear. The nonlinear Hodge theory conjectured by Bers and realized by Sibner and Sibner [34] introduces Hodge-like equations which model conservative velocity fields associated with steady, ideal compressible flow. Further extensions of this approach, specifically to 2-forms, model nonlinear electromagnetic fields [23], Born-Infeld fields [39], and certain twisted variants of these [26], [36]. In those extensions, the requirement of classical Hodge theory that the solution ω be co-closed under exterior differentiation is weakened to the requirement that only the product of ω and a possibly nonlinear term ρ must have this property.

Classical fields are frequently characterized by vortices. So although most conservative field theories are quasilinear, most quasilinear field theories are not strictly conservative, and it is reasonable to study the analytic properties of equations in which the requirement that the solution be closed under exterior differentiation is also weakened. Thus in a recent paper [21] we studied the invariantly defined system ([26], Sec. VI; [27], Sec. 4)

$$\delta\left(\rho(Q)\omega\right)=0, \tag{1.1}$$

$$d\omega=\Gamma\wedge\omega \tag{1.2}$$

for unknown $\omega\in\Lambda^k$, $k\in\mathbb{Z}^+$, and prescribed, continuously differentiable $\Gamma\in\Lambda^1$. Here $Q=|\omega|^2=*\left(\omega\wedge*\omega\right)$, where $*:\Lambda^k\to\Lambda^{n-k}$ is the Hodge involution; ρ is a positive, Hölder-continuously differentiable function of Q, which is generally given by the physical or geometric context. We call (1.1), (1.2) *nonlinear Hodge–Frobenius equations,* as they generalize the nonlinear Hodge equations

$$\delta\left(\rho(Q)\omega\right)=d\omega=0 \tag{1.3}$$

introduced in [34].

Equation (1.1) represents a rather generic field theory, as special cases of the mass density ρ are ubiquitous in models of classical fields (Sec. 2). Equation (1.2) represents the mildest natural weakening of the conservation hypothesis $d\omega=0$ in eq. (1.3). The resulting field is no longer conservative, but is completely integrable in the sense of Frobenius in the cases $k=1,n-1$, or for general k if Γ is exact; see, *e.g.*, [9], Sec. 4-2. The effect of replacing eqs. (1.3) by eqs. (1.1), (1.2) is exactly analogous to the effect in ordinary differential equations of replacing an exact differential equation with an equation that is exact when multiplied by an integrating factor; see the discussions in Secs. 2.1 and 2.2 of [21] and in Sec. 1 of [22].

Equations (1.1, 1.2) have an interpretation as the Euler-Lagrange equations of the nonlinear Hodge energy

$$E_{NH}=\frac{1}{2}\int_M\int_0^Q\rho(s)ds\,dM \tag{1.4}$$

where, in general, M is an n-dimensional Riemannian manifold. But in that interpretation, an extra term is introduced on the right-hand side of eq. (1.1), leading to the expression

$$\delta\left[\rho(Q)\omega\right]=(-1)^{n(k+1)}*\left(\Gamma\wedge*\rho(Q)\omega\right); \tag{1.5}$$

see [21], Sec. 5.1. Both forms of the nonlinear Hodge–Frobenius equations – the variational form (1.5), (1.2) and the original form (1.1), (1.2) – will be studied in this paper. In addition, the linear form of the equations, which results from taking ρ in (1.1) to be constant, will arise in Sec. 3.

Because of the close relation between the system (1.1), (1.2) and the nonlinear Hodge equations (1.3), it is natural to ask whether decomposition theorems exist for Hodge–Frobenius equations as they do for the nonlinear Hodge equations [35] and the equations for A-harmonic forms [15]. In Sec. 3 we derive an approach that leads to decomposition theorems in a limited sense. These decomposition theorems will be applied, not only to *a priori* arguments for the existence of solutions, but also to a variety of sufficient conditions for generating them.

It turns out that Hodge–Frobenius equations provide a natural context for generalizing the Bäcklund transformations of classical analysis. For a reference on conventional Bäcklund transformations, see, *e.g.*, [30]. The Hodge–Bäcklund transformation was defined in [21] to be a map taking a solution a of a nonlinear Hodge–Frobenius equation having mass density ρ_A into a solution b of a nonlinear Hodge–Frobenius equation having mass density ρ_B and vice-versa, where B may equal A. Many of the superficially different models for classical fields reviewed in the next section will be shown in Sec. 4 to be Hodge–Bäcklund transforms of each other.

2 A zoo of nonlinear Hodge densities

In this expository section we list some applications of the nonlinear Hodge equations (1.3). More detailed discussions of all these examples can be found in Sec. 2.7, and (especially) Chs. 5 and 6, of [29]. Initially, we restrict the degree k of the differential form ω to be 1.

If $n = 3$, and we choose

$$\rho(Q) = \left(1 - \frac{\gamma - 1}{2}Q\right)^{1/(\gamma-1)}, \ \gamma > 1, \tag{2.1}$$

then eqs. (1.3) admit an interpretation as the continuity equations for a *steady, compressible, irrotational ideal flow* with speed $|\omega|^2$ and adiabatic constant γ; see, *e.g.*, [34].

If $n = \gamma = 2$,a slight variation of this model can be used to represent the *passage to turbulence in shallow water* of depth ρ; see Sec. 10.12 of [37].

If $n = 2$ and

$$\rho(Q) = \frac{1}{\sqrt{1+Q}}, \ \omega = df, \tag{2.2}$$

then (1.3) are the equations for a *nonparametric minimal surface* in $\mathbb{R}^3$, where f is the graph of the surface; see, *e.g.*, [35].

If $n = 2$ and

$$\rho(Q) = \frac{1}{\sqrt{1-Q}}, \ \omega = df, \ Q < 1, \tag{2.3}$$

then (1.3) are the equations for a *maximal spacelike hypersurface* in Minkowski 3-space, where the surface is the graph of the function f; see, *e.g.*, [8]. If the denominator in (2.3) is replaced by $\sqrt{Q-1}$ for $Q > 1$, then the extremal surfaces are time-like.

If $n = 2$ and

$$\rho(Q) = \frac{1}{\sqrt{|1-Q|}}, \omega = df, \ Q \neq 1, \tag{2.4}$$

then (1.3) are the equations for an *elliptic-hyperbolic variational theory for extremal surfaces* in Minkowski 3-space, where f is the graph of the surface; see, *e.g.*, [12].

If $n = 2$ and

$$\rho(Q) = \left|1 - \frac{\tau^2}{Q}\right|^{1/2}, \; Q > 0, \tag{2.5}$$

then eqs. (1.3) are satisfied by the imaginary part of a complex eikonal equation associated to a wave propagating through a medium of refractive index τ [17].

If $n = 3$ and $\rho(Q)$ is a power of Q, then we obtain from eqs. (1.3) a qualitative model for *non-Newtonian pseudo-plastic flow*; see, *e.g.*, [5].

Let $n = 3$. Assume that atmospheric conductivity, represented by ρ, depends on the square Q of an atmospheric electric field. Then under appropriate boundary conditions, eqs. (1.3) correspond to the *Finkelstein–Rubinstein nonlinear conductivity model for ball lightning* [10].

Letting $n = 3$ and choosing ρ to have quadratic dependence on the magnetic field, eqs. (1.3) provide a model of ferromagnetism [23].

Of course, if ρ is constant, then eqs. (1.3) are Helmholtz's original vector formulation ("*no sources or sinks*") of what later became the Hodge–Kodaira equations for 1-forms; *c.f.* [13] and, *e.g.*, [14].

If we now take ω to be a 2-form, then if $n = 4$ and $\rho(Q)$ is taken to be the minimal surface density (2.2) or the maximal spacelike hypersurface density (2.3), then one obtains, respectively, the Euclidean or Lorentzian *Born–Infeld models for electromagnetism*:

$$\rho(Q) = \frac{1}{\sqrt{1 \pm Q}}, \; Q = |dA|^2, \tag{2.6}$$

where the 1-form A is an electromagnetic vector potential. This model was originally introduced in [7] to remove the fundamental singularity of conventional electromagnetic theory, but has attained new interest in connection with brane theories; for recent treatments see [11], [36] and [39]. (A somewhat different equation results if Q itself is defined on a pseudo-Euclidean metric; see, *e.g.*, [4].)

Regarding applications to Born–Infeld models, we recall that eqs. (1.5, 1.2) are, formally, a variant of the Yang-Mills equations with connection 1-form $-\Gamma$, in which the Yang-Mills curvature F is replaced by the k-form $\rho(Q)\omega$. Thus in particular, any Yang-Mills connection A is a solution to eqs. (1.5, 1.2) in the special case $\rho = 1$, $k = 2$, n even, $\omega = F_A$, and $\Gamma = -A$. If on the other hand, $k = 2$ and $n = 4$ in (1.5, 1.2) and the density $\rho(Q)$ is taken to be either the density in (2.2) or the density in (2.3), then we obtain, formally, a "twisted" variant of the Born–Infeld equations having the form

$$D_A^* \left(\rho(Q) F_A\right) = D_A F_A = 0, \tag{2.7}$$

where D_A is the exterior covariant derivative in the direction of A, having formal adjoint D_A^*, and $A = -\Gamma$; *c.f.* Sec. 1 of [26], Sec. 4 of [21] and Sec. 5.1 of [28]. (The equations (2.7) themselves were introduced in [25].) These variants are only meaningful geometrically if A is a Lie-algebra-valued 1-form. The corresponding Lie group is the structure group of a principal bundle for which

A is a connection 1-form and F_A is its curvature. Boundary value problems for this generalization of the Yang–Mills equations are non-standard, but can be formulated along the lines of [19, 20].

We also note that *coupled* variants of the nonlinear Hodge equations arise in Born–Infeld theory [39] and in nonlinear models of traffic flow [6].

Thus the nonlinear Hodge equations are reasonably generic: they apply, under various additional hypotheses, to a wide variety of models and it is reasonable to study their analytic properties, as we do here and in [21], without focusing on any particular application.

2.1 The effects of replacing conservative fields by completely integrable ones

If the condition $d\omega = 0$ is replaced by condition (1.2), solutions will no longer lie in a cohomology class but will generate a closed ideal.

The physical effect of replacing the condition $d\omega = 0$ in (1.3) by the Frobenius condition (1.2) is different for each of the preceding examples, and the replacement makes more sense in some cases than in others. In applications to extremal surfaces Σ, the condition $d\omega = 0$ implies that ω is locally the differential of a scalar function ζ, and we take Σ to be the graph of ζ. If we assume instead condition (1.2), then we are in effect multiplying that differential by a conformal factor $\exp[\eta]$, where η is a scalar such that $\Gamma = d\eta$; see the discussions in Sec. 2.1 of [21], Sec. 3.1 of [22], and Sec. 4.2 of [9]. Whereas integrals in the conservative-field case are independent of path, in the Hodge–Frobenius case only the integrals of $e^{-\eta}\omega$ are independent of path. One may also say that replacing the condition $d\omega = 0$ with (1.2) has the effect of replacing eq. (1.1) for ω with the following equation for du:

$$\delta\left(\tilde{\rho}(\eta, |du|)du\right) = 0\,,$$

with $\tilde{\rho}\,(\eta, |du|) \equiv e^{\eta}\,\rho\left(e^{2\eta}|du|^2\right)$; *i.e.*, it has the effect of extending ρ to a more general class of functions.

3 The existence and construction of solutions

3.1 Relation to A-harmonic forms

The Frobenius equation (1.2) emerges as a natural weakening of the conservation hypothesis $d\omega = 0$ in the field theory represented by (1.1). The linear Hodge–Frobenius equations – that is, (1.1), (1.2) with $\rho \equiv 1$ – also arise naturally from the nonlinear Hodge equations (1.3) in a completely different way, as a kind of dual, or conjugate form of the equations. The use of conjugate functions in nonlinear Hodge theory goes back at least to [34], but the conjugates in this section are not used in the same way that they are used in [34].

If $u \in \Lambda^{k-1}$ and $v \in \Lambda^{k+1}$, then the Cauchy-Riemann equations can be written in the form $du = \delta v$. More generally, we may consider *conjugate* A-

harmonic forms; see, *e.g.*, [1] for an exposition. We will call the differential forms $u \in \Lambda^{k-1}$ and $v \in \Lambda^{k+1}$ *A-harmonic* if they satisfy the equation

$$A\,(x, du) = \delta v, \tag{3.1}$$

where $A : \Omega \times \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ is a differential operator and Ω is a domain of $\mathbb{R}^n$. In the literature, A is required to satisfy a host of structural assumptions; and indeed, we will specify A to be given by (3.3) and impose further conditions as we require them. We will also place conditions on Ω. Our immediate goal is to define Hodge–Frobenius fields in terms of A-harmonic k-forms.

We say that the operator $A : \Omega \times \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ is *invertible* if there exists an operator $B : \Omega \times \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ such that

$$\begin{aligned} &B\,(x, A(x, \omega)) = \omega\,, \\ &A\,(x, B(x, \tilde{\omega})) = \tilde{\omega} \quad \forall \omega, \tilde{\omega} \in \Lambda^k(\Omega) \end{aligned} \tag{3.2}$$

Theorem 3.1. *Let $u \in \Lambda^{k-1}(\Omega)$ and $v \in \Lambda^{k+1}(\Omega)$ be sufficiently smooth, conjugate A-harmonic forms satisfying (3.1) with*

$$A\,(x, du) = A(du) = \rho(|du|^2)du\,, \tag{3.3}$$

for $\rho \in C^1(\mathbb{R}^+ \cup \{0\})$, $\rho(|du|) > 0$, on a domain Ω of $\mathbb{R}^n$, $n \geq 2$. Assume A to be invertible and Ω to be bounded, with sufficiently smooth boundary ($\partial\Omega \in C^0$ will do [24]). Then $\tilde{\omega} \equiv A(du) \in \Lambda^k(\Omega)$ is a solution to the Hodge–Frobenius equations

$$\begin{cases} \delta\tilde{\omega} = 0 \\ d\tilde{\omega} = d\tilde{\eta} \wedge \tilde{\omega} \end{cases}$$

with $\tilde{\eta} = \tilde{\eta}(|\tilde{\omega}|^2) = \log \rho(|B\,(\tilde{\omega})\,|^2)$, where $B \equiv A^{-1}$.

Proof. Having defined the operator B to be the inverse of A on Ω, (3.2), (3.3) imply

$$B\,(\delta v) = du = \frac{1}{\rho(|du|^2)} A(du) = \eta(|\delta v|^2)\delta v\,, \tag{3.4}$$

where $\eta(|\delta v|^2)$ is (well) defined by the formula $\eta(|\delta v|^2)\rho(|B\,(\delta v)\,|^2) = 1$. We conclude that $\eta(|\delta v|) > 0$, as $\rho(|du|) > 0$ by hypothesis. Having set $\tilde{\omega} \equiv A(du) = \delta v$, (3.4) becomes the identity $\eta\tilde{\omega} = du$, implying

$$0 = d^2u = d\,(\eta\tilde{\omega}) = d\eta \wedge \tilde{\omega} + \eta d\tilde{\omega}\,.$$

As $\eta(|\delta v|^2) > 0$, this is equivalent to the Frobenius equation (1.2) for $\tilde{\omega}$ in the form

$$d\tilde{\omega} = d\tilde{\eta} \wedge \tilde{\omega}\,, \tag{3.5}$$

with

$$\tilde{\eta}(|\tilde{\omega}|^2) \equiv -\log \eta(|\tilde{\omega}|^2) = \log \rho(|B\,(\delta v)\,|^2)\,.$$

Because

$$0 = \delta^2 v = \delta\tilde{\omega}\,, \tag{3.6}$$

we have recovered the Hodge–Frobenius equations (1.1, 1.2) for $\tilde{\omega}$, having the form (3.5), (3.6), with $\tilde{\eta}\left(|\tilde{\omega}|^2\right) = \log\rho\left(|B\left(\tilde{\omega}|^2\right)\right)$. This is a dual form of the nonlinear Hodge equations (1.3) for ω. □

Remark 3.1. We proved Theorem 3.1 under the hypothesis that the operator A is invertible, in which case we were able to determine the function

$$\eta = \eta(|\delta v|^2) = \frac{1}{\rho(|B\,(\delta v)\,|^2)}.$$

It is possible to weaken this hypothesis by assuming that there exists an exact form du (not necessarily unique), such that $A(du) \equiv \rho(|du|^2)du = \delta v \equiv \tilde{\omega}$ with $\rho(|du|^2) > 0$. Under that assumption,

$$d\tilde{\omega} = d\rho(|du|^2) \wedge du = \rho^{-1}(|du|^2)d\rho(|du|^2) \wedge \rho(|du|^2)du = d\,\log\rho(|du|^2) \wedge \tilde{\omega}\,,$$

and one can regard $\tilde{\eta} = \tilde{\eta}(x) = \log\rho(|du|^2(x))$ as a function of x if $du(x)$, and thus $|du|^2(x)$, is known. In fact, one can no longer write $\tilde{\eta}$ as a function of $|\delta v|^2$ because $\log\rho\left(|du|^2\right)$ would depend on the particular choice of du such that

$$\rho\left(|du|^2\right) du = \delta v\,. \tag{3.7}$$

Remark 3.2. If a solution to (3.7) exists, two such solutions, say du_1 and du_2, would satisfy

$$\rho\left(|du_1|^2\right) du_1 = \rho\left(|du_2|^2\right) du_2\,,$$

thus the condition $|du_1|^2 = |du_2|^2$ would be sufficient to guarantee $du_1 = du_2$. For this reason, taking the absolute values in both sides of (3.7) and squaring yields a function, namely the function $\Phi(t)$ in (3.10), the invertibility of which is related to the invertibility of A in the sense of Theorem (3.2).

3.2 Existence of solutions on the elliptic range

The invertibility of A gives a well-defined $\tilde{\eta}$ for the dual problem. Conversely, knowing $\tilde{\eta}$ for the dual problem yields the invertibility of A, using the idea contained in Remark 3.2. In this way we obtain a constructive method for finding explicit solutions to (1.1). In Theorem 3.2 we will apply this idea to the more general case of a k-form ω which is not necessarily closed. That more general result will allow us to construct explicit solutions to eqs. (1.1–1.3).

Theorem 3.1 assumes the existence of the conjugates $\omega = du$ and $\tilde{\omega} = \delta v$. However, if we assume alternatively that the k-form ω satisfies the nonlinear Hodge system of equations (1.3) on a contractible domain Ω, then the Poincarè Lemma implies the local existence of a $(k-1)$-form u such that $\omega = du$, as well as the local existence of a $(k+1)$-form v such that $A(du) = \rho(|du|^2)du = \delta v$. If we impose the ellipticity condition

$$0 < \rho^2(Q) + 2Q\rho'(Q)\rho(Q)\,, \tag{3.8}$$

then, given a Dirichlet or Neumann problem for a nonlinear Hodge system having density ρ on a suitable domain Ω, a solution $\omega = du$ exists by the decomposition theorems of [35], Secs. 1 and 5. These are briefly reviewed in Appendix A. See also [15] for the A-harmonic case. This yields by Remark 3.1 a solution $\tilde{\omega} = \rho(|\omega|^2)\omega = \delta v$ to the Hodge–Frobenius equations in the form (3.5, 3.6), with $\tilde{\eta} = \tilde{\eta}(x) \equiv \log \rho(|du(x)|^2)$. Homogeneous Dirichlet boundary conditions require the tangential component $(du)_T$ to vanish on $\partial\Omega$. In that case, eqs. (3.1), (3.3) imply the identities

$$\left(\rho(|du|^2)du\right)_T = (A\,(du))_T = (\delta v)_T = (\tilde{\omega})_T = 0,$$

and homogeneous Dirichlet conditions for the system (1.3) become homogeneous Dirichlet conditions for the system (3.5, 3.6). An exactly analogous argument applies to homogeneous Neumann boundary conditions, which require the normal component $(du)_N$ to vanish of $\partial\Omega$. See, *e.g.*, [35] for details. Thus the decomposition theorems for nonlinear Hodge equations in [35], which imply the existence of solutions to suitably defined boundary value problems for the system (1.3) in the elliptic regime, also imply the existence of solutions to analogous boundary value problems for the linear Hodge–Frobenius equations, on smooth, simply connected subdomains of $\mathbb{R}^n$.

3.3 "Transonic" solutions

We now want to use the operator

$$A \equiv \rho(|\omega|^2)\omega, \qquad \omega \in \Lambda^k(\Omega)$$

to prove results which are independent of type. The following assertions are applicable to the elliptic, hyperbolic and, in various cases, elliptic-hyperbolic regimes for the nonlinear Hodge or Hodge–Frobenius equations. In what follows, the k-forms ω are not assumed to be exact.

Theorem 3.2. *Let $A : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$, be defined via the formula*

$$A(\omega) \equiv \rho(|\omega|^2)\omega\,, \tag{3.9}$$

where in this case $\rho : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ is a fixed, prescribed, C^1 function. Assume that ρ is such that the function

$$\phi(t) \equiv t\rho^2(t)\,, \tag{3.10}$$

when restricted to the connected interval (t_1, t_2), satisfies

$$\frac{d\phi}{dt} > 0 \text{ or } \frac{d\phi}{dt} < 0\,. \tag{3.11}$$

Let $\Lambda^k(\Omega)_{t_1,t_2}$ denote the set of differential k-forms ω such that $t_1 \le |\omega|^2 \le t_2$, and let (r_1, r_2) be the image under ϕ of the interval (t_1, t_2). Then the operator A satisfies

$$A_{|\Lambda^k(\Omega)_{t_1,t_2}} \; : \; \Lambda^k(\Omega)_{t_1,t_2} \to \Lambda^k(\Omega)_{r_1,r_2} \,, \tag{3.12}$$

and its restriction to $\Lambda^k(\Omega)_{t_1,t_2}$ is invertible with inverse

$$B \, : \, \Lambda^k(\Omega)_{r_1,r_2} \to \Lambda^k(\Omega)_{t_1,t_2} \tag{3.13}$$

$$\tilde{\omega} \to \frac{\tilde{\omega}}{\rho(\psi(|\tilde{\omega}|^2))} \,, \tag{3.14}$$

where

$$\psi \, : \, (r_1, r_2) \to (t_1, t_2) \,, \tag{3.15}$$

is the (smooth) inverse of the function $\phi_{|(t_1,t_2)}$.

Proof. Condition (3.11) implies by monotonicity that there exists an inverse $\psi \, : \, (r_1, r_2) \to (t_1, t_2)$ of the map ϕ defined in (3.10) on the interval (t_1, t_2). Condition (3.12) is satisfied because

$$|A(\omega)|^2 \equiv |\rho(|\omega|^2)\omega|^2 = \rho^2(|\omega|^2)|\omega|^2 \equiv \phi(|\omega|^2) \,,$$

with $t_1 \le |\omega|^2 \le t_2$, and $\phi \, : \, (t_1, t_2) \to (r_1, r_2)$. Similarly, for k-forms $\tilde{\omega} \in \Lambda^k(\Omega)_{\tau_1,\tau_2}$ one has

$$|B\left(\tilde{\omega}\right)|^2 = \frac{|\tilde{\omega}|^2}{\rho^2\left(\psi\left(|\tilde{\omega}|^2\right)\right)} = \frac{|\tilde{\omega}|^2 \psi\left(|\tilde{\omega}|^2\right)}{\rho^2\left(\psi\left(|\tilde{\omega}|^2\right)\right)\psi\left(|\tilde{\omega}|^2\right)}. \tag{3.16}$$

Because ψ is the inverse of ϕ on (τ_1, τ_2), for every $\tau \in (\tau_1, \tau_2)$ there is a unique $t \equiv \psi(\tau) \in (t_1, t_2)$ such that $t\rho^2(t) = \tau$. Thus

$$\psi\left(|\tilde{\omega}|^2\right)\rho\left(\psi\left(|\tilde{\omega}|^2\right)\right) = |\tilde{\omega}|^2$$

and (3.16) becomes $|B\left(\tilde{\omega}\right)|^2 = \psi\left(|\tilde{\omega}|^2\right) \in (t_1, t_2)$. That is, eq. (3.13) is satisfied. For k-forms $\omega \in \Lambda^k(\Omega)_{t_1,t_2}$ we have

$$B\left(A(\omega)\right) = \frac{A(\omega)}{\rho(\psi(|A(\omega)|^2))} =$$

$$\frac{\rho(|\omega|^2)\omega}{\rho(\psi(\rho^2(|\omega|^2)|\omega|^2))} = \frac{\rho(|\omega|^2)\omega}{\rho(\psi(\phi(|\omega|^2)))} = \omega \,.$$

Likewise, for k-forms $\tilde{\omega} \in \Lambda^k(\Omega)_{r_1,r_2}$ one has

$$A\left(B(\tilde{\omega})\right) = \rho(|B(\tilde{\omega})|^2)B(\tilde{\omega}) = \rho\left(\frac{|\tilde{\omega}|^2}{\rho^2(\psi(|\tilde{\omega}|^2))}\right)\frac{\tilde{\omega}}{\rho(\psi(|\tilde{\omega}|^2))} = \tilde{\omega} \,,$$

as

$$|\tilde{\omega}|^2 = \phi\left(\psi(|\tilde{\omega}|^2)\right) = \rho^2(\psi(|\tilde{\omega}|^2)\psi(|\tilde{\omega}|^2) \,;$$

dividing both sides of this equation by $\rho^2(\psi(|\tilde{\omega}|^2)$, we have

$$\frac{|\tilde{\omega}|^2}{\rho^2(\psi(|\tilde{\omega}|^2))} = \psi(|\tilde{\omega}|^2)\,.$$

This concludes the proof. □

Remark 3.3. Note that $\delta(A(\omega)) = \delta\left(\rho(|\omega|^2)\omega\right)$ and that condition (3.11) is precisely the condition that makes the nonlinear Hodge system (1.1, 1.2) either elliptic (corresponding to $d\phi/dt > 0$), or hyperbolic (corresponding to $d\phi/dt < 0$).

Remark 3.4. Because Theorem 3.2 gives conditions under which the operator (3.12) can be inverted, it can be used to construct explicit k-form-valued solutions to the nonlinear Hodge–Frobenius equations which change from elliptic to hyperbolic type. Briefly, one argues by the Poincaré Lemma that a solution ω to (1.1) on a simply connected domain of $\mathbb{R}^n$ always admits a "stream $(n-k-1)$-form" f, that is, a form f satisfying

$$\rho(Q)\,\omega = *df\,. \tag{3.17}$$

On the other hand, given a (suitable) $(n-k-1)$-form f, the function ϕ can be inverted on the elliptic and hyperbolic regions individually in order to obtain solutions ω to (1.1) in terms of f, ρ, and ψ. Special care must be taken to assure the continuity of ω across the *sonic curve* dividing the elliptic from the hyperbolic regime. One can also write the formula (3.17), for singular stream forms f (that is, allowing more general domains, not necessarily simply connected), which implies (1.1) except at the singularities of f. More precisely, eqs. (3.10), (3.17) imply

$$\phi(Q) = |df|^2\,,$$

and the inversion of the operator (3.12) guaranteed by Theorem 3.2 allows us to construct solutions ω to eq. (1.1) by means of the explicit formula

$$\omega = \frac{*df}{\rho\left(\psi\left(|df|^2\right)\right)}\,. \tag{3.18}$$

These solutions are defined on $\Omega_f \equiv \{(x,y) \in \Omega \;:\; |\nabla f|^2\,(x,y) \in Im\,\phi\}$, except possibly on the *sonic curve* and at the singularities of f. Satisfaction of (1.2) for some one-form Γ can easily be shown, and is equivalent to the existence of an integrating factor in the cases $k=1$, or $k=n-1$. Details and examples for the special case of vectorial ω, corresponding to $k=1$, are given in [22]. Note that the hypothesis in Theorem 3.2 that ρ be a strictly positive function of the independent variable t is not necessary when applying the Theorem to construct explicit solutions as outlined in the present remark. In fact, for the validity of eq. (3.18) it is sufficient that $\rho\left(\psi\left(|df|^2\right)\right)$ be non-zero.

4 A zoo of Hodge-Bäcklund transformations

One finds in the literature a bewildering redundancy of choices for the mass density ρ. Whenever one choice is selected, others seem to appear as well. See, for example, Sec. 1 of [17], Sec. 2 of [18], and the pairs of densities that arise in Sec. 2 of this paper in connection with the Born–Infeld and extremal surface equations. It is natural to wonder whether there is a mathematical operation underlying the varieties of density. In this section we construct a mechanism for relating many of the densities in Sec. 2. That mechanism extends Theorem 6.1 of [21], which related two *particular* densities by an application of the Hodge–Bäcklund transformation. (See also the special cases studied in [39], [36], [3], [2], [8], and [16].) We find in this section that this class of transformations is of rather general applicability to systems of the form (1.2), (1.5).

Theorem 4.1. *Denote by Σ a given, continuously differentiable 1-form. The k-form ω satisfies the nonlinear Hodge–Frobenius system*

$$\begin{aligned} d * (\rho(Q)\omega) &= \Sigma \wedge * (\rho(Q)\omega) \\ d\omega &= \Gamma \wedge \omega \,, \end{aligned} \tag{4.1}$$

where $Q \equiv |\omega|^2$ and $\rho(Q)$ is an assigned density, if and only if the $(n-k)$-form $\xi \equiv * (\rho(Q)\omega)$ satisfies the nonlinear Hodge–Frobenius system

$$\begin{aligned} d * (\hat\rho(R)\xi) &= \Gamma \wedge * (\hat\rho(R)\xi) \\ d\xi &= \Sigma \wedge \xi \,, \end{aligned} \tag{4.2}$$

where $R \equiv |\xi|^2$ and $\hat\rho$, which we refer to as a density `dual` *to ρ, satisfies the identity*

$$\rho(Q)\hat\rho(R) \equiv 1 \,. \tag{4.3}$$

Proof. Multiplying ξ by $\hat\rho(R)$ and applying the Hodge operator $*$ to both sides of the definition $\xi \equiv * (\rho(Q)\omega)$, and using (4.3), we obtain

$$(*_k)^2\omega = *\hat\rho(R)\xi \,, \tag{4.4}$$

where $(*_k)^2$ is the square of the Hodge operator on k-forms; thus $(*_k)^2$ is either 1 or -1. Multiplying both sides by $(*_k)^2$ and using $(*_k)^4 = 1$, we also obtain

$$\omega = (*_k)^2 * \hat\rho(R)\xi \,. \tag{4.5}$$

By (4.4) and the second equation in (4.1),

$$d(*\hat\rho(R)\xi) = d\left((*_k)^2\omega\right) = (*_k)^2 d\omega = (*_k)^4\Gamma \wedge *(\hat\rho(R)\xi) = \Gamma \wedge *(\hat\rho(R)\xi) \,, \tag{4.6}$$

which is the first equation in the system (4.2) and the second equation in (4.1) with a change in notation. The second equation in (4.2) is precisely the first equation in the system (4.1) with a change in notation. □

Corollary 4.2. *The k-form ω satisfies the nonlinear Hodge–Frobenius system* (4.1) *with*

$$\rho(Q) = \left(1 - \frac{\gamma - 1}{2} Q\right)^{1/(\gamma-1)} \tag{4.7}$$

if and only if the $(n-k)$-form $\xi \equiv *(\rho(Q)\omega)$ satisfies the nonlinear Hodge–Frobenius system (4.2) *with the density $\hat{\rho}$ dual to ρ satisfying*

$$\frac{\gamma - 1}{2} R\hat{\rho}(R)^{\gamma+1} - \hat{\rho}(R)^{\gamma-1} + 1 \equiv 0\,. \tag{4.8}$$

Proof: Because the Hodge operator $*$ is an isometry, one obtains

$$R \equiv |\xi|^2 = \rho^2(Q)|\omega|^2 \equiv Q\rho^2(Q). \tag{4.9}$$

Likewise,

$$Q = R\hat{\rho}^2(R), \tag{4.10}$$

having used $\hat{\rho}(R)\rho(Q) \equiv 1$. Thus, applying Theorem 4.1 to the class of densities (4.7), we obtain

$$1 \equiv \hat{\rho}(R)\rho\left(R\hat{\rho}^2(R)\right) = \hat{\rho}(R)(1 - \frac{\gamma-1}{2}R\hat{\rho}^2(R)\,)^{\frac{1}{\gamma-1}}.$$

Taking the $\gamma - 1$ power of both sides, we obtain

$$1 = \hat{\rho}(R)^{\gamma-1}(1 - \frac{\gamma-1}{2}R\hat{\rho}^2(R)).$$

This yields the equation (4.8). □

Remark 4.1. The case of the Chaplygin flow density, defined for 1-forms ω, and the case of the Euclidean Born–Infeld model, defined for 2-forms ω, have in common that $\gamma = -1$ in (4.7). In these cases Corollary 4.2 yields

$$-R\hat{\rho}^0 - \hat{\rho}(R)^{-2} + 1 = -R - \hat{\rho}(R)^{-2} + 1 \equiv 0. \tag{4.11}$$

Thus

$$\hat{\rho} = \frac{1}{\sqrt{1-R}},$$

which is the density for the Lorentzian Born–Infeld model.

Remark 4.2. Comparing eqs. (4.1) and (4.2), we observe that Σ and Γ are interchanged. So, for example, if ω satisfies the nonlinear Hodge–Frobenius system (4.1) with $\Sigma \equiv 0$, then ξ satisfies a conventional nonlinear Hodge system (*i.e.*, $d\xi = 0$).

Remark 4.3. In some cases there is more than one mass density dual to ρ. This is the case for $\gamma = 3$. Indeed, if $\gamma = 3$, then by solving (4.8) in the form

$$R\hat{\rho}^4(R) - \hat{\rho}^2(R) + 1 = 0,$$

we obtain the following dual densities – of which only the first two are positive – defined for $R \in (0, \frac{1}{4}]$:

$$\hat{\rho} = \sqrt{\frac{1-\delta}{2R}}\,; \tag{4.12}$$

$$\hat{\rho} = \sqrt{\frac{1+\delta}{2R}}\,; \tag{4.13}$$

$$\hat{\rho} = -\sqrt{\frac{1-\delta}{2R}}\,; \tag{4.14}$$

$$\hat{\rho} = -\sqrt{\frac{1+\delta}{2R}}\,, \tag{4.15}$$

$$\tag{4.16}$$

where $\delta \equiv \sqrt{1-4R}$.

An initial density which will include perhaps all the known applications could have the form

$$\rho(x, Q) = \left(1 - at^2(x)Q^b\right)^{1/2a}\,, \tag{4.17}$$

where x is a vector; a and b are constants (which may be negative); $t(x)$ is a given, bounded function; and the order of the form is either $k = 1$ or $k = 2$. Then the following holds:

Corollary 4.3. *The k-form ω satisfies the nonlinear Hodge–Frobenius system* (4.1) *with*

$$\rho(x, Q) = \left(1 - at^2(x)Q^b\right)^{1/2a} \tag{4.18}$$

if and only if the $(n-k)$-form $\xi \equiv *(\rho(Q)\omega)$ satisfies the nonlinear Hodge–Frobenius system (4.2) *with dual density $\hat{\rho}$ satisfying*

$$at^2(x)R^b\hat{\rho}(R)^{2a+2b} - \hat{\rho}(R)^{2a} + 1 = 0. \tag{4.19}$$

Proof: The argument is identical to the proof of Corollary 4.2. Notice that here we are allowing the initial density ρ to depend explicitly on x. Nonetheless, the explicit dependence of ρ on x does not affect any steps in the proof of Theorem 4.1, nor of the straightforward calculation needed to obtain (4.19). □

Note that in particular, we recover by this method the transformation between the two densities (2.5) derived in [17], by taking $a = 1 = -b$ in eqs. (4.17) and (4.19).

A Decomposition theorems for nonlinear Hodge fields and related results

The material on nonlinear Hodge theory is taken from [35]. For details of the underlying linear theory see, *e.g.*, Ch. 7 of [24] and the references therein.

Let M be an oriented, finite Riemannian manifold with Lipschitz boundary. Let $n = \dim\,(M)$. For a k-form ω, denote by $T\omega$ its tangential component and by $N\omega$ its normal component, where we assume that $T\omega$ and $N\omega$ are differential forms defined on ∂M.

Denoting by $\mathcal{L}_2$ the L^2-completion of the space of smooth p-forms on M, we have

$$\mathcal{L}_2 = \mathcal{E}_T \oplus \mathcal{E}_N^* \oplus \mathcal{H} = \mathcal{E} \oplus \mathcal{E}_N^* \oplus \mathcal{H}_N,$$

for

$$\mathcal{E} = \{d\nu\}\,; \;\; \mathcal{E}^* = \{\delta\nu\}\,;$$

$$\mathcal{H} = \{\sigma | d\sigma = \delta\sigma = 0, N\sigma = 0\}\,;$$

$$\mathcal{E}_T = \{d\nu | T\nu = 0\}\,;$$

$$\mathcal{E}_N^* = \{\delta\tau | N\tau = 0\}\,.$$

Standard convexity arguments imply, by the lower semicontinuity of the nonlinear Hodge energy functional under suitable hypotheses on ρ, that for any closed subspace V for which $\mathcal{L}_2 = V \oplus V^\perp$, and any given k-forms α and β on T^*M, there exists a unique k-form τ such that $\tau - \alpha \in V$ and $\rho\,(x, Q(\tau))\,\tau - \beta \in V^\perp$, where ρ is defined as in Sec. 1 and $Q = *\,(\tau \wedge *\tau)$. See, for example, Lemma 4.2 of [34] for details of this argument.

If $H'(Q)$ is bounded below away from zero for all Q, where H is given by [21], [38]

$$H(Q) = \int_0^Q \left[\frac{1}{2}\rho(s) + s\rho'(s)\right] ds,$$

then ρ is said to be *regular*; *c.f.* (3.8). If we assume only that $H'(Q)$ is positive for all $Q \in [0, Q_{crit})$, where Q_{crit} is some critical value, then ρ is said to be *admissible.* Let

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

where $\mathcal{D}_1 = \text{Ker } d$ and $\mathcal{D}_2 = C^{1+\alpha}(\overline{M})$. If $n < 3$, let

$$\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2,$$

where $N_1 = \text{Ker } d$ and $\mathcal{N}_2 = \text{Ker } \delta$. If $n = 3$ assume also that the boundary of M is of class C^2. If $n > 3$ assume that the boundary data are homogeneous.

Theorem A.1 (L. M. Sibner and R. J. Sibner [35], Sec. 5)**.** *Assume that ρ is regular. Let $(\gamma, \sigma) \in \mathcal{D}$. Then there is a unique $\omega \in C^{1+\alpha}(\overline{M})$ which satisfies $d\omega = 0$, $\delta\rho\omega = \delta\sigma$, and $\omega - \gamma \in \mathcal{E}_T$. The solution depends continuously on γ and σ. Now let $(\gamma, \kappa) \in \mathcal{N}$. Then there is a unique $\omega \in C^{1+\alpha}(\overline{M})$ which satisfies $d\omega = 0$, $\delta\rho\omega = 0$, $\omega - \gamma \in \mathcal{E}$, and $N\rho\omega = N\kappa$. The solution depends continuously on γ and κ.*

Proof. The idea of the proof is as follows. We first to show that weak solutions exist $\forall\; k \in \mathbb{Z}^+$. The convexity arguments mentioned above will yield, for $V = \mathcal{E}_T$, $\omega - \gamma \in \mathcal{E}_T$ and $\rho\omega - \sigma \in \mathcal{E}_T^\perp$, a weak form the first assertion of the theorem, as γ is d-closed. A weak form of the second assertion of the theorem is obtained in the same way, but for a conjugate function ν. In this case we take $V = \mathcal{E}^{*\perp}$ and obtain $\omega - \gamma \in \mathcal{E}$ and $\rho\omega - \kappa \in \mathcal{E}^{\perp}$. Now De Giorgi–Nash–Moser methods can be applied to show that if $k = 1$, the weak solutions obtained by the lower-semicontinuity of the energy functional are in fact differentiable. This completes our brief outline of the proof. □

Corollary A.2 (L. M. Sibner and R. J. Sibner [35], Sec. 5)**.** *Theorem A.1 extends to the case of admissible* ρ.

Proof. The corollary follows immediately from the technique known as *Shiffman regularization*, which was introduced in [32] and for which an elementary reformulation is given in R. J. Sibner's appendix to [33]. □

References

[1] R. P. Agarwal, S. Ding, Advances in differential forms and the A-harmonic equations, Math. Comput. Modelling 37 (2003) 1393-1426.

[2] A. L. Albujer, L. J. Alías, Calabi–Bernstein results for maximal surfaces in Lorentzian product spaces, J. Geom. Physics 59 (2009) 620-631.

[3] L. J. Alías, B. Palmer, A duality result between the minimal surface equation and the maximal surface equation, An. Acad. Brasil. Ciênc 73 (2001) 161-164.

[4] B. M. Barbashov, N. A. Chernikov, Scattering of two plane electromagnetic waves in the nonlinear Born-Infeld electrodynamics, Commun. Math. Phys. 3 (1966) 313-322.

[5] H. Beirão de Veiga, On non-Newtonian p-fluids. The pseudo-plastic case, J. Math. Anal. Appl. 344 (2008) 175-185.

[6] J. H. Bick, G. F. Newell, A continuum model for two-dimensional traffic flow, Quart. Appl. Math. 18 (1960) 191-204.

[7] M. Born, L. Infeld, Foundation of a new field theory, Proc. R. Soc. London Ser. A 144 (1934) 425-451.

[8] E. Calabi, Examples of Bernstein problems for some nonlinear equations, Proceedings of the Symposium on Global Analysis, University of California at Berkeley, 1968, pp. 223-230.

[9] D. G. B. Edelen, Applied Exterior Calculus, Wiley, New York, 1985.

[10] D. Finkelstein, J. Rubinstein, Ball lightning, Phys. Rev. 135 (1964) A390-A396.

[11] G. W. Gibbons, Born-Infeld particles and Dirichlet p-branes, Nucl. Phys. B 514 (1998) 603-639.

[12] C. Gu, Complete extremal surfaces of mixed type in 3-dimensional Minkowski space, Chin. Ann. of Math. 15B (1994) 385-400.

[13] H. Helmholtz, Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, J. Reine Angew. Math. 55 (1858) 25-55.

[14] W. V. D. Hodge, A Dirichlet problem for harmonic functionals with applications to analytic varieties, Proc. London Math. Soc. 36 (1934) 257-303.

[15] T. Iwaniec, C. Scott, B. Stroffolini, Nonlinear Hodge theory on manifolds with boundary, Annali Mat. Pura Appl. 177 (1999) 37-115.

[16] H. Lee, Extensions of the duality between minimal surfaces and maximal surfaces, Geom. Dedicata 151 (2011) 373-386.

[17] R. Magnanini, G. Talenti, On complex-valued solutions to a 2D eikonal equation. Part one: qualitative properties, Contemporary Math. 283 (1999) 203-229.

[18] R. Magnanini, G. Talenti, Approaching a partial differential equation of mixed elliptic-hyperbolic type, in Ill-posed and Inverse Problems, Yu. E. Anikonov, A. L. Bukhageim, S. I. Kabanikhin and V. G. Romanov, eds., VSP 2002, pp. 263-276.

[19] A. Marini, Dirichlet and Neumann Boundary Value Problems for Yang-Mills Connections, Commun. Pure Appl. Math. 45 (1992) 1015-1050.

[20] A. Marini, Regularity theory for the generalized Neumann problem. Nontrivial examples in dimensions 3 and 4, Mathemathische Annalen 317 (2000) 173-193.

[21] A. Marini, T. H. Otway, Nonlinear Hodge–Frobenius equations and the Hodge–Bäcklund transformation, Proc. R. Soc. Edinburgh 140A (2010) 787-819.

[22] A. Marini, T. H. Otway, Constructing completely integrable fields by the method of generalized streamlines, preprint.

[23] A. Milani, R. Picard, Decomposition and their application to non-linear electro- and magnetostatic boundary value problems, in: S. Hildebrandt and R. Leis, eds., Partial Differential Equations and Calculus of Variations, Lecture Notes in Mathematics, vol. 1357, Springer, Berlin, 1988.

[24] C. B. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, Berlin, 1966.

[25] T. H. Otway, Yang-Mills fields with nonquadratic energy, J. Geom. Phys. 19 (1996) 379-398.

[26] T. H. Otway, Nonlinear Hodge maps, J. Math. Phys. 41 (2000) 5745-5766.

[27] T. H. Otway, Maps and fields with compressible density, Rend. Sem. Mat. Univ. Padova 111 (2004) 133-159.

[28] T. H. Otway, Variational equations on mixed Riemannian-Lorentzian metrics, J. Geom. Phys. 58 (2008) 1043-1061.

[29] T. H. Otway, The Dirichlet Problem for Elliptic-Hyperbolic Equations of Keldysh Type, Lecture Notes in Mathematics, Vol. 2043, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 2012.

[30] C. Rogers, W. K. Schief, Bäcklund and Darboux Transformations, Geometry and Modern Applications of Soliton Theory, Cambridge University Press, Cambridge, 2002.

[31] G. Schwarz, Hodge decomposition: a method for solving boundary value problems, Springer, Berlin, 1995.

[32] M. Shiffman, On the existence of subsonic flows of a compressible fluid, J. Rat. Mech. Anal. 1 (1952) 605-652.

[33] L. M. Sibner, An existence theorem for a non-regular variational problem, Manuscripta Math. 43 (1983) 45-72.

[34] L. M. Sibner, R. J. Sibner, A nonlinear Hodge-de Rham theorem, Acta Math. 125 (1970) 57-73.

[35] L. M. Sibner, R. J. Sibner, Nonlinear Hodge theory: Applications, Advances in Math. 31 (1979) 1-15.

[36] L. M. Sibner, R. J. Sibner, Y. Yang, Generalized Bernstein property and gravitational strings in Born-Infeld theory, Nonlinearity 20 (2007) 1193-1213.

[37] J. J. Stoker, Water Waves, Interscience, New York, 1987.

[38] K. Uhlenbeck, Regularity for a class of nonlinear elliptic systems, Acta Math. 138 (1977) 219-240.

[39] Y. Yang, Classical solutions in the Born-Infeld theory, Proc. R. Soc. Lond. Ser. A 456 (2000) 615-640.