A Diophantine problem with a prime and three squares of primes

Alessandro Languasco and Alessandro Zaccagnini

Abstract

We prove that if λ_1, λ_2, λ_3 and λ_4 are non-zero real numbers, not all of the same sign, λ_1/λ_2 is irrational, and ϖ is any real number then, for any $\varepsilon > 0$ the inequality $\bigl|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \varpi\bigr| \le \bigl(\max_j p_j\bigr)^{-1/18+\varepsilon}$ has infinitely many solution in prime variables $p_1, \dots, p_4$.

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1 Introduction

This paper deals with an improvement of the recent result of Li and Wang [4] concerning Diophantine approximation by means of a prime and three squares of primes. We prove the following Theorem.

Theorem 1 *Assume that λ_1, λ_2, λ_3 and λ_4 are non-zero real numbers, not all of the same sign and that λ_1/λ_2 is irrational. Let ϖ be any real number. For any $\varepsilon > 0$ the inequality*

$$\bigl|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \varpi\bigr| \le \bigl(\max_j p_j\bigr)^{-1/18+\varepsilon} \tag{1}$$

has infinitely many solution in prime variables $p_1, \dots, p_4$.

Li and Wang [4] had 1/28 in place of 1/18. Our improvement of their result derives from a more efficient use of Ghosh's bound for exponential sums over squares of primes in [1] to bound the contribution of the so-called "intermediate arc." This enables us to use a wider "major arc" and yields a stronger result. The exponent 1/18 arises from there. We also avoid estimating exponential integrals too early, and we evaluate them as far as possible, in order to prevent crucial losses of precision. We point out that we can not follow the argument leading to the upper bound for the error term in formula (3) of [4]: it does not seem to follow from a suitable form of the explicit formula by a simple partial integration. See also the proof of Lemma 5 of Vaughan [10] or Lemma 7 of [11].

We may change the hypothesis in Theorem 1 to the assumption that λ_2/λ_3 is irrational, say, and the result is the same, with minor changes in detail. Furthermore, since the role of λ_2, λ_3 and λ_4 in our statement above is symmetrical, the assumption that λ_1/λ_2 is irrational is not restrictive.

The same kind of argument for the intermediate arc can be used to improve the result in Languasco and Zaccagnini [3]. For brevity, we simply state the final result, with a very short sketch of the proof, at the end of this paper.

2 Outline of the proof

We use the variant of the circle method introduced by Davenport and Heilbronn to deal with Diophantine problems. In order to prove that (1) has infinitely many solutions, it is sufficient to construct an increasing sequence X_n with limit $+\infty$ such that (1) has at least a solution with $\max_j p_j \in [\delta X_n, X_n]$, where δ is a small, fixed positive constant that depends on the coefficients λ_j. This sequence actually depends on rational approximations for λ_1/λ_2: more precisely, there are infinitely many pairs of integers a and q such that $(a,q)=1$, $q>0$ and

$$\Big|\frac{\lambda_1}{\lambda_2}-\frac{a}{q}\Big| \le \frac{1}{q^2}.$$

We take the sequence $X = q^{9/5}$ (dropping the useless suffix n) and then, as customary, define all of the circle-method parameters in terms of X. We may obviously assume that q is sufficiently large. The choice of the exponent 9/5 is justified in the discussion following the proof of Lemma 3. Let

$$S_1(\alpha) = \sum_{\delta X \le p \le X} \log p \; e(p\alpha) \qquad \text{and} \qquad S_2(\alpha) = \sum_{\delta X \le p^2 \le X} \log p \; e(p^2\alpha),$$

where $e(\alpha) = e^{2\pi i\alpha}$. As usual, we approximate to S_1 and S_2 using the functions

$$T_1(\alpha) = \int_{\delta X}^{X} e(t\alpha)\,\mathrm{d}t \qquad \text{and} \qquad T_2(\alpha) = \int_{(\delta X)^{1/2}}^{X^{1/2}} e(t^2\alpha)\,\mathrm{d}t$$

and notice the simple inequalities

$$T_1(\alpha) \ll_\delta \min\bigl(X, |\alpha|^{-1}\bigr) \quad \text{and} \quad T_2(\alpha) \ll_\delta X^{-1/2} \min\bigl(X, |\alpha|^{-1}\bigr). \tag{2}$$

We detect solutions of (1) by means of the function

$$\widehat{K}_\eta(\alpha) = \max(0, \eta - |\alpha|)$$

for $\eta > 0$, which, as the notation suggests, is the Fourier transform of

$$K_\eta(\alpha) = \Bigl(\frac{\sin(\pi\eta\alpha)}{\pi\alpha}\Bigr)^2$$

for $\alpha \neq 0$, and, by continuity, $K_\eta(0) = \eta^2$. This relation transforms the problem of counting solutions of the inequality (1) into estimating suitable integrals. We recall the trivial property

$$K_\eta(\alpha) \ll \min\Bigl(\eta^2, |\alpha|^{-2}\Bigr). \tag{3}$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$ let

$$I(\eta, \varpi, \mathfrak{X}) = \int_{\mathfrak{X}} S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_2(\lambda_3\alpha) S_2(\lambda_4\alpha) K_\eta(\alpha) e(\varpi\alpha)\,\mathrm{d}\alpha.$$

In practice, we take as $\mathfrak{X}$ either an interval or a half line, or the union of two such sets. The starting point of the method is the observation that

$$\begin{aligned} I(\eta,\varpi,\mathbb{R}) &= \sum_{\substack{\delta X\le p_1\le X\\ \delta X\le p_j^2\le X}} \log p_1 \log p_2 \log p_3 \log p_4 \\ &\qquad\times \int_{\mathbb{R}} K_\eta(\alpha) e\bigl((\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \varpi)\alpha\bigr)\,\mathrm{d}\alpha \\ &= \sum_{\substack{\delta X\le p_1\le X\\ \delta X\le p_j^2\le X}} \log p_1 \log p_2 \log p_3 \log p_4 \\ &\qquad\times \max(0, \eta - |\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \varpi|) \\ &\le \eta(\log X)^4 \mathcal{N}(X), \end{aligned}$$

where $\mathcal{N}(X)$ denotes the number of solutions of the inequality (1) with $p_1 \in [\delta X, X]$ and $p_j^2 \in [\delta X, X]$ for $j = 2$, 3 and 4. We now give the definitions that we need to set up the method. More definitions will be given at appropriate places later. We let $P = P(X) = X^{2/5}/\log X$, $\eta = \eta(X) = X^{-1/18+\varepsilon}(\log X)^2$, and $R = R(X) = \eta^{-2}(\log X)^2$. The choice for P is justified at the end of §3.3, the one for η at the end of §4 and the one for R at the end of §5. We now decompose $\mathbb{R}$ as $\mathfrak{M}\cup\mathfrak{m}\cup\mathfrak{t}$ where

$$\mathfrak{M} = \Bigl[-\frac{P}{X}, \frac{P}{X}\Bigr], \qquad \mathfrak{m} = \Bigl(-R, -\frac{P}{X}\Bigr) \cup \Bigl(\frac{P}{X}, R\Bigr), \qquad \mathfrak{t} = \mathbb{R}\setminus(\mathfrak{M}\cup\mathfrak{m}),$$

so that

$$I(\eta,\varpi,\mathbb{R}) = I(\eta,\varpi,\mathfrak{M}) + I(\eta,\varpi,\mathfrak{m}) + I(\eta,\varpi,\mathfrak{t}).$$

These sets are called the major arc, the intermediate (or minor) arc and the trivial arc respectively. In §3 we prove that the major arc yields the main term for $I(\eta,\varpi,\mathbb{R})$. In order to show that the contribution of the intermediate arc does not cancel the main term, we exploit the hypothesis that λ_1/λ_2 is irrational to prove that $|S_1(\lambda_1\alpha)|$ and $|S_2(\lambda_2\alpha)|^2$ can not both be large for $\alpha\in\mathfrak{m}$: see §4, and in particular Lemma 3, for the details. The trivial arc, treated in §5, only gives a rather small contribution.

In the following sections, implicit constants may depend on the coefficients λ_j, on δ and on ϖ.

3 The major arc

We write

$$\begin{aligned} I(\eta,\varpi,\mathfrak{M}) &= \int_{\mathfrak{M}} S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_2(\lambda_3\alpha) S_2(\lambda_4\alpha) K_\eta(\alpha) e(\varpi\alpha)\,\mathrm{d}\alpha \\ &= \int_{\mathfrak{M}} T_1(\lambda_1\alpha) T_2(\lambda_2\alpha) T_2(\lambda_3\alpha) T_2(\lambda_4\alpha) K_\eta(\alpha) e(\varpi\alpha)\,\mathrm{d}\alpha \\ &\quad+ \int_{\mathfrak{M}} \bigl(S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha)\bigr) T_2(\lambda_2\alpha) T_2(\lambda_3\alpha) T_2(\lambda_4\alpha) K_\eta(\alpha) e(\varpi\alpha)\,\mathrm{d}\alpha \\ &\quad+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha) \bigl(S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha)\bigr) T_2(\lambda_3\alpha) T_2(\lambda_4\alpha) K_\eta(\alpha) e(\varpi\alpha)\,\mathrm{d}\alpha \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)\bigl(S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)\bigr)T_2(\lambda_4\alpha)K_\eta(\alpha)e(\varpi\alpha)\,\mathrm{d}\alpha \\
&+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)\bigl(S_2(\lambda_4\alpha) - T_2(\lambda_4\alpha)\bigr)K_\eta(\alpha)e(\varpi\alpha)\,\mathrm{d}\alpha \\
= {}& J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned}
$$

say. We will give a lower bound for J_1 and upper bounds for $J_2, \dots, J_5$. For brevity, since the computations for J_3 and J_4 are similar to, but simpler than, the corresponding ones for J_2 and J_5, we will skip them.

3.1 Lower bound for J_1

Apart from very small changes, the lower bound $J_1 \gg \eta^2 X^{3/2}$ is contained in Lemma 8 of Li and Wang [4]. Here we give the required result only in one case, the other ones being similar. We have

$$
\begin{aligned}
J_1 &= \int_{\mathfrak{M}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_2(\lambda_4\alpha)K_\eta(\alpha)e(\varpi\alpha)\,\mathrm{d}\alpha \\
&= \int_{\mathbb{R}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_2(\lambda_4\alpha)K_\eta(\alpha)e(\varpi\alpha)\,\mathrm{d}\alpha \\
&\quad + \mathcal{O}\biggl(\int_{P/X}^{+\infty} |T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_2(\lambda_4\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha\biggr).
\end{aligned}
$$

Using inequalities (2) and (3), we see that the error term is

$$
\ll \eta^2 X^{-3/2} \int_{P/X}^{+\infty} \frac{\mathrm{d}\alpha}{\alpha^4} \ll \eta^2 X^{3/2} P^{-3} = o\bigl(\eta^2 X^{3/2}\bigr).
$$

For brevity, we set $\mathfrak{D} = [\delta X, X] \times [(\delta X)^{1/2}, X^{1/2}]^3$. We can rewrite the main term in the form

$$
\begin{aligned}
&\int \cdots \int_{\mathfrak{D}} \int_{\mathbb{R}} e\bigl((\lambda_1 t_1 + \lambda_2 t_2^2 + \lambda_3 t_3^2 + \lambda_4 t_4^2 + \varpi)\alpha\bigr)\, K_\eta(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 \\
&= \int \cdots \int_{\mathfrak{D}} \max(0, \eta - |\lambda_1 t_1 + \lambda_2 t_2^2 + \lambda_3 t_3^2 + \lambda_4 t_4^2 + \varpi|)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4.
\end{aligned}
$$

We now proceed to show that the last integral is $\gg \eta^2 X^{3/2}$. Apart from trivial changes of sign, there are essentially three cases:

1. $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, $\lambda_4 < 0$.
2. $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\lambda_4 < 0$.
3. $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 < 0$.

We briefly deal with the second case. A suitable change of variables shows that

$$
J_1 \gg \int \cdots \int_{\mathfrak{D}'} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|)\,\frac{\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\,\mathrm{d}u_4}{(u_2 u_3 u_4)^{1/2}}
$$

$$\gg X^{-3/2} \iiiint_{\mathfrak{D}'} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|) \, \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \mathrm{d}u_4,$$

where $\mathfrak{D}' = [\delta X, (1-\delta)X]^4$, for large X. For $j = 1$, 2 and 3 let $a_j = 4|\lambda_4|\delta/|\lambda_j|$, $b_j = 3a_j/2$ and $\mathfrak{I}_j = [a_j X, b_j X]$. Notice that if $u_j \in \mathfrak{I}_j$ for $j = 1$, 2 and 3 then

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \bigl[2|\lambda_4|\delta X, 8|\lambda_4|\delta X\bigr]$$

so that, for every such choice of (u_1, u_2, u_3), the interval $[a, b]$ with endpoints $\pm\eta/|\lambda_4| + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)/|\lambda_4|$ is contained in $[\delta X, (1-\delta)X]$. In other words, for $u_4 \in [a, b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ cover the whole interval $[-\eta, \eta]$. Hence, for any $(u_1, u_2, u_3) \in \mathfrak{I}_1 \times \mathfrak{I}_2 \times \mathfrak{I}_3$ we have

$$\begin{aligned}\int_{\delta X}^{(1-\delta)X} & \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|) \, \mathrm{d}u_4 \\ &= |\lambda_4|^{-1} \int_{-\eta}^{\eta} \max(0, \eta - |u|) \, \mathrm{d}u \gg \eta^2.\end{aligned}$$

Finally,

$$J_1 \gg \eta^2 X^{-3/2} \iiint_{\mathfrak{I}_1 \times \mathfrak{I}_2 \times \mathfrak{I}_3} \mathrm{d}u_1 \, \mathrm{d}u_2 \, \mathrm{d}u_3 \gg \eta^2 X^{3/2},$$

which is the required lower bound.

3.2 Bound for J_2

Let

$$U_1(\alpha) = \sum_{\delta X \le n \le X} e(n\alpha) \qquad \text{and} \qquad U_2(\alpha) = \sum_{\delta X \le n^2 \le X} e(n^2\alpha).$$

By the Euler summation formula we have

$$T_j(\alpha) - U_j(\alpha) \ll 1 + |\alpha| X \qquad \text{for } j = 1,\, 2. \tag{4}$$

Using (3) we see that

$$\begin{aligned} J_2 &\ll \eta^2 \int_{\mathfrak{m}} \bigl|S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha)\bigr| \, |T_2(\lambda_2\alpha)| \, |T_2(\lambda_3\alpha)| \, |T_2(\lambda_4\alpha)| \, \mathrm{d}\alpha \\ &\le \eta^2 \int_{\mathfrak{m}} \bigl|S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)\bigr| \, |T_2(\lambda_2\alpha)| \, |T_2(\lambda_3\alpha)| \, |T_2(\lambda_4\alpha)| \, \mathrm{d}\alpha \\ &\quad + \eta^2 \int_{\mathfrak{m}} \bigl|U_1(\lambda_1\alpha) - T_1(\lambda_1\alpha)\bigr| \, |T_2(\lambda_2\alpha)| \, |T_2(\lambda_3\alpha)| \, |T_2(\lambda_4\alpha)| \, \mathrm{d}\alpha \\ &= \eta^2 (A_2 + B_2), \end{aligned}$$

say. In order to estimate A_2 we connect it to the Selberg integral as in Lemma 6 of Languasco and Zaccagnini [3]. We set

$$J(X, h) = \int_{\delta X}^{X} \bigl(\theta(x + h) - \theta(x) - h\bigr)^2 \, \mathrm{d}x,$$

where θ is the usual Chebyshev function. By the Cauchy inequality and (2) above, for any fixed $A > 0$ we have

$$
\begin{aligned}
A_2 &\ll \Bigl(\int_{-P/X}^{P/X} \bigl|S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)\bigr|^2 \,\mathrm{d}\alpha\Bigr)^{1/2} \\
&\qquad \times \Bigl(\int_{-P/X}^{P/X} |T_2(\lambda_2\alpha)|^2 \, |T_2(\lambda_3\alpha)|^2 \, |T_2(\lambda_4\alpha)|^2 \,\mathrm{d}\alpha\Bigr)^{1/2} \\
&\ll \frac{P}{X} J\Bigl(X, \frac{X}{P}\Bigr)^{1/2} \Bigl(\int_0^{1/X} X^3 \,\mathrm{d}\alpha + \int_{1/X}^{P/X} \frac{\mathrm{d}\alpha}{X^3\alpha^6}\Bigr)^{1/2} \\
&\ll_A \Bigl(\frac{X}{(\log X)^A}\Bigr)^{1/2} X \ll_A \frac{X^{3/2}}{(\log X)^{A/2}}
\end{aligned}
$$

by the Theorem in §6 of Saffari and Vaughan [9], which we can use provided that $X/P \geq X^{1/6+\varepsilon}$, that is, $P \leq X^{5/6-\varepsilon}$. This proves that $\eta^2 A_2 = o\bigl(\eta^2 X^{3/2}\bigr)$. Furthermore, using the inequalities (2) and (4) we see that

$$
\begin{aligned}
B_2 &\ll \int_0^{1/X} |T_2(\lambda_2\alpha)| \, |T_2(\lambda_3\alpha)| \, |T_2(\lambda_4\alpha)| \,\mathrm{d}\alpha \\
&\qquad + X \int_{1/X}^{P/X} \alpha \, |T_2(\lambda_2\alpha)| \, |T_2(\lambda_3\alpha)| \, |T_2(\lambda_4\alpha)| \,\mathrm{d}\alpha \\
&\ll \frac{1}{X} X^{3/2} + X \int_{1/X}^{P/X} \alpha X^{-3/2} \frac{\mathrm{d}\alpha}{\alpha^3} \ll X^{1/2} + X^{-1/2} \int_{1/X}^{P/X} \frac{\mathrm{d}\alpha}{\alpha^2} \ll X^{1/2},
\end{aligned}
$$

so that $\eta^2 B_2 = o\bigl(\eta^2 X^{3/2}\bigr)$.

3.3 Bound for J_5

Inequality (3) implies that

$$
\begin{aligned}
J_5 &\ll \eta^2 \int_{\mathfrak{M}} \bigl|S_1(\lambda_1\alpha)\bigr| \, \bigl|S_2(\lambda_2\alpha)\bigr| \, \bigl|S_2(\lambda_3\alpha)\bigr| \, \bigl|S_2(\lambda_4\alpha) - T_2(\lambda_4\alpha)\bigr| \,\mathrm{d}\alpha \\
&\ll \eta^2 \int_{\mathfrak{M}} \bigl|S_1(\lambda_1\alpha)\bigr| \, \bigl|S_2(\lambda_2\alpha)\bigr| \, \bigl|S_2(\lambda_3\alpha)\bigr| \, \bigl|S_2(\lambda_4\alpha) - U_2(\lambda_4\alpha)\bigr| \,\mathrm{d}\alpha \\
&\qquad + \eta^2 \int_{\mathfrak{M}} \bigl|S_1(\lambda_1\alpha)\bigr| \, \bigl|S_2(\lambda_2\alpha)\bigr| \, \bigl|S_2(\lambda_3\alpha)\bigr| \, \bigl|U_2(\lambda_4\alpha) - T_2(\lambda_4\alpha)\bigr| \,\mathrm{d}\alpha \\
&= \eta^2 (A_5 + B_5),
\end{aligned}
$$

say. Now let

$$
J^*(X, h) = \int_{\delta X}^{X} \bigl(\theta(\sqrt{x+h}) - \theta(\sqrt{x}) - (\sqrt{x+h} - \sqrt{x})\bigr)^2 \,\mathrm{d}x.
$$

The Parseval inequality and trivial bounds yield, for any fixed $A > 0$,

$$
A_5 \ll X \Bigl(\int_{\mathfrak{M}} \bigl|S_1(\lambda_1\alpha)\bigr|^2 \,\mathrm{d}\alpha\Bigr)^{1/2} \Bigl(\int_{\mathfrak{M}} \bigl|S_2(\lambda_4\alpha) - U_2(\lambda_4\alpha)\bigr|^2 \,\mathrm{d}\alpha\Bigr)^{1/2}
$$

$$
\begin{aligned}
&\ll X(X\log X)^{1/2}\frac{P}{X}J^*\Big(X,\frac{X}{P}\Big)^{1/2}\\
&\ll_A X^{3/2}(\log X)^{1/2-A/2}
\end{aligned}
$$

by Lemmas 3.12 and 3.13 of Languasco and Settimi [2], which we can use provided that $X/P \geq X^{7/12+\varepsilon}$, that is, $P \leq X^{5/12-\varepsilon}$. This proves that $\eta^2 A_5 = o\big(\eta^2 X^{3/2}\big)$. Furthermore, using (4), the Cauchy inequality and trivial bounds we see that

$$
\begin{aligned}
B_5 &\ll \int_0^{1/X} \big|S_1(\lambda_1\alpha)\big|\,\big|S_2(\lambda_2\alpha)\big|\,\big|S_2(\lambda_3\alpha)\big|\,\mathrm{d}\alpha\\
&\qquad + X\int_{1/X}^{P/X} \alpha\big|S_1(\lambda_1\alpha)\big|\,\big|S_2(\lambda_2\alpha)\big|\,\big|S_2(\lambda_3\alpha)\big|\,\mathrm{d}\alpha\\
&\ll \frac{1}{X}X^2 + X\Big(\int_{1/X}^{P/X}\alpha^4\,\mathrm{d}\alpha\Big)^{1/4}\Big(\int_{1/X}^{P/X}\big|S_1(\lambda_1\alpha)\big|^2\,\mathrm{d}\alpha\Big)^{1/2}\\
&\qquad \times \max_{\alpha\in[1/X,P/X]}\big|S_2(\lambda_2\alpha)\big|\Big(\int_{1/X}^{P/X}\big|S_2(\lambda_3\alpha)\big|^4\,\mathrm{d}\alpha\Big)^{1/4}\\
&\ll X + X\Big(\frac{P}{X}\Big)^{5/4}(X\log X)^{1/2}\max_{\alpha\in[1/X,P/X]}\big|S_2(\lambda_2\alpha)\big|\\
&\qquad \times \Big(\int_0^1\big|S_2(\lambda_3\alpha)\big|^4\,\mathrm{d}\alpha\Big)^{1/4}\\
&\ll X + X^{3/4}P^{5/4}(\log X)^{1/2}\Big(\int_0^1\big|S_2(\lambda_3\alpha)\big|^4\,\mathrm{d}\alpha\Big)^{1/4}.
\end{aligned}
$$

In order to estimate the integral at the far right we borrow (4.7) from Languasco and Settimi [2], that gives the bound $\ll X(\log X)^2$.
Hence $B_5 \ll XP^{5/4}\log X$, so that $\eta^2 B_5 = o\big(\eta^2 X^{3/2}\big)$ provided that $P = o\big(X^{2/5}(\log X)^{-4/5}\big)$. We may therefore choose $P = X^{2/5}/(\log X)$.

4 The intermediate arc

We need to show that $|S_1(\lambda_1\alpha)|$ and $|S_2(\lambda_2\alpha)|^2$ can not both be large for $\alpha \in \mathfrak{m}$, exploiting the fact that λ_1/λ_2 is irrational. We do this using two famous results by Vaughan about $S_1(\alpha)$ and by Ghosh about $S_2(\alpha)$.

Lemma 1 (Vaughan [12], Theorem 3.1) *Let α be a real number and a, q be positive integers satisfying $(a,q) = 1$ and $|\alpha - a/q| < q^{-2}$. Then*

$$
S_1(\alpha) \ll \Big(\frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{4/5}\Big)\log^4 X.
$$

Lemma 2 (Ghosh [1], Theorem 2) *Let α be a real number and a, q be positive integers satisfying $(a,q) = 1$ and $|\alpha - a/q| < q^{-2}$. Let moreover $\epsilon > 0$. Then*

$$
S_2(\alpha) \ll_\epsilon X^{1/2+\epsilon}\left(\frac{1}{q} + \frac{1}{X^{1/4}} + \frac{q}{X}\right)^{1/4}.
$$

Lemma 3 *Assume that λ_1/λ_2 is irrational and let $X = q^{9/5}$, where q is the denominator of a convergent of the continued fraction for λ_1/λ_2. Let $V(\alpha) = \min\bigl(|S_1(\lambda_1\alpha)|^{1/2}, |S_2(\lambda_2\alpha)|\bigr)$. Then, for arbitrary $\varepsilon > 0$, we have*

$$\sup_{\alpha\in\mathfrak{m}} V(\alpha) \ll X^{4/9+\varepsilon}.$$

Proof. Let $\alpha \in \mathfrak{m}$ and $Q = X^{2/9}/\log X \le P$. By Dirichlet's Theorem, there exist integers a_i, q_i with $1 \le q_i \le X/Q$ and $(a_i, q_i) = 1$, such that $|\lambda_i\alpha q_i - a_i| \le Q/X$, for $i = 1, 2$. We remark that $a_1a_2 \ne 0$ otherwise we would have $\alpha \in \mathfrak{M}$. Now suppose that $q_i \le Q$ for $i = 1, 2$. In this case we get

$$a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2 = (\lambda_1\alpha q_1 - a_1)\frac{a_2}{\lambda_2\alpha} - (\lambda_2\alpha q_2 - a_2)\frac{a_1}{\lambda_2\alpha}$$

and hence

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \le 2\left(1 + \left|\frac{\lambda_1}{\lambda_2}\right|\right)\frac{Q^2}{X} < \frac{1}{2q} \tag{5}$$

for sufficiently large X. Then, from the law of best approximation and the definition of $\mathfrak{m}$, we obtain

$$X^{5/9} = q \le |a_2q_1| \ll q_1q_2R \le Q^2R \le X^{5/9-2\varepsilon}\log^{-4} X, \tag{6}$$

which is absurd. Hence either $q_1 > Q$ or $q_2 > Q$. Assume first that $q_2 > Q$. Using Lemma 2 on $S_2(\lambda_2\alpha)$, we have

$$\begin{aligned} V(\alpha) \le |S_2(\lambda_2\alpha)| &\ll_\varepsilon X^{1/2+\varepsilon} \sup_{Q<q_2\le X/Q}\left(\frac{1}{q_2} + \frac{1}{X^{1/4}} + \frac{q_2}{X}\right)^{1/4} \\ &\ll_\varepsilon X^{4/9+\varepsilon}(\log X)^{1/4}. \end{aligned} \tag{7}$$

Assume now that $q_1 > Q$. Using Lemma 1 on $S_1(\lambda_1\alpha)$, we have

$$\begin{aligned} V(\alpha) \le |S_1(\lambda_1\alpha)|^{1/2} &\ll \sup_{Q<q_1\le X/Q}\left(\frac{X}{\sqrt{q_1}} + \sqrt{Xq_1} + X^{4/5}\right)^{1/2}\log^2 X \\ &\ll X^{4/9}(\log X)^3. \end{aligned} \tag{8}$$

Lemma 3 follows combining (7) and (8). □

The constraint on the choice $X = q^{9/5}$ arises from the bounds (5) and (6). Their combination prevents us from choosing the optimal value $X = q^2$.

Lemma 4 *We have*

$$\int_{\mathfrak{m}} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta X\log X$$

and

$$\int_{\mathfrak{m}} |S_2(\lambda_j\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha \ll \eta X(\log X)^2$$

for $j = 2$, 3 and 4.

Proof. The proof is achieved arguing as in §5 below where we bound the quantities A and B, the main difference being the fact that we have to split the range $[P/X, R]$ into two intervals in order to use (3) efficiently. See also the proof of Lemma 12 of [4]. For the sake of brevity we skip the details. □

Now let

$$\mathfrak{X}_1 = \{\alpha \in [P/X, R] \colon |S_1(\lambda_1\alpha)|^{1/2} \le |S_2(\lambda_2\alpha)|\}$$
$$\mathfrak{X}_2 = \{\alpha \in [P/X, R] \colon |S_1(\lambda_1\alpha)|^{1/2} \ge |S_2(\lambda_2\alpha)|\}$$

so that $[P/X, R] = \mathfrak{X}_1 \cup \mathfrak{X}_2$ and

$$\Bigl|I(\eta, \varpi, \mathfrak{m})\Bigr| \ll \Bigl(\int_{\mathfrak{X}_1} + \int_{\mathfrak{X}_2}\Bigr) \bigl|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_2(\lambda_4\alpha)\bigr| K_\eta(\alpha)\,\mathrm{d}\alpha.$$

Hölder's inequality gives

$$\begin{aligned}
\int_{\mathfrak{X}_1} &\le \Bigl(\int_{\mathfrak{X}_1} |S_1(\lambda_1\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \prod_{j=2}^{4} \Bigl(\int_{\mathfrak{X}_1} |S_2(\lambda_j\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\le \max_{\alpha\in\mathfrak{X}_1} |S_1(\lambda_1\alpha)|^{1/2} \Bigl(\int_{\mathfrak{m}} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\qquad \times \prod_{j=2}^{4} \Bigl(\int_{\mathfrak{m}} |S_2(\lambda_j\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\ll X^{4/9+\varepsilon} (\eta X \log X)^{1/4} (\eta X (\log X)^2)^{3/4} \\
&\ll \eta X^{13/9+\varepsilon} (\log X)^{7/4}
\end{aligned}$$

by Lemmas 3 and 4. The computation on $\mathfrak{X}_2$ is similar: we have

$$\begin{aligned}
\int_{\mathfrak{X}_2} &\le \Bigl(\int_{\mathfrak{X}_2} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/2} \max_{\alpha\in\mathfrak{X}_2} |S_2(\lambda_2\alpha)| \\
&\qquad \times \prod_{j=3}^{4} \Bigl(\int_{\mathfrak{X}_2} |S_2(\lambda_j\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\ll (\eta X \log X)^{1/2} X^{4/9+\varepsilon} (\eta X (\log X)^2)^{1/2} \\
&\ll \eta X^{13/9+\varepsilon} (\log X)^{3/2},
\end{aligned}$$

again by Lemmas 3 and 4. Summing up,

$$\Bigl|I(\eta, \varpi, \mathfrak{m})\Bigr| \ll \eta X^{13/9+\varepsilon} (\log X)^{7/4},$$

and this is $o\bigl(\eta^2 X^{3/2}\bigr)$ provided that $\eta \ge X^{-1/18+\varepsilon} (\log X)^2$.

5 The trivial arc

Using the Cauchy inequality and a trivial bound for $S_2(\lambda_4\alpha)$ we see that

$$\Bigl|I(\eta, \varpi, \mathfrak{t})\Bigr| \le 2 \int_R^{+\infty} |S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha)|\,|S_2(\lambda_3\alpha)|\,|S_2(\lambda_4\alpha)|\,K_\eta(\alpha)\,\mathrm{d}\alpha$$

$$
\begin{aligned}
&\ll \sup_{\alpha\in(R,+\infty)} |S_2(\lambda_4\alpha)| \Bigl(\int_R^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/2} \\
&\qquad \times \Bigl(\int_R^{+\infty} |S_2(\lambda_2\alpha)|^2\, |S_2(\lambda_3\alpha)|^2\, K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/2} \\
&\ll X^{1/2} \Bigl(\int_R^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/2} \Bigl(\int_R^{+\infty} |S_2(\lambda_2\alpha)|^4\, K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\qquad \times \Bigl(\int_R^{+\infty} |S_2(\lambda_3\alpha)|^4\, K_\eta(\alpha)\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\ll X^{1/2} \Bigl(\int_{|\lambda_1|R}^{+\infty} \frac{|S_1(\alpha)|^2}{\alpha^2}\,\mathrm{d}\alpha\Bigr)^{1/2} \Bigl(\int_{|\lambda_2|R}^{+\infty} \frac{|S_2(\alpha)|^4}{\alpha^2}\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\qquad \times \Bigl(\int_{|\lambda_3|R}^{+\infty} \frac{|S_2(\alpha)|^4}{\alpha^2}\,\mathrm{d}\alpha\Bigr)^{1/4} \\
&\ll X^{1/2} A^{1/2} B^{1/2},
\end{aligned}
$$

say, where in the last but one line we used the inequality (3), and we set

$$
A = \int_{|\lambda_1|R}^{+\infty} \frac{|S_1(\alpha)|^2}{\alpha^2}\,\mathrm{d}\alpha \qquad \text{and} \qquad B = \int_{\min(|\lambda_2|,|\lambda_3|)R}^{+\infty} \frac{|S_2(\alpha)|^4}{\alpha^2}\,\mathrm{d}\alpha.
$$

Using periodicity we have

$$
A \ll \sum_{n\ge|\lambda_1|R} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_1(\alpha)|^2\,\mathrm{d}\alpha \ll \frac{X\log X}{|\lambda_1|R}
$$

by the Prime Number Theorem, while

$$
B \ll \sum_{n\ge\min(|\lambda_2|,|\lambda_3|)R} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_2(\alpha)|^4\,\mathrm{d}\alpha \ll \frac{X(\log X)^2}{\min(|\lambda_2|,|\lambda_3|)R}.
$$

The last estimate follows from Satz 3 of Rieger [8], which is used to bound "non-diagonal" solutions of $p_1^2+p_2^2=p_3^2+p_4^2$, and the Prime Number Theorem for the remaining solutions. See also the bound for H_{12} in Liu [5]. Collecting these estimates, we conclude that

$$
\Bigl|I(\eta,\varpi,\mathfrak{t})\Bigr| \ll \frac{X^{3/2}(\log X)^{3/2}}{R}. \tag{9}
$$

Hence, the choice $R=\eta^{-2}(\log X)^2$ is admissible.

6 Proof of Theorem 2

In our paper [3] we dealt with a similar problem, with two primes and s powers of 2. The goal was to approximate any real number by means of values of the form

$$
\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s}, \tag{10}
$$

where λ_1 and λ_2 are real numbers of opposite sign, with an irrational ratio, and the non-zero coefficients μ_1, …, μ_s satisfy suitable conditions, p_1 and p_2 are prime numbers and

$m_1, \ldots, m_s$ are positive integers. The result is an upper bound on the least value s_0 that ensures the existence of an approximation of the form (10) for all $s \ge s_0$. The quality of the result depends on rational approximations to λ_1/λ_2: we let $\mathfrak{R}$ denote the set of irrational numbers ξ such that the denominators q_m of the convergents to ξ, arranged in increasing order of magnitude, satisfy $q_{m+1} \ll q_m^{1+\varepsilon}$. By Roth's Theorem, all algebraic numbers belong to $\mathfrak{R}$, and almost all real numbers, in the sense of the Lebesgue measure, also belong to $\mathfrak{R}$. We denote by $\mathfrak{R}'$ the set of irrational numbers that do not belong to $\mathfrak{R}$. For λ_1/λ_2 belonging to this set, we have the following improvement of our result in [3].

Theorem 2 *Suppose that λ_1 and λ_2 are real numbers such that λ_1/λ_2 is negative and irrational with $\lambda_1 > 1$, $\lambda_2 < -1$ and $|\lambda_1/\lambda_2| \ge 1$. Further suppose that $\mu_1, \ldots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$ for $i \in \{1, 2\}$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(\lambda_1/a_1; |\lambda_2/a_2|)$. Finally, for $\lambda_1/\lambda_2 \in \mathfrak{R}'$, let*

$$s_0 = 2 + \left\lceil \frac{\log(C(q_1,q_2)\lambda_1) - \log\eta}{-\log(0.884472132)} \right\rceil.$$

Then for every real number γ and every integer $s \ge s_0$ the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \gamma| < \eta$$

has infinitely many solutions in primes p_1,p_2 and positive integers $m_1, \ldots, m_s$, where $C(q_1,q_2) = \Big(\log 2 + C\cdot\mathfrak{S}'(q_1)\Big)^{1/2}\Big(\log 2 + C\cdot\mathfrak{S}'(q_2)\Big)^{1/2}$, $C = 10.0219168340$ and

$$\mathfrak{S}'(n) = \prod_{\substack{p\mid n\\ p>2}} \frac{p-1}{p-2}.$$

We can improve our previous treatment of the intermediate arc in §7 of [3]. We let $V(\alpha) = \min\big(|S_1(\lambda_1\alpha)|, |S_1(\lambda_2\alpha)|\big)$ and recall that $\mathfrak{m}_2$ is the subset of $[X^{-2/3}, (\log X)^2]$ where the exponential sum $G(\alpha) = \sum_{n\le L} e(2^n\alpha)$ is "large" in absolute value. Here $L = (\log(\varepsilon X/2M))/\log 2$ where $M = \max_j |\mu_j|$. The technique due to Pintz and Ruzsa [7] ensures that its measure is comparatively small. In the following computation, implicit constants may depend on λ_1 and λ_2. We have

$$\begin{aligned}
&\Big|\int_{\mathfrak{m}_2} S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)\prod_{j=1}^{s} G(\mu_j\alpha)K_\eta(\alpha)\,\mathrm{d}\alpha\Big| \\
&\quad\ll \eta^2(\log X)^s \int_{\mathfrak{m}_2} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|\,\mathrm{d}\alpha \\
&\quad\ll \eta^2(\log X)^s \sup_{\alpha\in\mathfrak{m}_2} V(\alpha) \int_{\mathfrak{m}_2} |S_1(\lambda_2\alpha)|\,\mathrm{d}\alpha \\
&\quad\ll \eta^2(\log X)^s \sup_{\alpha\in\mathfrak{m}_2} V(\alpha) \Big(\int_{\mathfrak{m}_2} \mathrm{d}\alpha\Big)^{1/2}\Big(\int_{\mathfrak{m}_2} |S_1(\lambda_2\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \\
&\quad\ll \eta^2(\log X)^s |\mathfrak{m}_2|^{1/2}\big(X(\log X)^3\big)^{1/2} \sup_{\alpha\in\mathfrak{m}_2} V(\alpha)
\end{aligned}$$

$$\ll \eta^2(\log X)^s(\log X)s^{1/2}X^{-c/2}X^{1/2}(\log X)^{3/2}\sup_{\alpha\in\mathfrak{m}_2}V(\alpha)$$
$$\ll \eta^2 s^{1/2}X^{1/2-c/2}(\log X)^{s+5/2}\sup_{\alpha\in\mathfrak{m}_2}V(\alpha).$$

The proof of Lemma 4 of Parsell [6] implies that

$$\sup_{\alpha\in\mathfrak{m}_2}V(\alpha)=\sup_{\alpha\in\mathfrak{m}_2}\min\bigl(|S_1(\lambda_1\alpha)|,|S_1(\lambda_2\alpha)|\bigr)\ll X^{7/8}(\log X)^5.$$

Hence the integral above is bounded by

$$\eta^2 s^{1/2}X^{11/8-c/2}(\log X)^{s+15/2}$$

It is therefore sufficient to take $c>\frac{3}{4}$ (instead of the bound $c>\frac{4}{5}$ that we had in [3]). Taking $c=\frac{3}{4}+10^{-20}$, the method due to Pintz and Ruzsa (see for example Lemma 5 of [3]) yields $\nu=0.884472132\ldots$ Hence we can replace the value $-\log(0.91237810306)$ that we had in §7 of [3] with $-\log(0.884472132)$ in the denominator of the definition of s_0 in the case where $\lambda_1/\lambda_2\in\mathfrak{R}'$.

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Alessandro LANGUASCO
Università di Padova
Dipartimento di Matematica
Via Trieste 63
35121 Padova, Italy
E-mail: languasco@math.unipd.it

Alessandro ZACCAGNINI
Università di Parma
Dipartimento di Matematica
Parco Area delle Scienze, 53/a
Campus Universitario
43124 Parma, Italy
E-mail: alessandro.zaccagnini@unipr.it