A CESÀRO AVERAGE OF HARDY-LITTLEWOOD NUMBERS

ALESSANDRO LANGUASCO & ALESSANDRO ZACCAGNINI

ABSTRACT. Let Λ be the von Mangoldt function and $r_{HL}(n) = \sum_{m_1+m_2^2=n} \Lambda(m_1)$, be the counting function for the Hardy-Littlewood numbers. Let N be a sufficiently large integer. We prove that

$$\sum_{n\le N} r_{HL}(n)\frac{(1-n/N)^k}{\Gamma(k+1)} = \frac{\pi^{1/2}}{2}\frac{N^{3/2}}{\Gamma(k+5/2)} - \frac{1}{2}\frac{N}{\Gamma(k+2)} - \frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{1/2+\rho}$$
$$+ \frac{1}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{\rho} + \frac{N^{3/4-k/2}}{\pi^{k+1}}\sum_{\ell\ge1}\frac{J_{k+3/2}(2\pi\ell N^{1/2})}{\ell^{k+3/2}}$$
$$- \frac{N^{1/4-k/2}}{\pi^{k}}\sum_{\rho}\Gamma(\rho)\frac{N^{\rho/2}}{\pi^{\rho}}\sum_{\ell\ge1}\frac{J_{k+1/2+\rho}(2\pi\ell N^{1/2})}{\ell^{k+1/2+\rho}} + \mathcal{O}_k(1).$$

for $k > 1$, where ρ runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$ and $J_\nu(u)$ denotes the Bessel function of complex order ν and real argument u.

1. Introduction

We continue our recent work on additive problems with prime summands. In [9] we studied the *average* number of representations of an integer as a sum of two primes, whereas in [10] we considered individual integers. In this paper we study a Cesàro weighted *explicit* formula for Hardy-Littlewood numbers (integers that can be written as a sum of a prime and a square) and the goal is similar to the one in [8], that is, we want to obtain an asymptotic formula with the expected main term and one or more terms that depend explicitly on the zeros of the Riemann zeta-function. Letting

$$r_{HL}(n) = \sum_{m_1+m_2^2=n} \Lambda(m_1), \tag{1}$$

the main result of the paper is the following theorem.

Theorem 1. *Let N be a sufficiently large integer. We have*

$$\sum_{n\le N} r_{HL}(n)\frac{(1-n/N)^k}{\Gamma(k+1)} = \frac{\pi^{1/2}}{2}\frac{N^{3/2}}{\Gamma(k+5/2)} - \frac{1}{2}\frac{N}{\Gamma(k+2)} - \frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{1/2+\rho}$$
$$+ \frac{1}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{\rho} + \frac{N^{3/4-k/2}}{\pi^{k+1}}\sum_{\ell\ge1}\frac{J_{k+3/2}(2\pi\ell N^{1/2})}{\ell^{k+3/2}}$$
$$- \frac{N^{1/4-k/2}}{\pi^{k}}\sum_{\rho}\Gamma(\rho)\frac{N^{\rho/2}}{\pi^{\rho}}\sum_{\ell\ge1}\frac{J_{k+1/2+\rho}(2\pi\ell N^{1/2})}{\ell^{k+1/2+\rho}} + \mathcal{O}_k(1).$$

for $k > 1$, where ρ runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$ and $J_\nu(u)$ denotes the Bessel function of complex order ν and real argument u.

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Similar averages of arithmetical functions are common in the literature, see, e.g., Chandrasekharan-Narasimhan [2] and Berndt [1] who built on earlier classical works (Hardy, Landau, Walfisz and others). In their setting the generalized Dirichlet series associated to the arithmetical function satisfies a suitable functional equation and this leads to an asymptotic formula containing Bessel functions of real order and argument. In our case we have no functional equation, and, as far as we know, it is the first time that Bessel functions with complex order arise in a similar problem. Moreover, from a technical point of view, the estimates of such Bessel functions are harder to perform than the ones already present in the Number Theory literature since the real argument and the complex order are both unbounded while, in previous papers, either the real order or the argument is bounded.

The method we will use in this additive problem is based on a formula due to Laplace [11], namely

$$\frac{1}{2\pi i}\int_{(a)} v^{-s}e^{v}\,\mathrm{d}v = \frac{1}{\Gamma(s)}, \tag{2}$$

where $\Re(s) > 0$ and $a > 0$, see, e.g., formula 5.4(1) on page 238 of [4]. In the following we will need the general case of (2) which can be found in de Azevedo Pribitkin [3], formulae (8) and (9):

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{iDu}}{(a+iu)^{s}}\,\mathrm{d}u = \begin{cases} \dfrac{D^{s-1}e^{-aD}}{\Gamma(s)} & \text{if } D > 0,\\ 0 & \text{if } D < 0, \end{cases} \tag{3}$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$, and

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{(a+iu)^{s}}\,\mathrm{d}u = \begin{cases} 0 & \text{if } \Re(s) > 1,\\ 1/2 & \text{if } s = 1, \end{cases} \tag{4}$$

for $a \in \mathbb{C}$ with $\Re(a) > 0$. Formulae (3)-(4) enable us to write averages of arithmetical functions by means of line integrals as we will see in §2 below.

We will also need Bessel functions of complex order ν and real argument u. For their definition and main properties we refer to Watson [14]. In particular, equation (8) on page 177 gives the Sonine representation:

$$J_\nu(u) := \frac{(u/2)^{\nu}}{2\pi i}\int_{(a)} s^{-\nu-1}e^{s}e^{-u^2/4s}\,\mathrm{d}s, \tag{5}$$

where $a > 0$ and $u, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$. We will use also a Poisson integral formula (see eq. (3) on page 48 of [14]), i.e.,

$$J_\nu(u) := \frac{2(u/2)^{\nu}}{\pi^{1/2}\Gamma(\nu+1/2)}\int_0^1 (1-t^2)^{\nu-1/2}\cos(ut)\,\mathrm{d}t \tag{6}$$

which holds for $\Re(\nu) > -1/2$ and $u \in \mathbb{C}$. An asymptotic estimate we will need is

$$J_\nu(u) = \Big(\frac{2}{\pi u}\Big)^{1/2}\cos\Big(u - \frac{\pi\nu}{2} - \frac{\pi}{4}\Big) + \mathcal{O}_{|\nu|}\big(u^{-5/2}\big) \tag{7}$$

which follows from eq. (1) on page 199 of Watson [14].

As in [8], we combine this approach with line integrals with the classical methods dealing with infinite sums over primes, exploited by Hardy & Littlewood (see [6] and [7]) and by Linnik [12]. The main difference here is that the problem naturally involves the modular relation for the complex theta function, see eq. (9); the presence of the Bessel functions in our statement strictly depends on such modularity relation. It is worth mentioning that it is not clear how to get such "modular" terms using the finite sums approach for the Hardy-Littlewood function $r_{HL}(n)$.

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2. Settings

We need $k > 0$ in this section. Let $z = a + iy$ with $a > 0$,

$$\widetilde{S}(z) = \sum_{m\geq 1} \Lambda(m) e^{-mz} \quad \text{and} \quad \omega_2(z) = \sum_{m\geq 1} e^{-m^2 z}. \tag{8}$$

Letting further $\theta(z) = \sum_{m=-\infty}^{+\infty} e^{-m^2 z}$, we notice that $\theta(z) = 1 + 2\omega_2(z)$ and, recalling the functional equation for θ (see, e.g., Proposition VI.4.3 of Freitag-Busam [5, page 340]):

$$\theta(z) = \Big(\frac{\pi}{z}\Big)^{1/2} \theta\Big(\frac{\pi^2}{z}\Big), \tag{9}$$

we immediately get

$$\omega_2(z) = \frac{1}{2}\Big(\frac{\pi}{z}\Big)^{1/2} - \frac{1}{2} + \Big(\frac{\pi}{z}\Big)^{1/2} \omega_2\Big(\frac{\pi^2}{z}\Big). \tag{10}$$

Recalling (1), we can write

$$\widetilde{S}(z)\omega_2(z) = \sum_{m_1\geq 1}\sum_{m_2\geq 1} \Lambda(m_1) e^{-(m_1+m_2^2)z} = \sum_{n\geq 1} r_{HL}(n) e^{-nz}$$

and, by (3)-(4), we see that

$$\sum_{n\leq N} r_{HL}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \sum_{n\geq 1} r_{HL}(n) \Big(\frac{1}{2\pi i}\int_{(a)} e^{(N-n)z} z^{-k-1}\, \mathrm{d}z\Big). \tag{11}$$

Our first goal is to exchange the series with the line integral in (11). To do so we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 1 below, to the statement

$$\widetilde{S}(a) \sim a^{-1} \qquad \text{for } a \to 0+,$$

which is classical: for the proof see for instance Lemma 9 in Hardy & Littlewood [7]. We will also use the inequality

$$|\omega_2(z)| \leq \omega_2(a) \leq \int_0^\infty e^{-at^2} \mathrm{d}t \leq a^{-1/2} \int_0^\infty e^{-v^2} \mathrm{d}v \ll a^{-1/2} \tag{12}$$

from which we immediately get

$$\sum_{n\geq 1} \big| r_{HL}(n) e^{-nz} \big| = \sum_{n\geq 2} r_{HL}(n) e^{-na} = \widetilde{S}(a)\omega_2(a) \ll a^{-3/2}.$$

Taking into account the estimates

$$|z|^{-1} \asymp \begin{cases} a^{-1} & \text{if } |y| \leq a, \\ |y|^{-1} & \text{if } |y| \geq a, \end{cases} \tag{13}$$

where $f \asymp g$ means $g \ll f \ll g$, and

$$|e^{Nz} z^{-k-1}| \asymp e^{Na} \begin{cases} a^{-k-1} & \text{if } |y| \leq a, \\ |y|^{-k-1} & \text{if } |y| \geq a, \end{cases}$$

we have

$$\int_{(a)} |e^{Nz} z^{-k-1}|\, |\widetilde{S}(z)\omega_2(z)|\, |\mathrm{d}z| \ll a^{-3/2} e^{Na} \Big(\int_{-a}^{a} a^{-k-1}\, \mathrm{d}y + 2\int_a^{+\infty} y^{-k-1}\, \mathrm{d}y\Big)$$

$$\ll a^{-3/2}e^{Na}\Big(a^{-k}+\frac{a^{-k}}{k}\Big),$$

but the last estimate is valid only if $k>0$. So, for $k>0$, we can exchange the line integral with the sum over n in (11) thus getting

$$\sum_{n\le N} r_{HL}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}(z)\omega_2(z)\,\mathrm{d}z. \tag{14}$$

This is the fundamental relation for the method.

3. Inserting zeros and modularity

We need $k>1/2$ in this section. The treatment of the integral at the right hand side of (14) requires Lemma 1. Letting $E(a,y)$ be the error term in (29), formula (14) becomes

$$\begin{aligned}\sum_{n\le N} r_{HL}(n)\frac{(N-n)^k}{\Gamma(k+1)} &= \frac{1}{2\pi i}\int_{(a)}\Big(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\omega_2(z)e^{Nz}z^{-k-1}\,\mathrm{d}z\\ &\quad+\mathcal{O}\Big(\int_{(a)}|E(a,y)|\,|e^{Nz}|\,|z|^{-k-1}|\omega_2(z)|\,|\mathrm{d}z|\Big).\end{aligned}$$

Using (12)-(13) and (29) we see that the error term is

$$\begin{aligned}&\ll a^{-1/2}e^{Na}\Big(\int_{-a}^{a} a^{-k-1/2}\mathrm{d}y+\int_{a}^{+\infty} y^{-k-1/2}\log^2(y/a)\,\mathrm{d}y\Big)\\ &\ll_k e^{Na}a^{-k}\Big(1+\int_{1}^{+\infty} v^{-k-1/2}\log^2 v\,\mathrm{d}v\Big)\ll_k e^{Na}a^{-k},\end{aligned}$$

provided that $k>1/2$. Choosing $a=1/N$, the previous estimate becomes $\ll_k N^k$. Summing up, for $k>1/2$, we can write

$$\sum_{n\le N} r_{HL}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(1/N)}\Big(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\omega_2(z)e^{Nz}z^{-k-1}\,\mathrm{d}z+\mathcal{O}_k\big(N^k\big). \tag{15}$$

We now insert (10) into (15), so that the integral on the right-hand side of (15) becomes

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(1/N)}\Big(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\Big(\frac{1}{2}\Big(\frac{\pi}{z}\Big)^{1/2}-\frac{1}{2}\Big)e^{Nz}z^{-k-1}\,\mathrm{d}z\\ &\quad+\frac{1}{2\pi i}\int_{(1/N)}\Big(\frac{\pi}{z}\Big)^{1/2}\Big(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\omega_2\Big(\frac{\pi^2}{z}\Big)e^{Nz}z^{-k-1}\,\mathrm{d}z\\ &=\mathcal{I}_1+\mathcal{I}_2,\end{aligned} \tag{16}$$

say. We now proceed to evaluate $\mathcal{I}_1$ and $\mathcal{I}_2$.

4. Evaluation of $\mathcal{I}_1$

We need $k>1/2$ in this section. By a direct computation we can write that

$$\begin{aligned}\mathcal{I}_1&=\frac{1}{4\pi i}\int_{(1/N)}\Big(\frac{\pi^{1/2}}{z^{1/2}}-1\Big)e^{Nz}z^{-k-2}\,\mathrm{d}z-\frac{\pi^{1/2}}{4\pi i}\int_{(1/N)}\sum_{\rho}\Gamma(\rho)e^{Nz}z^{-k-\rho-3/2}\,\mathrm{d}z\\ &\quad+\frac{1}{4\pi i}\int_{(1/N)}\sum_{\rho}\Gamma(\rho)e^{Nz}z^{-k-\rho-1}\,\mathrm{d}z=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3,\end{aligned}$$

say. We see now how to evaluate $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$.

4.1. **Evaluation of $\mathcal{J}_1$.** Using the substitution $s = Nz$, by (2) we immediately have

$$\mathcal{J}_1 = \frac{\pi^{1/2}}{2} N^{k+3/2} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-5/2}\, \mathrm{d}s - \frac{1}{2} N^{k+1} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-2}\, \mathrm{d}s$$
$$= \frac{\pi^{1/2}}{2} \frac{N^{k+3/2}}{\Gamma(k+5/2)} - \frac{1}{2} \frac{N^{k+1}}{\Gamma(k+2)}. \tag{17}$$

4.2. **Evaluation of $\mathcal{J}_2$.** Exchanging the sum over ρ with the integral (this can be done for $k > 0$, see §7) and using the substitution $s = Nz$, we have

$$\mathcal{J}_2 = -\frac{\pi^{1/2}}{2} \sum_\rho \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-\rho-3/2}\, \mathrm{d}z$$
$$= -\frac{\pi^{1/2}}{2} \sum_\rho \Gamma(\rho) N^{k+\rho+1/2} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-\rho-3/2}\, \mathrm{d}s$$
$$= -\frac{\pi^{1/2}}{2} \sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho}, \tag{18}$$

again by (2). By the Stirling formula (28), we remark that the series in $\mathcal{J}_2$ converges absolutely for $k > -1/2$.

4.3. **Evaluation of $\mathcal{J}_3$.** Arguing as in §7 with $-k-1$ which plays the role of $-k-3/2$ there, we see that we can exchange the sum with the integral provided that $k > 1/2$. Hence, performing again the usual substitution $s = Nz$, we can write

$$\mathcal{J}_3 = \frac{1}{2} \sum_\rho \Gamma(\rho) N^{k+\rho} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-1-\rho}\, \mathrm{d}s = \frac{1}{2} \sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho}. \tag{19}$$

By the Stirling formula (28), we remark that the series in $\mathcal{J}_3$ converges absolutely for $k > 0$.

Summing up, by (17)-(19) and for $k > 1/2$ we get

$$\mathcal{I}_1 = \frac{\pi^{1/2}}{2} \frac{N^{k+3/2}}{\Gamma(k+5/2)} - \frac{1}{2} \frac{N^{k+1}}{\Gamma(k+2)} - \frac{\pi^{1/2}}{2} \sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho}$$
$$+ \frac{1}{2} \sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho}. \tag{20}$$

5. Evaluation of $\mathcal{I}_2$ and conclusion of the proof of Theorem 1

We need $k > 1$ in this section. Using the definition of $\omega_2(\pi^2/z)$, see (8), we have

$$\mathcal{I}_2 = \frac{1}{2\pi i} \int_{(1/N)} \Big(\frac{\pi}{z}\Big)^{1/2} \Big(\sum_{\ell \geq 1} e^{-\ell^2 \pi^2 / z} \Big) e^{Nz} z^{-k-2}\, \mathrm{d}z$$
$$- \frac{1}{2\pi i} \int_{(1/N)} \Big(\frac{\pi}{z}\Big)^{1/2} \Big(\sum_{\ell \geq 1} e^{-\ell^2 \pi^2 / z} \Big) \Big(\sum_\rho z^{-\rho} \Gamma(\rho) \Big) e^{Nz} z^{-k-1}\, \mathrm{d}z = \mathcal{J}_4 + \mathcal{J}_5, \tag{21}$$

say. We see now how to evaluate $\mathcal{J}_4$ and $\mathcal{J}_5$.

5.1. **Evaluation of $\mathcal{J}_4$.** By means of the substitution $s = Nz$, since the exchange is justified in §8 for $k > -1/2$, we get

$$\mathcal{J}_4 = \pi^{1/2} N^{k+3/2} \sum_{\ell \geq 1} \frac{1}{2\pi i} \int_{(1)} e^s e^{-\ell^2 \pi^2 N/s} s^{-k-5/2} \, \mathrm{d}s.$$

Setting $u = 2\pi \ell N^{1/2}$ in (5) we obtain

$$J_\nu\bigl(2\pi \ell N^{1/2}\bigr) = \frac{(\pi \ell N^{1/2})^\nu}{2\pi i} \int_{(a)} e^s e^{-\pi^2 \ell^2 N/s} s^{-\nu-1} \, \mathrm{d}s, \tag{22}$$

and hence we have

$$\mathcal{J}_4 = \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{\ell \geq 1} \frac{J_{k+3/2}(2\pi \ell N^{1/2})}{\ell^{k+3/2}}. \tag{23}$$

The absolute convergence of the series in $\mathcal{J}_4$ is studied in §10.

5.2. **Evaluation of $\mathcal{J}_5$.** With the same substitution used before, since the double exchange between sums and the line integral is justified in §9 for $k > 1$, we see that

$$\mathcal{J}_5 := -\pi^{1/2} \sum_{\rho} \Gamma(\rho) N^{k+1/2+\rho} \sum_{\ell \geq 1} \Bigl(\frac{1}{2\pi i} \int_{(1)} e^s e^{-\ell^2 \pi^2 N/s} s^{-k-3/2-\rho} \, \mathrm{d}s \Bigr).$$

Using (22) we get

$$\mathcal{J}_5 = -\frac{N^{k/2+1/4}}{\pi^k} \sum_{\rho} \Gamma(\rho) \frac{N^{\rho/2}}{\pi^\rho} \sum_{\ell \geq 1} \frac{J_{k+1/2+\rho}(2\pi \ell N^{1/2})}{\ell^{k+1/2+\rho}}. \tag{24}$$

In this case the absolute convergence of the series in $\mathcal{J}_5$ is more delicate; such a treatment is again described in §10.

Substituting (23)-(24) in (21) we have

$$\mathcal{I}_2 = \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{\ell \geq 1} \frac{J_{k+3/2}(2\pi \ell N^{1/2})}{\ell^{k+3/2}} - \frac{N^{k/2+1/4}}{\pi^k} \sum_{\rho} \frac{\Gamma(\rho) N^{\rho/2}}{\pi^\rho} \sum_{\ell \geq 1} \frac{J_{k+1/2+\rho}(2\pi \ell N^{1/2})}{\ell^{k+1/2+\rho}}. \tag{25}$$

Finally, inserting (20) and (25) into (16) and (15) we finally obtain

$$\begin{aligned}
\sum_{n \leq N} r_{HL}(n) \frac{(N-n)^k}{\Gamma(k+1)} &= \frac{\pi^{1/2}}{2} \frac{N^{k+3/2}}{\Gamma(k+5/2)} - \frac{1}{2} \frac{N^{k+1}}{\Gamma(k+2)} - \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho} \\
&\quad + \frac{1}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho} + \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{\ell \geq 1} \frac{J_{k+3/2}(2\pi \ell N^{1/2})}{\ell^{k+3/2}} \\
&\quad - \frac{N^{k/2+1/4}}{\pi^k} \sum_{\rho} \Gamma(\rho) \frac{N^{\rho/2}}{\pi^\rho} \sum_{\ell \geq 1} \frac{J_{k+1/2+\rho}(2\pi \ell N^{1/2})}{\ell^{k+1/2+\rho}} + \mathcal{O}_k\bigl(N^k\bigr),
\end{aligned} \tag{26}$$

for $k > 1$. Theorem 1 follows dividing (26) by N^k.

6. Lemmas

We recall some basic facts in complex analysis. First, if $z = a + iy$ with $a > 0$, we see that for complex w we have

$$\begin{aligned} z^{-w} &= |z|^{-w} \exp(-iw \arctan(y/a)) \\ &= |z|^{-\Re(w)-i\Im(w)} \exp((-i\Re(w) + \Im(w)) \arctan(y/a)) \end{aligned}$$

so that

$$|z^{-w}| = |z|^{-\Re(w)} \exp(\Im(w) \arctan(y/a)). \tag{27}$$

We also recall that, uniformly for $x \in [x_1, x_2]$, with x_1 and x_2 fixed, and for $|y| \to +\infty$, by the Stirling formula we have

$$|\Gamma(x + iy)| \sim \sqrt{2\pi} e^{-\pi|y|/2} |y|^{x-1/2}, \tag{28}$$

see, e.g., Titchmarsh [13, §4.42].

We will need the following lemmas from Languasco-Zaccagnini [8].

Lemma 1 (See Lemma 1 of [8]). *Let $z = a + iy$, where $a > 0$ and $y \in \mathbb{R}$. Then*

$$\widetilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(a, y)$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a, y) \ll |z|^{1/2} \begin{cases} 1 & \text{if } |y| \leq a \\ 1 + \log^2(|y|/a) & \text{if } |y| > a. \end{cases} \tag{29}$$

Lemma 2 (See Lemma 2 of [8]). *Let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function and $\alpha > 1$ be a parameter. The series*

$$\sum_{\rho\colon \gamma > 0} \gamma^{\beta - 1/2} \int_1^{+\infty} \exp\Bigl(-\gamma \arctan \frac{1}{u}\Bigr) \frac{\mathrm{d}u}{u^{\alpha + \beta}}$$

converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $(\log u)^c$, for any fixed $c \geq 0$.

Lemma 3 (See Lemma 3 of [8]). *Let $\alpha > 1$, $z = a + iy$, $a \in (0, 1)$ and $y \in \mathbb{R}$. Let further $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function. We have*

$$\sum_{\rho} |\gamma|^{\beta - 1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp\Bigl(\gamma \arctan \frac{y}{a} - \frac{\pi}{2}|\gamma|\Bigr) \frac{\mathrm{d}y}{|z|^{\alpha + \beta}} \ll_{\alpha} a^{-\alpha},$$

where $\mathbb{Y}_1 = \{y \in \mathbb{R} \colon y\gamma \leq 0\}$ and $\mathbb{Y}_2 = \{y \in [-a, a] \colon y\gamma > 0\}$. The result remains true if we insert in the integral a factor $(\log(|y|/a))^c$, for any fixed $c \geq 0$.

7. Interchange of the series over zeros with the line integral

We need $k > 0$ in this section. We have to establish the convergence of

$$\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| |z|^{-k-3/2} |z^{-\rho}| \, |\mathrm{d}z|, \tag{30}$$

where, as usual, $\rho = \beta + i\gamma$ runs over the non-trivial zeros of the Riemann zeta-function. By (27) and the Stirling formula (28), we are left with estimating

$$\sum_{\rho} |\gamma|^{\beta - 1/2} \int_{\mathbb{R}} \exp\Bigl(\gamma \arctan(Ny) - \frac{\pi}{2}|\gamma|\Bigr) \frac{\mathrm{d}y}{|z|^{k + 3/2 + \beta}}. \tag{31}$$

We have just to consider the case $\gamma y > 0$, $|y| > 1/N$ since in the other cases the total contribution is $\ll_k N^{k+1}$ by Lemma 3 with $\alpha = k+3/2$ and $a = 1/N$. By symmetry, we may assume that $\gamma > 0$. We have that the integral in (31) is

$$
\begin{aligned}
&\ll \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{+\infty} \exp\Bigl(-\gamma \arctan \frac{1}{Ny}\Bigr) \frac{\mathrm{d}y}{y^{k+3/2+\beta}} \\
&= N^k \sum_{\rho\colon \gamma>0} N^{\beta}\gamma^{\beta-1/2} \int_{1}^{+\infty} \exp\Bigl(-\gamma \arctan \frac{1}{u}\Bigr) \frac{\mathrm{d}u}{u^{k+3/2+\beta}}.
\end{aligned}
$$

For $k > 0$ this is $\ll_k N^{k+1}$ by Lemma 2. This implies that the integrals in (31) and in (30) are both $\ll_k N^{k+1}$ and hence this exchange step is fully justified.

8. Interchange of the series over ℓ with the line integral

We need $k > -1/2$ in this section. We have to establish the convergence of

$$
\sum_{\ell\ge 1} \int_{(1/N)} |e^{Nz}||z|^{-k-5/2} e^{-\pi^2\ell^2\Re(1/z)}\, |\mathrm{d}z|. \tag{32}
$$

A trivial computation gives

$$
\Re(1/z) = \frac{N}{1+N^2y^2} \gg \begin{cases} N & \text{if } |y| \le 1/N \\ 1/(Ny^2) & \text{if } |y| > 1/N. \end{cases} \tag{33}
$$

By (33), we can write that the quantity in (32) is

$$
\ll \sum_{\ell\ge 1} \int_0^{1/N} \frac{e^{-\ell^2 N}}{|z|^{k+5/2}}\, \mathrm{d}y + \sum_{\ell\ge 1} \int_{1/N}^{+\infty} \frac{e^{-\ell^2/(Ny^2)}}{|z|^{k+5/2}}\, \mathrm{d}y = U_1 + U_2, \tag{34}
$$

say, since the π^2 factor in the exponential function is negligible. Using (12)-(13), we have

$$
U_1 \ll N^{k+3/2}\omega_2(N) \ll N^{k+1} \tag{35}
$$

and

$$
\begin{aligned}
U_2 &\ll \sum_{\ell\ge 1} \int_{1/N}^{+\infty} \frac{e^{-\ell^2/(Ny^2)}}{y^{k+5/2}}\, \mathrm{d}y \ll N^{k/2+3/4} \sum_{\ell\ge 1} \frac{1}{\ell^{k+3/2}} \int_0^{\ell^2 N} u^{k/2-1/4} e^{-u}\, \mathrm{d}u \\
&\le \Gamma\Bigl(\frac{2k+3}{4}\Bigr) N^{k/2+3/4} \sum_{\ell\ge 1} \frac{1}{\ell^{k+3/2}} \ll_k N^{k/2+3/4}
\end{aligned} \tag{36}
$$

provided that $k > -1/2$, where we used the substitution $u = \ell^2/(Ny^2)$.

Inserting (35)-(36) into (34) we get, for $k > -1/2$, that the quantity in (32) is $\ll N^{k+1}$.

9. Interchange of the double series over zeros with the line integral

We need $k > 1$ in this section. We first have to establish the convergence of

$$
\sum_{\ell\ge 1} \int_{(1/N)} \Bigl|\sum_{\rho} \Gamma(\rho) z^{-\rho}\Bigr| |e^{Nz}||z|^{-k-3/2} e^{-\pi^2\ell^2\Re(1/z)}\, |\mathrm{d}z|. \tag{37}
$$

Using the PNT and (29), we first remark that

$$
\Bigl|\sum_{\rho} z^{-\rho}\Gamma(\rho)\Bigr| = \Bigl|\widetilde{S}(z) - \frac{1}{z} - E(y, \frac{1}{N})\Bigr| \ll N + \frac{1}{|z|} + \Bigl|E(y, \frac{1}{N})\Bigr|
$$

$$\ll \begin{cases} N & \text{if } |y| \le 1/N, \\ |z|^{-1} + |z|^{1/2}\log^2(2N|y|) & \text{if } |y| > 1/N. \end{cases} \tag{38}$$

By (33) and (38), we can write that the quantity in (37) is

$$\begin{aligned} &\ll N \sum_{\ell\ge1} \int_0^{1/N} \frac{e^{-\ell^2 N}}{|z|^{k+3/2}}\,\mathrm{d}y \quad + \sum_{\ell\ge1} \int_{1/N}^{+\infty} \frac{e^{-\ell^2/(Ny^2)}}{|z|^{k+5/2}}\,\mathrm{d}y + \sum_{\ell\ge1} \int_{1/N}^{+\infty} \log^2(2Ny) \frac{e^{-\ell^2/(Ny^2)}}{|z|^{k+1}}\,\mathrm{d}y \\ &= V_1 + V_2 + V_3, \end{aligned} \tag{39}$$

say. V_1 and V_2 can be estimated exactly as U_1, U_2 in §8; hence we have

$$V_1 + V_2 \ll_k N^{k+1} \tag{40}$$

provided that $k > -1/2$.

Using the substitution $u = \ell^2/(Ny^2)$, we obtain

$$V_3 \ll \sum_{\ell\ge1} \int_{1/N}^{+\infty} \log^2(2Ny) \frac{e^{-\ell^2/(Ny^2)}}{y^{k+1}}\,\mathrm{d}y = \frac{N^{k/2}}{8} \sum_{\ell\ge1} \frac{1}{\ell^k} \int_0^{\ell^2 N} u^{k/2-1} \log^2\Big(\frac{4\ell^2 N}{u}\Big) e^{-u}\,\mathrm{d}u.$$

Hence a direct computation shows that

$$\begin{aligned} V_3 &\ll N^{k/2} \sum_{\ell\ge1} \frac{\log^2(\ell N)}{\ell^k} \int_0^{\ell^2 N} u^{k/2-1} e^{-u}\,\mathrm{d}u + N^{k/2} \sum_{\ell\ge1} \frac{1}{\ell^k} \int_0^{\ell^2 N} u^{k/2-1} \log^2(u)\, e^{-u}\,\mathrm{d}u \\ &\ll_k \Gamma(k/2) N^{k/2} \sum_{\ell\ge1} \frac{\log^2(\ell N)}{\ell^k} + N^{k/2} \ll_k N^{k/2}\log^2 N \end{aligned} \tag{41}$$

provided that $k > 1$. Inserting (40)-(41) into (39) we get, for $k > 1$, that the quantity in (37) is $\ll N^{k+1}$.

Now we have to establish the convergence of

$$\sum_{\ell\ge1} \sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| |z|^{-k-3/2} |z^{-\rho}| e^{-\pi^2\ell^2 \Re(1/z)}\,|\mathrm{d}z|. \tag{42}$$

By symmetry, we may assume that $\gamma > 0$. For $y \in (-\infty, 0]$ we have $\gamma \arctan(y/a) - \frac{\pi}{2}\gamma \le -\frac{\pi}{2}\gamma$. Using (33), (13) and the Stirling formula (28), the quantity we are estimating becomes

$$\begin{aligned} &\ll \sum_{\ell\ge1} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{2}\gamma\Big) \Big(\int_{-1/N}^0 N^{k+3/2+\beta}\ e^{-\ell^2 N}\,\mathrm{d}y + \int_{-\infty}^{-1/N} \frac{e^{-\ell^2/(Ny^2)}}{|y|^{k+3/2+\beta}}\,\mathrm{d}y \Big) \\ &\ll_k N^{k+3/2} \sum_{\ell\ge1} e^{-\ell^2 N} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{2}\gamma\Big) \\ &\qquad + N^{k/2+1/4} \sum_{\ell\ge1} \frac{1}{\ell^{k+1/2}} \sum_{\rho\colon \gamma>0} \frac{N^{\beta/2}}{\ell^\beta} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{2}\gamma\Big) \int_0^{\ell^2 N} u^{k/2-3/4+\beta/2} e^{-u}\,\mathrm{d}u \\ &\ll_k N^{k+1} + \Big(\max_\beta \Gamma\Big(\frac{\beta}{2} + \frac{k}{2} + \frac{1}{4}\Big)\Big) N^{k/2+3/4} \sum_{\ell\ge1} \frac{1}{\ell^{k+1/2}} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{2}\gamma\Big) \\ &\ll_k N^{k+1} \end{aligned} \tag{43}$$

provided that $k > 1/2$, where we used the substitution $u = -\ell^2/(Ny^2)$, (12) and standard density estimates.

Let now $y > 0$. Using the Stirling formula (28) and (33) we can write that the quantity in (42) is

$$\ll \sum_{\ell\geq 1} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \exp\Bigl(-\frac{\pi}{4}\gamma\Bigr) \int_0^{1/N} \frac{e^{-\ell^2 N}}{|z|^{k+3/2+\beta}} \,\mathrm{d}y$$

$$+ \sum_{\ell\geq 1} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{+\infty} \exp\Bigl(\gamma(\arctan(Ny) - \frac{\pi}{2})\Bigr) \frac{e^{-\ell^2/(Ny^2)}}{|z|^{k+3/2+\beta}} \,\mathrm{d}y$$

$$= W_1 + W_2, \tag{44}$$

say. Using (13) and (12), we have that

$$W_1 \ll N^{k+3/2} \sum_{\ell\geq 1} e^{-\ell^2 N} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \exp\Bigl(-\frac{\pi}{4}\gamma\Bigr) \ll N^{k+1} \tag{45}$$

by standard density estimates. Moreover we get

$$W_2 \ll \sum_{\ell\geq 1} \sum_{\rho\colon \gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{+\infty} y^{-k-3/2-\beta} \exp\Bigl(-\frac{\gamma}{Ny} - \frac{\ell^2}{Ny^2}\Bigr) \,\mathrm{d}y$$

$$= N^{k/2+1/4} \sum_{\ell\geq 1} \frac{1}{\ell^{k+1/2}} \sum_{\rho\colon \gamma>0} \frac{N^{\beta/2}\gamma^{\beta-1/2}}{\ell^\beta} \int_0^{\ell\sqrt{N}} v^{k-1/2+\beta} \exp\Bigl(-\frac{\gamma v}{\ell\sqrt{N}} - v^2\Bigr) \,\mathrm{d}v,$$

in which we used the substitution $v^2 = \ell^2/(Ny^2)$. Remark now that, for $k > 1$, we can set $\varepsilon = \varepsilon(k) = (k-1)/2 > 0$ and that $k - \varepsilon = (k+1)/2 > 1$. We further remark that $\max_v(v^{k-\varepsilon}e^{-v^2})$ is attained at $v_0 = ((k-\varepsilon)/2)^{1/2}$, and hence we obtain, for N sufficiently large, that

$$W_2 \ll_k N^{k/2+1/4} \sum_{\ell\geq 1} \frac{1}{\ell^{k+1/2}} \sum_{\rho\colon \gamma>0} \frac{N^{\beta/2}\gamma^{\beta-1/2}}{\ell^\beta} \int_0^{\ell\sqrt{N}} v^{\beta-1/2+\varepsilon} \exp\Bigl(-\frac{\gamma v}{\ell\sqrt{N}}\Bigr) \,\mathrm{d}v.$$

Making the substitution $u = \gamma v/(\ell\sqrt{N})$ we have

$$W_2 \ll_k N^{k/2+1/2+\varepsilon/2} \sum_{\ell\geq 1} \frac{1}{\ell^{k-\varepsilon}} \sum_{\rho\colon \gamma>0} \frac{N^\beta}{\gamma^{1+\varepsilon}} \int_0^\gamma u^{\beta-1/2+\varepsilon} e^{-u} \,\mathrm{d}u$$

$$\ll_k N^{k/2+3/2+\varepsilon/2} \sum_{\ell\geq 1} \frac{1}{\ell^{k-\varepsilon}} \sum_{\rho\colon \gamma>0} \frac{1}{\gamma^{1+\varepsilon}} \Bigl(\max_\beta \Gamma\Bigl(\beta + \frac{1}{2} + \varepsilon\Bigr)\Bigr) \ll_k N^{k/2+3/2+\varepsilon/2}, \tag{46}$$

by standard density estimates.

Inserting (45)-(46) into (44) and recalling (43), we get, for $k > 1$, that the quantity in (42) is $\ll N^{k+1}$.

10. Absolute convergence of $\mathcal{J}_4$ and $\mathcal{J}_5$

To study the absolute convergence of the series in $\mathcal{J}_4$ we first remark that, by (5) and (23), we get

$$\sum_{\ell\geq 1} \frac{|J_{k+3/2}(2\pi\ell N^{1/2})|}{\ell^{k+3/2}} \ll_k N^{-k/2-3/4} \sum_{\ell\geq 1} \int_{(1/N)} |e^{Nz}||z|^{-k-5/2} e^{-\pi^2\ell^2\Re(1/z)} \,|\mathrm{d}z|$$

which is the quantity in (32). So the argument in §8 also proves that the series in $\mathcal{J}_4$ converges absolutely for $k > -1/2$.

In fact a more direct argument leads to a better estimate on k. Using, for $\nu > 0$ fixed, $u \in \mathbb{R}$ and $u \to +\infty$, the estimate

$$|J_\nu(u)| \ll_\nu u^{-1/2}$$

which immediately follows from (7) (or from eq. (2.4) of Berndt [1]), and performing a direct computation, we obtain that $\mathcal{J}_4$ converges absolutely for $k > -1$ (and for N sufficiently large) and that $\mathcal{J}_4 \ll_k N^{(k+1)/2}$.

For the study of the absolute convergence of the series in $\mathcal{J}_5$ we have a different situation. In this case the direct argument needs a more careful estimate of the Bessel functions involved since both ν and u are not fixed and, in fact, unbounded. In fact it is easy to see that (7) can be used only if $\nu \in \mathbb{C}$ is bounded, but we are not in this case since $\nu = k + 1/2 + \rho$, where ρ is a nontrivial zero of the Riemann ζ-function. On the other hand, (6) can be used only for u bounded, but again this is not our case since $u = 2\pi\ell N^{1/2}$ and ℓ runs up to infinity. Moreover, the use of the asymptotic relations for $J_\nu(u)$ when $\nu \in \mathbb{C}$ and $u \in \mathbb{R}$ are both "large" seems to be very complicated in this setting.

So it turned out that the best direct approach we are able to perform is the following. By a double partial integration on (6), we immediately get

$$\begin{aligned}
J_\nu(u) &= \frac{2(u/2)^\nu(2\nu-1)}{\pi^{1/2}u^2\Gamma(\nu+1/2)} \int_0^1 \Bigl(1 - \frac{(2\nu-3)t^2}{1-t^2}\Bigr)(1-t^2)^{\nu-3/2}\cos(ut)\,\mathrm{d}t \\
&\ll_{\Re(\nu)} \frac{|u|^{\Re(\nu)-2}|2\nu-1|}{|\Gamma(\nu+1/2)|} \int_0^1 \Bigl(1 + |2\nu-3|\Bigr)|\cos(ut)|\,\mathrm{d}t \\
&\ll_{\Re(\nu)} \frac{|\nu|^2|u|^{\Re(\nu)-2}}{|\Gamma(\nu+1/2)|}, \qquad (47)
\end{aligned}$$

where the last two estimates hold for $\Re(\nu) > 3/2$ and $u > 0$. Inserting (47) into (24) and using the Stirling formula (28), a direct computation shows the absolute convergence of the double sum in $\mathcal{J}_5$ for $k > 2$ (and for N sufficiently large).

Unfortunately, such a condition on k is worse than the one we have in §9. So, coming back to the Sonine representation of the Bessel functions (5) on the line $\Re(s) = 1$ and using the usual substitution $s = Nz$, to study the absolute convergence of the double sum in $\mathcal{J}_5$ we are led to consider the quantity

$$\begin{aligned}
&\sum_\rho \Bigl|\Gamma(\rho)\frac{N^{\rho/2}}{\pi^\rho}\Bigr| \sum_{\ell\ge1} \Bigl|\frac{J_{k+1/2+\rho}(2\pi\ell N^{1/2})}{\ell^{k+1/2+\rho}}\Bigr| \\
&\qquad\qquad \ll_k N^{-k/2-1/4} \sum_\rho |\Gamma(\rho)| \sum_{\ell\ge1} \int_{(1/N)} |e^{Nz}||z|^{-k-3/2}|z^{-\rho}|e^{-\pi^2\ell^2\Re(1/z)}\,|\mathrm{d}z|,
\end{aligned}$$

which is very similar to the one in (42) (the sums are interchanged). It is not hard to see that the argument used in (42)-(46) can be applied in this case too. It shows that the double series in $\mathcal{J}_5$ converges absolutely for $k > 1$ and this condition fits now with the one we have in §9.

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Alessandro Languasco
Università di Padova
Dipartimento di Matematica
Via Trieste 63
35121 Padova, Italy
e-mail: languasco@math.unipd.it

Alessandro Zaccagnini
Università di Parma
Dipartimento di Matematica
Parco Area delle Scienze, 53/a
43124 Parma, Italy
e-mail: alessandro.zaccagnini@unipr.it