The Asymptotic Behavior of Compositions of the Euler and Carmichael Functions

Vishaal Kapoor
vkapoor@google.com

June 4, 2012

Abstract

We compare the asymptotic behavior of $\lambda(\varphi(n))$ and $\lambda(\lambda(n))$ on a set of positive integers n of asymptotic density 1, where λ is Carmichael's λ-function and φ is Euler's totient function. We prove that $\log \lambda(\varphi(n))/\lambda(\lambda(n))$ has normal order $\log\log n \log\log\log n$.

1 Introduction

Euler's totient function $\varphi(n)$ is defined to be the cardinality of the multiplicative group modulo n, for any positive integer n. *Carmichael's λ-function* [2] denotes the cardinality of the largest cycle in the multiplicative group modulo n. In other words, $\lambda(n)$ is the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$ for all reduced residues $a \pmod{n}$. We notice that when the multiplicative group modulo n is cyclic, namely when $n = 1, 2, 4, p^a$ or $2p^a$ where p is an odd prime and $a \geq 1$, both $\varphi(n)$ and $\lambda(n)$ are equal.

One may compute $\varphi(n)$ with the aid of the Chinese remainder theorem by using the formula

$$\varphi(n) = |(\mathbb{Z}/p_1^{a_1}\mathbb{Z})^\times| \times \cdots \times |(\mathbb{Z}/p_k^{a_k}\mathbb{Z})^\times| = p_1^{a_1-1}(p_1-1)\cdots p_k^{a_k-1}(p_k-1).$$

where n has the prime decomposition $n = p_1^{a_1}\cdots p_k^{a_k}$. For Carmichael's function we note

$$\lambda(p^a) = \begin{cases} p^{a-1}(p-1) & \text{if } p \geq 3 \text{ or } a \leq 2, \text{ and} \\ 2^{a-2} & \text{if } p = 2 \text{ and } a \geq 3, \end{cases} \tag{1}$$

together with

$$\lambda(n) = \text{lcm}(\lambda(p_1^{a_1}), ..., \lambda(p_k^{a_k})). \tag{2}$$

In what follows we introduce the following notation. Given two functions $f(n)$ and $g(n)$, we will frequently drop the outer parentheses from the expression $f(g(n))$, instead writing the composition as $fg(n)$. Additionally for $f(n)$ denoting $\lambda(n), \varphi(n)$ or $\log(n)$, we define $f_1(n) = f(n)$ and $f_{k+1}(n) = f(f_k(n))$ for $k \geq 1$. We will use the expression "for almost all n" to mean for n in a set of positive integers of asymptotic density 1, and the expression "for almost all $n \leq x$" to be analogous, but restricting $n \leq x$. We recall that for arithmetic functions $f(n)$ and $g(n)$, we say $f(n)$ has normal order $g(n)$ if $f(n)$ is asymptotic to $g(n)$ for almost all n, or equivalently if $f(n) = (1 + o(1))g(n)$ for almost all n.

The theorem that we prove in this article is:

Theorem 1. *The normal order of* $\log(\lambda\varphi(n)/\lambda\lambda(n))$ *is* $\log_2 n \log_3 n$.

More precisely, we show that for almost all $n \leq x$,

$$\log \frac{\lambda\varphi(n)}{\lambda\lambda(n)} = \log_2 n \log_3 n + O(\psi(x) \log_2 x), \tag{3}$$

where $\psi(x)$ is a function tending to infinity slower than $\log_3 x$. We also show that the exceptional set of positive integers n for equation (3) is of asymptotic density $O(x/\psi(x))$. This work is part of the author's PhD thesis (see [7]).

There has been extensive study on the asymptotic behavior of $\varphi(n)$ and $\lambda(n)$ and their compositions. In 1928, Schoenberg [9] established that the quotient $n/\varphi(n)$ has a continuous distribution function. In other words:

Proposition 2. *The limit*

$$\Phi(t) = \lim_{N \to \infty} |\{n \leq N : n/\varphi(n) \geq t\}|/N$$

exists and is continuous for any real t.

Recently Weingartner [10] studied the asymptotic behavior of $\Phi(t)$ showing that as t tends to infinity, $\log \Phi(t) = -\exp(te^{-\gamma})(1 + O(t^{-2}))$, where $\gamma = 0.5722...$ is Euler's constant.

We mention that higher iterates of $\varphi(n)$ have been studied by Erdős, Granville, Pomerance and Spiro in [4]. They established:

Proposition 3. *The normal order of the* $\varphi_k(n)/\varphi_{k+1}(n)$ *is* $ke^{\gamma} \log_3 n$, *for* $k \geq 1$.

In 1955 Erdős established the normal order of $\log(n/\lambda(n))$ in [3]. This result was refined by Erdős, Pomerance, and Schmutz in [5] where they proved the following result.

Proposition 4. *For almost all $n \leq x$,*

$$\log \frac{n}{\lambda(n)} = \log_2 n(\log_3 n + A + O((\log_3 n)^{-1+\varepsilon}),$$

where

$$A = -1 + \sum_{q \ prime} \frac{q}{(q-1)^2} = .2269688...,$$

and $\varepsilon > 0$ is fixed but arbitrarily small.

The author is undertaking the analysis of Theorem 1 to obtain a more accurate asymptotic formula of a form more closely resembling the previous proposition.

Martin and Pomerance subsequently considered the question of understanding the behavior of $\lambda\lambda(n)$. In [8] they proved

Proposition 5. *For almost all n,*

$$\log \frac{n}{\lambda\lambda(n)} = (1 + o(1))(\log_2 n)^2 \log_3 n. \tag{4}$$

Recently Harland [6] proved a conjecture of Martin and Pomerance concerning the behavior of the higher iterates of $\lambda(n)$:

Proposition 6. *For each $k \geq 1$,*

$$\log \frac{n}{\lambda_k(n)} = \left(\frac{1}{(k-1)!} + o(1)\right)(\log_2 n)^k \log_3 n,$$

for almost all n.

Banks, Luca, Saidak, and Stănică [1] studied the the compositions of λ and φ. In particular, they studied set of n on which $\lambda\varphi(n) = \varphi\lambda(n)$. In their paper, they also established the following:

Proposition 7. *For almost all n,*

$$\log \frac{n}{\varphi\lambda(n)} = (1 + o(1)) \log_2 n \log_3 n, \ \textit{and} \tag{5}$$

$$\log \frac{n}{\lambda\varphi(n)} = (1 + o(1))(\log_2 n)^2 \log_3 n. \tag{6}$$

Consequently, $\log \dfrac{\varphi\lambda(n)}{\lambda\varphi(n)}$ has normal order $(\log_2 n)^2 \log_3 n$.

The proof of Proposition 7 uses a simple clever argument that rests on the theorem of Martin and Pomerance. It is interesting to see what we may obtain trivially from Propositions 5 and 7. Subtracting (5) from (4) gives an asymptotic formula for the comparison between $\varphi\lambda(n)$ and $\lambda\lambda(n)$,

$$\log \frac{\varphi\lambda(n)}{\lambda\lambda(n)} \sim (\log_2 n)^2 \log_3 n,$$

for almost all n. However, if we subtract (6) from (4), the main terms cancel and we are left with

$$\log \frac{\lambda\varphi(n)}{\lambda\lambda(n)} = o((\log_2 n)^2 \log_3 n),$$

for almost all n. This relation is interesting because it leads one to seek a more accurate asymptotic formula. This more accurate result is the content of Theorem 1.

2 Notation and Useful Results

Let $a, n \in \mathbb{Z}$. Then the Brun-Titchmarsh inequality is the asymptotic relationship that

$$\pi(z; n, a) \ll \frac{z}{\varphi(n) \log(z/n)} \qquad (z > n), \tag{7}$$

where $\pi(z; n, a)$ is the number of primes congruent to $a \pmod n$ up to z. We will be primarily concerned with implications of the Brun-Titchmarsh inequality in the case that $a = 1$. For convenience, define $\mathcal{P}_n$ to be the set of primes congruent to $1 \pmod n$, and for a given integer m, define the greatest common divisor of m and $\mathcal{P}_n$, denoted $(m, \mathcal{P}_n)$, to be the product of the primes congruent to $1 \pmod n$ that divides m, or 1 if none exist. We will frequently use the following weaker form of (7) without mention.

Lemma 8 (A Brun-Titchmarsh Inequality)**.** *For all $z > e^e$,*

$$\sum_{\substack{p \le z \\ p \in \mathcal{P}_n}} \frac{1}{p} \ll \frac{\log \log z}{\varphi(n)}. \tag{8}$$

One may obtain (8) from (7) by partial summation. We will also use the following prime estimates stated in [8].

Lemma 9. *Let $z > e$. Then we have the following:*

$$\sum_{p \le z} \log p \ll z, \qquad \sum_{p \le z} \frac{\log p}{p} \ll \log z, \qquad \sum_{p \le z} \frac{\log^2 p}{p} \ll \log^2 z,$$
$$\sum_{p > z} \frac{\log p}{p^2} \ll \frac{1}{z}, \text{ and } \qquad \sum_{p > z} \frac{1}{p^2} \ll \frac{1}{z \log z},$$

These estimates follow via partial summation applied to Mertens' estimate $M(z) = \sum_{p \le z} (\log p)/p = \log z + O(1)$. We illustrate the derivation of the first tail estimate. One writes the Riemann-Steltjies integral

$$\begin{aligned}\sum_{p>z} (\log p)/p^2 &= \int_z^\infty 1/t \, \mathrm{d}M(t) = M(t)/t \Big|_z^\infty + \int_z^\infty M(t)/t^2 \, \mathrm{d}t \\ &= (\log z)/z + O(1/z) + \int_z^\infty (\log t)/t^2 + O(1/t^2) \, \mathrm{d}t \\ &\ll 1/z^2,\end{aligned}$$

as required.

We remind the reader that we will be writing the composition of two arithmetic functions $f(n)$ and $g(n)$ as $fg(n)$, and subscripts will be used with functions to indicate the number of times a function will be composed with itself (ie $\log_2 n = \log\log n$). The multiplicity to which a prime q divides n is denoted by $\nu_q(n)$. In what follows, the variables p, q, r will be reserved for primes. *Throughout, we denote* $y = y(x) = \log_2 x$. The function $\psi(x)$ denotes a function tending to infinity, but slower than $\log y$. When we use the expression "for almost all $n \le x$", we will mean for all positive integers $n \le x$ except those in an exceptional set of asymptotic density $O(x/\psi(x))$. We will make use of two parameters $Y = Y(x)$ and $Z = Z(x)$ in the course of the proof of Theorem 1 which we now define as

$$\begin{aligned} Y &= 3cy, \text{ and} \\ Z &= y^2, \end{aligned}$$

where c is the implicit constant appearing in the Brun-Titchmarsh theorem (7) and (8).

3 The Proof of Theorem 1

We intend to establish an asymptotic formula for

$$\log \frac{\lambda\varphi(n)}{\lambda\lambda(n)} = \sum_q (\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n))) \log q, \tag{9}$$

valid for n in a set of natural density 1. We will consider the "large" q and the "small" q separately. The cut-off for this distinction is the parameter Y giving the cases $q > Y$ and $q \le Y$, respectively.

For $q > Y$, it will be unusual for $\nu_q(\lambda\varphi(n))$ to be strictly larger than $\nu_q(\lambda\lambda(n))$ and so the contribution in (9) from large q will be negligible. We bound the sum in (9) by the two cases,

$$\sum_{q>Y}(\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n)))\log q \le \sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))\ge 2}} \nu_q(\lambda\varphi(n))\log q + \sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))=1}} (\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n)))\log q. \tag{10}$$

We prove the two bounds:

Proposition 10. *For almost all $n \le x$,*

$$\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))=1}} (\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n)))\log q \ll y\psi(x),$$

and

Proposition 11. *For almost all $n \le x$,*

$$\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))\ge 2}} \nu_q(\lambda\varphi(n))\log q \ll y\psi(x). \tag{11}$$

Combining Propositions 10 and 11 gives the upper bound we seek:

Proposition 12. *For almost all $n \le x$,*

$$\sum_{q>Y}(\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n)))\log q \ll y\psi(x). \tag{12}$$

We now consider with those primes $q \le Y$. It will turn out that the main term comes from the quantity $\sum_{q\le Y}\nu_q(\lambda\varphi(n))$ with the sum $\sum_{q\le Y}\nu_q(\lambda\lambda(n))$ sufficiently small.

Proposition 13. *For almost all $n \le x$,*

$$\sum_{q\le Y}\nu_q(\lambda\lambda(n))\log q \ll y\psi(x).$$

We are left with the final piece of establishing the asymptotic behavior of $\sum_{q\le Y}\nu_q(\lambda\varphi(n))$. This will involve a case-by-case analysis of the various ways that q can divide $\lambda\varphi(n)$ with multiplicity. Two functions $g(n)$ and $h(n)$ arise from this analysis:

$$g(n) = \sum_{q \le Y} \sum_{\substack{\alpha \ge 1 \\ q^{\alpha+1} | \varphi(n)}} \log q,$$
$$h(n) = \sum_{q \le Y} \sum_{\substack{\alpha \ge 1 \\ \omega(n, Q_{q^\alpha}) > 0}} \log q, \text{ and}$$
$$Q_{q^\alpha} = \{r \le x : \exists p \in P_{q^\alpha} \text{ st } r \in P_p\}.$$

We will show that $g(n)$ is a good approximation to $\sum_{q \le Y} \nu_q(\lambda\varphi(n))$. To deal with $g(n)$, we will choose a suitably close additive function to approximate $g(n)$ and employ the Turán-Kubilius inequality to find the normal order of $g(n)$.

Proposition 14. *For almost all* $n \le x$,

$$g(n) = y \log y + O(y).$$

Proposition 15. *For almost all* $n \le x$,

$$h(n) \ll \psi(x) y.$$

We will combine these propositions to show

Proposition 16.

$$\sum_{q \le Y} (\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n))) \log q = y \log y + O(\psi(x) y).$$

Summing the results from Propositions 12 and 16 gives

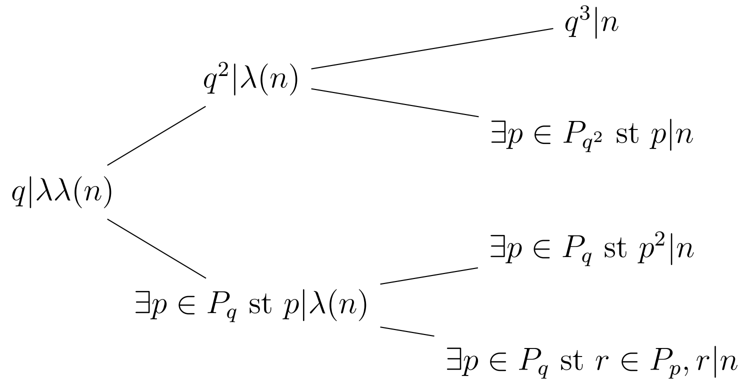
$$\sum_{q} (\nu_q(\lambda\varphi(n)) - \nu_q(\lambda\lambda(n))) \log q = y \log y + O(\psi(x) y),$$

which proves Theorem 1. In the following two sections, we will establish all of the propositions of this section except proposition 12 which we have established.

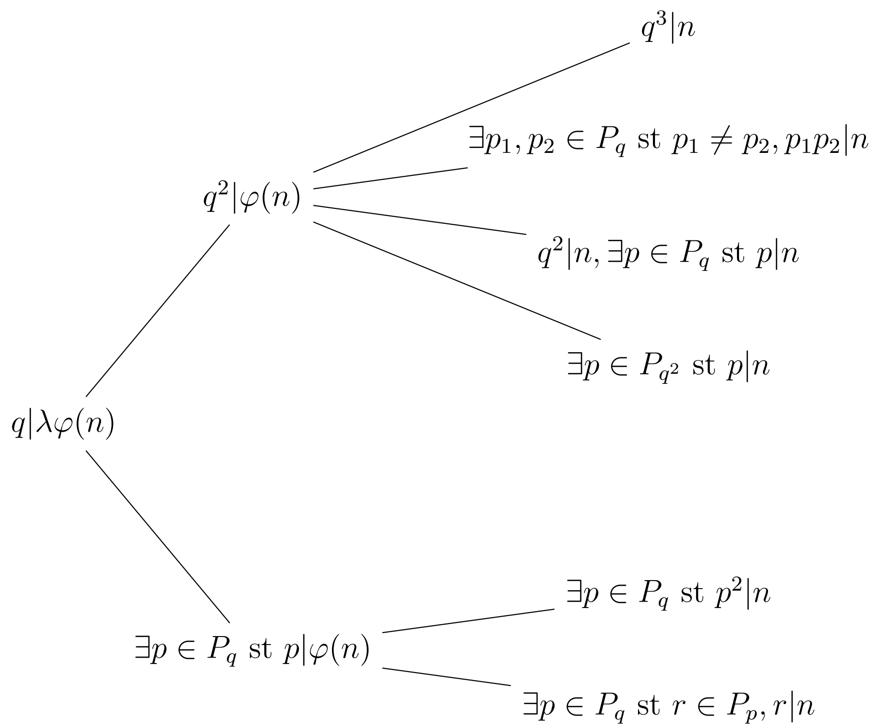
4 Large Primes $q > Y$

In this section we prove Propositions 10 and 11. In order to proceed, we must first understand the different ways in which prime powers can divide $\lambda\lambda(n)$ and $\lambda\varphi(n)$. We assume $Y \ge 2$ so all primes q under consideration are odd.

From the definition $\lambda(n)$ (see (1) and (2)), one sees that $\lambda\lambda(n)$ has q as a prime divisor if q^2 divides $\lambda(n)$ or if n is divisible by some prime in $\mathcal{P}_q$. We emphasize that these conditions are not exclusive. We may expand these conditions in turn. If $q^2|\lambda(n)$, then the higher power q^3 divides n, or a prime in $\mathcal{P}_{q^2}$ divides n; while if some prime $p \in \mathcal{P}_q$ divides $\lambda(n)$, then $p^2|n$, or $(n, \mathcal{P}_p) > 1$. We summarize these cases in the tree diagram below.



We proceed with a similar analysis on the ways that q can be a divisor of $\lambda\varphi(n)$. We saw that either q^2 or some prime in $\mathcal{P}_q$ must divide the argument $\varphi(n)$ of $\lambda\varphi(n)$. If two copies of q divide $\varphi(n)$, then their presence can come from the cube q^3 dividing n, two distinct primes dividing n with each prime in $\mathcal{P}_q$ contributing one factor of q, both $q^2|n$ and a prime $p \in \mathcal{P}_q$ dividing n, or a single prime in $\mathcal{P}_{q^2}$ dividing n. In the other case, if a prime $p \in \mathcal{P}_q$ divides $\varphi(n)$, then $p^2|n$ or $(n, \mathcal{P}_p) > 1$.



Now we turn to the proof of Proposition 10.

Proof of Proposition 10. One sees from the above analysis that $q|\lambda\varphi(n)$ whenever $q|\lambda\lambda(n)$, so the only way $(\nu_q(\lambda\varphi(n))-\nu_q(\lambda\lambda(n)))$ can be nonzero is if $q|\lambda\varphi(n)$ and $q \nmid \lambda\lambda(n)$. Moreover, there are only two ways that q can divide $\lambda\varphi(n)$ but not $\lambda\lambda(n)$; namely, two distinct primes $p_1, p_2 \in P_q$ could divide n, or both q^2 and a single prime $p \in P_q$ could divide n. Thus

$$\begin{aligned}\frac{1}{x}\sum_{n\le x}\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))=1}}(\nu_q(\lambda\varphi(n))-\nu_q(\lambda\lambda(n)))\log q &\le \frac{1}{x}\sum_{q>Y}\sum_{\substack{p_1,p_2\in P_q\\ p_1p_2|n\\ n\le x}}\log q+\frac{1}{x}\sum_{q>Y}\sum_{\substack{n\le x\\ p\in P_q\\ pq^2|n}}\log q\\ &\ll \frac{1}{x}\sum_{q>Y}\left(\frac{xy^2}{q^2}+\frac{xy}{q^3}\right)\log q\\ &\ll y^2/Y,\end{aligned}$$

where we used Lemmata 8 and 9. Plugging in $Y = 3cy$ the upper bound is $\ll y$. We deduce that for almost all $n \le x$,

$$\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))=1}}(\nu_q(\lambda\varphi(n))-\nu_q(\lambda\lambda(n)))\log q \ll y\psi(x).$$

□

Now we would like to show that

$$\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))\ge 2}}\nu_q(\lambda\varphi(n))\log q \ll y^2\psi(x)/Y \tag{13}$$

holds normally.

Proof of Proposition 11. Define $S_q = S_q(x) = \{n \le x : q^2|n \text{ or } p|n \text{ for some } p \in P_{q^2}\}$ and $S = \cup_{q>Y} S_q$. A simple estimate shows that the cardinality of S is $O(xy/(Y\log Y))$. We will choose Y to be of asymptotic order $\gg y$, thus the number of elements in S is $O(x/\psi(x))$. As we are interested in a normality result, we may safely ignore the positive integers in S. Consequently, to establish (13) for almost all n, it suffices to establish the mean value estimate

$$\frac{1}{x}\sum_{\substack{n\le x\\ n\notin S}}\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))\ge 2}}\nu_q(\lambda\varphi(n))\log q \ll y^2/Y. \tag{14}$$

To this end we write

$$\frac{1}{x}\sum_{\substack{n\le x\\ n\notin S}}\sum_{\substack{q>Y\\ \nu_q(\lambda\varphi(n))\ge 2}}\nu_q(\lambda\varphi(n))\log q \le \frac{2}{x}\sum_{\substack{q>Y\\ \alpha\ge 2}}\sum_{\substack{n\le x\\ n\notin S\\ q^\alpha|\lambda\varphi(n)}}\log q$$

$$\le \frac{2}{x}\sum_{\substack{q>Y\\ \alpha\ge 2}}\Bigg(\sum_{\substack{n\le x\\ p\in P_{q^\alpha}\\ p|\varphi(n)}} + \sum_{\substack{n\le x\\ n\notin S\\ q^{\alpha+1}|\varphi(n)}}\Bigg)\log q.$$

In order for the prime p to be a divisor of $\varphi(n)$, one of: p^2 divides n, or $r\in P_p$ and r divides n for some prime r must occur. Thus,

$$\sum_{\substack{n\le x\\ p\in P_{q^\alpha}\\ p|\varphi(n)}} 1 = \sum_{\substack{p\le x\\ p\in P_{q^\alpha}}}\sum_{\substack{n\le x\\ p|\varphi(n)}} 1 \ll \sum_{\substack{p\le x\\ p\in P_{q^\alpha}}}\left(\frac{x}{p^2}+\sum_{\substack{r\le x\\ r\in P_p}}\frac{x}{r}\right) \ll \sum_{p>q^\alpha}\frac{x}{p^2}+\sum_{\substack{p\le x\\ p\in P_{q^\alpha}}}\frac{xy}{p} \ll \frac{x}{\alpha q^\alpha \log q}+\frac{xy^2}{q^\alpha}. \tag{15}$$

Summing over $q>Y$ and $\alpha\ge 2$ and weighting by $\log q$ we have the asymptotic upper bound

$$\frac{1}{x}\sum_{\substack{q>Y\\ \alpha\ge 2}}\sum_{\substack{n\le x\\ p\in P_{q^\alpha}\\ p|\varphi(n)}}\log q \ll y^2/Y.$$

Now we would like to establish

$$\frac{1}{x}\sum_{\substack{q>Y\\ \alpha\ge 2}}\sum_{\substack{n\le x\\ n\notin S\\ q^{\alpha+1}|\varphi(n)}}\log q \ll y^2/Y.$$

We note that the contribution of prime powers of q dividing $\varphi(n)$ for $n\notin S$ can only come from distinct primes in P_q dividing n. We then have

$$\sum_{\substack{n\le x\\ n\notin S\\ q^{\alpha+1}|\varphi(n)}} 1 \ll \frac{1}{(\alpha+1)!}\sum_{p_1,\ldots,p_{\alpha+1}\in P_q}\sum_{p_1\cdots p_{\alpha+1}|n\le x} 1 \ll \frac{x(cy)^{\alpha+1}}{(\alpha+1)!q^{\alpha+1}}, \tag{16}$$

where we intentionally omit the condition that the primes $p_i\in P_q$ are distinct and where c is the constant appearing in the Brun-Titchmarsh theorem. As $Y\ge 2cy$ we have $cy/q\le 1/2$. Thus summing the LHS of (16) over $\alpha\ge 2$ and $q>Y$ and weighting by $\log q$ gives

$$\sum_{q>Y}\sum_{\alpha\ge 3}\frac{xc^\alpha y^\alpha}{\alpha! q^\alpha}\log q \le xc^2y^2\sum_{\alpha\ge 1}\frac{1}{\alpha! 2^\alpha}\sum_{q>Y}\frac{\log q}{q^2} \ll xy^2/Y \tag{17}$$

as required. □

5 Small primes $q \le Y$

In this section we will be concerned with estimates for small primes; namely, we will prove Propositions 13, 14, 15 and 16. The main term in our asymptotic formula will come from Proposition 14 which concerns the sum

$$\sum_{q \le Y} \nu_q(\lambda\varphi(n)) \log q. \tag{18}$$

The remaining two Propositions provide us with error terms.

We restate a Lemma 11 from [8] which we will use:

Lemma 17. *For a power of a prime q^a, the number of positive integers $n \le x$ with q^a dividing $\lambda\lambda(n)$ is $O(xy^2/q^a)$.*

Proof of Proposition 13. We break the summation up into two parts depending on the size of q^α,

$$\begin{aligned}\sum_{q \le Y} \nu_q(\lambda\lambda(n)) \log q &= \sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha | \lambda\lambda(n)}} 1 \\ &\ll \sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha \le Z}} 1 + \sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha > Z \\ q^\alpha | \lambda\lambda(n)}} 1.\end{aligned}$$

We may bound the first sum as

$$\sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha \le Z}} 1 \ll Y \log Z / \log Y.$$

We use an average estimate to bound the second sum. Note

$$\frac{1}{x} \sum_{n \le x} \sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha > Z \\ q^\alpha | \lambda\lambda(n)}} 1 = \frac{1}{x} \sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha > Z}} \sum_{\substack{n \le x \\ q^\alpha | \lambda\lambda(n)}} 1. \tag{19}$$

From Lemma 17, we see (19) is

$$\ll \frac{1}{x} \sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha > Z}} \frac{xy^2}{q^\alpha} \ll \sum_{q \le Y} \frac{y^2 \log q}{Z} \ll \frac{y^2 Y}{Z}.$$

Therefore

$$\sum_{q \le Y} \log q \sum_{\substack{\alpha \ge 1 \\ q^\alpha > Z \\ q^\alpha | \lambda\lambda(n)}} 1 \ll y^2 Y \psi(x)/Z,$$

for almost all $n \le x$. Combining our upper bounds gives

$$\sum_{q \le Y} \nu_q(\lambda\lambda(n)) \log q \ll (Y \log Z/\log Y + y^2 Y/Z)\psi(x),$$

for almost all $n \le x$. Substituting $Y = 3cy$ and $Z = y^2$ gives the theorem. □

Recall q^α divides $\lambda\varphi(n)$ if one of

- $q^{\alpha+1} | \varphi(n)$
- $q^\alpha | p-1, p | r-1, r | n$
- $q^\alpha | p-1, p^2 | n$

occurs. Note that these conditions are not mutually exclusive. We write (18) as

$$\sum_{q \le Y} \nu_q(\lambda\varphi(n)) \log q = g(n) + O\Bigg(h(n) + \sum_{q \le Y} \sum_{\substack{p \in P_{q^\alpha} \\ p^2 | n}} \log q\Bigg),$$

where

$$g(n) = \sum_{q \le Y} \sum_{\substack{\alpha \ge 1 \\ q^{\alpha+1} | \varphi(n)}} \log q,$$

$$h(n) = \sum_{q \le Y} \sum_{\substack{\alpha \ge 1 \\ \omega(n, Q_{q^\alpha}) > 0}} \log q, \text{ and}$$

$$Q_{q^\alpha} = \{r \le x : \exists p \in P_{q^\alpha} \text{ st } r \in P_p\}.$$

Thus, for almost all $n \le x$,

$$\sum_{q \le Y} \nu_q(\lambda\varphi(n)) \log q = g(n) + O(h(n) + \psi(x) \log_2 Y). \tag{20}$$

In the next two sections, we prove Propositions 14 and 15. We see that Proposition 16 follows immediately by applying these two propositions to equation (20) giving

$$\sum_{q \le Y} \nu_q(\lambda\varphi(n)) \log q = y \log y + O(y\psi(x))$$

for almost all $n \le x$, as required.

5.1 Normal order of $g(n)$

Our strategy is to approximate $g(n)$ from above and below by an additive arithmetic function, thus indirectly making $g(n)$ amenable to the Turán-Kubilius inequality. To start, write $g(n)$ as

$$\begin{aligned} g(n) &= \sum_{q \le Y} \sum_{\substack{\alpha \ge 1 \\ q^{\alpha+1} | \varphi(n)}} \log q \\ &= \sum_{q \le Y} (\nu_q(\varphi(n)) - 1) \log q \\ &= \sum_{q \le Y} \sum_{p|n} \nu_q(p-1) \log q - Y(1+o(1)) + O\left(\sum_{q \le Y} \nu_q(n) \log q \right), \end{aligned} \tag{21}$$

where we used the double inequality

$$\sum_{p|n} \nu_q(p-1) \le \nu_q(\varphi(n)) \le \sum_{p|n} \nu_q(p-1) + \nu_q(n).$$

We will use the Turán-Kubilius inequality:

Lemma 18 (The Turán-Kubilius Inequality)**.** *There exists an absolute constant C such that for all additive functions $f(n)$ and all $x \ge 1$ the inequality*

$$\sum_{n \le x} |f(n) - A(x)|^2 \le CxB(x)^2 \tag{22}$$

holds where

$$\begin{aligned} A(x) &= \sum_{p \le x} f(p)/p, \text{ and} \\ B(x)^2 &= \sum_{p^k \le x} |f(p^k)|^2/p^k. \end{aligned}$$

Proof of Proposition 14. We will use Lemma 18 for the additive function $g_0(n) = \sum_{q \le Y} \sum_{p|n} \nu_q(p-1) \log q$. Let $A(x)$ and $B(x)$ be the first and second moments:

$$\begin{aligned} A(x) &= \sum_{r \le x} g_0(r)/r, \text{ and} \\ B(x) &= \sum_{r^k \le x} g_0(r^k)^2/r^k. \end{aligned}$$

Notice that $g_0(r^k) = g_0(r) = \sum_{q\le Y} \nu_q(r-1)\log q$ leading to

$$A(x) = \sum_{r\le x} \frac{1}{r} \sum_{q\le Y} \sum_{p\mid r} \nu_q(p-1)\log q = \sum_{q\le Y} \log q \sum_{r\le x} \frac{\nu_q(r-1)}{r}$$
$$= \sum_{q\le Y} \log q \sum_{\alpha\ge 1} \sum_{\substack{r\le x\\ r\in P_{q^\alpha}}} \frac{1}{r}.$$

We split the sum over α into

$$\sum_{1\le\alpha\le w_q} \sum_{\substack{r\le x\\ r\in P_{q^\alpha}}} \frac{1}{r} + \sum_{\alpha>w_q} \sum_{\substack{r\le x\\ r\in P_{q^\alpha}}} \frac{1}{r},$$

with w_q to be determined later. The first we estimate with Page's theorem and the second we bound with the Brun-Titchmarsh bound

$$\sum_{\substack{r\le x\\ r\equiv 1 \pmod d}} 1/r \ll y/\varphi(d).$$

$$\sum_{\alpha=1}^{\infty} \frac{y}{\varphi(q^\alpha)} + O\left(\frac{y}{q^{w_q}} + w_q\right) = \frac{yq}{(q-1)^2} + O\left(\frac{y}{q^{w_q}} + w_q\right) \tag{23}$$

Note used the bound $1/q^{\lfloor w_q\rfloor+1} = O(1/q^{w_q})$. Taking $w_q = \log y/\log q$ gives an error term of $O(w_q) = O(\log y/\log q)$. Summing (23) over $q \le Y$ weighted by $\log q$ gives the asymptotic formula

$$A(x) = y\sum_{q\le Y} \frac{q\log q}{(q-1)^2} + O\left(\frac{Y\log y}{\log Y} + Y\right)$$
$$= y\log Y + O\left(\frac{Y\log y}{\log Y} + Y\right). \tag{24}$$

Expanding the square, write the second moment $B(x)$ as

$$B(x) = \sum_{q_1,q_2\le Y} \log q_1 \log q_2 \sum_{r\le x} \nu_{q_1}(r-1)\nu_{q_2}(r-1) \sum_{\substack{k\le 1\\ r^k\le x}} 1/r^k.$$

Uniformly in primes r, $\sum_{k\ge 1} 1/r^k \ll 1/r$. We may also express $\nu_{q_i}(r-1)$ $(i=1,2)$ as

$$\nu_{q_i}(r-1) = \sum_{\substack{\alpha_i\ge 1\\ r\in P_{q_i^{\alpha_i}}}} 1,$$

giving the expanded

$$B(x) \ll \sum_{q_1,q_2 \le Y} \log q_1 \log q_2 \sum_{\alpha_1,\alpha_2 \ge 1} \sum_{\substack{r \le x \\ r \in P_{q_1^{\alpha_1}} \cap P_{q_2^{\alpha_2}}}} \frac{1}{r}.$$

We split the sum in q_1, q_2 into the two cases: $q_1 = q_2$ and $q_1 \neq q_2$. For the q_1, q_2 with $q = q_1 = q_2$ we have

$$\begin{aligned} \sum_{q \le Y} (\log q)^2 \sum_{\alpha_1,\alpha_2 \ge 1} \sum_{\substack{r \le x \\ r \in P_{q^{\max(\alpha_1,\alpha_2)}}}} \frac{1}{r} &= \sum_{q \le Y} (\log q)^2 \sum_{\alpha \ge 1} \sum_{\substack{r \le x \\ r \in P_{q^\alpha}}} \frac{\alpha}{r} \\ &\ll \sum_{q \le Y} (\log q)^2 \sum_{\alpha \ge 1} \frac{\alpha y}{q^\alpha} \\ &\ll y \sum_{q \le Y} \frac{(\log q)^2}{q} \\ &\ll y (\log Y)^2. \end{aligned} \tag{25}$$

If q_1 and q_2 are distinct then we have an upper bound (intentionally ignoring the condition that $q_1 \neq q_2$ in the sum)

$$\begin{aligned} \sum_{q_1,q_2 \le Y} \log q_1 \log q_2 \sum_{\alpha_1,\alpha_2 \ge 1} \sum_{\substack{r \le x \\ r \in P_{q_1^{\alpha_1} q_2^{\alpha_2}}}} \frac{1}{r} &\ll \sum_{q_1,q_2 \le Y} \log q_1 \log q_2 \sum_{\alpha_1,\alpha_2 \ge 1} \frac{y}{q_1^{\alpha_1} q_2^{\alpha_2}} \\ &\ll y \sum_{q_1,q_2 \le Y} \frac{\log q_1 \log q_2}{q_1 q_2} \\ &\ll y (\log Y)^2. \end{aligned} \tag{26}$$

Combining (25) and (26) gives

$$B(x) \ll y (\log Y)^2. \tag{27}$$

Using Lemma 18 we may conclude that The statement of Lemma 18 gives us the equation

$$\sum_{n \le x} |g_0(n) - A(x)|^2 \le C x B(x)^2. \tag{28}$$

Thus the set of $n \le x$ on which $g_0(n)$ differs from $A(x)$ by more than y is $O(x(\log Y)^2/y) = O(x/\psi(x))$.

The mean value of $\sum_{q \le Y} \nu_q(n) \log q$ for $n \le x$ is $\ll 1/x \sum_{q \le Y} x \log q/q \ll \sum_{q \le Y} \log q/q \sim \log Y$, so $\sum_{q \le Y} \nu_q(n) \log q \ll \log^2 Y$ for almost all $n \le x$. Thus from (21), we see that for almost all $n \le x$,

$$g(n) = y \log Y + O\left(\frac{Y \log y}{\log Y} + Y\right), \tag{29}$$

Substituting $Y = 3cy$ gives the theorem. □

5.2 Normal order of $h(n)$

Proof of Proposition 15. In order to find an upper bound on a set of asymptotic density 1, we will compute the first moment of $h(n)$:

$$\begin{aligned} H(x) := \frac{1}{x}\sum_{n \le x} h(n) &= \frac{1}{x}\sum_{\substack{q \le Y \\ \alpha \ge 1}} \sum_{\substack{n \le x \\ \omega(n, Q_{q^\alpha}) > 0}} \log q \\ &= \frac{1}{x}\sum_{\substack{q^\alpha \le Z \\ q \le Y \\ \alpha \ge 1}} \sum_{\substack{n \le x \\ \omega(n, Q_{q^\alpha}) > 0}} \log q + \frac{1}{x}\sum_{\substack{q^\alpha > Z \\ q \le Y}} \sum_{\substack{n \le x \\ \omega(n, Q_{q^\alpha}) > 0 \\ \alpha \ge 1}} \log q. \end{aligned}$$

We deal with the two sums in turn.

Small q^α The first part is for small powers of q:

$$\frac{1}{x}\sum_{\substack{q^\alpha \le Z \\ q \le Y}} \sum_{\substack{n \le x \\ \omega(n, Q_{q^\alpha}) > 0}} \log q \le \frac{1}{x}\sum_{\substack{q^\alpha \le Z \\ q \le Y}} \log q \sum_{n \le x} 1 \le \sum_{\substack{q^\alpha \le Z \\ q \le Y}} \log q = \frac{Y \log Z}{\log Y}. \tag{30}$$

Large q^α The second part is for large powers of q. In this case we use a crude estimate that is sufficient for our needs:

$$\begin{aligned}\frac{1}{x}\sum_{\substack{q^\alpha>Z\\ q\le Y}}\sum_{\substack{n\le x\\ \omega(n,Q_{q^\alpha})>0}}\log q &\ll \frac{1}{x}\sum_{\substack{q^\alpha>Z\\ q\le Y}}\log q\sum_{r\in Q_{q^\alpha}}\sum_{\substack{n\le x\\ r\mid n}}1\\ &\ll \frac{1}{x}\sum_{\substack{q^\alpha>Z\\ q\le Y}}\log q\sum_{r\in Q_{q^\alpha}}\frac{x}{r}\\ &\ll \sum_{\substack{q^\alpha>Z\\ q\le Y}}\log q\sum_{p\in P_{q^\alpha}}\sum_{r\in P_p}\frac{1}{r}\\ &\ll y^2\sum_{\substack{q^\alpha>Z\\ q\le Y}}\frac{\log q}{q^\alpha}. \qquad (31)\end{aligned}$$

The RHS of (31) is less than $\sum_{q\le Y}\sum_{\alpha>\log Z/\log q}\log q/q^\alpha \le 2\sum_{q\le Y}\log q/q^{\log Z/\log q} \ll Y/Z$, or alternatively $q^\alpha \ge Z$ and $\sum_{q\le Y}\log q \sim Y$.

Thus

$$\frac{1}{x}\sum_{\substack{q^\alpha>Z\\ q\le Y}}\sum_{\substack{n\le x\\ \omega(n,Q_{q^\alpha})>0}}\log q = O(y^2 Y/Z). \qquad (32)$$

Summing (30) and (32) gives

$$H(x) \ll Y\log Z/\log Y + y^2Y/Z \ll y,$$

where we substituted the values of Y and Z. Thus, for almost all $n\le x$,

$$h(n) \ll y\psi(x).$$

□

Acknowledgements I would like to thank my supervisor, Greg Martin, for his support and encouragement during the preparation of this article.

References

[1] W. D. Banks, F. Luca, F. Saidak, and P. Stănică. Compositions with the Euler and Carmichael functions. *Abh. Math. Sem. Univ. Hamburg*, 75:215–244, 2005.

[2] R. D. Carmichael. On Composite Numbers P Which Satisfy the Fermat Congruence $a^{P-1} \equiv 1 \bmod P$. *Amer. Math. Monthly*, 19(2):22–27, 1912.

[3] P. Erdös. On pseudoprimes and Carmichael numbers. *Publ. Math. Debrecen*, 4:201–206, 1956.

[4] P. Erdős, A. Granville, C. Pomerance, and C. Spiro. On the normal behavior of the iterates of some arithmetic functions. In *Analytic number theory (Allerton Park, IL, 1989)*, volume 85 of *Progr. Math.*, pages 165–204. Birkhäuser Boston, Boston, MA, 1990.

[5] Paul Erdős, Carl Pomerance, and Eric Schmutz. Carmichael's lambda function. *Acta Arith.*, 58(4):363–385, 1991.

[6] Nick Harland. The iterated carmichael lambda function. preprint.

[7] Vishaal Kapoor. *Asymptotic formulae for arithmetic functions*. PhD thesis, The University of British Columbia, April 2011.

[8] Greg Martin and Carl Pomerance. The iterated Carmichael λ-function and the number of cycles of the power generator. *Acta Arith.*, 118(4):305–335, 2005.

[9] Isac Schoenberg. Über die asymptotische Verteilung reeller Zahlen mod 1. *Math. Z.*, 28(1):171–199, 1928.

[10] Andreas Weingartner. The distribution functions of $\sigma(n)/n$ and $n/\varphi(n)$. *Proc. Amer. Math. Soc.*, 135(9):2677–2681 (electronic), 2007.