ARITHMETIC, GEOMETRIC AND HARMONIC MEAN FOR ACCRETIVE-DISSIPATIVE MATRICES

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Abstract. The concept of Löwner (partial) order for general complex matrices is introduced. After giving the definition of arithmetic, geometric and harmonic mean for accretive-dissipative matrices, we study their basic properties, in particular, A-G-H mean inequality is established for two accretive-dissipative matrices in the sense of this extended Löwner order.

1. Introduction

Let $M_n(\mathbf{C})$ be the space of complex matrices of size $n \times n$. For any $T \in M_n(\mathbf{C})$, we can write

$$T = A + iB, \tag{1.1}$$

in which $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$ are both Hermitian. This is called the Toeplitz decomposition (sometimes also called Cartesian decomposition) of T. For Hermitian matrices, there is an important partial order called Löwner (partial) order which says that for two Hermitian matrices $A, B \in M_n(\mathbf{C})$, $A > (\geq) B$ provided $A - B$ is positive (semi)definite. However, a similar partial order for general complex matrices seems lacking. The unique decomposition (1.1) enables us to give a natural extension of the Löwner order for general complex matrices.

Let $T, S \in M_n(\mathbf{C})$, with their Toeplitz decompositions

$$T = A + iB,\ S = C + iD, \tag{1.2}$$

we define the partial order $T \succ (\succeq) S$ provided that both $A > (\geq) C$ and $B > (\geq) D$.

Recall that a matrix $T \in M_n(\mathbf{C})$ is said to be accretive-dissipative if, in its Toeplitz decomposition (1.1), both matrices A and B are positive definite[1]. The set of accretive-dissipative matrices of order n will be denoted by M_n^{++}. The symbol M_n^{+} will be used to denote the broader set of accretive-dissipative matrices defined by the conditions $A \geq 0$ and $B \geq 0$. Obviously, both M_n^{++} and M_n^{+} are cones. There are several recent work devoted to studying this kind of matrix (see [7, 9, 10, 14]) and more generally, matrices with positive real part (see [2, 3, 15]).

Our main focus is the possible extension of the properties of Hermitian positive definite matrices in the set of accretive-dissipative matrices. The latter behaves differently from the former. For example, the set of accretive-dissipative matrices is not closed under inversion. The partial order for general complex matrices is also not a trivial extension of the Löwner order for Hermitian matrices.

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[1]Since the numerical range of T is easily seen to be in the first quadrant, another terminology for this kind of matrix may be *first quadrant matrix*.

Let $T \in M_n^{++}$, then we know (see e.g.,kato) there is a unique square root of T, denoted by $T^{\frac{1}{2}}$, that belongs to M_n^{++}.

The famous Löwner-Heniz theorem (see e.g., [16]) states that

For Hermitian positive definite matrices $A, C \in M_n(\mathbf{C})$, Then

$$A \geq C \text{ ensures } A^r \geq C^r \quad \text{for any } r \in [0,1].$$

But this fails for accretive-dissipative matrices as the following example shows,

Example 1.1. *Let $T = 32I + 24iI$, $S = 7I + 24iI$ (throughout, I denotes the identity matrix of an appropriate size). Obviously, $T \succeq S$. However, $T^{\frac{1}{2}} = 6I + 2iI$, $S^{\frac{1}{2}} = 4I + 3iI$, so we don't have $T^{\frac{1}{2}} \succeq S^{\frac{1}{2}}$.*

The paper is organized as follows: in Section 2, we define the arithmetic, geometric and harmonic mean for accretive-dissipative matrices and study their basic properties; in Section 3, we present the partial order between these three means, including a partial order between harmonic mean and the parallel sum of two accretive-dissipative matrices; in the final section, some concluding remarks are given.

2. Arithmetic, Geometric and Harmonic mean

For two Hermitian positive definite matrices $A, C \in M_n(\mathbf{C})$, we use the following notation for the arithmetic, geometric and harmonic means of A and C, respectively (see [12]):

$$\begin{aligned} A \nabla C &= \frac{A+C}{2}, \\ A \sharp C &= A^{\frac{1}{2}} (A^{-\frac{1}{2}} C A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}, \\ A ! C &= 2(A^{-1} + C^{-1})^{-1}. \end{aligned}$$

In this section, we extend the above three means to the cone of accretive-dissipative matrices. For $T, S \in M_n^{++}$, the arithmetic mean T and S is naturally defined by

$$\frac{T+S}{2}.$$

Next, we define the geometric mean of T and S (also denoted by $T \sharp S$) by the maximum (in the sense of partial order) of an $X \in M_n(\mathbf{C})$ such that

$$\begin{bmatrix} T & X \\ X & S \end{bmatrix} \succeq 0.$$

Write $X = Y + iZ$ to be the Toeplitz decomposition of X, then it means

$$\begin{bmatrix} A & Y \\ Y & C \end{bmatrix} + i \begin{bmatrix} B & Z \\ Z & D \end{bmatrix} \succeq 0,$$

or the maximum of Hermitian matrices Y and Z such that

$$\begin{bmatrix} A & Y \\ Y & C \end{bmatrix} \geq 0 \text{ and } \begin{bmatrix} B & Z \\ Z & D \end{bmatrix} \geq 0.$$

The maximum of such Y, Z are $A \sharp C$, $B \sharp D$, respectively (e.g., [13]). Thus, we can write

$$T \sharp S = A \sharp C + i B \sharp D.$$

Our harmonic mean of T and S (also denoted by $T!S$) is defined by the maximum of all $X \in M_n(\mathbf{C})$ for which

$$\begin{bmatrix} 2T & 0 \\ 0 & 2S \end{bmatrix} \succeq \begin{bmatrix} X & X \\ X & X \end{bmatrix}.$$

Write $X = Y + iZ$ to be the Toeplitz decomposition of X, then it means the maximum of Hermitian matrices Y and Z such that

$$\begin{bmatrix} 2A & 0 \\ 0 & 2C \end{bmatrix} \succeq \begin{bmatrix} Y & Y \\ Y & Y \end{bmatrix} \text{ and } \begin{bmatrix} 2B & 0 \\ 0 & 2D \end{bmatrix} \succeq \begin{bmatrix} Z & Z \\ Z & Z \end{bmatrix}$$

It also turns out that

$$T!S = A!C + iB!D.$$

Proposition 2.1. *Let $T, S \in M_n^{++}$, then for any nonsingular $Q \in M_n(\mathbf{C})$, we have*

$$(Q^*TQ)\sharp(Q^*SQ) = Q^*(T\sharp S)Q.$$

Proof. We write T, S as in (1.2), then

$$\begin{aligned} (Q^*TQ)\sharp(Q^*SQ) &= (Q^*AQ + iQ^*BQ)\sharp(Q^*CQ + iQ^*DQ) \\ &= (Q^*AQ)\sharp(Q^*CQ) + i(Q^*BQ)\sharp(Q^*DQ) \\ &= Q^*(A\sharp C)Q + iQ^*(B\sharp D)Q \\ &= Q^*(A\sharp C + iB\sharp D)Q \\ &= Q^*(T\sharp S)Q. \end{aligned}$$

□

Proposition 2.2. *Let $T, S \in M_n^{++}$. If $TS = ST$ and either S or T is normal, then*

$$T\sharp S = A^{\frac{1}{2}}C^{\frac{1}{2}} + iB^{\frac{1}{2}}D^{\frac{1}{2}}.$$

Proof. We may assume that S is normal and write T, S as in (1.2). Since $TS = ST$, then by Fuglede's theorem [6], we have $TS^* = S^*T$ and so $ST^* = T^*S$. Hence

$$(T + T^*)(S + S^*) = (S + S^*)(T + T^*)$$

$$(T - T^*)(S - S^*) = (S - S^*)(T - T^*)$$

i.e.,

$$AC = CA, \; BD = DB.$$

Therefore, $T\sharp S = A\sharp C + iB\sharp D = A^{\frac{1}{2}}C^{\frac{1}{2}} + iB^{\frac{1}{2}}D^{\frac{1}{2}}$. □

The following example shows that generally we don't have $T\sharp S = T^{\frac{1}{2}}S^{\frac{1}{2}}$ even when both T, S are normal and commute.

Example 2.3. *Let $T = 3I + 4iI$, $S = 15I + 8iI$, then $T^{\frac{1}{2}} = 2I + iI$, $S^{\frac{1}{2}} = 4I + iI$. Obviously, T, S are normal and commute. However,*

$$T\sharp S = 3\sqrt{5}I + 4\sqrt{2}iI \neq 7I + 6iI = T^{\frac{1}{2}}S^{\frac{1}{2}}.$$

3. A-G-H mean inequality and more

With the notion of arithmetic, geometric and harmonic mean of accretive-dissipative matrices developed in Section 2, we can write down the A-G-H mean inequality for accretive-dissipative matrices:

Theorem 3.1. *Let $T, S \in M_n^{++}$, then*

$$T \nabla S \succeq T \sharp S \succeq T!S. \tag{3.1}$$

The inequality (3.1) in turn demonstrates the our definition is a faithful one.

Let $T \in M_n^{++}$ as in (1.1), set

$$T^{-1} = E + iF, \ E = E^*, F = F^*.$$

Then [15]

$$E = (A + BA^{-1}B)^{-1}, \ F = -(B + AB^{-1}A)^{-1}.$$

In a similar manner, we can show,

$$\begin{aligned} T = (T^{-1})^{-1} &= (E + iF)^{-1} \\ &= (E + FE^{-1}F)^{-1} - i(F + EF^{-1}E)^{-1}. \end{aligned}$$

Thus, comparing the real and imaginary part, we have the following identities:

$$\begin{aligned} A^{-1} &= (A + BA^{-1}B)^{-1} \\ &\quad +(B + AB^{-1}A)^{-1}(A + BA^{-1}B)(B + AB^{-1}A)^{-1}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} B^{-1} &= (B + AB^{-1}A)^{-1} \\ &\quad +(A + BA^{-1}B)^{-1}(B + AB^{-1}A)(A + BA^{-1}B)^{-1}. \end{aligned} \tag{3.3}$$

Using Sherman-Morrison-Woodbury matrix identity [8], we have $(A+BA^{-1}B)^{-1} = A^{-1} - A^{-1}B(A + BA^{-1}B)^{-1}BA^{-1}$, after some rearrangement, one could indeed verify the above two identities.

Recall that for Hermitian positive definite matrices $A, C \in M_n(\mathbf{C})$, the parallel sum $A : C$ is given by

$$A : C = (A^{-1} + C^{-1})^{-1}.$$

Thus, $A!C = 2(A : C)$ for Hermitian positive definite matrices.

Using a property of accretive-dissipative matrices (see [7, Property 1]), we know that $T : S \in M_n^{++}$, provided both $T, S \in M_n^{++}$. It is curious to know the relation between $T!S$ and $T : S$ for $T, S \in M_n^{++}$. The remaining of this section is devoted to this problem.

Let $A, B \in M_n(\mathbf{C})$ be Hermitian, and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}, \ B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} \tag{3.4}$$

be comformably partitioned such that the diagonal blocks are square. If A_{22} is invertible, then the Schur complement of A_{22} in A is denoted by $A/A_{22} := A_{11} - A_{12}A_{22}^{-1}A_{12}^*$.

Lemma 3.2. *Consider the partition as in (3.4). If A_{22}, B_{22} and $A_{22} + B_{22}$ are invertible, then*

$$(A + B)/(A_{22} + B_{22}) = A/A_{22} + B/B_{22} + X(A_{22}^{-1} + B_{22}^{-1})^{-1}X^*, \tag{3.5}$$

where $X = A_{12}A_{22}^{-1} - B_{12}B_{22}^{-1}$.

Proof.

$$
\begin{aligned}
& X(A_{22}^{-1}+B_{22}^{-1})^{-1}X^* \\
= \; & (A_{12}A_{22}^{-1}-B_{12}B_{22}^{-1})B_{22}(A_{22}+B_{22})^{-1}A_{22}(A_{22}^{-1}A_{12}^*-B_{22}^{-1}B_{12}^*) \\
= \; & (A_{12}A_{22}^{-1}B_{22}-B_{12})(A_{22}+B_{22})^{-1}(A_{12}^*-A_{22}B_{22}^{-1}B_{12}^*) \\
= \; & (A_{12}A_{22}^{-1}B_{22}+A_{12}-A_{12}-B_{12})(A_{22}+B_{22})^{-1}(A_{12}^*-A_{22}B_{22}^{-1}B_{12}^*) \\
= \; & (A_{12}A_{22}^{-1}B_{22}+A_{12})(A_{22}+B_{22})^{-1}(A_{12}^*-A_{22}B_{22}^{-1}B_{12}^*) \\
& -(A_{12}+B_{12})(A_{22}+B_{22})^{-1}(A_{12}^*-A_{22}B_{22}^{-1}B_{12}^*) \\
= \; & A_{12}A_{22}^{-1}(A_{12}^*-A_{22}B_{22}^{-1}B_{12}^*) \\
& -(A_{12}+B_{12})(A_{22}+B_{22})^{-1}(A_{12}^*+B_{12}^*-B_{12}^*-A_{22}B_{22}^{-1}B_{12}^*) \\
= \; & A_{12}A_{22}^{-1}A_{12}^*-A_{12}B_{22}^{-1}B_{12}^*-(A_{12}+B_{12})(A_{22}+B_{22})^{-1}(A_{12}+B_{12})^* \\
& +(A_{12}+B_{12})(A_{22}+B_{22})^{-1}(B_{12}^*+A_{22}B_{22}^{-1}B_{12}^*) \\
= \; & A_{12}A_{22}^{-1}A_{12}^*-A_{12}B_{22}^{-1}B_{12}^*-(A_{12}+B_{12})(A_{22}+B_{22})^{-1}(A_{12}+B_{12})^* \\
& +(A_{12}+B_{12})B_{22}^{-1}B_{12}^* \\
= \; & A_{12}A_{22}^{-1}A_{12}^*+B_{12}B_{22}^{-1}B_{12}^*-(A_{12}+B_{12})(A_{22}+B_{22})^{-1}(A_{12}+B_{12})^*.
\end{aligned}
$$

It becomes clear after writing down $(A+B)/(A_{22}+B_{22})$, A/A_{22} and B/B_{22} explicitly. □

Remark 3.3. *The formula (3.5) presents the sum of two Schur complements, for the difference of two Schur complements, see* [4]. *A similar formula can also be found in* [14]. *By Lemma 3.2, if A, B are positive definite, then $(A+B)/(A_{22}+B_{22}) \geq A/A_{22}+B/B_{22}$, see* [5, Theorem 1].

Theorem 3.4. *Let $T, S \in M_n^{++}$ as in (1.2), then*

$$2(T:S) \succeq T!S.$$

Proof. Denoted by $M=(A+BA^{-1}B)^{-1}+(C+DC^{-1}D)^{-1}$, $N=(B+AB^{-1}A)^{-1}+(D+CD^{-1}C)^{-1}$, then

$$T:S=(M+NM^{-1}N)^{-1}+i(N+MN^{-1}M)^{-1}.$$

From the expression

$$
\begin{aligned}
\begin{bmatrix} M & N \\ N & -M \end{bmatrix} \; = \; & \begin{bmatrix} (A+BA^{-1}B)^{-1} & (B+AB^{-1}A)^{-1} \\ (B+AB^{-1}A)^{-1} & -(A+BA^{-1}B)^{-1} \end{bmatrix} \\
& + \begin{bmatrix} (C+DC^{-1}D)^{-1} & (D+CD^{-1}C)^{-1} \\ (D+CD^{-1}C)^{-1} & -(C+DC^{-1}D)^{-1} \end{bmatrix}
\end{aligned}
$$

and by Lemma 3.2, we have

$$
\begin{aligned}
M+NM^{-1}N \; = \; & (A+BA^{-1}B)^{-1}+(B+AB^{-1}A)^{-1}(A+BA^{-1}B)(B+AB^{-1}A)^{-1} \\
& +(C+DC^{-1}D)^{-1}+(D+CD^{-1}C)^{-1}(C+DC^{-1}D)(D+CD^{-1}C)^{-1} \\
& +\text{a Hermitian negative definite matrix, say } R \\
= \; & A^{-1}+C^{-1}+R \\
\leq \; & A^{-1}+C^{-1},
\end{aligned}
$$

where the second equality is by (3.2) and (3.3).

Thus $(A^{-1}+C^{-1})^{-1} \leq (M+NM^{-1}N)^{-1}$. The role of A, C, M and B, D, N are symmetric, so we also have $(B^{-1}+D^{-1})^{-1} \leq (N+MN^{-1}M)^{-1}$. This completes the proof. □

The following example shows that there is not a partial order for $2(T:S)$ and $T\sharp S$ for $T, S \in M_n^{++}$.

Example 3.5. *Let $T = I + iI$, $S = I + 2iI$, then*

$$T\sharp S = I + \sqrt{2}iI, \ 2(T:S) = \frac{14}{13}I + \frac{6}{13}iI.$$

There is no ordering between $2(T:S)$ and $T\sharp S$ in this case.

4. Concluding remarks

This paper defines the arithmetic, geometric and harmonic mean of two accretive-dissipative matrices. It opens a door to study the Löwner order of complex matrices, in particular the Löwner order of accretive-dissipative matrices. We have seen though not all properties in Hermitian positive definite matrices have its (direct) counterpart in the set of accretive-dissipative matrices, some connection and analogy still exist. It is expected that many interesting results on this aspect can be found in the near future.

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