EXPECTED PATTERNS IN PERMUTATION CLASSES

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ABSTRACT. In the set of all patterns in S_n, it is clear that each k-pattern occurs equally often. If we instead restrict to the class of permutations avoiding a specific pattern, the situation quickly becomes more interesting. Miklós Bóna recently proved that, surprisingly, if we consider the class of permutations avoiding the pattern 132, all other non-monotone patterns of length 3 are equally common. In this paper I examine the class Av(123), and give exact formula for the occurrences of each length 3 pattern. While this class does not break down as nicely as Av(132), we find some interesting similarities between the two and prove that the number of 231 patterns is the same in each.

1. BACKGROUND

Let $p = p_1 p_2 \ldots p_n$ be a permutation in the symmetric group S_n, written in one-line notation. Given a permutation $q \in S_k$, say that p contains q as a pattern (denoted $q \prec p$) if there exist k indices $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$ such that the entries $p_{i_1} p_{i_2} \ldots p_{i_k}$ are in the same relative order as the entries of q. If p does not contain q as a pattern, we say that p avoids q.

The set of all permutations, equipped with this ordering, can be viewed as a partially ordered set which is graded with respect to permutations length. With this in mind, we define a *permutation class* to be a downset (or ideal) of this poset. That is, a class is a collection of permutations $\mathcal{C}$ for which, if $p \in \mathcal{C}$ and $q \prec p$, then $q \in \mathcal{C}$.

Given a pattern q, the set $\mathrm{Av}(q)$ of all permutations avoiding q forms a natural permutation class, and much study has been devoted to understanding and enumerating these classes. An early result in the area, due to Knuth, is that the number of n-permutations avoiding any length three pattern is equal to the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. A more comprehensive introduction to permutation patterns can be found in [1].

A question of Joshua Cooper and a result of Mikós Bóna have opened up a new line of research: in a given permutation class, what can be said about the average number of occurences of each pattern? It is simple to show that in the class of all permutations, all patterns are equally common. The situation becomes much more interesting as we restrict our attention to smaller classes.

2. Preliminaries

Definition 1. Let p, q be permutations. Denote by $f_q(p)$ the number of occurences of q in p as a pattern.

For example, $f_{213}(462513) = 2$ since the first third and fourth entries as well as the third fifth and sixth entries form 213 patterns. Also, for any permutation p, $f_{21}(p)$ counts the number of inversions of p. Note that every permutation statistic can be expressed through combinations of permutation patterns, as described in [4].

We'll be concerned primarily with the total number of patterns in a class of permutations. For simplicity, we use similar notation.

Definition 2. Let $\mathcal{C}$ denote a permutation class, and q a pattern. Define $f_q(\mathcal{C}) = \sum_{p \in \mathcal{C}} f_q(p)$. We omit the $\mathcal{C}$ when the class in question is unambiguous.

Example 3. Let $q \in S_k$. Then it follows by linearity of expectation that

$$f_q(S_n) = \frac{n!}{k!}\binom{n}{k}.$$

However, if we restrict ourselves to proper classes, the situation becomes more interesting.

The Catalan numbers will appear frequently in our enumeration, and so it will be useful to establish some standard notation and a few simple identities.

Definition 4. Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number. Also, let

$$C(x) = \sum_{n \geq 0} c_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Fact 5. The following identities follow directly from the recurrence $C(x) = xC(x)^2 + 1$.

$$C(x)^2 = \frac{C(x)}{1 - xC(x)} = \frac{1}{(1 - xC(x))^2} \quad \text{and} \quad \frac{C(x) - 1}{C(x)} = xC(x).$$

In [2] and [3], Miklós Bóna studied the class $Av_n(132)$ and found some interesting symmetries. He also gave exact formula and generating functions for the expectation of all length three patterns. In this paper we give a similar classification of the class $Av_n(132)$, with some equally surprising connections to $Av_n(132)$.

Because 132 is an involution and $231^{-1} = 312$, we have the following identity. Further identites, however, require considerably more effort.

Fact 6. In $Av_n(132)$, $f_{231} = f_{312}$.

For a given integer k, inversion provides a simple bijection from the set of permutations containing exactly k 231 patterns to the set containing exactly k 312 patterns, proving not only that the total number of each pattern is the

same, but that these numbers are equidistributed across the class $\mathrm{Av}_n(132)$. Bóna showed that equidistribution is not required for the total number of patterns to be equal.

Theorem 7 (Bóna). *In* $\mathrm{Av}_n(132)$, *the total numbers of* 231, 213, *and* 312 *patterns are equal, and their enumerations are given by the generating function*

$$\frac{x^2 C(x)^3}{(1 - 2xC(x))(1 - 4x)^{\frac{3}{2}}}.$$

Furthermore, 321 *is the most common pattern and* 123 *is the least common.*

Note that f_{231} and f_{312} are *not* equidistributed as statistics in $\mathrm{Av}_n(132)$. Theorem 7 was proved with a bijection from patterns to patterns, not necessarily respecting the underlying permutation.

Turning our attention to the class $\mathrm{Av}_n(123)$, we will similarly classify the expectation of all length 3 patterns and provide both generating functions and exact formula. In addition, we show some interesting and surprising connections to patterns in $\mathrm{Av}_n(132)$.

3. THE CLASS $\mathrm{Av}(123)$

3.1. **Patterns of length** 2. The simplest place to start is with patterns of length 2. Since 21 patterns correspond to inversions, these numbers have already been studied, most notably in [5]. Clearly, the total number of 12 patterns plus the number of 12 patterns gives the total number of pairs of entries in all permutations in the class, which is given by $\binom{n}{2} c_n$.

Theorem 8 (Cheng, Eu, Fu). *Let* $\mathcal{C} = \mathrm{Av}_n^*(123)$. *Then*

$$\mathrm{f}_{12}(\mathcal{C}) = 4^{n-1} - \binom{2n-1}{n}.$$

$$F = \sum_{n \geq 0} f_{12}(\mathcal{C}) x^n = \frac{x^2 C(x)^2}{1 - 4x}.$$

Corollary 9. *The number of* 21 *patterns in the class of all n-permutations avoiding* 123 *is given by*

$$\mathrm{f}_{21}(\mathrm{Av}_n(123)) = \binom{n}{2} c_n + \binom{2n-1}{n} - 4^{n-1}.$$

3.2. **Patterns of length** 3. We turn our attention now to patterns of length 3, and provide a similar classification. To start, using the fact that 123 is an involution provides some immediate identities, since inversion provides a natural map from the set $\mathrm{Av}_n(123)$ to itself.

Fact 10. *In* $\mathrm{Av}_n(123)$, $\mathrm{f}_{132} = \mathrm{f}_{213}$ *and* $\mathrm{f}_{231} = \mathrm{f}_{312}$.

Numerical data and intuition suggest that $f_{132} < f_{231} < f_{321}$. We begin by establishing some basic relationship between these numbers, which will eventually combine to give us exact formulae. First, note that the total number of length k patterns is exactly $\binom{n}{k} c_n$. For $k = 3$ this gives the following fact.

Fact 11. In the class $\mathrm{Av}_n(123)$ we have that

$$f_{132} + f_{213} + f_{231} + f_{312} + f_{321} = \binom{n}{3} c_n.$$

Note that since $f_{132} = f_{213}(\mathcal{C})$ and $f_{312} = f_{231}$, we can rewrite this as

$$2 f_{213} + 2 f_{231} + f_{321} = \binom{n}{3} c_n.$$

Our next uses Theorem 8 to provide another linear relationship between these three numbers.

Proposition 12. *In the class $\mathrm{Av}_n(123)$, we have*

$$4 f_{213} + 2 f_{231} = (n-2) f_{12}.$$

Proof. Rewrite the equation as

$$(n-2) f_{12} - (f_{132} + f_{213}) = f_{132} + f_{213} + f_{231} + f_{312}.$$

Claim that both sides count the number of ordered triples of entries which contain at least one 12 pattern. Indeed, the right hand side counts all triplets which do not form a 321 pattern. The left hand side first takes a 12 pattern and adds another entry to it. However, this double counts each triple which has two 12 patterns, and these are exactly the patterns 132 and 213. $\square$

Note that we now have two linear relationships between the three unknown quantities, and so some new information would completely solve the system. We summarize this in the following lemma.

Lemma 13. *Let $\mathcal{C} = \mathrm{Av}_n(123)$, and let $a_n = f_{132}(\mathcal{C}) = f_{213}(\mathcal{C})$, $b_n = f_{231}(\mathcal{C}) = f_{312}(\mathcal{C})$, and $d_n = f_{321}(\mathcal{C})$. Then we have*

$$\begin{aligned} 2a + 2b + d &= \binom{n}{3} c_n \\ 4a + 2b &= 4^{n-1} - \binom{2n-1}{n} \end{aligned}.$$

We note that Proposition 12 has a complementary analogue, obtained by counting inversions instead of noninversions. However, this leads to a relation which is linearly dependent on the first two. It takes a new approach to yield new information. We need a few new definitions before proceeding.

Definition 14. A permutation $p = p_1 p_2 \ldots p_n$ is *decomposable* if there exists $k \in [n]$ such that for all $i \le k$ and all $j < k$, we have that $p_i > p_j$.
An *indecomposable* permutation is one for which no such k exists.

Definition 15. Denote the class of all indecomposable 123 avoiding permutations by $\mathrm{Av} *(123)$, and $\mathrm{Av}^*(123) \cap S_n$ by $\mathrm{Av}_n^*(123)$.

In general, for simplicity of notation, indecomposability will be denoted with a star. Our first step, naturally, is to enumerate the size of the set $\mathrm{Av}_n^*(123)$.

Proposition 16. *For all $n \geq 1$,*

$$|\mathrm{Av}_n^*(123)| = \frac{1}{n}\binom{2n-2}{n-2} = c_{n-1}.$$

Proof. We know that $|\mathrm{Av}_n(123)| = c_n$, so

$$\sum_{n \geq 0}|\mathrm{Av}_n(123)|x^n = \frac{1 - \sqrt{1-4x}}{2x} = C(x).$$

Let $C^*(x) = \sum_{n \geq 1}|\mathrm{Av}_n^*(123)|x^n$. Every permutation in $\mathrm{Av}_n^*(123)$ can be expressed as a skew sum of indecomposable 123 avoiding permutations, so it follows that

$$C(x) = 1 + C^*(x) + (C^*(x))^2 + (C^*(x))^3 + \ldots = \frac{1}{1 - C^*(x)}.$$

Solving this algebraically gives that $C^*(x) = \frac{C(x)-1}{C(x)} = xC(x)$, which finishes the proof. $\square$

Lemma 13 now has an immediate indecomposable analogue, and the numbers $\mathrm{f}_q(\mathrm{Av}_n(123))$ and $\mathrm{f}_q(\mathrm{Av}_n^*(123))$ can be related relatively easily. However, this alone does not allow us to solve for an exact formula.

Our new information will come from exactly counting the number of 213 patterns in the set $\mathrm{Av}_n^*(123)$ by building a bijection to Dyck paths. We start by defining these paths, which are counted by the Catalan numbers.

Definition 17. A *Dyck path* of length $2n$ (or of semilength n) is defined as a sequence of steps from the set $\{(1,1),(-1,1)\}$ which begins at $(0,0)$, ends at $(2n,0)$, and never steps below the line $x = 0$.

Lemma 18. *The generating function A^* for the number of 213 patterns in $\mathrm{Av}_n^*(123)$ is given by*

$$A^*(x) = \frac{x^3C(x)}{(1-4x)^{3/2}} = \frac{x^2}{2(1-4x)^{3/2}} - \frac{x^2}{2(1-4x)}.$$

Proof. The proof consists of three parts: First, we examine the structure of permutations in $\mathrm{Av}_n^*(123)$, and find a way of counting the number of 213 patterns. Second, we build a bijection onto Dyck paths which maps 213 patterns to a path statistic. Finally, we find the weighted sum of all Dyck paths with respect to this statistic.

Fix n, and let a permutation $p \in \mathrm{Av}_n^*(123)$. Note that since p avoids 123, p can be viewed as a union of two descending sequences, so every entry in p is a left-to-right minima or a right-to-left maxima, and by indecomposability no entry is both. Graph p on an $n \times n$ lattice by plotting $(i, p(i))$ for each $i \in [n]$, and color each left-to-right minima red and each right-to-left maxima

blue. Denote the sequence of red entries (ordered from left to right) by $\mathcal{R} = (r_1, r_2, \ldots r_j)$, and the sequence of blue entries by $\mathcal{B} = (b_1, b_2, \ldots b_k)$. Denote by $\text{Span}\, b_i$ the number of red entries below and to the left of b_i. Now, we count the number of 213 patterns in p. It follows that for any such pattern q, the 2 entry and 1 entry must be red, and the 3 entry blue. It is also clear that each blue entry is contained in $\binom{\text{Span}\, b_i}{2}$ 213 patterns. Therefore we have that

$$\mathrm{f}_{213}\, p = \sum_{i=1}^{k} \binom{\text{Span}\, b_i}{2}.$$

Now we are ready to build our bijection $\phi : \mathrm{Av}_n^*(123) \to \mathcal{D}_{n-1}$, where $\mathcal{D}_{n-1}$ denotes the set of Dyck paths of semilength $n - 1$. From each blue vertex, extend a vertical line to the x-axis and a horizontal line to the y-axis, and color each point of intersection of these lines green. Define a path P' from $(1, n)$ to $(n, 1)$ by the following rules:

1) Begin by walking east from $(1, n)$
2) At a blue vertex, turn south and continue walking
3) At a green vertex, turn eash and continue walking
4) End at $(n, 1)$

Rotate the path P' by $\pi/4$ radians counter-clockwise to obtain a Dyck path P. This path is a slight modification of the path given by Krattenthaler's bijection ([8]), taking advantage of the indecomposability of the permutation. This geometric interpretation of the bijection gives some additional insight into the number of 213 patterns.
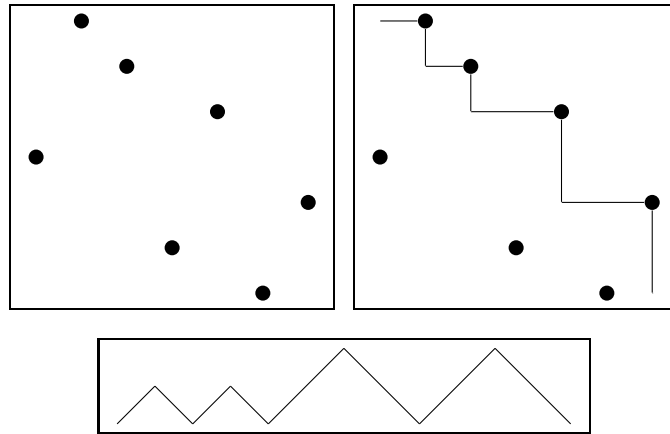

FIGURE 1. $\phi(4762513) = UDUDUUDDUUDD$

Note that each blue entry in p produces a peak in P. Furthermore, b_i corresponds to a peak of height $\text{Span}\, b_i$ above the x-axis in P. Therefore, if we let $h_{n,k}$ denote the total number of peaks of height k in all Dyck paths

of semilength n, we have that

$$\mathrm{f}_{213}(\mathrm{Av}_n^*(123)) = \sum_{k=1}^{n-1} \binom{k}{2} h_{n-1,k}.$$

Finally, we can compute $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$ as follows. First, note that since each Dyck path begins with an upstep, it has a unique first point at which the path returns to the x-axis, so we can decompose each path P of length n into the concatenation of two shorter paths Q and R, with $P = uQdR$, where u denotes an upstep and d a downstep. Then each peak of height $k - 1$ in Q and height k in R leads to a peak of height k in P. With this in mind, we have the following generating function relation:

$$H(x, u) = ux(H(x, u) + 1)C(x) + xH(x, u)C(x).$$

Here the first term counts the peaks from the uQd part, including the case when Q is empty. The second term counts the contribution from the R part. Rearranging leads to

$$H(x, u) = \frac{uxC(x)}{1 - uxC(x) - xC(x)}.$$

Now, to count 213 patterns, we need to count each peak with weight $\binom{k}{2}$. By taking derivatives twice with respect to u, setting $u = 1$, dividing by two and scaling by x, we find that

$$\sum_{n,k \geq 0} \binom{k}{2} h_{n-1,k} x^n = x \frac{\partial_u^2 H\big|_{u=1}}{2} = \frac{x^3 C(x)}{(1 - 4x)^{\frac{3}{2}}}$$

$$= x^3 + 7x^4 + 38x^5 + 187x^6 + 874x^7 + \dots .$$

The sequence $0, 0, 1, 7, 38, 187 \dots$ is entry A000531 in the OEIS. Finally, the correspondence between peaks and 213 patterns completes the proof. $\square$

Now, it is relatively simple to move from the class of indecomposable 123 avoiding permutations to the larger class of all 123 avoiding permutations.

Theorem 19. *Let a_n be the number of 213 patterns in Av_n 123. Then*

$$\sum_{n \geq 0} a_n x^n = \frac{x^3 C(x)^3}{(1 - 4x)^{3/2}} = \frac{x - 1}{2(1 - 4x)} - \frac{3x - 1}{2(1 - 4x)^{3/2}}.$$

Proof. Let $A(x)$ be the generating function for the numbers a_n, and let $A^*(x)$ denote the generating function for the number of 213 patterns in *indecomposable* 123 avoiding permutations.

Now, any permutation p in $\mathrm{Av}(123)$ can be written uniquely as a skew sum of a nonempty indecomposable 123 avoiding permutation q and another, possibly empty, 123 avoiding permutation r. Now, it is clear that any 213 pattern in p must be contained entirely in either q or r. This leads to the following relation:

$$A(x) = A^*(x)C(x) + xC(x)A(x).$$

Solving for A gives

$$A(x) = \frac{A^*(x)C(x)}{1 - xC(x)} = C^2(x)A^*(x).$$

Lemma 18 now implies

$$A(x) = \frac{x^3 C(x)^3}{(1 - 4x)^{3/2}}.$$

$\square$

Theorem 19 combined with Lemma 13 allows us to obtain both generating functions and exact formula for the occurrence of all length 3 patterns in $\mathrm{Av}_n(123)$. We start with 231 patterns, which reveal a striking connection to the class $\mathrm{Av}(132)$.

Corollary 20. *Let b_n denote the number of 231 (or 312) patterns in all 123 avoiding n-permutations. Then*

$$\sum_{n \geq 0} b_n x^n = \frac{3x - 1}{(1 - 4x)^2} - \frac{4x^2 - 5x + 1}{(1 - 4x)^{5/2}}.$$

Proof. Let $B(x)$ be the generating function for the numbers b_n, let $A(x)$ be the generating function for the number of 213 patterns, and let f_n be the number of 12 patterns with corresponding generating function F. We know from Lemma 13 that

$$4A(x) + 2B(x) = \sum_{n \geq 0} (n - 2) f_n x^n = (F(x)/x^2)' x^3.$$

Solving this for $B(x)$ using elementary algebra and a bit of calculus yields

$$\begin{aligned} B(x) &= \frac{x^2 C(x)^3}{(1 - 2xC(x))(1 - 4x)^{3/2}} \\ &= \frac{3x - 1}{(1 - 4x)^{3/2}} - \frac{4x^2 - 5x + 1}{(1 - 4x)^{5/2}} \end{aligned}$$

$\square$

Corollary 21. *Theorem 20 together with Theorem 7 imply immediately that the total number of 231 patterns in $\mathrm{Av}_n(123)$ is equal to the total number of 231 patterns in $\mathrm{Av}_n(132)$.*

We can similarly apply Lemma 13 to 321 patterns.

Corollary 22. *Let $d_n = f_{321}(\mathrm{Av}_n(123))$. Then we have that*

$$\sum_{n \geq 0} d_n x^n = \frac{8x^3 - 20x^2 + 8x - 1}{(1 - 4x)^2} - \frac{36x^3 - 34x^2 + 10x - 1}{(1 - 4x)^{5/2}}.$$

Before analyzing these generating functions, we note also that Lemma 13 and its indecomposable analogue produce several other identities. We summarize some of them here for completeness.

Corollary 23. *The following identities hold.*

$$\mathrm{f}_{21}(\mathrm{Av}_n(123)) = 2\,\mathrm{f}_{213}(\mathrm{Av}_n^*(123))$$

$$\mathrm{f}_{213}(\mathrm{Av}_n(123)) + \mathrm{f}_{231}(\mathrm{Av}_n(123)) = \mathrm{f}_{231}(\mathrm{Av}_{n-1}^*(123))$$

$$C(x)\left(\sum_{n\geq 0}\mathrm{f}_{213}(\mathrm{Av}_n(123))x^n\right) = xC'(x)\left(\sum_{n\geq 0}\mathrm{f}_{12}(\mathrm{Av}_n(123))x^n\right)$$

$$\sum_{n\geq 0}\mathrm{f}_{213}(\mathrm{Av}_n^*(132)x^n) = \sum_{n\geq 0}\left(\mathrm{f}_{132}(\mathrm{Av}_n^*(123)) + \mathrm{f}_{231}(\mathrm{Av}_n^*(123))\right)x^n.$$

Note that each of these identities are equivalent. That is, combined with Lemma 13, any of them would imply all of the others (including Lemma 18).

Now we can do a bit of analysis of the main sequences. Using some standard generating function analysis ([7]), we find that the asymptotic growth of the number of lenght 3 patterns are as follows:

$$\mathrm{f}_{213}(\mathrm{Av}_n(123)) \sim \sqrt{\frac{n}{\pi}}4^n$$

$$\mathrm{f}_{231}(\mathrm{Av}_n(123)) \sim \frac{n}{2}4^n$$

$$\mathrm{f}_{321}(\mathrm{Av}_n(123)) \sim \frac{8}{3}\sqrt{\frac{n^3}{\pi}}4^n.$$

We see that the three sequences each differ by a factor of approximately $\sqrt{n}$. Surprisingly, this is the same factor that the sequences $\mathrm{f}_{123}, \mathrm{f}_{231}, \mathrm{f}_{321}$ differ by in the class $\mathrm{Av}(132)$, as seen in [2].

Each of these generating functions are simple enough that exact formulas can be obtained with relatively little hassle. One could argue that the asymptotic values are more interesting and provide more insight than the complicated formulas, but we present them here for completeness.

Corollary 24. *Let $a_n = \mathrm{f}_{132}(\mathrm{Av}_n(123))$, $b_n = \mathrm{f}_{213}(\mathrm{Av}_n(123))$, and $d_n = \mathrm{f}_{321}(\mathrm{Av}_n(123))$. Then we have that*

$$a_n = \frac{n+2}{4}\binom{2n}{n} - 3\cdot 2^{2n-3}$$

$$b_n = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4)\cdot 2^{2n-3}$$

$$d_n = \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2}$$
$$- 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1)\cdot 4^{n-1}.$$

3.3. **Larger Patterns.** Some of these same techniques are modifiable for larger patterns. For example, we can easily modify Lemma 13 to for patterns of all sizes. These leads to increasingy complicated expressions, but this idea can be used to prove the following proposition.

Proposition 25. *Let $k, l \in \mathbb{Z}^+$, and q be any permutation in S_k other than the decreasing permutation. Then for n large enough, we have that*

$$\mathrm{f}_{k\ldots321}(\mathrm{Av}_n(123)) > \mathrm{f}_q(\mathrm{Av}_n(123)).$$

Proof. Let T be the set of permutation in S_k which are not the descending permutation. As in Fact 11 and Proposition 12, we can express $(n - k + 1)\,\mathrm{f}_{12}(\mathrm{Av}_n(123))$ as a positive linear combination of all of $\mathrm{f}_q(\mathrm{Av}_n(123))$ where $q \in T$, and we can express $\binom{n}{k}c_n$ as the sum of all $\mathrm{f}_r(\mathrm{Av}_n(123))$ where $r \in S_n$. It follows that there is a positive integer m and positive integers e_i such that

$$\binom{n}{k}c_n - (n - m + 1)\,\mathrm{f}_{12}(\mathrm{Av}_n(123)) = \mathrm{f}_{k\ldots321} - \sum_{q \in T} e_i\,\mathrm{f}_q(\mathrm{Av}_n(123)).$$

Asymptotic analysis shows that the left hand side is eventually positive, and so the first term on the right side eventually beats the second term, which completes the proof. □

4. Further Directions

The numbers $\mathrm{f}_q(\mathrm{Av}\,p)$ for permutations p, q exhibit numerous symmetries and produce many new questions. All of the generating presented here and in [2] are almost rational, in the sense that they lie in the ring $\mathbb{Q}(x, \sqrt{1 - 4x})$. This allows for easy asymptotic analysis, and leaves open the possibility of bijections to other Catalan-related objects.

Building on what was mentioned in [3], we have instances of the same sequence of numbers which correspond sums of statistics with different distributions in objects counted by the Catalan numbers. Do these sequences and statistics have anologues in other such objects?

Thus far, to the author's knowledge, the distribution of patterns has only been studied for the classes $\mathrm{Av}(123)$ and $\mathrm{Av}(132)$ (and their symmetries). Applying these to more general classes could yield similarly interesting identities. Note that the increasing and decreasing permutation do not always provide the extreme case: it is simple to show that $\mathrm{f}_{123}(\mathrm{Av}(2413)) = \mathrm{f}_{321}(\mathrm{Av}(2413))$. This leads to the natural question: in the set of permutations avoiding a specific pattern (or a set of patterns), can we easily determine what pattern is most common?

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