# Extreme values for two-dimensional discrete Gaussian free field

Jian Ding
Stanford University
& MSRI

Ofer Zeitouni*
University of Minnesota
& Weizmann institute

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**Abstract**

We consider in this paper the collection of near maxima of the discrete, two dimensional Gaussian free field in a box with Dirichlet boundary conditions. We provide a rough description of the geometry of the set of near maxima, estimates on the gap between the two largest maxima, and precise (in the exponential scale) estimate for the right tail on the law of the centered maximum.

## 1 Introduction

The discrete Gaussian free field (GFF) $\{\eta_v : v \in V_N\}$ on a 2D box $V_N$ of side length $N$ with Dirichlet boundary condition, is a mean zero Gaussian process which takes the value 0 on $\partial V_N$ and satisfies the following Markov field condition for all $v \in V_N \setminus \partial V_N$: $\eta_v$ is distributed as a Gaussian variable with variance 1 and mean equal to the average over the neighbors given the GFF on $V_N \setminus \{v\}$ (see later for more formal definitions). One facet of the GFF that has received intensive attention is the behavior of its maximum. In this paper, we prove a number of results involving the maximum and near maxima of the GFF. Our first result concerns the geometry of the set of near maxima and states that the vertices of large values are either close to or far away from each other.

**Theorem 1.1.** *There exists an absolute constant $c > 0$,*

$$\lim_{r\to\infty}\lim_{N\to\infty} \mathbb{P}(\exists v, u \in V_N : r \leqslant |v-u| \leqslant N/r \text{ and } \eta_u, \eta_v \geqslant m_N - c\log\log r) = 0\,, \tag{1}$$

*where $m_N = \mathbb{E}\max_{v\in V_N}\eta_v$.*

(The asymptotic behavior of $m_N$ is recalled in (4) below.) In addition, we show that the number of particles within distance $\lambda$ from the maximum grows exponentially.

**Theorem 1.2.** *For $\lambda > 0$, let $A_{N,\lambda} = \{v \in V_N : \eta_v \geqslant m_N - \lambda\}$ for $\lambda > 0$. Then there exist absolute constants $c, C$ such that*

$$\lim_{\lambda\to\infty}\lim_{N\to\infty} \mathbb{P}(c\mathrm{e}^{c\lambda} \leqslant |A_{N,\lambda}| \leqslant C\mathrm{e}^{C\lambda}) = 1\,.$$

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Another important characterization of the joint behavior for the near maxima is the spacings of the ordered statistics, out of which the gap between the largest two values is of particular interest. We show that the right tail of this gap is of Gaussian type, as well as that the gap is of order 1.

**Theorem 1.3.** *Let $\Gamma_N$ be the gap between the largest and the second largest values in $\{\eta_v : v \in V_N\}$. Then, there exists absolute constant $C > 0$ such that for all $\lambda > 0$ and all $N \in \mathbb{N}$*

$$c\mathrm{e}^{-C\lambda^2} \leqslant \mathbb{P}(\Gamma_N \geqslant \lambda) \leqslant C\mathrm{e}^{-c\lambda^2}\,, \tag{2}$$

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}(\Gamma_N \leqslant \delta) = 0\,. \tag{3}$$

Finally, we compute the exponent in the right tail for the maximum. Set $M_N = \max_v \eta_v$.

**Theorem 1.4.** *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that for all $\lambda \in [0, \sqrt{\log N})$,*

$$C_\varepsilon^{-1}\mathrm{e}^{-(\sqrt{2\pi}+\varepsilon)\lambda} \leqslant \mathbb{P}(M_N > m_N + \lambda) \leqslant C_\varepsilon\mathrm{e}^{-(\sqrt{2\pi}-\varepsilon)\lambda}\,.$$

**Related work.** The study on the maximum of the GFF goes back at least to Bolthausen, Deuschel and Giacomin [10] who established the law of large numbers for $M_N/\log N$ by associating with the GFF an appropriate branching structure. Afterwards, the main focus has shifted to the study of fluctuations of the maximum. Using hypercontractivity estimates, Chatterjee [15] showed that the variance of the maximum is $o(\log n)$, thus demonstrating a better concentration than that guaranteed by the Borell-Sudakov-Tsirelson isoperimetric inequality, which is however still weaker than the correct $O(1)$ behavior. Later, Bolthausen, Deuschel and Zeitouni [11] proved that $(M_n - \mathbb{E}M_n)$ is tight along a deterministic subsequence $(n_k)_{k\in\mathbb{N}}$; they further showed that in order to get rid of the susbequence, it suffices to compute a precise estimate (up to additive constant) on the expectation of the maximum. An estimate in such precision was then achieved by Bramson and Zeitouni [14], by comparing the GFF with the modified branching random walk (MBRW) introduced therein. They showed that the sequence of random variables $M_N - m_N$ is tight, where

$$m_N = 2\sqrt{2/\pi}\big(\log N - \tfrac{3}{8\log 2}\log\log N\big) + O(1)\,. \tag{4}$$

Using the "sprinkling method", this was later improved by Ding [19], who showed that there exist absolute constants $C, c > 0$ so that for all $N \in \mathbb{N}$ and $0 \leqslant \lambda \leqslant (\log N)^{2/3}$

$$c\mathrm{e}^{-C\lambda} \leqslant \mathbb{P}(M_N \geqslant m_N + \lambda) \leqslant C\mathrm{e}^{-c\lambda}\,,\ \text{and}\ c\mathrm{e}^{-C\mathrm{e}^{C\lambda}} \leqslant \mathbb{P}(M_N \leqslant m_N - \lambda) \leqslant C\mathrm{e}^{-c\mathrm{e}^{c\lambda}}\,. \tag{5}$$

Note that our Theorem 1.4 gives a partial improvement upon (5).

In contrast with the reasearch activity concerning the maximum of the GFF, not much has been done concerning its near maxima. To our knowledge, the only work in literature is due to Daviaud [16] who studied the geometry of the set of large values of the GFF which are within a *multiplicative* constant from the expected maximum, i.e. those values above $\eta m_N$ with $\eta \in (0, 1)$.

In contrast with the GFF, much more is known concerning both the location of the maximum and the structure of near maxima for the model of branching Brownian motions. The study of the maximum of the BBM dated back to a classical paper by Kolmogorov, Petrovskii, and Piscounov [26], where they studied its connection with the so-called KPP-equation. The probabilistic interpretation of the KPP-equation in terms of BBM, described in McKean [30], was further exploited by Bramson [12, 13]. It was then proved that both the left and right tails exhibit exponential decay and the precise exponents were computed. See, e.g., Bramson [13] and Harris [24] for the right tail, and see Arguin, Bovier and Kistler [8] for the left tail (the argument is due to De Lellis). In addition, Lalley and Sellke [27] obtained an integral representation for the limiting law of the centered maximum.

More recently, the structure of the point process of maxima of the BBM was described in great detail, in a series of papers by Arguin, Bovier and Kistler [8, 7, 6] and in a paper by Aïdékon, Berestycki, Brunet and Shi [3]. These papers describe the scaling limit of the process of extremes of BBM, as a certain Poisson point process with exponential density where each atom is decorated by an independent copy of an auxiliary point process.

Our results in this work are a first step in the study of the process of extrema for the GFF. In particular, Theorem 1.1 is a precise analog of results in [8], while Theorems 1.2 and 1.3 provide weaker results than those of [7].

Finally, a connection between the maximum of the GFF and the cover time for the random walk has been shown in Ding, Lee and Peres [21] and Ding [18]. In particular, an upper bound on the fluctuation of the cover time for 2D lattice was shown in [20] using such a connection, improving a previous work of Dembo, Peres, Rosen and Zeitouni [17]. It is worthwhile emphasizing that the precise estimate on the expectation of the maximum of the GFF in [14] plays a crucial role in [20].

**A word on proof strategy.** A general approach in the study of the maximum of the GFF, which we also follow, is to compare the maxima of the GFF and of Gaussian processes of relative amenable structures; this is typically achieved using comparison theorems for Gaussian processes (see Lemmas 2.2 and 2.5). The first natural "comparable" process is the branching random walk (BRW) which admits a natural tree structure (although [10] do not use directly Gaussian comparisons, the BRW features implicitly in their approach). In [14], the modified branching random walk (see Subsection 2.1) was introduced as a finer approximation of the GFF, based on which a precise (up to additive constant) estimate on the expectation of the maximum was achieved.

Our work also uses comparisons of the GFF with the MBRW/BRW. One obstacle we have to address is the lack of effective, direct comparisons for the collection of near maxima of two Gaussian processes. We get around this issue by comparing a certain functional of the GFF, which could be written as the maximum of a certain associated Gaussian process. Various such comparisons between the GFF and the MBRW/BRW are employed in Section 2. In particular, we use a variant of Slepian's inequality that allows one to compare the sum of the $m$-largest values for two Gaussian processes. Afterwards, estimates for the aforementioned functionals of MBRW/BRW are computed in Section 3. Finally, based on the estimates of these functionals of the GFF (obtained via comparison), we deduce our main theorems in Section 4.

Along the way, another method that was used often is the so-called *sprinkling* method, which in our case could be seen as a two-level structure. Sprinkling method was initialized by Ajtai, Komlós and Szemerédi [4] in the study of percolation, and found its applications later in that area (see, e.g., [5, 9]). In the context of the study of the maximum of the GFF, this method was first successfully applied in [19]; an application to the study of cover times of random walks appears in [18].

**Discussions and open problems.** There are a number of natural open problems in this line of research on the GFF, of which establishing the limiting law of the maximum and the scaling limit of the extreme process are of great interest. Even partial progresses toward these goals could be interesting. For instance, it would be of interest to provide more information on the joint behavior of the maxima by characterizing other important statistics, or to obtain more refined estimates on the tails of the maximum. Let us also point out that we computed the exponent only for the right tail as in Theorem 1.4, but not for the left tail. A conceptual difficulty in computing the exponent in the left tail is that the MBRW has Gaussian type left tail (analogous to BRW) as opposed to doubly-exponential tail in (5) — the top levels in the MBRW could shift the value of the whole process to the left with a Gaussian type cost in probability, while in GFF the Dirichlet boundary condition decouples the GFF near the boundary such that the GFF behaves almost independently close to the boundary. Therefore, it is possible that a new approximation needs to be introduced in order to study the left tail of the maximum in higher precision (merely using the sprinkling method as done in [18] seems unlikely to yield the exponent).

**Three perspectives of Gaussian free field.** A quick way to rigourously define GFF is to give the density function. Denoting by $f$ be the density function of $\eta_v$, we have

$$f((x_v)) = Z\mathrm{e}^{-\frac{1}{16}\sum_{u\sim v}(x_u - x_v)^2}\,, \tag{6}$$

where $Z$ is a normalizing constant and $x_v = 0$ for $v \in \partial V_N$. (Note that each edge appears *twice* in (6).)

Alternatively, a slower but more informative way to define GFF is using the connection with random walks (in particular, Green functions). Consider a connected graph $G = (V, E)$. For $U \subset V$, the Green function $G_U(\cdot,\cdot)$ of the discrete Laplacian is given by

$$G_U(x, y) = \mathbb{E}_x(\textstyle\sum_{k=1}^{\tau_U - 1}\mathbf{1}\{S_k = y\})\,,\ \text{for all } x, y \in V\,, \tag{7}$$

where $\tau_U$ is the hitting time of the set $U$ for random walk $(S_k)$, defined by (the notation applies throughout the paper)

$$\tau_U = \min\{k \geqslant 0 : S_k \in U\}\,. \tag{8}$$

The GFF $\{\eta_v : v \in V\}$ with Dirichlet boundary on $U$ is then defined to be a mean zero Gaussian process indexed by $V$ such that the covariance matrix is given by Green function $(G_U(x, y))_{x,y\in V}$. It is clear to see that $\eta_v = 0$ for all $v \in U$. For equivalence of definitions in (6) and (7), c.f., [25].

Finally, we give a connection between the GFF and the electric network. We can view the 2D box $V_N$ as an electric network where each edge is placed a unitary resistor and the boundary is wired together. We then associate a classic quantity to the network, the so-called *effective resistance*, which is denoted by $R_{\mathrm{eff}}(\cdot,\cdot)$. The following well-known identity relates the GFF to the electric network, see, e.g., [25, Theorem 9.20].

$$\mathbb{E}\,(\eta_u - \eta_v)^2 = 4R_{\mathrm{eff}}(u, v). \tag{9}$$

Note that the factor of 4 above is due to a different normalization we are selecting in the 2D lattice (in general, this factor is 1 with a standard normalization).

## 2 Comparisons with modified branching random walk

In this section, we compare the maxima of Gaussian free field with those of the so-called modified branching random walk (MBRW), which was introduced in [14].

## 2.1 A short review on MBRW

Consider $N = 2^n$ for some positive integer $n$. For $k \in [n]$, let $\mathcal{B}_k$ be the collection of squared boxes in $\mathbb{Z}^2$ of side length $2^k$ with corners in $\mathbb{Z}^2$, and let $\mathcal{BD}_k$ denote the subsets of $\mathcal{B}_k$ consisting of squares of the form $([0, 2^k - 1] \cap \mathbb{Z})^2 + (i2^k, j2^k)$. For $v \in \mathbb{Z}^2$, let $\mathcal{B}_k(v) = \{B \in \mathcal{B}_k : v \in B\}$ be the collection of boxes in $\mathcal{B}_k$ that contains $v$, and define $\mathcal{BD}_k(v)$ be the (unique) box in $\mathcal{BD}_k$ that contains $v$. Furthermore, denote by $\mathcal{B}_k^N$ the subset of $\mathcal{B}_k$ consisting of boxes whose lower left corners are in $V_N$. Let $\{a_{k,B}\}_{k \geqslant 0, B \in \mathcal{BD}_k}$ be i.i.d. standard Gaussian variables, and define the branching random walk to be

$$\vartheta_v = \textstyle\sum_{k=0}^{n} a_{k, \mathcal{BD}_k(v)} \,. \tag{10}$$

For $k \in [n]$ and $B \in \mathcal{B}_k^N$, let $b_{k,B}$ be independent centered Gaussian variables with $\mathrm{Var}(b_{k,B}) = 2^{-2k}$, and define

$$b_{k,B}^N = b_{k,B'} , \text{ for } B \sim_N B' \in \mathcal{B}_k^N \,, \tag{11}$$

where $B \sim_N B'$ if and only if there exist $i, j \in \mathbb{Z}$ such that $B = (iN, jN) + B'$ (note that for any $B \in \mathcal{B}_k$, there exists a unique $B' \in \mathcal{B}_k^N$ such that $B \sim_N B'$). In a manner compatible with definition in (11), we let $d_N(u, v) = \min_{w \sim_N v} \|u - w\|$ be the $\ell^2$ distance between $u$ and $v$ when considering $V_N$ as a torus, for all $u, v \in V_N$. Finally, we define the MBRW $\{\xi_v^N : v \in V_N\}$ such that

$$\xi_v^N = \textstyle\sum_{k=0}^{n} \sum_{B \in \mathcal{B}_k(v)} b_{k,B}^N \,. \tag{12}$$

The motivation of the above definition is that the MBRW approximates the GFF with high precision. That is to say, the covariance structure of the MBRW approximates that of the GFF well. This is elaborated in the next lemma which compares their covariances (see [14, Lemma 2.2] for a proof).

**Lemma 2.1.** *There exists a constant $C$ such that the following holds with $N = 2^n$ for all $n$.*

$$|\mathrm{Cov}(\xi_u^N, \xi_v^N) - (n - \log_2(d^N(u, v)))| \leqslant C, \text{ for all } u, v \in V_N \,,$$
$$\left|\mathrm{Cov}(\eta_u^{4N}, \eta_v^{4N}) - \tfrac{2\log 2}{\pi}(n - (0 \vee \log_2 \|u - v\|))\right| \leqslant C, \text{ for all } u, v \in (2N, 2N) + V_N \,.$$

## 2.2 Comparison of the maximal sum over restricted pairs

In this subsection, we approximate the GFF by the MBRW based on the following comparison theorem on the expected maximum of Gaussian process (See e.g., [23] for a proof).

**Lemma 2.2** (Sudakov-Fernique)**.** *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a\}_{a \in \mathcal{A}}$ and $\{Y_a\}_{a \in \mathcal{A}}$ be two centered Gaussian processes such that*

$$\mathbb{E}(X_a - X_b)^2 \geqslant \mathbb{E}(Y_a - Y_b)^2 \,, \text{ for all } a, b \in \mathcal{A} \,. \tag{13}$$

*Then $\mathbb{E} \max_{a \in \mathcal{A}} X_a \geqslant \mathbb{E} \max_{a \in \mathcal{A}} Y_a$.*

Instead of directly comparing the expected maximum as in [14], we compare the following two functionals for GFF and MBRW respectively. For an integer $r$, define

$$\begin{aligned} \eta_{N,r}^{\diamond} &= \max\{\eta_v^N + \eta_u^N : u, v \in V_N, r \leqslant \|u - v\| \leqslant N/r\} \,, \\ \xi_{N,r}^{\diamond} &= \max\{\xi_v^N + \xi_u^N : u, v \in V_N, r \leqslant \|u - v\| \leqslant N/r\} \,. \end{aligned} \tag{14}$$

The main goal in this subsection is to prove the following.

**Proposition 2.3.** *There exists constant $\kappa \in \mathbb{N}$ such that for all $r, n$ with $N = 2^n$*

$$\sqrt{\tfrac{2\log 2}{\pi}}\mathbb{E}\xi^{\diamond}_{2^{-\kappa}N,r} \leqslant \mathbb{E}\eta^{\diamond}_{N,r} \leqslant \sqrt{\tfrac{2\log 2}{\pi}}\mathbb{E}\xi^{\diamond}_{2^{\kappa}N,r}\,.$$

In order to prove the preceding proposition, it is convenient to consider

$$\tilde{\eta}^{\diamond}_{N,r} = \max\{\eta^{4N}_{v+(2N,2N)} + \eta^{4N}_{u+(2N,2N)} : u, v \in V_N, r \leqslant \|u - v\| \leqslant N/r\}\,.$$

We start with proving the next useful lemma.

**Lemma 2.4.** *Using the above notation, we have*
*(i)* $\mathbb{E}\eta^{\diamond}_{N,r} \leqslant \mathbb{E}\tilde{\eta}^{\diamond}_{N,r}$*;*
*(ii)* $\mathbb{P}(\max_{v\in V_N} \eta^N_v \geqslant \lambda) \leqslant 2\mathbb{P}(\max_{v\in V_N} \eta^{4N}_{v+(2N,2N)} \geqslant \lambda)$ *for all* $\lambda \in \mathbb{R}$.

*Proof.* Denote by $V'_N = \{v+(2N, 2N) : v \in V_N\}$, and consider the process $\eta^N_\cdot$ as indexed over the set $V'_N$. Note that the conditional covariance matrix of $\{\eta^{4N}_v\}_{v\in V'_N}$ given the values of $\{\eta^{(4N)}_v\}_{v\in V_{4N}\setminus V'_N}$ corresponds to the covariance matrix of $\{\eta^N_v\}_{v\in V'_N}$. This implies that

$$\{\eta^{4N}_v : v \in V'_N\} \overset{law}{=} \{\eta^N_v + \mathbb{E}(\eta^{4N}_v \mid \{\eta^{4N}_u : u \in V_{4N} \setminus V'_N\}) : v \in V'_N\}\,, \tag{15}$$

where on the right hand side $\{\eta^N_v : v \in V'_N\}$ is independent of $\{\eta^{4N}_u : u \in V_{4N} \setminus V'_N\}$. Write

$$\phi_v = \mathbb{E}(\eta^{4N}_v \mid \{\eta^{4N}_u : u \in V_{4N} \setminus V'_N\}) = \mathbb{E}(\eta^{4N}_v \mid \{\eta^{4N}_u : u \in \partial V'_N\}).$$

Note that $\phi_v$ is a linear combination of $\{\eta^{4N}_u : u \in \partial V'_N\}$, and thus a mean zero Gaussian variable. By the above identity in law and the independence, we derive that

$$\mathbb{E}\tilde{\eta}^{\diamond}_{N,r} \geqslant \mathbb{E}(\eta^{\diamond}_{N,r} + \phi_{\tau_1} + \phi_{\tau_2}) \geqslant \mathbb{E}\eta^{\diamond}_{N,r}\,,$$

where $(\tau_1, \tau_2)$ is the pair at which the sum in the definition of $\eta^{\diamond}_{N,r}$ is maximized. This completes the proof of Part (i). Part (ii) follows from the same argument, by noting that

$$\max_{v\in V'_N} \eta^{4N}_v \geqslant \max_{v\in V_N} \eta^N_v + \phi_\tau\,,$$

where $\tau \in V'_N$ is the maximizer for $\{\eta^{4N}_v : v \in V'_N\}$. The desired bound follows from the fact that $\phi_\tau$ is a centered Gaussian variable independent of $\max_{v\in V_N} \eta^N_v$. □

**Proof of Proposition 2.3.** For the upper bound, by the preceding lemma, it suffices to prove that $\mathbb{E}\tilde{\eta}^{\diamond}_{N,r} \leqslant \mathbb{E}\xi^{\diamond}_{2^\kappa N,r}$. For this purpose, define the mapping $\psi_N : V_N \mapsto V_{2^\kappa N}$ by

$$\psi_N(v) = (2^{\kappa-2}N, 2^{\kappa-2}N) + 2^{\kappa-3}v\,, \text{ for } v \in V_N\,. \tag{16}$$

Applying Lemma 2.1, we obtain that there exists sufficiently large $\kappa$ (that depends only on the universal constant $C$ in Lemma 2.1) such that for all $v, u, v', u' \in V_N$,

$$\begin{aligned}&\mathbb{E}(\eta^{4N}_{v+(2N,2N)} + \eta^{4N}_{u+(2N,2N)} - \eta^{4N}_{v'+(2N,2N)} - \eta^{4N}_{u'+(2N,2N)})^2\\ \leqslant&\tfrac{2\log 2}{\pi}\mathbb{E}(\xi^{2^\kappa N}_{\psi_N(v)} + \xi^{2^\kappa N}_{\psi_N(u)} - \xi^{2^\kappa N}_{\psi_N(v')} - \xi^{2^\kappa N}_{\psi_N(u')})^2\,.\end{aligned} \tag{17}$$

A key observation in order to verify (17) is that the variance of $\xi^{2^\kappa N}_{\psi_N(v)}$ grows with $\kappa$ while the covariance between $\xi^{2^\kappa N}_{\psi_N(v)}$ and $\xi^{2^\kappa N}_{\psi_N(u)}$ does not, for all $u, v \in V_N$ (this allows us to select $\kappa$ large to increase the right hand side in (17)). Now, an application of Lemma 2.2 on the processes

$$\begin{aligned}
&\{\eta^{4N}_{v+(2N,2N)} + \eta^{4N}_{u+(2N,2N)} : u, v \in V_N, r \leqslant \|v-u\| \leqslant N/r\}\,,\\
\text{and}\qquad &\{\sqrt{\tfrac{2\log 2}{\pi}}(\xi^{2^\kappa N}_{\psi_N(v)} + \xi^{2^\kappa N}_{\psi_N(u)}) : u, v \in V_N, r \leqslant \|v-u\| \leqslant N/r\}
\end{aligned}$$

yields that $\mathbb{E}\tilde{\eta}^{\diamond}_{N,r} \leqslant \sqrt{\frac{2\log 2}{\pi}}\mathbb{E}\xi^{\diamond}_{2^\kappa N,r}$. Here we used the fact that $r \leqslant \|\psi_N(v) - \psi_N(u)\| \leqslant 2^\kappa N/r$ for all $u, v \in V_N$ such that $r \leqslant \|v-u\| \leqslant N/r$.

The lower bound follows along the same line, which we now sketch. Analogous to (17), we can derive that for all $u, v, u', v' \in V_{2^{-\kappa}N}$

$$\mathbb{E}(\eta^N_{\psi_{2^{-\kappa}N}(v)} + \eta^N_{\psi_{2^{-\kappa}N}(u)} - \eta^N_{\psi_{2^{-\kappa}N}(v')} - \eta^N_{\psi_{2^{-\kappa}N}(u')})^2 \geqslant \tfrac{2\log 2}{\pi}\mathbb{E}(\xi^{2^{-\kappa}N}_v + \xi^{2^{-\kappa}N}_u - \xi^{2^{-\kappa}N}_{v'} - \xi^{2^{-\kappa}N}_{u'})^2\,.$$

Combined with the fact that $r \leqslant \|\psi_{2^{-\kappa}N}(v) - \psi_{2^{-\kappa}N}(u)\| \leqslant N/r$ for all $u, v \in V_{2^{-\kappa}N}$ such that $r \leqslant \|u-v\| \leqslant 2^{-\kappa}N/r$, another application of Lemma 2.2 completes the proof.

### 2.3 Comparison of the right tail for the maximum

In this subsection, we compare the maximum of GFF with that of MBRW in the sense of "stochastic domination", for which we will use Slepian's [31] comparison lemma.

**Lemma 2.5** (Slepian)**.** *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a\}_{a\in\mathcal{A}}$ and $\{Y_a\}_{a\in\mathcal{A}}$ be two centered Gaussian processes such that* (13) *holds and* $\operatorname{Var} X_a = \operatorname{Var} Y_a$ *for all $a \in \mathcal{A}$. Then $\mathbb{P}(\max_{a\in\mathcal{A}} X_a \geqslant \lambda) \geqslant \mathbb{P}(\max_{a\in\mathcal{A}} Y_a \geqslant \lambda)$, for all $\lambda \in \mathbb{R}$.*

**Remark.** The additional assumption on the identical variance allows for a comparison beyond the expectation, and meanwhile requires a careful treatment when carrying out the comparison.

The main result of this subsection is the following.

**Lemma 2.6.** *There exists a universal integer $\kappa > 0$ such that for all $N$ and $\lambda \in \mathbb{R}$*

$$\tfrac{1}{2}\mathbb{P}(\max_{v\in V_{2^{-\kappa}N}}\sqrt{\tfrac{2\log 2}{\pi}}\xi^{2^{-\kappa}N}_v \geqslant \lambda) \leqslant \mathbb{P}\big(\max_{v\in V_N}\eta^N_v \geqslant \lambda\big) \leqslant 4\mathbb{P}\big(\max_{v\in V_{2^\kappa N}}\sqrt{\tfrac{2\log 2}{\pi}}\xi^{2^\kappa N}_v \geqslant \lambda\big)\,.$$

*Proof.* We first prove the upper bound in the comparison. In light of Part (ii) of Lemma 2.4, it suffices to consider the maximum of GFF in a smaller central box (of half size), with the convenience that the variance is almost uniform therein. Indeed, by Lemma 2.1, we see that for a universal constant $C > 0$

$$|\operatorname{Var}\eta^{4N}_u - \operatorname{Var}\eta^{4N}_v| \leqslant C\,, \text{ for all } u, v \in (2N, 2N) + V_N\,. \tag{18}$$

Let $\psi_N$ be defined as in (16). It is clear that for $\kappa$ sufficiently large (independent of $N$), we have $\operatorname{Var}\eta^{4N}_{v+(2N,2N)} \leqslant \frac{2\log 2}{\pi}\operatorname{Var}\xi^{2^\kappa N}_{\psi_N(v)}$ for all $v \in V_N$. Therefore, we can choose a collection of positive numbers $\{a_v\}_{v\in V_N}$ such that

$$\operatorname{Var}(\eta^{4N}_{v+(2N,2N)} + a_v X) = \tfrac{2\log 2}{\pi}\operatorname{Var}\xi^{2^\kappa N}_{\psi_N(v)}\,, \tag{19}$$

where $X$ is an independent standard Gaussian variable. Furthermore, due to (18) and the fact that the MBRW has precisely uniform variance over all vertices, we have for a universal constant $C > 0$

$$|a_u - a_v| \leqslant C\,, \text{ for all } u, v \in V_N\,.$$

This implies that

$$\mathbb{E}((\eta^{4N}_{v+(2N,2N)} + a_v X) - (\eta^{4N}_{u+(2N,2N)} + a_u X))^2 \leqslant \mathbb{E}((\eta^{4N}_{v+(2N,2N)} - \eta^{4N}_{u+(2N,2N)})^2 + C^2\,, \text{ for all } u, v \in V_N\,.$$

Combined with the fact that $\mathbb{E}(\xi^{2^\kappa N}_{\psi_N(u)} - \xi^{2^\kappa N}_{\psi_N(v)})^2$ grows (linearly) with $\kappa$ and Lemma 2.1, it follows that for $\kappa$ sufficiently large (independent of $N$) and for all $u, v \in V_N$

$$\mathbb{E}((\eta^{4N}_{v+(2N,2N)} + a_v X) - (\eta^{4N}_{u+(2N,2N)} + a_u X))^2 \leqslant \tfrac{2\log 2}{\pi}\mathbb{E}(\xi^{2^\kappa N}_{\psi_N(u)} - \xi^{2^\kappa N}_{\psi_N(v)})^2\,. \tag{20}$$

Combined with (19), an application of Lemma 2.5 yields that

$$\mathbb{P}\big(\max_{v\in V_N} \eta^{4N}_{v+(2N,2N)} + a_v X \geqslant \lambda\big) \leqslant \mathbb{P}\big(\sqrt{\tfrac{2\log 2}{\pi}} \max_{v \in V_N} \xi^{2^\kappa N}_{\psi_N(v)} \geqslant \lambda\big)\,, \text{ for all } \lambda \in \mathbb{R}\,. \tag{21}$$

It is clear that

$$\begin{aligned}\mathbb{P}\big(\max_{v\in V_N} \eta^{4N}_{v+(2N,2N)} + a_v X \geqslant \lambda\big) &\geqslant \mathbb{P}\big(\max_{v\in V_N} \eta^{4N}_{v+(2N,2N)} \geqslant \lambda, X \geqslant 0\big)\\ &= \tfrac{1}{2}\mathbb{P}\big(\max_{v\in V_N} \eta^{4N}_{v+(2N,2N)} \geqslant \lambda\big)\,.\end{aligned}$$

Combined with (21), the desired upper bound follows.

We now turn to the proof of the lower bound, which shares the same spirit with the proof of the upper bound. Recall the definition of $\psi_{2^{-\kappa}N}$ as in (16). Using Lemma 2.1 again, we obtain that

$$|\operatorname{Var} \eta^N_{\psi_{2^{-\kappa}N}(v)} - \operatorname{Var} \eta^N_{\psi_{2^{-\kappa}N}(u)}| \leqslant C\,, \text{ for all } u, v \in V_{2^{-\kappa}N}\,.$$

It is also clear from Lemma 2.1 that $\operatorname{Var} \eta^N_{\psi_{2^{-\kappa}N}(v)} \geqslant \frac{2\log 2}{\pi} \operatorname{Var} \xi^{2^{-\kappa}N}_v$, for $\kappa$ sufficiently large (independent of $N$) and for all $v \in V_{2^{-\kappa}N}$. Continue to denote by $X$ an independent standard Gaussian variable. We can then choose a collection of positive numbers $\{a'_v : v \in V_{2^{-\kappa}N}\}$ satisfying $|a'_v - a'_u| \leqslant C$ such that

$$\operatorname{Var} \eta^N_{\psi_{2^{-\kappa}N}(v)} = \tfrac{2\log 2}{\pi} \operatorname{Var}(\xi^{2^{-\kappa}N}_v + a'_v X)\,, \text{ for all } v \in V_{2^{-\kappa}N}\,.$$

Analogous to the derivation of (20), we get that for $\kappa$ sufficiently large (independent of $N$),

$$\mathbb{E}((\eta^N_{\psi_{2^{-\kappa}N}(v)} - \eta^N_{\psi_{2^{-\kappa}N}(u)})^2 \geqslant \tfrac{2\log 2}{\pi}\mathbb{E}((\xi^{2^{-\kappa}N}_v + a'_v X) - (\xi^{2^{-\kappa}N}_u + a'_u X))^2\,, \text{ for all } u, v \in V_{2^{-\kappa}N}\,.$$

Another application of Lemma 2.5 yields that for all $\lambda \in \mathbb{R}$

$$\begin{aligned}\mathbb{P}\big(\max_{v\in V_{2^{-\kappa}N}} \eta^N_{\psi_{2^{-\kappa}N}(v)} \geqslant \lambda\big) &\geqslant \mathbb{P}\big(\sqrt{\tfrac{2\log 2}{\pi}} \max_{v\in V_{2^{-\kappa}N}} (\xi^{2^{-\kappa}N}_v + a'_v X) \geqslant \lambda\big)\\ &\geqslant \mathbb{P}\big(\sqrt{\tfrac{2\log 2}{\pi}} \max_{v\in V_{2^{-\kappa}N}} \xi^{2^{-\kappa}N}_v \geqslant \lambda, X \geqslant 0\big)\\ &= \tfrac{1}{2}\mathbb{P}\big(\sqrt{\tfrac{2\log 2}{\pi}} \max_{v\in V_{2^{-\kappa}N}} \xi^{2^{-\kappa}N}_v \geqslant \lambda\big)\,.\end{aligned}$$

Combined with the fact that $\psi_{2^{-\kappa}N}(v) \in V_N$ for all $v \in V_{2^{-\kappa}N}$, this completes the proof. $\square$

### 2.4 Comparison of the sum of large particles

We conclude this section with a comparison between the Gaussian free field and branching random walk, which will be used in the proof of Theorem 1.2.

We need the following variant of Slepian's inequality.

**Lemma 2.7.** *Let $\mathbf{X} = (X_i : i \in [n])$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two mean-zero Gaussian processes such that $\mathbb{E}X_i^2 = \mathbb{E}Y_i^2$ and $\mathbb{E}X_iX_j \leqslant \mathbb{E}Y_iY_j$ for all $i, j \in [n]$. Fix $1 \leqslant m \leqslant n$, and define $S_m(\mathbf{x}) = \max\{\sum_{i \in A} x_i : A \subseteq [n], |A| = m\}$ for $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbb{E}S_m(\mathbf{X}) \geqslant \mathbb{E}S_m(\mathbf{Y})$.*

*Proof.* For $\beta > 0$, define $F_\beta : \mathbb{R}^n \mapsto \mathbb{R}$ by

$$F_\beta(\mathbf{x}) = \beta^{-1} \log \sum_{A \in \Omega_m} \mathrm{e}^{\beta \mathbf{x}_A} ,$$

where we denote by $\Omega_m = \{A \subseteq [n] : |A| = m\}$ and $\mathbf{x}_A = \sum_{i \in A} x_i$. We prove below that

$$\partial^2 F_\beta / \partial x_i \partial x_j \leqslant 0 , \quad i \neq j . \tag{22}$$

Then, by [29, Theorem 3.11], one has that

$$EF_\beta(\mathbf{X}) \geqslant EF_\beta(\mathbf{Y}) .$$

Taking $\beta \to \infty$ yields the lemma.

It remains to prove (22). For $k \in [n]$ and $I \subseteq [n]$, we set $\Omega_k^{\setminus I} = \{B \subseteq [n] \setminus I : |B| = k\}$. Then, for $i \neq j$,

$$\frac{\partial^2 F_\beta}{\partial x_i \partial x_j} = \frac{\beta \mathrm{e}^{\beta(x_i + x_j)} \sum_{B \in \Omega_{m-2}^{\setminus\{i,j\}}} \mathrm{e}^{\beta \mathbf{x}_B}}{\sum_{A \in \Omega_m} \mathrm{e}^{\beta \mathbf{x}_A}} - \frac{\beta \mathrm{e}^{\beta(x_i + x_j)} \sum_{B \in \Omega_{m-1}^{\setminus i}} \mathrm{e}^{\beta \mathbf{x}_B} \sum_{B' \in \Omega_{m-1}^{\setminus j}} \mathrm{e}^{\beta \mathbf{x}_{B'}}}{(\sum_{A \in \Omega_m} \mathrm{e}^{\beta \mathbf{x}_A})^2} .$$

The inequality (22) follows from the following combinatorial claim. □

**Claim 2.8.** *For all $i, j, m \in [n]$ and $\beta > 0$, we have*

$$\sum_{A \in \Omega_m} \mathrm{e}^{\beta \mathbf{x}_A} \sum_{B \in \Omega_{m-2}^{\setminus\{i,j\}}} \mathrm{e}^{\beta \mathbf{x}_B} \leqslant \sum_{B \in \Omega_{m-1}^{\setminus i}} \mathrm{e}^{\beta \mathbf{x}_B} \sum_{B' \in \Omega_{m-1}^{\setminus j}} \mathrm{e}^{\beta \mathbf{x}_{B'}} .$$

*Proof.* Fix a sequence $(a_1, \dots, a_n)$ such that $a_\ell \in \{0, 1, 2\}$ for all $\ell \notin \{i, j\}$, $a_i, a_j \in \{0, 1\}$ and $\sum_\ell a_\ell = 2m - 2$. We count the multiplicity of the term $\mathrm{e}^{\sum_\ell \beta a_\ell x_\ell}$ in the left (denoted by $L$) and right hand sides (denoted by $R$), respectively. Let $k = |\{\ell \in [n] \setminus \{i, j\} : a_\ell = 1\}|$. It is straightforward to verify that

$$L = \begin{cases} \binom{k}{k/2+1} , & \text{if } a_i + a_j = 0 , \\ \binom{k-1}{(k-1)/2} , & \text{if } a_i + a_j = 1 , \\ \binom{k-2}{(k-2)/2} , & \text{if } a_i + a_j = 2 ; \end{cases} \quad \text{and} \quad R = \begin{cases} \binom{k}{k/2} , & \text{if } a_i + a_j = 0 , \\ \binom{k-1}{(k-1)/2} , & \text{if } a_i + a_j = 1 , \\ \binom{k-2}{(k-2)/2} , & \text{if } a_i + a_j = 2 . \end{cases}$$

Therefore, we always have $L \leqslant R$, completing the proof of the claim. □

We now demonstrate a comparison for the sum of large values between the GFF and the BRW.

**Lemma 2.9.** *For $N = 2^n$ with $n \in \mathbb{N}$, let $\{\eta_v : v \in V_N\}$ be the Gaussian free field and $\{\vartheta_v : v \in V_N\}$ the branching random walk as defined in* (10)*. For $\ell \in \mathbb{N}$, define*

$$\mathcal{S}_{\ell,N} = \max\{\textstyle\sum_{v\in A}\eta_v : |A| = \ell, A \subset V_N\}\,,\ \textit{and}\ \mathcal{R}_{\ell,N} = \sqrt{\tfrac{2\log 2}{\pi}}\max\{\textstyle\sum_{v\in A}\vartheta_v : |A| = \ell, A \subset V_N\}\,.$$

*Then, there exists absolute constant $\kappa \in \mathbb{N}$ such that $\mathbb{E}\mathcal{S}_{\ell,N} \leqslant \mathbb{E}\mathcal{R}_{\ell,N2^\kappa}$.*

*Proof.* Consider $\vartheta_v^* = \vartheta_v + \kappa X_v$ where $X_v$ are i.i.d. standard Gaussian variables, and define $\mathcal{R}^*_{\ell,N} = \sqrt{2\log 2/\pi}\max\{\sum_{v\in A}\vartheta_v^* : |A| = \ell, A \subset V_N\}$. Clearly, $\mathbb{E}\mathcal{R}^*_{\ell,N} \leqslant \mathbb{E}R_{\ell,N2^\kappa}$. Let $X$ be another independent standard Gaussian variable and choose a non-negative sequence $\{a_v : v \in (2N,2N) + V_N\}$ such that

$$\mathrm{Var}(\eta_v^{4N} + a_v X) = \mathrm{Var}\,\vartheta_v^*\,,\ \text{for all } v \in (2N,2N) + V_N\,. \tag{23}$$

By Lemma 2.1, we see that $|a_u - a_v| \leqslant C$ for an absolute constant $C > 0$. Further define

$$\mathcal{S}^*_{\ell,N} = \max\{\textstyle\sum_{v\in A+(2N,2N)}\eta_v^{4N} + a_v X : |A| = \ell, A \subset V_N\}\,.$$

Using similar arguments as in the proof of Lemma 2.4, we deduce that $\mathbb{E}\mathcal{S}_{\ell,N} \leqslant \mathbb{E}\mathcal{S}^*_{\ell,N}$. Therefore, it remains to prove $\mathbb{E}\mathcal{S}^*_{\ell,N} \leqslant \mathbb{E}\mathcal{R}^*_{N,\ell}$. To this end, note that we can select $\kappa = 4C$ such that for all $u, v \in V_N$

$$\mathbb{E}(\vartheta_v^*\vartheta_u^*) \leqslant \mathbb{E}((\eta^{4N}_{v+(2N,2N)} + a_{v+(2N,2N)}X)(\eta^{4N}_{v+(2N,2N)} + a_{v+(2N,2N)}X))\,.$$

Combined with (23) and Lemma 2.7, it completes the proof. ∎

# 3 Maxima of the modified branching random walk

This section is devoted to the study of the maxima of MBRW, from which we will deduce properties for the maxima of GFF.

## 3.1 The maximal sum over pairs

The following lemma is the key to controlling the maximum over pairs. Set $\tilde m_N = \sqrt{\pi/2\log 2}\cdot m_N$.

**Lemma 3.1.** *There exist constants $c_1, c_2 > 0$ so that*

$$2\tilde m_N - c_2\log\log r \leqslant \mathbb{E}\xi^\diamond_{N,r} \leqslant 2\tilde m_N - c_1\log\log r\,.$$

We consider first a branching random walk $\{X_i^n : i = 1, \ldots, 4^n\}$, with four descendants per particle and standard normal increments. Note that $\{\vartheta_v : v \in V_N\}$ as defined in (10) is a BRW with four descendants per particle and $n$ generations. We use different notation in this subsection that allows us to ignore the unnecessary underlying "lattice" structure for BRW. Let $T_n$ be the maximum of the BRW after $n$ generations. Let $c^* = 2\sqrt{\log 2}$, $\bar c = (3/2)/c^*$ and $t_n = c^*n - \bar c\log n$. The following estimates are standard. We refer e.g. to [1] and to [32, (2.5.11),(2.5.13)].

**Lemma 3.2.** *The expectation $ET_n$ satisfies*

$$\mathbb{E}T_n = c^*n - \bar c\log n + O(1)\,. \tag{24}$$

*Further, there exist constants $c, C > 0$ so that, for $y \in [0, \sqrt{n}]$,*

$$c\mathrm{e}^{-c^*y} \leqslant \mathbb{P}(T_n \geqslant t_n + y) \leqslant C(1+y)^2\mathrm{e}^{-c^*y}\,, \tag{25}$$

*with the upper bound holding for any $y \geqslant 0$.*

(In fact, more is known through the convergence results of [2], but we only need the crude estimates (25).) We remark that the upper bound in (25) for $y > \sqrt{n}$ is an immediate consequence of a union bound. Further, (25) implies that with $T'_n$ an independent copy of $T_n$, there exists a constant $C$ such that

$$\mathbb{P}(T_n + T'_n \geqslant 2t_n + 2y) \leqslant C(1+y)^4 \mathrm{e}^{-2c^* y} \tag{26}$$

for any $y \geqslant 0$ and any positive integer $n$.

For $x \in \mathbb{Z}$, let

$$\Xi_n(x) = \#\{1 \leqslant i \leqslant 4^n : X_i^n \in [t_n - x - 1, t_n - x]\}$$

be the number of particles in the BRW at distance roughly $x$ behind the leader. The following is essentially folklore, we include a proof since we have not been able to find an appropriate reference.

**Proposition 3.3.** *For some universal constant $C$, and all $x \in \mathbb{Z}$,*

$$\mathbb{E}\Xi_n(x) \leqslant Cn\mathrm{e}^{c^* x - x^2/2n}\,. \tag{27}$$

*Further, for any $u > -x$ so that $0 < x + u \leqslant \sqrt{n/2}$,*

$$\mathbb{P}(\Xi_n(x) \geqslant \mathrm{e}^{c^*(x+u)}) \leqslant C\mathrm{e}^{-c^* u + C \log_+(x+u)}\,. \tag{28}$$

Note that the interest in (28) is only in situations in which $x+u$ is at most at logarithmic scale (in $n$).

*Proof.* The estimate (27) is a simple union bound: with $G$ a zero mean Gaussian with variance $n$ we have

$$E\Xi_n(x) = 4^n P(G \in [t_n - x - 1, t_n - x])\,.$$

Using standard estimates for the Gaussian distribution and the value of $t_n$, the estimate (27) follows.

We write the proof of (28) in case $x \geqslant 0$, the general case is similar. We use Lemma 3.2. Fix $\delta > 0$, $r = 2(x+u)^2$ and $y = u - \bar{c}\log r$. Note that $\bar{c}\log r + y + x < \sqrt{r}$. With $K$ an arbitrary positive integer,

$$\begin{aligned}\mathbb{P}(T_{n+r} \geqslant t_{n+r} + y) &\geqslant \mathbb{P}(\Xi_n(x) \geqslant K)\left[1 - \left(\mathbb{P}\left(T_r \leqslant t_r + \bar{c}\log r + y + x - \bar{c}\log(1 + r/n)\right)\right)^K\right] \\ &\geqslant \mathbb{P}(\Xi_n(x) \geqslant K)\left[1 - \left(1 - C\mathrm{e}^{-c^*(y + x + \bar{c}\log r)}\right)^K\right],\end{aligned} \tag{29}$$

where in the last inequality we used the *lower bound* in (25). Taking $K = \mathrm{e}^{c^*(x+u)}$ we have that $\mathrm{e}^{-c^*(y+x+\bar{c}\log r)}K$ is uniformly bounded below and therefore

$$\mathbb{P}(T_{n+r} \geqslant t_{n+r} + y) \geqslant c\mathbb{P}(\Xi_n(x) \geqslant K)\,.$$

Using the *upper bound* in (25) we get that

$$\mathbb{P}(\Xi_n(x) \geqslant K) \leqslant C\mathrm{e}^{-c^* y}(1+y)^2\,.$$

This yields (28). ∎

In what follows, we write $i \sim_s j$ if the particles $X_i^n$ and $X_i^n$ had a common ancestor at generation $n - s$.

**Corollary 3.4.** *There exists a constant $C>0$ such that, for any $s \leqslant n/2$ positive integer, and any $z$ positive,*

$$\mathbb{P}(\exists i \sim_s j : X_i^n + X_j^n \geqslant 2t_n - \bar{c}\log s + z) \leqslant C[\mathrm{e}^{-0.9c^* z} + \mathrm{e}^{-0.45c^* z - 0.7\log s}]\,. \tag{30}$$

*Similarly,*

$$\mathbb{P}(\exists i \sim_{n-s} j : X_i^n + X_j^n \geqslant 2t_n - \bar{c}\log s + z) \leqslant C[\mathrm{e}^{-0.9c^* z} + \mathrm{e}^{-0.45c^* z - 0.7\log s}]\,. \tag{31}$$

*In particular, there exists an $r_0$ such that for all $r > r_0$ and all $n$ large,*

$$\mathbb{E} \max_{i \sim_q j, s \in [r, n-r]} (X_i^n + X_j^n) \leqslant 2t_n - (\bar{c}/4)\log r\,. \tag{32}$$

*Proof.* We first provide the proof of (30); the claim (31) follows similarly and (32) will then be an easy consequence.

In what follows we set $u^* = u^*(x,z) = \max(|x|, z)$ and $j^* = j^*(x,z) = \lceil u^* \rceil$. We also define $\mathbb{Z}_-^{(1)} = \mathbb{Z}_- \cap \{x : |x| \leqslant (z + \bar{c}\log s)/2\}$, $\mathbb{Z}_-^{(2)} = \mathbb{Z}_- \cap \{x : |x| > (z + \bar{c}\log s)/2\}$ and $\mathcal{Z}_n = \{x \in \mathbb{Z} : 0 \leqslant x + u^* \leqslant \sqrt{n/4}\}$.

The starting point of the proof of (30) is the following estimate, obtained by decomposing over the location of particles at generation $n-s$.

$$\begin{aligned}
&\mathbb{P}(\exists i \sim_s j : X_i^n + X_j^n \geqslant 2t_n - \bar{c}\log s + z) \\
&\quad \leqslant \sum_{x \in \mathbb{Z}} \mathbb{P}(\Xi_{n-s}(x) \geqslant \mathrm{e}^{c^*(x+u^*)}) + \\
&\binom{4}{2} \sum_{x \in \mathbb{Z}_+ \cap \mathcal{Z}_n} \sum_{j=0}^{j^*(x,z)} \mathbb{P}(\Xi_{n-s}(x) \geqslant \mathrm{e}^{c^*(x+j)}) e^{c^*(x+j+1)} \mathbb{P}(T_s + T_s' \geqslant 2t_s + z + 2x + \bar{c}\log_+ s) + \\
&\binom{4}{2} \sum_{x \in \mathbb{Z}_-^{(1)} \cap \mathcal{Z}_n} \sum_{j=|x|}^{j^*(x,z)} \mathbb{P}(\Xi_{n-s}(x) \geqslant \mathrm{e}^{c^*(x+j)}) \mathrm{e}^{c^*(x+j+1)} \mathbb{P}(T_s + T_s' \geqslant 2t_s + z + 2x + \bar{c}\log_+ s) + \\
&\sum_{x \in \mathbb{Z}_-^{(1)} \cap \mathcal{Z}_n^c} \mathbb{E}(\Xi_{n-s}(x)) \mathbb{P}(T_s + T_s' \geqslant 2t_s + z + 2x + \bar{c}\log_+ s) + \\
&\sum_{x \in \mathbb{Z}_+ \cap \mathcal{Z}_n^c} \mathbb{E}(\Xi_{n-s}(x)) \mathbb{P}(T_s + T_s' \geqslant 2t_s + z + 2x + \bar{c}\log_+ s) + \sum_{x \in \mathbb{Z}_-^{(2)}} \mathbb{P}(\Xi_{n-s}(x) \geqslant 1) \\
&\quad =: \sum_{x \in \mathbb{Z}} A_1(x) + \sum_{x \in \mathbb{Z}_+ \cap \mathcal{Z}_n} A_2(x) + \sum_{x \in \mathbb{Z}_-^{(1)} \cap \mathcal{Z}_n} A_3(x) + \sum_{x \in \mathbb{Z}_-^{(1)} \cap \mathcal{Z}_n^c} A_4(x) + \sum_{x \in \mathbb{Z}_+ \cap \mathcal{Z}_n^c} A_5(x) + \sum_{x \in \mathbb{Z}_-^{(2)}} A_6(x) \\
&\quad =: A_1 + A_2 + A_3 + A_4 + A_5 + A_6\,,
\end{aligned} \tag{33}$$

where $T_s'$ is an independent copy of $T_s$. The contribution to $A_1$ from $x \in \mathcal{Z}_n$ can be estimated using (28) and one finds

$$\sum_{x \in \mathcal{Z}_n} A_1(x) \leqslant C \sum_{|x| \leqslant z} \mathrm{e}^{-c^* z + C\log_+ z} + 2C \sum_{x=z}^{\infty} \mathrm{e}^{-c^* x + C\log_+ x} \leqslant C(z+1)\mathrm{e}^{-c^* z}\,. \tag{34}$$

A similar computation using (28) and (26) yields

$$\sum_{x\in\cap\mathbb{Z}_+\cap\mathcal{Z}_n^+} A_2(x) \leqslant C \sum_{x\in\mathbb{Z}_+\cap\mathcal{Z}_n}\sum_{j=0}^{u^*} \mathrm{e}^{-c^*j+C\log_+(x+j)}e^{c^*(x+j+1)}\mathrm{e}^{-c^*(z+2x+\bar{c}\log s)}(z+|x|+\bar{c}\log s)^4$$
$$\leqslant C(1+\log s)^4\mathrm{e}^{C\log z}e^{-c^*z}\,. \tag{35}$$

To control $A_3$, we repeat the last computation and obtain

$$\sum_{x\in\mathbb{Z}_-^{(1)}\cap\mathcal{Z}_n} A_3(x) \leqslant C \sum_{x\in\mathbb{Z}_-^{(1)}\cap\mathcal{Z}_n}\sum_{j=0}^{u^*} \mathrm{e}^{-c^*j+C\log_+(x+j)}e^{c^*(x+j+1)}\mathrm{e}^{-c^*(z+2x+\bar{c}\log s)}(z+|x|+\bar{c}\log s)^4$$
$$\leqslant C(1+\log s)^4\mathrm{e}^{C\log z}\mathrm{e}^{-c^*z/2-c^*\bar{c}\log s/2}\,. \tag{36}$$

To control $A_6$ over $\mathcal{Z}_n$, we repeat the estimate as in controlling $A_1$ and obtain

$$\sum_{x\in\mathbb{Z}_-^{(2)}\cap\mathcal{Z}_n} A_6(x) \leqslant C(1+\log s+z)\mathrm{e}^{-c^*z/2+c^*\bar{c}\log s/2}\,. \tag{37}$$

The estimate for $x\notin\mathcal{Z}_n$ is easier, using this time (27). Indeed, in such a situation either $|x|$ or $z$ are at least of order $\sqrt{n}$. One has

$$\sum_{x\notin\mathcal{Z}_n} A_1(x) \leqslant C\sum_{x\notin\mathcal{Z}_n}(E\Xi_{n-s}(x))\mathrm{e}^{-c^*(x+u^*)} \leqslant \sum_{x\notin\mathcal{Z}_n} Cn\mathrm{e}^{-c^*u^*-x^2/n} \leqslant \mathrm{e}^{-0.9c^*z-2\log n}\,.$$

In particular, since $\log s<\log n$ we get

$$\sum_{x\notin\mathcal{Z}_n} A_1(x) \leqslant Cs^{-2}\mathrm{e}^{-0.9c^*z}\,. \tag{38}$$

Similarly,

$$\sum_{x\in\mathbb{Z}_+\cap\mathcal{Z}_n^c} A_5(x) \leqslant C\sum_{x\in\mathbb{Z}_+\cap\mathcal{Z}_n^c}(1+z+x+\bar{c}\log s)^4n\mathrm{e}^{-c^*(x+z)-x^2/2n-c^*\bar{c}\log s} \leqslant \mathrm{e}^{-0.9c^*z}\,. \tag{39}$$

For negative $x$ one has to exercise some care, this is the reason for the definition of $Z_-^{(1)}$ and $Z_-^{(2)}$. One has

$$\sum_{x\in\mathbb{Z}_-^{(1)}\cap\mathcal{Z}_n^c} A_4(x) \leqslant C\sum_{x\in\mathbb{Z}_-^{(1)}\cap\mathcal{Z}_n^c} n(1+z+|x|+\bar{c}\log s)^4\mathrm{e}^{-c^*(x+z+\bar{c}\log s)}$$
$$\leqslant C\mathrm{e}^{-0.45c^*z-0.99c^*\bar{c}\log s} \leqslant \mathrm{e}^{-0.45c^*z-0.7\log s}\,, \tag{40}$$

where we have used that $c^*\bar{c}=3/2$. Finally, just using (27), we get similarly

$$\sum_{x\in\mathbb{Z}_-^{(2)}\cap\mathcal{Z}_n^c} A_6(x) \leqslant \sum_{x\in\mathbb{Z}_-^{(2)}\cap\mathcal{Z}_n^c} Cn\mathrm{e}^{c^*x-x^2/2n} \leqslant \mathrm{e}^{-0.45c^*z-0.7\log s}\,. \tag{41}$$

Summing (34)-(41) yields (30). As mentioned before, the proof of (31) is similar. Because $c^*\bar{c} = 3/2$ and $0.9 \cdot 3/2 > 1$ we also have then that

$$\begin{aligned}&\mathbb{P}(\exists s \in \{r, \dots, n/2\}, \exists i \sim_s j : X_i^n + X_j^n \geqslant 2t_n - (\bar{c}/4)\log r + z)\\ &\leqslant \sum_{s=r}^{n/2} C[\mathrm{e}^{-0.9c^*(z+\bar{c}\log(s/r^{1/4}))} + \mathrm{e}^{-0.45c^*(z+\bar{c}\log(s/r^{0.25}))-0.7\log s}] \leqslant C\mathrm{e}^{-0.45c^* z}\,.\end{aligned}$$

A similar estimates holds for the range $s \in \{n/2, \dots, n-r\}$. Summing those over $z$ yields (31). We omit further details. □

We can now provide the

**Proof of Lemma 3.1.** We begin with the upper bound. The argument is similar to what was done in the proofs in Section 2 and therefore we will not provide all details.

Let $S_v^N$ be a BRW of depth $n$ and set $R_v^N = (1-\varepsilon_N)S_v^N + G_v$ where $G_v$ is a collection of i.i.d. zero mean gaussians of variance $\sigma^2$ to be defined (independent of $N$) and $\varepsilon_N = O(1/n)$. Choosing $\sigma$ and $\varepsilon_N$ appropriately one can ensure that $E((R_u^N)^2) = E((\xi_u^N)^2)$ and that $E((R_u^N - R_v^N)^2 \geqslant E((\xi_u^N - \xi_v^N)^2)$. Applying Lemma 2.7 and Corollary 3.4, we deduce the upper bound in Lemma 3.1.

We now turn to the proof of the lower bound. The first step is the following proposition. In what follows, $\tilde{\xi}^\diamond_{N,r}$ is defined as $\xi^\diamond_{N,r}$ except that the maximum is taken only over pairs of vertices at distance at least $N/4$ from the boundary, and the top two levels of the MBRW are not added.

**Proposition 3.5.** *There exist constants $C_1, C_2 > 0$ such that for all $N$ large and all $r$,*

$$\mathbb{P}(\tilde{\xi}^\diamond_{N,r} \geqslant 2\tilde{m}_N - C_1 \log\log r) \geqslant C_2\,. \tag{42}$$

We postpone the proof of Proposition 3.5 and show how to deduce the lower bound in Lemma 3.1 from it. Fix $C = 2^c > 1$ integer and consider the MBRW $\xi_v^{N,C}$ in the box $V_{CN}$ with levels up to $n = \log_2(N/4)$ (that is, the last $c+2$ levels are not taken), and define $\xi^\diamond_{N,C,r}$ in a natural way. By independence of the field in sub-boxes of side $N/4$ that are at distance at least $N/2$ of each other, we get that

$$\mathbb{P}(\xi^\diamond_{N,C,r} \geqslant 2\tilde{m}_N - C_1\log\log r) \geqslant 1 - (1-C_2^2)^{C^2/2}\,.$$

Adding the missing $c+2$ levels we then obtain, by standard estimates for the Gaussian distribution,

$$\mathbb{P}(\xi^\diamond_{CN,r} \geqslant 2\tilde{m}_N - C_1\log\log r - y) \geqslant 1 - (1-C_2^2)^{C^2/2} - C_3\mathrm{e}^{-C_4 y^2/c}\,.$$

Renaming $N$, we rewrite the last estimate as

$$\mathbb{P}(\xi^\diamond_{N,r} \geqslant 2\tilde{m}_N - C_1\log\log r - y - C_5 c) \geqslant 1 - (1-C_2^2)^{C^2/2} - C_3\mathrm{e}^{-C_4 y^2/c}\,.$$

Choosing $y = C_5 c$ and summing over $c$ we obtain that $\mathbb{E}\xi^\diamond_{N,r} \geqslant 2\tilde{m}_N - C_6 \log\log r$, as claimed.

**Proof of Proposition 3.5.** We consider $V_N$ as being centered. There are two steps.

**Step 1** We consider the MBRW from level $n - \log r - 1$ to level 1. That is, with $r$ fixed define

$$\hat{\xi}_v^N = \sum_{k=0}^{n-\log_2 r-1} \sum_{B \in \mathcal{B}_k(v)} b_{k,B}^N\,, \text{ and } A_{n,r} = V_{N-r} \cap \left(\frac{N}{r}\mathbb{Z}\right)^2\,. \tag{43}$$

For each $x \in A_{n,r}$, let $V_{N,r}(x)$ denote the $\mathbb{Z}^2$ box centered at $x$ with side $N/2r$. We call $y \in A_{n,r}$ a *right neighbor* of $x \in A_{n,r}$ if $x_2 = y_2$ and $y_1 > x_1$ satisfies $y_1 = x_1 + N/r$, and we write $y = x_R$. Finally, we set, for $x \in A_{N,r}$,

$$\xi^*_{N,r,x} = \max_{v \in V_{N,r}(x)} \hat{\xi}^N_v \,.$$

Note that, by construction, the collection $\{\xi^*_{N,r,x}\}_{x \in A_{n,r}}$ is i.i.d.

A straight forward adaptation of [14] shows that

$$\mathbb{P}(\xi^*_{N,r,x} \geqslant \tilde{m}_{N/r} - c) \geqslant g(c)\,, \tag{44}$$

where $g(c) \to_{c\to\infty} 1$ is independent of $N, r$. Let $\zeta^*_{x,N}$ be the (unique) element of $V_{N,r}(x)$ such that $\xi^*_{N,r,x} = \hat{\xi}^N_{\zeta^*_{x,N}}$. Let

$$M_{N,r,c} = \{x \in A_{n,r} : \xi^*_{N,r,x} \geqslant \tilde{m}_{N/r} - c, \xi^*_{N,r,x_R} \geqslant \tilde{m}_{N/r} - c\}\,.$$

By independence, we get from (44) that there exists a constant $c$, independent of $N, r$, so that

$$\mathbb{P}(|M_{N,r,c}| \geqslant r^2/4) \geqslant \tfrac{1}{2}\,. \tag{45}$$

**Step 2.** For $x \in M_{N,r,c}$, set $\bar{\xi}^*_{N,r,x} = \xi^*_{N,r,x} + \xi^*_{N,r,x_R}$; note that for such $x$, one has $\bar{\xi}_{N,r,x} \geqslant 2\tilde{m}_{N/r} - 2c$. Define, for $v \in V_N$,

$$Y^N_v = \sum_{k=n-\log_2 r}^{n} \sum_{B \in \mathcal{B}_k(v)} b^N_{k,B}\,, \tag{46}$$

and for $x \in A_{N,r}$, set

$$Z^N_x = Y^N_{\zeta^*_{x,N}} + Y^N_{\zeta^*_{x_R,N}}\,.$$

Conditioned on the sigma algebra $\mathcal{F}_{N,r}$ generated by the collection of variables $\{\zeta^*_{x,N}\}$, the collection $\{Z^N_x\}_x$ is a zero mean Gaussian field, with (conditional) covariance satisfying

$$|\tilde{\mathbb{E}}(Z^N_x Z^N_y) - 4(\log_2 r - \log_2(|x-y|/(N/r))| \leqslant C\,,$$

for some constant $C$ independent of $N, r$; here, $\tilde{\mathbb{E}}$ denotes expectation conditioned on $\mathcal{F}_{N,r}$.

It is then straightforward, using the argument in the proof of Proposition 5.2 in [14], to verify that $Z^*_N = \max_{x \in M_{N,r,c}} Z^N_x$ is comparable to twice the maximum of MBRW run for $\log_2 r$ generations, i.e. that on the event $|M_{N,r,c}| \geqslant r^2/4$ there exist positive constants $c_1, c_2$ independent of $r, N$ (but dependent on $c$) such that

$$\tilde{\mathbb{P}}(Z^*_N \geqslant 2\tilde{m}_r - c_1) \geqslant c_2\,,$$

We now combine the two steps. Let $x^*_N$ be the (unique) random element of $M_{N,r,c}$ such that $Z^*_N = Z^N_{x^*_N}$. Then, on the event $|M_{N,r,c}| \geqslant r^2/4$, we have

$$\tilde{\xi}^\diamond_{N,r} \geqslant Z^N_{x^*_N} + 2\tilde{m}_{N/r} - 2c\,.$$

Therefore, with probability at least $g(c) \cdot c_2$, we get that

$$\tilde{\xi}^\diamond_{N,r} \geqslant 2(\tilde{m}_r + \tilde{m}_{N/r}) - c_4 \geqslant 2\tilde{m}_N - c_5 \log\log r\,,$$

completing the proof of the proposition. □

Combined with Proposition 2.3, Lemma 3.1 immediately gives the following consequence.

**Corollary 3.6.** *There exist absolute constants* $c_1, c_2, C > 0$ *so that*

$$2m_N - c_2 \log\log r - C \leqslant \mathbb{E}\eta^\diamond_{N,r} \leqslant 2m_N - c_1 \log\log r + C\,.$$

### 3.2 The right tail for the maximum

In this subsection, we compute the exponent in the right tail for the maximum of the MBRW.

**Lemma 3.7.** *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that for all $\lambda \in [0, \sqrt{n})$ and $n$ large enough,*

$$C_\varepsilon^{-1} \mathrm{e}^{-(2\sqrt{\log 2}+\varepsilon)\lambda} \leqslant \mathbb{P}(\max_v \xi_v^N > \tilde{m}_N + \lambda) \leqslant C_\varepsilon \mathrm{e}^{-(2\sqrt{\log 2}-\varepsilon)\lambda} .$$

*Proof.* The upper bound is an immediate comparison argument. Consider the MBRW $\xi_v^N$, and consider the associated BRW $\bar{\xi}_v^N$. As noted in [14, Prop. 3.2], $\mathbb{E}(\xi_v^N)^2 = \mathbb{E}(\bar{\xi}_v^N)^2$ and there exists a constant $C$ such that for $v \neq v'$,

$$\mathbb{E}\xi_v^N \xi_{v'}^N + C \geqslant E\bar{\xi}_v^N \bar{\xi}_{v'}^N .$$

Let $G, G_v$ be iid Gaussian variables of zero mean and variance $C$, independent of the fields $\{\xi, \bar{\xi}\}$. Set $\mu_v^N = \xi_v^N + G$ and $\bar{\mu}_v^N = \bar{\xi}_v^N + G_v$. Clearly, it is still the case that $\mathbb{E}(\mu_v^N)^2 = \mathbb{E}(\bar{\mu}_v^N)^2$, while now, for $v \neq v'$,

$$\mathbb{E}\mu_v^N \mu_{v'}^N \geqslant \mathbb{E}\bar{\mu}_v^N \bar{\mu}_{v'}^N .$$

We conclude from Slepian's lemma that

$$\mathbb{P}(\max_v \bar{\mu}_v^N \geqslant t) \geqslant \mathbb{P}(\max_v \mu_v^N \geqslant t) \geqslant \tfrac{1}{2}\mathbb{P}(\max_v \xi_v^N \geqslant t) .$$

(The last inequality because $\mathbb{P}(G \geqslant 0) = 1/2$.) On the other hand, $\max_v \mu_v^N$ is trivially stochastically dominated by $\max_v \bar{\xi}_v^{\lceil C \rceil N}$. Combining these with the upper bound in (25) yields the upper bound in the lemma.

The proof of the lower bound is very similar to the argument in [12], see Section 6 in [14]. One just replaces there $A_n$ by $A_n + y$ (which is allowed for any $y \geqslant 0$ independent of $n$). □

Combined with Lemma 2.6, the preceding lemma directly yields Theorem 1.4.

## 4 Maxima of the Gaussian free field

This section is devoted to the study of the maxima of the GFF, for which we will harvest results from previous sections.

### 4.1 Physical locations for large values in Gaussian free field

This subsection is devoted to the proof of Theorem 1.1. We first briefly explain the strategy for the proof. Suppose that there exists a number $\varepsilon, \lambda > 0$ such that the limiting probability in (1) is larger than $\varepsilon$ along a subsequence $\{r_k\}$. Then, we can take $N' \asymp N/\varepsilon$ such that the same limiting probability with $N$ replaced by $N'$ will approach almost 1. This would then (roughly) imply that the expected value of $\eta^{\diamond}_{N', r_k}$ will exceed $2m_N - 2\lambda - O(1)$, contradicting with Corollary 3.6 as $k \to \infty$. The details of the proof are carried out in what follows.

We start with the following preliminary lemma.

**Lemma 4.1.** *For $N' > 8N$, consider a discrete ball $B$ of radius $8N$ in a box $V_{N'}$ of side length $N'$. Let $B^* \subset B$ be a box of side length $N$ such that the centers of $B$ and $B^*$ coincide. Let $\{\eta_v : v \in V_{N'}\}$ be a GFF on $V_{N'}$ with Dirichlet boundary condition and let*

$$\psi_v = \mathbb{E}(\eta_v \mid \{\eta_u : u \in \partial B\}) .$$

*Then for $v \in B^*$, we have* $\operatorname{Var} \psi_v = O(\log(N'/N))$.

*Proof.* We need the following lemma, which implies that the harmonic measure on $\partial B$ with respect to any $v \in B^*$ is comparable to the uniform distribution.

**Lemma 4.2.** *[28, Lemma 6.3.7] Let $\mathcal{C}_n \subset \mathbb{Z}^2$ be a discrete ball of radius $n$ centered at the origin. There exist absolute constants $c, C > 0$ such that for all $x \in \mathcal{C}_{n/4}$ and $y \in \partial\mathcal{C}_n$*

$$c/n \leqslant \mathbb{P}_x(\tau_{\partial\mathcal{C}_n} = y) \leqslant C/n\,.$$

A specific property of GFF allows to write the conditional expectation for GFF at a vertex given values on the boundary as a harmonic mean for the values over the boundary (see e.g. [22, Thm. 1.2.2]). Combined with the preceding lemma, this implies that for $v \in B^* \subset B$, we have

$$\psi_v = \sum_{w \in \partial B} a_{v,w}\eta_w,\ \text{ where } c/N \leqslant a_{v,w} \leqslant C/N\,. \tag{47}$$

Therefore, we have

$$\operatorname{Var}\psi_v = \Theta(1/N^2) \sum_{u,w \in \partial B} G_{\partial V_{N'}}(u,w)\,. \tag{48}$$

In order to estimate the sum of Green functions, we use the next lemma.

**Lemma 4.3.** *[28, Prop. 6.4.1] For $\ell < n$ and $x \in \mathcal{C}_n \setminus \mathcal{C}_\ell$, we have*

$$\mathbb{P}_x(\tau_{\partial\mathcal{C}_n} < \tau_{\partial\mathcal{C}_\ell}) = \frac{\log|x| - \log\ell + O(1/\ell)}{\log n - \log \ell}\,.$$

By the preceding lemma, we have

$$\mathbb{P}_u(\tau_{\partial V_{N'}} < \tau^+_{\partial B}) \geqslant O(1/(N\log(N'/N))) \text{ for all } u \in \partial B\,,$$

where $\tau^+_{\partial B} = \min\{t \geqslant 1 : S_t \in \partial B\}$. Thus, $\sum_{w \in \partial B_i} G_{\partial V_{N'}}(u,w) = O(N\log(N'/N))$. Therefore,

$$\operatorname{Var}(\psi_v) = O(\log(N'/N))\,,\ \text{ for all } v \in B^*\,. \qquad \square$$

The following lemma, using the sprinkling idea, is the key to the proof of Theorem 1.1. In the lemma, for $\varepsilon, \delta > 0$ we set $C(\delta,\varepsilon) = 2\log\delta/\log(1-\varepsilon)$.

**Lemma 4.4.** *There exist a a constant $C > 0$ such that, if*

$$\mathbb{P}(\exists v, u \in V_N : r \leqslant |v-u| \leqslant N/r \text{ and } \eta_u, \eta_v \geqslant m_N - \lambda) \geqslant \varepsilon \tag{49}$$

*for some $\varepsilon, \lambda > 0$ and $N, r \in \mathbb{N}$, then for any $\delta > 0$, setting $N'$ to be the smallest power of 2 larger than or equal to $C(\delta,\varepsilon)N$ and $\gamma = C(\sqrt{\log C(\delta,\varepsilon)/\delta})$, the following holds*

$$\mathbb{P}(\eta^{\diamond}_{N',r} \geqslant 2m_N - 2\lambda - \gamma) \geqslant 1 - \delta\,.$$

*Proof.* Let $N' = N2^{k+3}$ with $k = \lceil \log_2 C(\delta,\varepsilon) - 3\rceil$. $B_1, \ldots, B_{2^k} \subset V_{N'}$ be disjoint discrete balls of radius $8N$, and for $i \in [2^k]$ let $B_i^* \subset B_i$ be a box of side length $N$ such that these two centers (of the ball and the box) coincide. Let $\{\eta'_v : v \in V_{N'}\}$ be a GFF on $V_{N'}$ with Dirichlet boundary condition, and for $i \in [2^k]$ let $\{\eta^{(i)}_v : v \in B_i\}$ be i.i.d. GFFs on $B_i$ with Dirichlet boundary condition. We first claim that for all $i \in [2^k]$

$$\mathbb{P}(\exists v, u \in B_i^* : r \leqslant |v-u| \leqslant N/r \text{ and } \eta^{(i)}_u + \eta^{(i)}_v \geqslant 2m_N - 2\lambda) \geqslant \varepsilon/2\,. \tag{50}$$

In order to prove the preceding inequality, we consider the decomposition of $\{\eta_v^{(i)} : v \in B_i^*\}$ (by conditioning on the values at $\partial B_i^*$ analogous to (15)) as

$$\eta_v^{(i)} = \eta_v^{(i),*} + \phi_v \text{ for all } v \in B_i^*$$

where $\{\eta_v^{(i),*} : v \in B_i^*\}$ is a GFF on $B_i^*$ with Dirichlet boundary condition and is independent of the centered Gaussian process $\{\phi_v : v \in B_i^*\}$. Note that $\phi_v$ here denotes the conditional expectation of $\eta_v^i$ given the values on $\partial B_i^*$. Let $\tau_1(i), \tau_2(i) \in B_i^*$ be the locations of maximizers of

$$\max\{\eta_v^{(i),*} + \eta_u^{(i),*} : u, v \in B_i^*, r \leqslant |v - u| \leqslant N/r\}\,.$$

By Assumption (49), we have

$$\mathbb{P}(\eta_{\tau_1(i)}^{(i),*} + \eta_{\tau_2(i)}^{(i),*} \geqslant 2m_N - 2\lambda) \geqslant \varepsilon\,.$$

Since $\phi_{\tau_1(i)} + \phi_{\tau_2(i)}$ is a centered Gaussian variable that is independent of $\eta_{\tau_1(i)}^{(i),*} + \eta_{\tau_2(i)}^{(i),*}$, we can deduce (50) as required.

Let us now consider the decomposition for $\{\eta'_v : v \in V_{N'}\}$. We can write

$$\eta'_v = \eta_v^{(i)} + \psi_v \text{ for } v \in B_i^* \text{ and } i \in [2^k]\,,$$

where $\{\psi_v : v \in B_i^*\}$ is a Gaussian process independent of $\{\eta_v^{(i)} : i \in [2^k], v \in B_i\}$, and furthermore

$$\psi_v = \mathbb{E}(\eta'_v \mid \{\eta'_u : u \in \partial B_i\})\,,\ \text{for } v \in B_i^*\,.$$

By Lemma 4.1, we obtain that $\operatorname{Var}\psi_v = O(k)$ for all $v \in B_i^*$ and $i \in [2^k]$.

Next, let $\iota \in [2^k]$ be the location of the maximizer of

$$\max\{\eta_{\tau_1(i)}^{(i)} + \eta_{\tau_2(i)}^{(i)} : i \in [2^k]\}\,.$$

By the independence of $\{\eta_\cdot^{(i)}\}$ for $i \in [2^k]$, we deduce that

$$\mathbb{P}(\eta_{\tau_1(\iota)}^{(\iota)} + \eta_{\tau_2(\iota)}^{(\iota)} \geqslant 2m_N - 2\lambda) \geqslant 1 - (1-\varepsilon)^{2^k}\,.$$

Conditioning on the location of $\iota$ and $\tau_1(\iota), \tau_2(\iota)$, we see that $\operatorname{Var}(\phi_{\tau_1(\iota)} + \phi_{\tau_2(\iota)}) = O(k)$. Therefore,

$$\mathbb{P}(\eta'_{\tau_1(\iota)} + \eta'_{\tau_2(\iota)} \geqslant 2m_N - 2\lambda - \gamma) \geqslant (1 - (1-\varepsilon)^{2^k})(1 - \tfrac{O(k)}{\gamma^2})\,.$$

With our choice of $k, \gamma$, this completes the proof. ∎

We next bound the lower tail on $\eta^\diamond_{N,r}$ from above. To this end, we first show that the maximal sum over pairs for the GFF has fluctuation at most $O(\log\log r)$.

**Lemma 4.5.** *For any $r \leqslant N$, let $\eta^\diamond_{N,r}$ be defined as in* (14). *Then sequence of random variables $\{(\eta^\diamond_{N,r} - \mathbb{E}\eta^\diamond_{N,r})/\log\log r\}_{N,r}$ is tight along $N \in \mathbb{N}$ and $r \in \{0, \ldots, N\}$.*

*Proof.* For simplicity of notation, we consider the sequence $N = 2^n$ in the proof (the tightness of the full sequence will follow from the same proof with slight modification by considering $n(N) = \max\{k \in \mathbb{N} : 2^k \leqslant N\}$). To this end, we first claim that

$$\mathbb{E}\eta^{\diamond}_{2N,r} \geqslant \mathbb{E}\max\{Z_1, Z_2\}\,, \tag{51}$$

where $Z_1, Z_2 \sim \eta^{\diamond}_{N,r}$ and $Z_1$ is independent of $Z_2$. The proof of (51) follows from the similar argument as in the proof of Lemma 2.4, as we sketch briefly in what follows. Consider $V_N, V'_N \subset V_{2N}$ where $V_N$ and $V'_N$ are two disjoint boxes of side length $N$. Using a similar decomposition as in (15), we can write $\eta^{2N}_v = \eta^N_v + \phi_v$ for $v \in V_N$ and $\eta^{2N}_v = \hat{\eta}^N_v + \phi_v$ for $v \in V'_N$, where $\eta^N_\cdot$ and $\hat{\eta}^N_\cdot$ are two independent copies of GFF in a 2D box of side length $N$. This would then yield (51). Now using the inequality $a \vee b = \frac{a+b+|a-b|}{2}$, we deduce that

$$\mathbb{E}|Z_1 - Z_2| \leqslant 2(\mathbb{E}\eta^{\diamond}_{2N,r} - \mathbb{E}Z_1) \leqslant 2C\log\log r\,,$$

where the last inequality follows from Corollary 3.6. This completes the proof of the lemma. ☐

Based on the preceding lemma, we prove a stronger result which will also imply that the number of point whose values in the GFF exceed $m_N - \lambda$ grows at least exponentially in $\lambda$. We will follow the proof for the upper bound on the lower tail of the maximum of GFF in [19, Sec. 2.4]. For $N, r \in \mathbb{N}$, define

$$\Xi_{N,r} = \{(u,v) \in V_N \times V_N : r \leqslant |u - v| \leqslant N/r\}\,.$$

**Lemma 4.6.** *There exists absolute constants $C, c > 0$ such that for all $N \in \mathbb{N}$ and $r, \lambda \geqslant C$*

$$\mathbb{P}(\exists A \subset \Xi_{N,r} \text{ with } |A| \geqslant \log r : \forall (u,v) \in A : \eta_u + \eta_v \geqslant 2m_N - 2\lambda\log\log r) \geqslant 1 - C\mathrm{e}^{-\mathrm{e}^{c\lambda\log\log r}}\,.$$

*Proof.* The proof idea is similar to [19], and thus we will be brief in what follows. Denote by $R = N(\log r)^{-\lambda/10}$ and $\ell = N(\log r)^{-\lambda/100}$. Assume that the left bottom corner of $V_N$ is the origin $o = (0,0)$. Define $o_i = (i\ell, 2R)$ for $1 \leqslant i \leqslant M = \lfloor N/2\ell \rfloor = (\log r)^{\lambda/100}/2$. Let $\mathcal{C}_i$ be a discrete ball of radius $r$ centered at $o_i$ and let $B_i \subset \mathcal{C}_i$ be a box of side length $R/8$ centered at $o_i$. We next regroup the $M$ boxes into $m$ blocks. Let $m = (\log r)^{\lambda/200}$, and let $\mathfrak{C}_j = \{\mathcal{C}_i : (j-1)m < i < jm\}$ and $\mathcal{B}_j = \{B_i : (j-1)m < i < jm\}$ for $j = 1, \ldots, M/m$.

Now we consider the maximal sum over pairs of the GFF in each $\mathcal{B}_j$. For ease of notation, we fix $j = 1$ and write $\mathcal{B} = \mathcal{B}_1$ and $\mathfrak{C} = \mathfrak{C}_1$. For each $B \in \mathcal{B}$, analogous to (15), we can write

$$\eta_v = g^B_v + \phi_v \text{ for all } v \in B \subseteq \mathcal{C} \in \mathfrak{C}\,,$$

where $\{g^B_v : v \in B\}$ is the projection of the GFF on $\mathcal{C}$ with Dirichlet boundary condition on $\partial\mathcal{C}$, and $\{\{g^B_v : v \in B\} : B \in \mathcal{B}\}$ are independent of each other and of $\{\eta_v : v \in \partial\mathfrak{C}\}$, and $\phi_v = \mathbb{E}(\eta_v \mid \{\eta_u : u \in \partial\mathfrak{C}\})$ is a convex combination of $\{\eta_u : u \in \partial\mathfrak{C}\}$. For every $B \in \mathcal{B}$, define $(\chi_{1,B}, \chi_{2,B}) \in B \times B \cap \Xi_{N,r}$ such that

$$g^B_{1,\chi_B} + g^B_{\chi_{2,B}} = \sup_{u,v \in B \times B \cap \Xi_{N,r}} g^B_v + g^B_u\,.$$

Since $\lambda$ is large enough, we get from Corollary 3.6 and Lemma 4.5 that

$$\mathbb{P}(g^B_{1,\chi_B} + g^B_{\chi_{2,B}} \geqslant 2m_N - \lambda\log\log r) \geqslant 1/4\,.$$

Let $W = \{(\chi_{1,B}, \chi_{2,B}) : g^B_{1,\chi_B} + g^B_{\chi_{2,B}} \geqslant 2m_N - \lambda \log\log r, B \in \mathcal{B}\}$. By independence, a standard concentration argument gives that for an absolute constant $c > 0$

$$\mathbb{P}(W \leqslant \tfrac{1}{8}m) \leqslant \mathrm{e}^{-cm}\,. \tag{52}$$

It remains to study the process $\{\phi_u + \phi_v : (u,v) \in W\}$. If $\phi_u + \phi_v \geqslant 0$ for $(u,v) \in W$ , we have $\eta_u + \eta_v \geqslant 2m_N - \lambda \log\log r$. The required estimate is summarized in the following lemma.

**Lemma 4.7.** *[19, Lemma 2.3] Let $U \subset \cup_{B\in\mathcal{B}} B \times B$ such that $|U \cap B \times B| \leqslant 1$ for all $B \in \mathcal{B}$. Assume that $|U| \geqslant m/8$. Then, for some absolute constants $C, c > 0$*

$$\mathbb{P}(\phi_u + \phi_v \leqslant 0 \textit{ for all } (u,v) \in U) \leqslant C\mathrm{e}^{-c(\log r)^{c\lambda}}\,.$$

Despite the fact that we are considering a sum over a pairs (instead of a single value $\phi_v$) in the current setting as well as slightly different choices of parameters, the proof of the preceding lemma goes exactly the same as that in [19]. The main idea is to control the correlations among $(\phi_u + \phi_v)$ for $(u,v) \in U$. Indeed, one can show that the correlation coefficient is uniformly bounded by $O(\lambda \log\log r\sqrt{R/\ell})$. Slepian's comparison theorem can then be invoked to complete the proof. Due to the similarity, we do not reproduce the proof here.

Altogether, the preceding lemma implies that

$$\mathbb{P}(\max_{B\in\mathcal{B}} \max_{v,v\in B\times B\cap\Xi_{N,r}} \eta_u + \eta_v \geqslant 2m_N - 2\lambda \log r \log r) \geqslant 1 - C\mathrm{e}^{-c(\log r)^{c\lambda}}\,.$$

Now, let $(\chi_{1,j}, \chi_{2,j}) \in \mathcal{B}_j \times \mathcal{B}_j \cap \Xi_{N,r}$ be such that

$$\eta_{\chi_{1,j}} + \eta_{\chi_{2,j}} = \max_{B\in\mathcal{B}_j} \max_{(u,v)\in B\times B\cap\Xi_{N,r}} \eta_u + \eta_v\,,$$

and let $A = \{(\chi_{1,j}, \chi_{2,j}) : 1 \leqslant j \leqslant M/m\}$. A union bound gives that $\min_{(u,v)\in A} \eta_u + \eta_v \geqslant 2m_N - 2\lambda \log\log r$ with probability at least $1 - C\mathrm{e}^{-c(\log r)^{c\lambda}}$, concluding the proof. □

The following is an immediate corollary of the preceding lemma.

**Corollary 4.8.** *There exist absolute constants $C, c > 0$ such that for all $N \in \mathbb{N}$ and $\lambda, r > 0$*

$$\mathbb{P}(\eta^{\diamond}_{N,r} \geqslant 2m_N - 2\lambda \log\log r) \geqslant 1 - C\mathrm{e}^{-c\mathrm{e}^{c\lambda \log\log r}}\,.$$

We are now ready to give

**Proof of Theorem 1.1.** Suppose otherwise that there exists $\varepsilon, \lambda > 0$ and a subsequence $\{r_k\}$ such that for all $k$

$$\limsup_{N\to\infty} \mathbb{P}(\exists v, u \in V_N : r_k \leqslant |v-u| \leqslant N/r_k \text{ and } \eta_u, \eta_v \geqslant m_N - c\log\log r_k) \geqslant \varepsilon\,.$$

Then by Lemma 4.4, for a $\delta > 0$ to be specified and $C(\varepsilon,\delta) > 0$, we have

$$\limsup_{N\to\infty} \mathbb{P}(\exists v, u \in V_{C(\varepsilon,\delta)N} : r_k \leqslant |v-u| \leqslant C(\varepsilon,\delta)N/r_k\,, \eta_u + \eta_v \geqslant 2m_N - 2c\log\log r_k - C(\varepsilon,\delta)) \geqslant 1-\delta\,.$$

Combined with Corollary 4.8, it is clear that for a certain $\delta > 0$ along a subsequence $\{N_j\}$

$$\mathbb{E}\eta^{\diamond}_{C(\varepsilon,\delta)N_j,r_k} \geqslant 2m_{N_j} - 2c\log\log r_k - C(\varepsilon,\delta) - (c_1/2)\log\log r_k \text{ for all } k \in \mathbb{N}\,, \tag{53}$$

where $c_1$ is the constant in Corollary 3.6. Setting $c = c_1/8$ and sending $k \to \infty$, we obtain a contradiction with Corollary 3.6. This completes the proof. $\square$

We conclude this subsection by providing

**Proof of Theorem 1.2.** The lower bound on $A_{\lambda,N}$ follows immediately from Lemma 4.6. A straightforward deduction from Theorem 1.1 together with a packing argument yields an upper bound of merely doubly-exponential on $A_{\lambda,N}$. In what follows, we strengthen the upper bound to exponential of $\lambda$. Continue denoting $\mathcal{S}_{\ell,N}$ and $\mathcal{R}_{\ell,N}$ as in Lemma 2.9. Following notations in Section 3.1, we see that

$$\mathcal{R}_{\ell,N} \leqslant \ell T_N - \tfrac{\ell}{4c^*} \log \ell \mathbf{1}_{\Xi^*_{N,(c^*)^{-1}\log \ell/2} \leqslant \ell/2}\,,$$

where $\Xi^*_{N,x} = \bigcup_{i=t_N-T_N}^{x} \Xi_N(i)$. Applying (25) and (28), we deduce that there exists a constant $c > 0$ such that for sufficiently large $\ell$

$$\mathbb{E}\mathcal{R}_{\ell,N} \leqslant \ell(\sqrt{2\log 2/\pi}m_N - c\log \ell)\,.$$

Combined with Lemma 2.9, it follows that for sufficiently large $\ell$

$$\mathbb{E}\mathcal{S}_{\ell,N} \leqslant \ell(\sqrt{2\log 2/\pi}m_N - c\log \ell)\,. \tag{54}$$

At this point, the proof can be completed analogous to the deduction of (53), as we sketch below. Suppose otherwise that for any $\alpha > 0$ there exists a subsequence $\{r_k\}$ such that for all $k$ there exists a subsequence $N_{k,i}$ with

$$\mathbb{P}(|A_{N_{k,i},r_k}| \geqslant \mathrm{e}^{\alpha r_k}) \geqslant \varepsilon\,, \text{ for all } i \in \mathbb{N}\,,$$

where $\varepsilon > 0$ is a positive constant. Then, following the same sprinkling idea in Lemma 4.4, we can show that for any $\delta > 0$, there exists $C(\varepsilon, \delta)$ such that for $N'_{k,i} = C(\delta,\varepsilon)N_{k,i}$ and $\gamma = \gamma(\varepsilon,\delta)$, the following holds

$$\mathbb{P}(|A_{N'_{k,i},r_k-\gamma}| \geqslant \mathrm{e}^{\alpha r_k}) \geqslant 1-\delta\,.$$

Combined with Lemma 4.6, it follows that

$$\mathbb{E}\mathcal{S}_{\mathrm{e}^{\alpha r_k},N'_{k,i}} \geqslant \mathrm{e}^{\alpha r_k}(\sqrt{2\log 2/\pi}m_{N'_{k,i}} - (1+c'\delta\alpha)r_k - \gamma)\,,$$

where $c' > 0$ is a constant that arise from the estimate in Lemma 4.6. Now, setting $\delta = (c/2c')$, $\alpha = 4/c$ and sending $r_k \to \infty$, we obtain a bound that contradicting with (54), completing the proof of Theorem 1.2.

## 4.2 The gap between the largest two values in Gaussian free field

In this subsection, we study the gap between the largest two values and prove Theorem 1.3.

**Upper bound on the right tail.** In order to show the upper bound in (2), it suffices to prove that for some absolute constants $C, c > 0$ and all $\lambda > 0$

$$\mathbb{P}(\lambda < \Gamma_N \leqslant \lambda + 1) \leqslant \mathbb{P}(\Gamma_N \leqslant 1)\cdot C\mathrm{e}^{-c\lambda^2}\,. \tag{55}$$

To this end, define

$$\Omega_\lambda = \{(x_v)_{v\in V_N} : \gamma((x_v)) \in (\lambda, \lambda+1]\} \text{ for all } \lambda \geqslant 0\,,$$

where $\gamma((x_v))$ is defined to be the gap between the largest two values in $\{x_v\}$. For $(x_v)_{v\in V_N} \in \Omega_\lambda$, let $\tau \in V_N$ be such that $\eta_\tau = \max_{v\in V_N} x_v$. We construct a mapping $\phi_\lambda : \Omega_\lambda \mapsto \Omega_0$ that maps $(x_v)_{v\in V_N} \in \Omega_\lambda$ to $(y_v)_{v\in V_N}$ such that

$$y_v = x_v \text{ if } v \neq \tau\,, \text{ and } y_\tau = x_\tau - \lambda\,.$$

It is clear that the mapping is 1-1 and $(y_v)_{v\in V_N} \in \Omega_0$. Furthermore, the Jacobian of the mapping $\phi_\lambda$ is precisely 1 on $\Omega_\lambda$. It remains to estimate the density ratio $f((x_v))/f((y_v))$. Using (6), we get that

$$f((x_v)) = Z\mathrm{e}^{-\frac{1}{16}\sum_{u\sim v}(x_u-x_v)^2} = Z\mathrm{e}^{-\frac{1}{16}\sum_{u\sim v}(y_u-y_v)^2}\mathrm{e}^{-\frac{1}{8}\sum_{u\sim\tau}((x_u-x_\tau)^2-(y_u-y_\tau)^2)} \leqslant f((y_v))\mathrm{e}^{-\frac{1}{2}\lambda^2}\,.$$

It then follows that

$$\mathbb{P}((\eta_v^N) \in \Omega_\lambda) \leqslant \mathrm{e}^{-\lambda^2/2}\mathbb{P}((\eta_v^N) \in \Omega_0)\,,$$

completing the proof of (55).

**Lower bound on the right tail.** In order to prove the lower bound on the right tail for the gap, we first show that with positive probability there exists a vertex such that all its neighbors in the GFF take values close to $m_N$ within a constant window. To this end, we consider a new Gaussian process $\{\zeta_v : v \in V_N\}$ defined by

$$\zeta_v = \tfrac{1}{4}{\textstyle\sum_{u\sim v}}\eta_v \text{ for } v \in V_N \setminus \partial V_N\,, \text{ and } \zeta|_{\partial V} = 0\,. \tag{56}$$

It is natural to suspect that $\sup_v \zeta_v \approx \sup_v \eta_v$, as stated in the next lemma.

**Lemma 4.9.** *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\mathbb{P}(\max\nolimits_{v\in V_N}\zeta_v \geqslant m_N - C_\varepsilon) \geqslant 1-\varepsilon\,.$$

*Proof.* We will apply Lemma 2.2. For $\kappa \in \mathbb{N}$ to be specified later, define $\phi_\kappa(\cdot) : V_{2^{-\kappa}N} \mapsto V_N$ by

$$\phi_\kappa(v) = 2^\kappa v\,, \text{ for all } v \in V_{2^{-\kappa}N}\,.$$

Let $\{\eta_v^{2^{-\kappa}N} : v \in V_{2^{-\kappa}N}\}$ be a GFF on $V_{2^{-\kappa}N}$. We claim that for large $\kappa$ (independent of $N$)

$$\mathbb{E}(\zeta_{\phi_\kappa(u)} - \zeta_{\phi_\kappa(v)})^2 \geqslant \mathbb{E}(\eta_u^{2^{-\kappa}N} - \eta_v^{2^{-\kappa}N})^2, \text{ for all } u, v \in V_{2^{-\kappa}N}\,. \tag{57}$$

In order to see this, we note that by (9) and the triangle inequality

$$\mathrm{Var}(\eta_v) = \mathrm{Var}\,\eta_u + O(1) = \mathrm{Cov}(\eta_v,\eta_u) + O(1)\,, \text{ for all } u \sim v\,.$$

This then implies that

$$\mathbb{E}(\zeta_v - \zeta_u)^2 = \mathbb{E}(\eta_u - \eta_v)^2 + O(1)\,, \text{ for all } u, v \in V_N\,.$$

Now again using the fact that

$$\mathbb{E}(\eta_{\phi_\kappa(u)} - \eta_{\phi_\kappa(v)})^2 - \mathbb{E}(\eta_u^{2^{-\kappa}N} - \eta_v^{2^{-\kappa}N})^2$$

grows with $\kappa$, we could select $\kappa$ large (though independent of $N$) to beat the $O(1)$ term, and thus obtain (57). At this point, an application of Lemma 2.2 and (4) yields that

$$\mathbb{E}\max_v \zeta_v = m_N + O(1)\,. \tag{58}$$

In addition, it is clear that $\max_v \zeta_v \leqslant \max_v \eta_v$. Therefore, (5) implies an exponential right tail for $\max_v \zeta_v$. Together with (58), this completes the proof of the lemma. $\square$

Combined with (5), the preceding lemma yields that

$$\mathbb{P}(\max_v \zeta_v \geqslant \max_v \eta_v - C) \geqslant 1/4\,,\ \text{for an absolute constant } C > 0\,. \tag{59}$$

In light of the above, we define

$$\Omega^\star = \{(x_v) : \max_v \tfrac{1}{4}\textstyle\sum_{u\sim v} x_u \geqslant \max_v x_v - C\}\,.$$

For $(x_v) \in \Omega^\star$, let $v^\star$ be such that

$$\tfrac{1}{4}\textstyle\sum_{u\sim v^\star} x_u = \max_v \tfrac{1}{4}\textstyle\sum_{u\sim v} x_u\,.$$

Let $\Omega^* = \{(x_v) \in \Omega^\star : x_{v^\star} - \frac{1}{4}\sum_{u\sim v^\star} x_u \in (-1, 0)\}$. By Lemma 4.9, we see that

$$\mathbb{P}((\eta_v) \in \Omega^*) \geqslant 1/100\,. \tag{60}$$

For $\lambda \geqslant 0$, define a map $\Psi_\lambda : \Omega^* \mapsto \mathbb{R}^{V_N}$ by $\Psi((x_v)) = (y_v)$ with

$$y_v = x_v \text{ for all } v \neq v^\star,\ \text{and } y_{v^\star} = 2(\max_v x_v + \lambda) - x_{v^\star}\,.$$

By definition, we have that $\gamma((\Psi_\lambda((x_v)))) \geqslant \lambda$ for all $(x_v) \in \Omega^*$. It is also obvious that $\Psi_\lambda$ is a bijective mapping and that the determinant of the Jacobian is 1. In addition, it is straightforward to check (by definition of $\Omega^*$) for some absolute constants $c, C^* > 0$

$$f(\Psi_\lambda((x_v))) \geqslant c e^{-C^*\lambda^2} f((x_v))\,,\ \text{for all } (x_v) \in \Omega^*\,.$$

Integrating over $\Omega^*$ and applying (60), we complete the proof for the lower bound in (2).

**Lower bound on the gap.** In this subsection, we prove the lower bound on the gap as described in (3). For any $\varepsilon > 0$, an application of Lemma 4.9 and (5) gives that there exists $C_\varepsilon > 0$ such that $\mathbb{P}((\eta_v) \in \Omega_\varepsilon) \geqslant 1 - \varepsilon$, where $\Omega_\varepsilon \triangleq \{(x_v) : \max_v \frac{1}{4}\sum_{u\sim v} x_u \geqslant \max_v x_v - C_\varepsilon\}$. Denote by $\tau$ the maximizer of $\max_v \sum_{u\sim v} x_u$ and by $\tau'$ the maximizer of $\max_v x_v$. It is then clear that there exists $C^*_\varepsilon > 0$ such that $\mathbb{P}(\Omega^*_\varepsilon) \geqslant 1 - 2\varepsilon$, where $\Omega^*_\varepsilon = \Omega_\varepsilon \cap \{(x_v) : x_\tau \geqslant x_{\tau'} - C^*_\varepsilon\}$. Also, by Theorem 1.2, there exists $C_\star > 0$ such that $\mathbb{P}(\Omega^\star_\varepsilon) \geqslant 1 - 3\varepsilon$, where

$$\Omega^\star_\varepsilon = \Omega^*_\varepsilon \cap \{(x_v) : |x_v \geqslant x_{\tau'} - C^*_\varepsilon - 1| \geqslant C^\star_\varepsilon\}\,.$$

Consider $0 < \delta < 1$, and define $\mathcal{C}_i = \{(x_v) : (i-1)\delta \leqslant \gamma((x_v)) < i\delta\}$ for all $i \geqslant 1$. We then construct a mapping $\Phi_i : \Omega^\star_\varepsilon \cap \mathcal{C}_1 \mapsto \mathcal{C}_i \cup \mathcal{C}_{i+1}$ by (say $\Phi_i$ maps $(x_v)$ to $(y_v)$) defining

$$y_v = x_v \text{ if } v \notin \{\tau, \tau'\}\,,\ \text{and } y_\tau = x_{\tau'} + i\delta\,,$$

and in addition $y_{\tau'} = x_\tau$ if $\tau \neq \tau'$. For all $(x_v) \in \Omega_\varepsilon$ and $i = 1, \ldots, 1/\delta$, it is clear that

$$f((x_v)) \leqslant C'_\varepsilon f(\Phi_i((x_v)))\,,$$

where $C'_\varepsilon$ is a constant that depends on $C_\varepsilon$ and $C^*_\varepsilon$. In addition, for all $(x_v) \in \Omega_\varepsilon \cap \mathcal{C}_1$ we see that $\Phi_i((x_v)) \in \mathcal{C}_i \cup \mathcal{C}_{i+1}$, where whether the image is in $\mathcal{C}_i$ or $\mathcal{C}_{i+1}$ depends on whether $\tau$ is the maximizer for $\max_v x_v$. Furthermore, every image has at most $C^\star_\varepsilon + 1$ pre-images in $\Omega^\star_\varepsilon \cap \mathcal{C}_1$. In order to see this, we note that there are two cases when trying to reconstruct $(x_v)$ from $(y_v)$: (1)

$\tau = \tau'$, in which we obtain one valid instance of $(x_v)$; (2) $\tau \neq \tau'$, in which we obtain at most $C^\star_\varepsilon$ valid instances of $(x_v)$. This is because by definition $\tau$ is the maximizer of $\max_v y_v$; and $\tau'$ satisfies that $y_{\tau'} = x_\tau \geqslant x_{\tau'} - C^*$, and there are at most $C^\star_\varepsilon$ locations whose values in $y.$ is no less than $x_{\tau'} - C^*$. Once we locate $\tau$ and $\tau'$, the sequence $(x_v)$ is uniquely determined by $(y_v)$. Altogether, we obtain

$$\mathbb{P}((\eta_v) \in \Omega^\star_\varepsilon \cap \mathcal{C}_1) \leqslant C'_\varepsilon(C^\star_\varepsilon + 1)\mathbb{P}((\eta_v) \in \mathcal{C}_i \cap \mathcal{C}_{i+1})$$

for all $1 \leqslant i \leqslant 1/\delta$. Since $\mathcal{C}_i$'s are disjoint, we obtain that

$$\mathbb{P}((\eta_v) \in \Omega^\star_\varepsilon \cap \mathcal{C}_1) \leqslant 2C'_\varepsilon(C^\star_\varepsilon + 1)\delta\,.$$

Now, sending $\delta \to 0$ and then $\varepsilon \to 0$ completes the proof.

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