Slowly oscillating wavefronts of the KPP-Fisher delayed equation

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Abstract

This paper concerns the semi-wavefronts (i.e. bounded solutions $u = \phi(x\cdot\nu + ct) > 0$, $|\nu| = 1$, satisfying $\phi(-\infty) = 0$) to the delayed KPP-Fisher equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t-\tau,x)), \ u \geq 0, \ x \in \mathbb{R}^m. \qquad (*)$$

First, we show that each semi-wavefront should be either monotone or slowly oscillating. Then a complete solution to the problem of existence of semi-wavefronts is provided. We prove next that the semi-wavefronts are in fact wavefronts (i.e. additionally $\phi(+\infty) = 1$) if $c \geq 2$ and $\tau \leq 1$; our proof uses dynamical properties of some auxiliary one-dimensional map with the negative Schwarzian. The analysis of the fronts' asymptotic expansions at infinity is another key ingredient of our approach. It allows to indicate the maximal domain $\mathcal{D}_n$ of (τ, c) where the existence of non-monotone wavefronts can be expected. Here we show that the problem of wavefront's existence is closely related to the Wright's global stability conjecture.

Key words: KPP-Fisher delayed reaction-diffusion equation, slow oscillations, non-monotone positive traveling front, existence, uniqueness.
2000 Mathematics Subject Classification: 34K12, 35K57, 92D25

1 Introduction and main results

The delayed KPP-Fisher equation or the diffusive Hutchinson's equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)(1-u(t-\tau,x)),\ u \geq 0,\ x \in \mathbb{R}^m, \qquad (1)$$

can be considered as one of the most important examples of delayed reaction-diffusion equations. In particular, during the past decade, this model has been studied by many authors, see [2,5,7,9,8,10,14,24] and the references therein. A significant part of the research dealt with the existence of traveling fronts connecting the trivial and positive steady states in (1) and in its non-local variant [3,6,11,23]

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\left(1 - \int_{\mathbb{R}} K(y)u(t,x-y)dy\right),\ \int_{\mathbb{R}} K(s)ds = 1. \qquad (2)$$

We recall that the classical solution $u(x,t) = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, is a wavefront (or a traveling front) for (1) or (2) propagating at the velocity $c \geq 0$, if the profile ϕ is non-negative and satisfies $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. By replacing condition $\phi(+\infty) = 1$ with less restrictive $0 < \liminf_{s\to+\infty} \phi(s) \leq \limsup_{s\to+\infty} \phi(s) < \infty$, we get the definition of a semi-wavefront. The non-negativity requirement $\phi \geq 0$ is due to the biological interpretation of u as of the concentration of a dominant gene that is reminiscent of the seminal works by Kolmogorov, Petrovskii, Piskunov and Fisher.

Recently, the wavefront existence problem for (1), (2) was considered by using quite different approaches. The first method was proposed by Wu and Zou in [24]. It uses the positivity and monotonicity properties of the integral operator

$$(A\phi)(t) = \frac{1}{z_2 - z_1}\left\{\int_{-\infty}^{t} e^{z_1(t-s)}(\mathcal{H}\phi)(s)ds + \int_{t}^{+\infty} e^{z_2(t-s)}(\mathcal{H}\phi)(s)ds\right\}, \qquad (3)$$

where $(\mathcal{H}\phi)(s) = \phi(s)(b+1-\phi(s-h))$, $h := c\tau$, is taken with some appropriate $b > 1$, and $z_1 < 0 < z_2$ satisfy $z^2 - cz - b = 0$. A direct verification shows that the profiles $\phi \in C(\mathbb{R}, \mathbb{R}_+)$ of semi-wavefronts can be also identified as positive bounded solutions of the integral equation $A\phi = \phi$ satisfying the above mentioned boundary conditions at $\pm\infty$. Unfortunately, the presence of positive delay in (3) strongly affects the monotonicity of A. In order to overcome this difficulty, two different orderings, the usual one and a non-standard Smith and Thieme ordering of $C(\mathbb{R}, \mathbb{R}_+)$, were combined in [24]. Even so the operator A was monotone with respect to each of these two orderings only for sufficiently small h and monotone ϕ.

The operator A is well defined when $b > 0$. Taking formally $b = -1$ in (3) and interpreting correctly the obtained expression for $c > 2$, instead of A we obtain

$$(B\varphi)(t) = \frac{1}{\mu - \lambda}\int_{t}^{+\infty} (e^{\lambda(t-s)} - e^{\mu(t-s)})\varphi(s)\varphi(s-h)ds, \qquad (4)$$

where $0 < \lambda < \mu$ are the roots of $z^2 - cz + 1 = 0$. Remarkably, all monotone wavefronts to equation (1) can be found via a monotone iterative algorithm which uses B (or its limit version B_2 if $c = 2$) and converges uniformly on $\mathbb{R}$, see [10]. Similar ideas were also successfully applied in [5,6,14]. However, our attempts to use the monotone operator B in the case of non-monotone waves were not fruitful.

Aiming to get rid of monotonicity requirements, Shiwang Ma achieved an important progress in [16,17]. He showed that operators similar to A, B have good compactness properties in suitable Banach spaces. Therefore, in certain situations, the Schauder fixed point theorem could be used instead of the iterative monotone scheme from [24]. Ma's idea was successfully applied to various reaction-diffusion models with bounded nonlinearities. Nevertheless, equation $A\phi = \phi$ with A defined by (3) has never been considered within the Ma's approach: this is mainly because of the considerable difficulties related to the construction of a nontrivial A-invariant set suitable for the application of the Schauder fixed point theorem.

It is therefore tempting, in order to avoid the construction of a non-trivial bounded A-invariant convex closed set Ω, to apply the Leray-Shauder continuation principle to equation $A\phi = \phi$. The main obstacle for the realization of such an idea is the apparent impossibility to have at the same time complete continuity of A and the non-empty interior of Ω. This problem was avoided in a nice way by Berestycki *et al.* in [3]. Working with equation (2), for a fixed $\delta > 0$, Berestycki *et al.* considered a family of associated boundary value problems, with the boundary conditions $\phi_n(-n) = 0, \phi_n(n) = 1, \phi_n(0) = \delta$. Fortunately, the above mentioned contradiction between the compactness of operator and the openness of its domain does not occur on finite intervals $[-n, n]$. Hence, the Leray-Shauder continuation principle (with corresponding calculation of *a priori* estimates, degrees etc) can be applied for each $n \in \mathbb{N}$. Finally, the wave profile ϕ was obtained in [3] as the limit of ϕ_n . The proof of the existence in [3] is rather technical and non-trivial. Regrettably, the conditions of C^1-smoothness of kernel K and especially the positivity of $K(0) > 0$ do not allow use the existence theorem from [3] to deduce a similar result for equation (1). Indeed, if we take some $\delta-$like sequence of kernels $K^{(j)}(s) \to \delta(s - h)$ then the corresponding sequence of traveling waves $\phi^{(j)}(s)$ could be eventually unbounded in view of *a priori* estimates obtained in [3].

Our short description of analytical tools used to prove the wave existence in (1), (2) would be incomplete without mentioning the Lin-Hale approach to heteroclinic solutions developed in [7,**?**]. This method allowed to obtain almost optimal existence results (i.e. $\tau \leq 3/2$ and $c \geq c'$, for some indefinite and large c': see also Fig. 1 and Conjecture 1 below) for rapidly traveling fronts. Nevertheless, the most interesting in applications critical waves were excluded in [7,**?**]. Surprisingly, as the recent work [9] shows, the Lin-Hale method still can be extended to give a complete solution to the problem of existence of monotone fronts in several models (including (1)). However, the monotonicity of waves is one of crucial assumptions in [9] and, at this moment, it is not clear whether it can be dropped.

After analyzing the above approaches to the existence problem and motivated by [3,16,24], we decided to work with the equation $A\phi = \phi$. As a result, we elaborated a framework suitable for the application of the Schauder fixed point theorem for an appropriately modified version of the operator A. Before stating the corresponding existence theorem, let us define several subsets of parameters $(\tau, c) \in \mathbb{R}_+^2$ (see also Figure 1 below):

$$\mathfrak{D}_s = \{(\tau, c) \in \mathbb{R}_+ : \text{there exists a semi-wavefront to Eq. (1)}\},$$

$$\mathfrak{D}_m = \{(\tau, c) \in \mathbb{R}_+ : \text{there exists a monotone wavefront to Eq. (1)}\},$$

$$\mathfrak{D}_n = \{(\tau, c) \in \mathbb{R}_+ : \text{there exists a non-monotone wavefront to Eq. (1)}\}.$$

Theorem 1 (Existence criterion for semi-wavefronts) $\mathfrak{D}_s = \{(\tau, c) \in \mathbb{R}_+^2 : c \geq 2\}$. *Furthermore, there exist continuous functions* $\delta_\pm : \mathbb{R}_+^2 \to (0, +\infty)$ *such that* $\delta_-(\tau, c) < \phi(t) < \delta_+(\tau, c),\ t \geq Q_0,\ \phi(t) < 1,\ t < Q_0$, *for each semi-wavefront* $u = \phi(x \cdot \nu + ct), |\nu| = 1$, *and some appropriate* $Q_0 = Q_0(\phi)$.

The proof of Theorem 1 requires a detailed study of oscillation/monotonicity properties of semi-wavefront profiles. Here we were inspired by geometrical descriptions from [22] of semi-wavefront profiles to the Mackey-Glass type delayed reaction-diffusion equation

$$u_t(t, x) = \Delta u(t, x) - u(t, x) + g(u(t - \tau, x)),\ u(t, x) \geq 0,\ x \in \mathbb{R}^m. \qquad (5)$$

It is known that in the ordinary case (i.e. when $u = u(t)$) models (1), (5) can be considered within the same family of differential equations governed by linear friction (possibly, degenerate) and negative delayed feedback. Inclusion of the diffusive terms, however, makes the similarity between (1) and (5) much less direct. Nevertheless, it is still possible to prove that the semi-wavefront profiles to (1) share all geometric properties established in the case of equation (5). Amazingly, the statements of corresponding assertions become even sharper while their proof simplifies: cf. Theorem 2, 4 below with Theorems 1,3 in [22].

Theorem 2 (Monotonicity of the leading edge of semi-wavefronts)
Let $u(x, t) = \phi(\nu \cdot x + ct),\ |\nu| = 1$, *be a non-negavite non constant (possibly, unbounded) solution of equation (1) satisfying* $\phi(-\infty) = 0$. *Then* $\phi(t) > 0$, $t \in \mathbb{R}$, *and* ϕ *has a monotone leading edge. The latter means that* $\phi'(s) > 0$ *on* $(-\infty, T_0) \cup (T_1, T_2)$ *and* $\phi'(s) < 0$ *on* (T_0, T_1) *for some* $T_2 \geq T_1 \geq T_0 \in \mathbb{R} \cup \{+\infty\}$. *Furthermore,* T_0 *is finite if and only if* $\phi(T_0) > 1$.

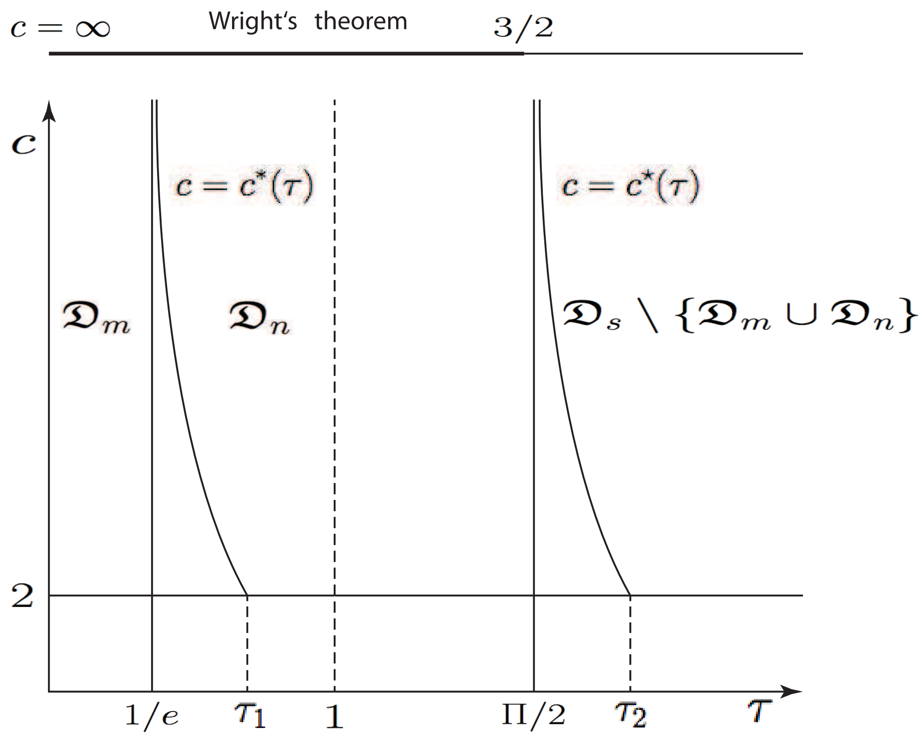

Fig. 1. Schematic presentation of the critical speeds and delays.

Before stating the next theorem, we need to introduce the concepts of critical speeds $c^* < c^\star$ and slowly oscillating semi-wavefronts for equation (1). Let $\psi(z,c) := z^2 - cz - \exp(-zc\tau)$ and set $\tau_1 := 0.560771160\ldots$

By [10, Lemma 3], there exists function $c^* = c^*(\cdot) : [0, \tau_1] \to [2, +\infty]$ such that $\psi(z, c),\ c \geq 2$, has has exactly two (counting multiplicity) negative zeros in the half plane $\{\Re z < 0\}$ if and only if $\tau \in [0, \tau_1]$ and $c \in [2, c^*(\tau)]$. Moreover, $c^*(\tau) = +\infty$ if and only if $\tau \leq 1/e$ while on the interval $(1/e, \tau_1]$ function $c^*(\tau)$ is decreasing and $c^*(\tau_1) = 2$, see Fig. 1. Similarly, we have the following

Lemma 3 *Let $c \geq 2,\ \tau \geq 0$. Then $\psi(z,c)$ has exactly one (counting multiplicity) zero in the right half-plane $\Re z > 0$ if and only if one of the following conditions holds*

(1) $0 \leq \tau \leq \pi/2$, *and* $c \leq c^\star(\tau) := +\infty$,
(2) $\pi/2 < \tau \leq 1.86173\ldots := \tau_2$ *and* $c \leq c^\star(\tau)$, *where* $c^\star$ *is given implicitly by*

$$\tau = \frac{\arccos(-w^2(c^\star))}{c^\star w(c^\star)}, \quad w^2(c) = \frac{\sqrt{c^4+4} - c^2}{2}. \tag{6}$$

Furthermore, if $c > c^\star(\tau)$ and $\psi(\lambda_j, c) = 0$, $\Re\lambda_j \leq 0$, then $|\Im\lambda_j| > 2\pi/(c\tau)$.

As in [22], we follow closely the definition of slow oscillations from [19,20]:

Definition 1 *Set* $h := c\tau, \mathbb{K} = [-h, 0] \cup \{1\}$. *For any* $v \in C(\mathbb{K}) \setminus \{0\}$ *we define the number of sign changes by*

$$\mathrm{sc}(v) = \sup\{k \geq 1 : \text{there are } t_0 < \ldots < t_k \text{ such that } v(t_{i-1})v(t_i) < 0 \text{ for } i \geq 1\}.$$

We set $\mathrm{sc}(v) = 0$ *if* $v(s) \geq 0$ *or* $v(s) \leq 0$ *for* $s \in \mathbb{K}$. *If* φ *is a non-monotone semi-wavefront profile to (1), we set* $(\bar{\varphi}_t)(s) = \varphi(t+s) - 1$ *if* $s \in [-h, 0]$, *and* $(\bar{\varphi}_t)(1) = \varphi'(t)$. *We will say that* $\varphi(t)$ *is sine-like slowly oscillating if graph of* φ *oscillates around* 1 *and has exactly one critical point between each two consecutive intersections with the level* 1, *and, in addition, for each* $t \geq T_0$ *(*T_0 *was defined in Theorem 2), it holds that either* $sc(\bar{\varphi}_t) = 1$ *or* $sc(\bar{\varphi}_t) = 2$.

Our next result is similar to [22, Theorem 3]. In fact, it is even stronger, since it excludes non-monotone but eventually monotone wavefronts to equation (1). As the numerical simulations of [3, Figure 1] show, this irregular behavior can occur in simple non-local KPP-Fisher equations. We also believe that such kind of irregular non-monotone wavefronts can be found in equation (5).

Theorem 4 (Semi-wavefronts are either monotone or slowly oscillating) *Let* $u = \phi(\nu \cdot x + ct)$ *be as in Theorem 2. Then one of the next options holds*

(1) ϕ *is monotonically converging to* 1*;*
(2) ϕ *is sine-like slowly oscillating around* 1 *on a finite maximal interval and, for some* $A > 0$, t_0, *it holds* $\phi'(t) > 0$, $\phi(t) > Ae^{ct}$, $t \geq t_0$*;*
(3) ϕ *is sine-like slowly oscillating around* 1 *and it is bounded.*

Remark 5 *By Theorem 10 below, each bounded profile* ϕ *has to develop non-decaying slow oscillations around* 1 *for each* $c > c^\star(\tau)$ *and then, due to [20], these oscillations should be asymptotically sine-like periodic.*

The final part of this section concerns the determination of domain $\mathfrak{D}_n \subset \mathbb{R}^2_+$. We recall that $\mathfrak{D}_s$ was already found in Theorem 1 while the complete description of $\mathfrak{D}_m$ was given in [10]:

Proposition 6 $\mathfrak{D}_m = \{(\tau, c) \in \mathbb{R}^2_+ : 2 \leq c \leq c^*(\tau)\}$. *Furthermore, for some appropriate* ϕ_- *(given explicitly), we have that* $\phi = \lim_{j\to+\infty} B^j\phi_-$ *(if* $c > 2$*), and* $\phi = \lim_{j\to+\infty} B_2^j\phi_-$ *(if* $c = 2$*), where the convergence is monotone and uniform on* $\mathbb{R}$. *Finally, for each fixed* $c \neq c^*(\tau)$, $\phi(t)$ *is the only possible monotone profile (modulo translation).*

As it was recently demonstrated by Fang and Wu in [5, Theorem 6.2], condition $c \neq c^*(\tau)$ of Proposition 6 can be dropped. In any case, the uniqueness in [5,10] was established only within the class of monotone fronts (see also [6] for a similar assertion concerning (2)). Here, by combining the Berestycki-Nirenberg sliding argument [4] with the approach of [10], we obtain the following

Theorem 7 *Suppose that* $(\tau, c) \in \mathfrak{D}_m$ *and* $u = \phi_1, \phi_2$ *are wavefronts to (1). Then* $\phi_1(t) \equiv \phi_2(t+\alpha)$ *for some* $\alpha \in \mathbb{R}$ *and* $\phi_j'(t) > 0,\ t \in \mathbb{R}$.

The sliding solutions method does not work when $(c, \tau) \notin \mathfrak{D}_m$. However, as the recent works [1,8] have showed, the uniqueness (up to translation) of the semi-waveronts to (1) is very likely to be true for large speeds. We believe that for each fixed pair (τ, c) the semi-wavefront solution to equation (1) is unique (up to a translation) whenever it exists.

Theorem 7 is instrumental in proving Theorem 4 and, whence, in establishing our last two results:

Theorem 8 (Existence of non-monotone wavefronts)

$$\mathfrak{D}_n \cup \mathfrak{D}_m \supset \mathcal{D} = \{(\tau, c) \in \mathbb{R}_+^2 : 0 \le \tau \le 1,\ c \ge 2\}.$$

Moreover if $(\tau, c) \in \mathcal{D}$ *then necessarily* $\phi(+\infty) = 1$. *Hence, for each* $\tau \le 1$ *equation (1) has at least one semi-wavefront which necessarily is a wavefront.*

Corollary 9 (Absolute uniqueness of monotone wavefronts) *Suppose that* $(\tau, c) \in \mathfrak{D}_m$ *and* $u = \phi_1, \phi_2$ *are semi-wavefronts to (1). Then* $\phi_1(t) \equiv \phi_2(t+\alpha)$ *for some* $\alpha \in \mathbb{R}$ *and* $\phi_j'(t) > 0,\ t \in \mathbb{R}$.

Theorem 10 (Admissible wavefront speeds and non-existence of fronts) *Eq. (1) does not have any travelling front (neither monotone nor non-monotone) propagating at velocity* $c > c^\star(\tau)$ *or* $c < 2$.

It can bee seen from Proposition 6 and Theorem 10 that $\mathfrak{D}_n \subset \{(\tau, c) \in \mathbb{R}_+^2 : c^*(\tau) < c \le c^\star(\tau)\}$. Moreover, by Theorem 8 and [8, Theorem 5.1], $\mathfrak{D}_n \cup \mathfrak{D}_m \supset [0, 3/2] \times [c', +\infty) \cup [0, 1] \times [2, +\infty)$ for some large c'. See also Figure 1. In this way, considerations of the present work suggest the following natural criterion for the existence of non-monotone wavefronts in (1):

Conjecture 1 $\mathfrak{D}_n = \{(\tau, c) \in \mathbb{R}_+^2 : c^*(\tau) < c \le c^\star(\tau)\}$.

It can be regarded as an extension of the famous Wright's global stability conjecture [13,15]. Therefore, in our opinion, it would be very interesting (and, perhaps, very difficult) to prove it. In particular, in the limit case $c = +\infty$, Conjecture 1 is true if the Wright's conjecture is true. An important partial result in proving Conjecture 1 would be the following analog of the Wright's 3/2-global stability theorem: $\mathfrak{D}_n \cup \mathfrak{D}_m \supset [0, 3/2] \times [2, +\infty)$.

The structure of the remainder of this paper is as follows. Section 2 contains the proof of Theorem 7. In the third section, we describe the geometrical form of semi-wavefronts. Theorems 8, 1 and 10 are proved in Sections 4, 5, 6 respectively. In Appendix, the characteristic function of the variational equation at the positive steady state is analyzed.

2 Absolute uniqueness of monotone wavefronts

Take some $(\tau, c) \in \mathfrak{D}_m$. Then by Proposition 6 and [5, Theorem 6.2] there exists a *unique* monotone wavefront $u = \psi_2(\nu \cdot x + ct)$. Suppose that $u = \psi_1(\nu \cdot x + ct)$ is a different (and therefore non-monotone) wavefront. Clearly, each profile $\psi_i(t)$ satisfies

$$\begin{aligned} &\phi''(t) - c\phi'(t) + \phi(t)(1 - \phi(t-h)) = 0, \ h := c\tau, \\ &\phi(-\infty) = 0, \quad \phi(t) \geq 0, \quad t \in \mathbb{R}, \end{aligned} \tag{7}$$

and therefore it is strongly positive due to

Lemma 11 *Let non-negative $\phi \not\equiv 0$ solve (7). Then $\phi(t) > 0, \ t \in \mathbb{R}$.*

PROOF. Suppose that, for some s, we have $\phi(s) = 0$. Since $\phi(t) \geq 0, \ t \in \mathbb{R}$, this yields $\phi'(s) = 0$. Therefore $y = \phi(t)$ satisfies the following initial value problem for a linear second order ordinary differential equation

$$y''(t) - cy'(t) + (1 - \phi(t-h))y(t) = 0, \quad y(s) = y'(s) = 0.$$

But then $y(t) \equiv 0$ due to the uniqueness theorem. □

We also will need the asymptotical description of profiles ψ_i at $\pm\infty$. Recall that $0 < \lambda \leq \mu$ denote the roots of $z^2 - cz + 1 = 0$, $c \geq 2$.

Lemma 12 *Let $c > 2, \ q \in \mathbb{R}$. Then, for sufficiently small $\epsilon > 0$, it holds*

$$\psi_i(t+q) = \frac{e^{\lambda(t+q)}}{\sqrt{c^2-4}} \int_{\mathbb{R}} e^{-\lambda s} \psi_i(s) \psi_i(s-h) ds + O(e^{(\lambda+\epsilon)t}), \quad t \to -\infty.$$

Similarly, if $c = 2$ then

$$\psi_i(t+q) = e^{\lambda(t+q)} \int_{\mathbb{R}} e^{-\lambda s} \psi_i(s) \psi_i(s-h) ds(-t + O(1)), \quad t \to -\infty.$$

PROOF. It is a straightforward consequence of [10, Lemma 28]. See also proof of Theorem 6 in [10]. □

Lemma 13 *Suppose that $(\tau, c) \in \mathfrak{D}_m$ and let $\lambda_0 < 0$ be as in Lemma 30. Let $c \in [2, c^*(\tau)), \ q \in \mathbb{R}$, and $\epsilon > 0$ be sufficiently small. Then*

$$\psi_i(t+q) = 1 - K_i e^{\lambda_0(t+q)} + O(e^{(\lambda_0 - \epsilon)t}), \quad t \to +\infty,$$

for some $K_2 > 0$ and $K_1 \in \mathbb{R}$ independent on q. Similarly, if $c = c^*(\tau)$ then

$$\psi_i(t+q) = 1 - e^{\lambda_0(t+q)}(K_i t + O(1)), \quad t \to +\infty, \quad K_2 > 0, \ K_1 \in \mathbb{R}.$$

PROOF. In the monotone case (i.e. $i = 2$), this statement follows from [10, Lemma 28] and Lemma 30 (see also [10, Theorem 6] for more details). Next, due to [10, Lemma 10], the condition $(\tau, c) \in \mathfrak{D}_m$ implies the hyperbolicity of the positive equilibrium of (7) and therefore $|\psi_1(t) - 1|$ converges exponentially to 0 at $+\infty$. With this observation, the analysis of the non-monotone wavefront is completely analogous to the monotone case considered in [10, Section 7]. The unique exception is the sign of K_1. Indeed, in virtue of non-monotonicity of the wavefront ψ_1, K_1 could take any real value including 0. $\square$

By applying a sliding argument, we are ready now to prove Theorem 7. Set

$$\mathcal{Q} := \{q : \psi_1(t) \geq \psi_2(t-q),\ t \in \mathbb{R}\}.$$

It follows from Lemmas 12, 13 that $\mathcal{Q} \neq \emptyset$. On the other hand, it is obvious that the set $\mathcal{Q}$ is closed, below bounded and connected (the latter is due to the monotonicity of ψ_2). Let $q_* = \inf \mathcal{Q}$, we claim that, for some finite t_*,

$$\psi_1(t_*) = \psi_2(t_* - q_*). \tag{8}$$

Indeed, otherwise

$$\psi_1(t) > \psi_3(t) := \psi_2(t - q_*), \quad t \in \mathbb{R}, \tag{9}$$

and therefore Lemma 12 (taken with $q = 0$ and applied to ψ_1 and ψ_3) implies that there are S_0 and $\delta_0 > 0$ such that $\psi_1(t) > \psi_3(t+\delta),\ t \leq S_0$ for all $\delta \in [0, \delta_0]$. Now, applying Lemma 13 (with $q = 0$) to the profiles ψ_3 and ψ_1 we obtain that necessarily $K_2 \geq K_1$. We claim that $K_2 > K_1$. Indeed, otherwise $K_2 = K_1 > 0$ and therefore the uniqueness proof of [10, Section 6.3] can be repeated for $c \leq c^*(\tau)$, see also [5, Theorem 6.2]. Hence, $K_2 > K_1$ and therefore there exist $S_1 \geq S_0, \delta_1 > 0$ such that $\psi_1(t) > \psi_3(t+\delta),\ t \geq S_1$ for all $\delta \in [0, \delta_1]$. Finally, considering inequality (9) on a fixed interval $[S_0, S_1]$, we find that, for some $\delta_2 > 0$, it holds $\psi_1(t) > \psi_3(t+\delta),\ t \in [S_0, S_1]$ for all $\delta \in [0, \delta_2]$. But then

$$\psi_1(t) > \psi_3(t+\delta),\ t \in \mathbb{R},\ \text{for all } \delta \in [0, \delta_*],\ \delta_* = \min\{\delta_j,\ j = 0, 1, 2\}.$$

Therefore $q_* - \delta_* \in \mathcal{Q}$, a contradiction.

Hence, (8) holds and therefore non-negative function $\theta(t) = \psi_1(t) - \psi_3(t)$ attains its zero minimum at t_*. Moreover, as $\theta(t) > 0$ for $t \leq S_0$, we may assume that t_* is the leftmost zero minimum of θ. Now, it is easy to see that bounded θ also satisfies the differential equation

$$\theta''(t) - c\theta'(t) + \theta(t) = \theta(t)\psi_1(t-h) + \theta(t-h)\psi_3(t) =: \Theta(t),$$

so that either

$$\theta(t) = \frac{1}{\mu - \lambda} \int_t^{+\infty} (e^{\lambda(t-s)} - e^{\mu(t-s)})\Theta(s)ds, \quad \text{if } c > 2, \text{ cf. (4)},$$

$$\text{or} \quad \theta(t) = \int_t^{+\infty} (s-t)e^{(t-s)}\Theta(s)ds, \quad \text{if } c = 2.$$

Considering the above relations with $t = t_*$, we deduce immediately that $\Theta(s) \equiv 0$ on $[t_*, +\infty)$. However, this can not happen because of the inequality $\Theta(s) \geq \theta(s-h)\psi_3(s) > 0,\ s \in [t_*, t_* + h)$. The obtained contradiction ends the proof of Theorem 7. $\square$

3 Semi-wavefront's shape

This section contains a detailed analysis of the oscillation and monotonicity properties of profiles ϕ corresponding to non constant non-negative solutions $u(t,x) = \phi(\nu\cdot x + ct), |\nu| = 1, \phi(-\infty) = 0$, of the delayed KPP-Fisher equation. The main conclusions of the section (see also Lemma 22 below) are presented as Theorem 2 and Theorem 4 in Introduction.

By Lemma 11, similarly to the case of the Hutchinson's equation, the change of variables $\phi(t) = e^{-x(t)}$ can be applied to (7). The obtained equation (see equation (10) below) is a unidirectional monotone cyclic feedback system with delay [20]. Therefore, analogously as it was done in [22], fundamental results from [19,20] can be used to demonstrate slowly oscillating character of the non-monotone semi-wavefronts. Nevertheless, here we have preferred to give short and self-contained direct proofs of this fact, additionally establishing sinusoidal shape of all (and not only periodic as in [20]) oscillating solutions. See also Remark 5 in the introduction.

Lemma 14 *Let Q_0 be such that $0 < \phi(s) < 1$ for $s < Q_0$ and $\phi(Q_0) = 1$. Then $\phi'(s) > 0$ for all $s \in (-\infty, Q_0]$.*

PROOF. If $\phi'(s) = 0, \phi(s) \leq 1$, for some $s \leq Q_0$, then necessarily $\phi''(s) < 0$ so that s is a critical point (local maximum) of ϕ. As a consequence, $\phi'(t) < 0$ for all $t > s$ since otherwise there exists $s_1 > s$ such that $\phi'(s_1) < 0, \phi''(s_1) = 0$, $\phi(s_1), \phi(s_1 - h) \in (0, 1]$, a contradiction. However, $\phi'(t) < 0,\ \phi(t) < 1, t > s$, yields $\phi''(t) < 0$ for all $t > s$, and therefore $\phi(t)$ can not be positive for large t, a contradiction. $\square$

Lemma 15 *Let Q_0 be as in Lemma 14 and Q_1 be such that $\phi(s) > 1$ for all s from some maximal open interval (Q_0, Q_1). Then the only options for the geometrical shape of ϕ on (Q_0, Q_1) are:*

- *(I) Q_1 is finite and $\phi(Q_1) = 1$. Equation $\phi'(t) = 0$ has only one solution $T_0 \in [Q_0, Q_1]$ which is the absolute maximum point of ϕ on $[Q_0, Q_1]$. Next, if $a > T_0$ is the leftmost point where $\phi'(a) = 0$ then $a > Q_1, a-h \in (Q_0, Q_1)$.*
- *(II) ϕ strongly increases on $(Q_0, +\infty)$, with at most one critical point $Q_0 + h$.*
- *(III) ϕ has exactly two critical points: strong local maximum at $T_0 \in (Q_0, Q_0 + h)$ and a strong local minimum at $t_m > Q_0 + h$, where $\phi(t_m) \geq 1$. On the interval $(t_m, +\infty)$, function ϕ is increasing with $\phi'(t) > 0, \phi''(t) > 0$.*

PROOF. Obviously, we get the second option if $\phi'(t) > 0$ for all $t \in \mathbb{R}$. Thus we may suppose that there exists some leftmost point $T_0 > Q_0$ where $\phi'(T_0) = 0$. This implies immediately that $\phi(T_0) > 1$, $\phi''(T_0) \leq 0$, and, consequently, $\phi(T_0 - h) \leq 1$.

(I) Suppose that Q_1 is finite so that $\phi(Q_1) = 1$. We claim that $\phi(T_0 - h) < 1$ and therefore $\phi''(T_0) < 0$ with T_0 being a local maximum point. Indeed, if $\phi(T_0 - h) = 1, \phi''(T_0) = 0, \phi'(T_0) = 0$, then $\phi'''(T_0) = \phi(T_0)\phi'(T_0 - h) > 0$ in virtue of Lemma 14. In consequence $\phi(t) = \phi(T_0) + \phi'''(T_0)(t - T_0)^3/6 + o((t - T_0)^3)$ and thus T_0 is not an absolute maximum point on $[Q_0, Q_1]$. Let $q > T_0$ be such a point, then $\phi(q - h) > 1, \phi''(q) \leq 0, \phi'(q) = 0$, a contradiction.

Hence, $\phi''(T_0) < 0, \phi(T_0 - h) < 1$. Let $a > T_0$ be the leftmost point where $\phi'(a) = 0$. Then a is finite, $\phi''(a) \geq 0$ and therefore $\phi(a - h) \geq 1$. Now, if $\phi''(a) = 0$ then $\phi(a - h) = 1$ and $\phi'''(a) = \phi(a)\phi'(a - h) > 0$, a contradiction (since ϕ is strictly decreasing on (T_0, a)). This means that $\phi''(a) > 0$ and $\phi(a - h) > 1$.

Suppose that $a < Q_1$. Then there is $b \in (a, Q_1)$ such that $\phi(b - h) > 1$, $\phi'(b) = 0$, $\phi''(b) \leq 0$, contradicting to equation (7). Next, if $a = Q_1$ then $\phi(a - h) > 1$, $\phi(a) = 1$, $\phi'(a) = 0, \phi''(a) > 0$. Therefore $\phi''(t) > 0, t \geq a$, so that the option (III) holds. (Indeed, otherwise there is $d > a$ such that $\phi'(t) > 0$ for all $t \in (a, d]$, and $\phi''(d) = 0, \phi(d - h) \geq 1$, a contradiction).

(II) Now, suppose that $Q_1 = +\infty$ and $\phi(T_0 - h) = 1$. Then, as it was shown in (I), we obtain $\phi'''(T_0) > 0$ that yields $\phi'(t) > 0$ for all $t > T_0$.

(III) Finally, consider the situation when $Q_1 = +\infty$ and $\phi(T_0 - h) < 1$ (i.e. ϕ reaches a strict local maximum at T_0). In such a case, ϕ should have subsequent leftmost critical point $q > T_0, \phi(q) > 1$. Indeed, otherwise $\phi'(t) < 0, \phi(t) > 1$, $t > T_0$, so that ϕ converges monotonically to 1 at $+\infty$. However, due to the proof of [10, Lemma 20], this is possible only when $(\tau, c) \in \mathfrak{D}_m$ and therefore this contradicts to Theorem 7. By the arguments in (I), we already know that $\phi'(q) = 0, \phi''(q) > 0$ and $\phi(q - h) > 1$. This makes impossible the existence of $p > q$, where $\phi''(p) = 0, \phi'(p) > 0$ and $\phi(p - h) > 1$. In particular, $\phi'(t) > 0, t > q$. □

Corollary 16 *Let $Q_0 < T_0$ be as in Lemma 15(I) or 15(III). Then*

$$\phi(T_0) = \max_{s \in [Q_0, Q_0 + h]} \phi(s) \leq e^{ch}$$

and $\phi(t) > e^{c(t - Q_0)}, t < Q_0, \phi'(Q_0) < c, \phi(t) < e^{c(t - Q_0)}, t \in (Q_0, Q_0 + h]$.

PROOF. Integrating equation (7) between $-\infty$ and $t \leq Q_0 + h$, and taking into account that $\phi(t)(1 - \phi(t - h)) > 0$ for all $t < Q_0 + h$, we find that $\phi'(t) - c\phi(t) < 0$, $t < Q_0 + h$. Hence $(\phi(t)e^{-ct})'$ is strictly decreasing on $(-\infty, Q_0 + h]$. In particular, $\phi'(Q_0) < c\phi(Q_0) = c$ and $\phi(T_0)e^{-cT_0} < \phi(Q_0)e^{-cQ_0} = e^{-cQ_0}$. Thus $\phi(T_0) < e^{c(T_0 - Q_0)} < e^{ch}$. The proof of other inequalities is similar. □

Lemma 17 *Assume that option (I) of Lemma 15 holds. Then there exists a finite number* $Q_2 > Q_1$ *such that* $\phi(Q_2) = 1, \phi'(Q_2) > 0$ *and* $\phi(t) < 1$ *on* (Q_1, Q_2)*. Moreover, equation* $\phi'(t) = 0$ *has only one solution* $T_1 \in [Q_1, Q_2]$ *which is the absolute minimum point on* (Q_1, Q_2)*. Next, if* $T_2 > T_1$ *is the finite leftmost point where* $\phi'(T_2) = 0$ *then* $T_2 - h \in (Q_1, Q_2)$*. Finally,* $Q_2 - Q_0 > h$*.*

PROOF. Let $T_1 > Q_1$ be the leftmost point where $\phi'(T_1) = 0$. By Lemma 15(I), we know that $\phi''(T_1) > 0,\ \phi(T_1 - h) > 1,\ \phi(T_1) < 1$. Next, let (Q_1, Q_2) denote the maximal open interval containing T_1 where $\phi(t) < 1$.

First, assume that $\phi'(t) > 0$ for $t > T_1$. Then $\phi(t)$ is unbounded since otherwise $\phi(t)$ converges monotonically to 1 that is possible only when $(\tau, c) \in \mathfrak{D}_m$ and therefore this contradicts to Theorem 7. As a consequence, there exists a finite Q_2 with the mentioned properties.

Suppose now that there exists some leftmost point $T_2 > T_1$ where $\phi'(T_2) = 0$. Then $\phi''(T_2) \leq 0$ and therefore $\phi(T_2 - h) \leq 1$. For an instance, suppose additionally that $T_2 \in (T_1, Q_2]$. If $\phi''(T_2) = 0$ then $\phi(T_2 - h) = 1$ and $\phi'''(T_2) = \phi(T_2)\phi'(T_2 - h) < 0$, a contradiction (since ϕ is strictly increasing on (T_1, T_2)). This means that $\phi''(T_2) < 0$ and $\phi(T_2 - h) < 1$. But then ϕ can not have any critical point $b > T_2, \phi(b) < 1$, since otherwise we get a contradiction: $\phi''(d) = 0, \phi'(d) < 0,\ \phi(d - h) \leq 1$ for some $d \in (T_2, b)$. Therefore $\phi'(t) < 0$ for $t > T_2$ so that $\phi''(t) < 0$ for $t > T_2$ and $\phi(t)$ can not be positive for large positive t. The latter contradiction shows that actually $T_2 > Q_2$ and thus Q_2 is finite and $\phi'(Q_2) > 0$. Finally, $Q_2 - Q_0 > T_1 - Q_0 > h$ while the inequality $T_2 - h < Q_2$ can be proved in the same way as the inequality $T_0 - h < Q_0$ in Lemma 15(I). □

Corollary 18 *Graph of each oscillating solution consists from the arcs similar to described in Lemmas 15(I),17 and therefore it is sine-like slowly oscillating.*

Finally, the following result describes behavior of positive unbounded waves:

Corollary 19 *Let profile* ϕ *be unbounded. Then, for some* $A > 0$ *and* $t_0 \geq Q_0$*, it holds that* $\phi'(t) > 0,\ \phi(t) > Ae^{ct},\ t \geq t_0$*.*

PROOF. By Lemmas 15, parts (II) and (III), for an appropriate t_0, each unbounded solution satisfies $\phi'(t) > 0, \phi(t - h) > 1,\ t \geq t_0$. This implies that $\phi''(t) - c\phi'(t) > 0,\ t \geq t_0$ and therefore $\phi'(t) > \phi'(t_0)e^{c(t-t_0)} > 0,\ t \geq t_0$. □

4 *A priori* estimates and the convergence of semi-wavefronts

With the change of variables $\phi(t) = e^{-x(t)}$, equation (7) is transformed into

$$x''(t) - cx'(t) - (x'(t))^2 + (e^{-x(t-h)} - 1) = 0,\ t \in \mathbb{R}. \tag{10}$$

Let $\phi(t) = e^{-x(t)}$ be an oscillating semi-wavefront and for the simplicity take $Q_0 = 0$. By Corollary 16, $0 < x(t) < -ct,\ t < 0$, and $x(t) > -ct > -ch$ for $t \in (0, T_0)$.

Our *a priori* estimates are based on the following key assertion:

Lemma 20 *Let y solve the boundary value problem*

$$y' - cy - y^2 + g(t) = 0, \;\; y(a) = y(b) = 0, \;\; -1 < A := \min_{s\in[a,b]} g(s) < 0,$$

where $c \geq 2$ and g is continuous. Then

$$\beta := \min_{s\in[a,b]} y(s) \geq \frac{2A}{c + \sqrt{c^2 + 4A}} =: f(A).$$

Similarly,

$$\gamma := \max_{s\in[a,b]} y(s) \leq \frac{2B}{c + \sqrt{c^2 + 4B}} = f(B), \;\; \text{where } B := \max_{s\in[a,b]} g(s).$$

PROOF. If $\beta = 0$, the conclusion of the first part of Lemma 20 is obvious. Thus we can suppose that $\beta =: y(s') < 0$ and that $y(t) < 0$ for all t from some maximal open interval $(a', b') \subset (a, b)$ containing s'. In particular, $y'(s') = 0$, $y(a') = y(b') = 0$, so that $\beta \in \{\lambda_1(s'), \lambda_2(s')\}$ where $\lambda_1(s) < f(A) \leq \lambda_2(s)$ are simple roots of the quadratic equation $y^2 + cy - g(s) = 0$. Observe that

$$f(A) \leq \lambda_2(s) = f(g(s)) \leq f(B),$$

since $f(u) = 0.5(\sqrt{c^2 + 4u} - c)$ is strictly increasing in u. It is clear that each $\lambda_j(s)$ depends continuously on s, and that $\lambda_1(s) < f(A) < 0$. Suppose for a moment that $\beta = \lambda_1(s')$. We claim that then $y(s) < f(A)$ for all $s \in [s', b']$. Indeed, let q be the minimal real number such that $y(q) = f(A)$. Then $y'(q) \geq 0$ and we have the following dichotomy: either (i) $y'(t) > 0$ on some maximal subinterval (p, q), $y'(p) = 0$, of (s', q), or (ii) there exists a sequence $\{t_j\}$, $t_j < q$, converging to q such that $y'(t_j) = 0$. In every case, $y(p) = \lambda_1(p)$, $y(t_j) = \lambda_1(t_j)$ due to $y(p), y(t_j) < y(q) = f(A)$. Therefore the case (i) is not possible because of the following contradiction: $y(q) = \lim y(t_j) = \lim \lambda_1(t_j) = \lambda_1(q) < f(A)$. Similarly, the case (ii) should also be discarded in virtue of the following argument: $y^2(s) + cy(s) - g(s) = y'(s) > 0$ on (p, q), $y(p) = \lambda_1(p)$, so that $y(s) < \lambda_1(s) < f(A)$, $s \in (p, q)$, whence $y(q) \leq \lambda_1(q) < f(A)$, a contradiction. Hence, assuming that $\beta = \lambda_1(s')$ we obtain that $y(s) < f(A)$ for all $s \in [s', b']$. In particular, $0 = y(b') < f(A) < 0$. This contradiction proves the first part of Lemma 20.

Next, it is clear that $\gamma \geq 0$. If $\gamma = 0$ then $B \geq g(a) = -y'(a) \geq 0$ and the claimed inequality is immediate. If $\gamma > 0$ then $\gamma \in \{\lambda_1(s''), \lambda_2(s'')\}$ for some $s'' \in (a, b)$. As a consequence, we obtain the second estimation of the lemma: $\lambda_1(s'') \leq \gamma \leq \lambda_2(s'') \leq f(B)$. □

Recall that the Schwarz derivative Sp of C^3-smooth function p is defined as

$$(Sp)(x) = p'''(x)(p'(x))^{-1} - (3/2)\left(p''(x)(p'(x))^{-1}\right)^2.$$

Lemma 21 *Let $c \geq 2$. Then real analytic function $(f \circ g)(x) = f(e^{-x} - 1)$, $x \in \mathbb{R}$, $(f \circ g)(0) = 0$, is well defined, strictly decreasing and has the negative Schwarz derivative on $\mathbb{R}$.*

PROOF. Since $f(u) = 0.5(\sqrt{1 + 4u/c^2} - 1)$, we find easily that $(Sf)(u) = 6(c^2+4u)^{-2}$. By the well known formula for the Schwarzian of the composition,

$$S(f \circ g)(x) = (Sf)(g(x))(g'(x))^2 + (Sg)(x) = \frac{6e^{-2x}}{(c^2 + 4(e^{-x} - 1))^2} - \frac{1}{2} =$$

$$= \frac{6}{(e^x(c^2 - 4) + 4)^2} - \frac{1}{2} \leq \frac{6}{4^2} - \frac{1}{2} = -\frac{1}{8}.$$

The other properties of $f \circ g$ are straightforward to verify. $\square$

Lemma 22 *Let $c \geq 2$ and $\phi(t), \phi(-\infty) = 0$, be a slowly oscillating on $[Q_0, +\infty)$ positive solution of equation (7). Then ϕ is bounded and*

$$0 < L_e(c, h) < \phi(t) < U_e(c, h), \; t \geq Q_0,$$

where $U_e(c, h) := \exp(-L(c, h)), \; L_e(c, h) := \exp(-U(c, h))$ *and*

$$U(c, h) = hf(e^{-L(c,h)} - 1), \quad L(c, h) := -h \max\left\{c, \frac{2}{c + \sqrt{c^2 - 4}}\right\}.$$

PROOF. Without the loss of generality, we can set $Q_0 = 0$. Then it suffices to prove the boundedness of $x(t) = -\ln \phi(t)$ on $[0, +\infty)$. Since $\phi(t)$ is slowly oscillating about 1, the transformed solution $x(t)$ oscillates slowly around the zero equilibrium of (10). This implies that there exists an increasing sequence $Q_j, j \geq 0, Q_0 = 0$, of zeros of $x(t)$ such that $x(t) < 0$ on $(Q_0, Q_1) \cup (Q_2, Q_3) \cup \dots$ and $x(t) > 0$ on $(Q_1, Q_2) \cup (Q_3, Q_4) \cup \dots$ We proceed by evaluating extremal values $V_j = x(T_j)$ of $x(t)$ on each interval (Q_j, Q_{j+1}). As we already have established, $|V_0| = -V_0 \leq ch$. Next, we have that $V_1 = x(T_1) > 0$ with $T_1 > h$ and $x'(T_1) = 0, \; x(Q_1) = 0, \; T_1 - Q_1 < h$. Hence,

$$V_1 = \int_{Q_1}^{T_1} x'(s)ds \leq h \max_{s \in [T_0, T_1]} x'(s) \leq h \max_{s \in [T_0, T_1]} f(e^{-x(s-h)} - 1) \leq hf(w(V_0)),$$

where $w(x) := e^{-x} - 1$. Next, consider $V_2 = x(T_2) < 0$, we have $x'(T_2) = 0$, $x(Q_2) = 0$ and $T_2 - Q_2 < h$. Recalling that $\phi(t)$ (and, consequently, $x(t)$) is sine-like slowly oscillating (so that $x'(t) < 0$ on (T_1, T_2)) and applying Lemma 20, we obtain

$$V_2 = \int_{Q_2}^{T_2} x'(s)ds \geq h \min_{s \in [T_1, T_2]} x'(s) \geq h \min_{s \in [T_1, T_2]} f(e^{-x(s-h)} - 1) \geq hf(w(V_1)).$$

Since $Q_{j+2} - Q_j > h$ for each j, we can repeat the above two steps to conclude that

$$V_{2j+1} \leq hf(w(V_{2j})),\ j \geq 0, \quad V_{2j} \geq hf(w(V_{2j-1})),\ j > 0. \tag{11}$$

As a consequence,

$$V_{2j} \geq hf(w(V_{2j-1})) > hf(w(+\infty)) = \frac{-2h}{c + \sqrt{c^2 - 4}} =: B_*(c, h),\ j > 0,$$

and therefore, after setting $L(c, h) = \min\{-ch, B_*(c, h)\}$, we obtain that

$$V_{2j+1} \leq hf(w(V_{2j})) \leq hf(w(L(c, h))),\ j \geq 0.$$

This ends the proof of Lemma 22. □

Suppose now that $\phi,\ \phi(-\infty) = 0$, is an unbounded positive solution of (7). By Lemmas 15, 17 and 22, function ϕ is either monotone or slowly oscillating around 1 on some interval $(-\infty, Q_m],\ \phi(Q_m) = 1$, and $\phi(t) > 1$ for $t > Q_m$. Let T_m denote the rightmost critical point of ϕ (whenever it exists) and set $S_m = \max\{T_m, Q_m\}$.

Corollary 23 *There exists a positive constant $\beta(c, h) > U_e(c, h)$ depending only on (c, h) such that $\phi(t) < \beta(c, h),\ t \leq S_m + 2h$. In this way, if $\phi(\bar{s}) = \beta(c, h)$ for some $\bar{s} \in \mathbb{R}$ then $\phi'(t) > 0,\ \phi(t) > 1$ for all $t \geq \bar{s} - h$.*

PROOF. Step I. Suppose first that $S_m = Q_m > T_m$. As the proof of Lemma 22 shows, we have that $\phi(t) < U_e(c, h)$ for all $t \leq Q_m$. Next, on the half-line $\mathcal{I} := (-\infty, Q_m + h]$, function $\phi(t)$ satisfies the homogeneous linear equation

$$y''(t) - cy'(t) + a(t)y(t) = 0, \tag{12}$$

whose coefficient $a(t) = 1 - \phi(t - h)$ is uniformly bounded on $\mathcal{I}$ by a constant depending only on c, h. We consider separately the cases $m = 0$ and $m > 0$.

If $m = 0$ then $\phi'(t) > 0$ for all t, and $\phi'(Q_0) < c, \phi(Q_0) = 1$, see Corollary 16. As a consequence, the solution $y(t) \equiv \phi(t)$ of the initial value problem $y(Q_0) = 1, y'(Q_0) = \phi'(Q_0)$, to equation (12) exists on $(-\infty, Q_0 + h]$ where it is bounded by some constant $\rho_0(c, h)$ depending only on c, h. Therefore the absolute value of $a(t) = 1 - \phi(t - h) = 1 - y(t - h),\ t \leq Q_0 + 2h$, is bounded by $\rho_0(c,h)+1$ and we can repeat the above argument to conclude that the solution $y(t) \equiv \phi(t)$ of the initial value problem $y(Q_0)=1, y'(Q_0)=\phi'(Q_0)$, of equation (12) exists on $(-\infty, Q_0+2h]$ where it is bounded by some constant $\rho_1(c,h)$.

Now we can assume that $m > 0$ and $\phi(t) < 1$ on some maximal interval (Q_{m-1}, Q_m). We also know that $\phi'(t) > 0$ on some maximal open subinterval (T_{m-1}, Q_m) of (Q_{m-1}, Q_m). Since $\phi'(T_{m-1}) = 0, \phi(T_{m-1}) < 1$, we can integrate equation (12) repeatedly (as it has been done in the case $m = 0$) to prove the existence of $\rho_2 = \rho_2(c,h)$ such that $\phi(t) < \rho_2,\ t \leq T_{m-1}+4h$. If $Q_m + 2h \leq T_{m-1}+4h$, the proof is finished. Otherwise $Q_m > T_{m-1}+2h$ and $\phi(t)$ is strictly increasing on $[Q_m - 2h, Q_m]$. In particular, $\phi'(\hat{s}) \in (0, (2h)^{-1})$, $\phi(\hat{s}) \in (0,1)$, at some point $\hat{s} \in [Q_m - 2h, Q_m]$. But then there exists $\rho_3 = \rho_3(c,h)$ depending only on c, h and such that $\phi(t) < \rho_3$ on $[\hat{s}, \hat{s}+4h] \supset [Q_m, Q_m+2h]$. Therefore, by taking $\beta(c,h) = \max\{\rho_j(c,h), j = 0,1,2,3\}$, we finalize the proof of Corollary 23 in the case when $S_m = Q_m > T_m$.

Step II. Suppose now that $S_m = T_m \geq Q_m$. This situation corresponds to the cases (II) and (III) of Lemma 15. Since $\phi'(S_m) = 0$ and $\phi(t) \leq U_e(c,h)$, $t \leq S_m$, we can again integrate equation (12) repeatedly to prove the existence of $\rho_4 = \rho_4(c,h)$ such that $\phi(t) < \rho_4,\ t \leq S_m + 2h$. □

For fixed $c \geq 2, h > 0$, we will consider also the following modified equation

$$\phi''(t) - c\phi'(t) + g(\phi(t))(1 - \phi(t-h)) = 0, \tag{13}$$

with $\beta(c,h)$ defined in Corollary 23 and with continuous piece-wise linear

$$g(u) = \begin{cases} u, & u \in [0, \beta(c,h)], \\ \max\{0, 2\beta(c,h) - u\}, & u > \beta(c,h). \end{cases}$$

Lemma 24 *Equations (13) and (7) share the same set of semi-wavefronts.*

PROOF. Due to Lemma 22 and the definition of $g(u)$, each semi-wavefront of (7) also satisfies (13). Conversely, suppose that ϕ is a semi-wavefront to (13). We will prove that then $\phi(t) < \beta(c,h)$. Indeed, otherwise $\phi(\bar{s}) = \beta(c,h)$ at some leftmost point $\bar{s}$. Since $\phi(t)$ is also satisfying (7) for all $t \leq \bar{s}$, Corollary 23 assures that $\phi'(\bar{s}) > 0$ and $\phi(t-h) > 1$ for all $t \in [\bar{s}, \bar{s}+h]$. Thus $\phi''(t) > c\phi'(t)$, $t \in [\bar{s}, \bar{s}+h]$, and consequently $\phi'(t) > \phi'(\bar{s})e^{c(t-\bar{s})}, t \in [\bar{s}, \bar{s}+h]$. Hence, $\phi''(t) \geq c\phi'(t) > 0$ on $[\bar{s}, \bar{s}+h]$. Using step by step continuation argument, we can conclude that $\phi(+\infty) = +\infty$, a contradiction. □

Lemma 25 *Let $\phi(t)$ be a slowly oscillating semi-wavefront to equation (7). If $\tau \leq 1,\ c \geq 2$, then $\phi(+\infty) = 1$.*

PROOF. Lemma 22 assures the existence of finite limits

$$0 \geq m_* = \liminf_{j \to +\infty} V_j = \liminf_{t \to +\infty} x(t), \quad 0 \leq M_* = \limsup_{j \to +\infty} V_j = \limsup_{t \to +\infty} x(t).$$

Clearly, the lemma will be proved if we show that $\tau \leq 1$ implies $M_* = 0$. From (11), we deduce that $M_* \leq hf(w(m_*)),\ m_* \geq hf(w(M_*))$ and therefore

$$M_* \leq (hf \circ w)^2(M_*) \leq (hf \circ w)^4(M_*) \leq \ldots \leq (hf \circ w)^{2k}(M_*) \leq \ldots$$

Here $f^k = f \circ \ldots \circ f$ denotes the k-times composition of f. Now, by Lemma 21, analytic function $hf \circ w$ is strictly decreasing, below bounded and has the negative Schwarzian. Therefore the inequality $|hf'(0)w'(0)| = h/c = \tau \leq 1$ assures the global stability of the fixed point 0 of the one-dimensional mapping $hf \circ w : \mathbb{R} \to \mathbb{R}$. See [15, Proposition 3.3] for more details. In particular, this means that $(hf \circ w)^{2k}(M_*) \to 0$ as $k \to +\infty$. Hence, $M_* = 0$ and Lemma 25 is proved. □

5 Existence of semi-wavefronts for $c \geq 2,\ h > 0$.

With $g(u)$ defined in (13) and with some $b > 1 + 2\beta(c,h)$ (to be specified later), let us consider $r(\phi(t), \phi(t-h)) := b\phi(t) + g(\phi(t))(1 - \phi(t-h))$. By Lemma 24, it suffices to prove that equation

$$\phi''(t) - c\phi'(t) - b\phi(t) + r(\phi(t), \phi(t-h)) = 0 \tag{14}$$

has a semi-wavefront. Observe that if some $\psi(t)$ satisfies $0 \leq \psi(t) \leq \beta(c,h)$ and $\psi(t-h) \leq 2\beta(c,h) < b$, then

$$r(\psi(t), \psi(t-h)) = \psi(t)(b + 1 - \psi(t-h)) \geq 0. \tag{15}$$

Now, if $\beta(c,h) \leq \psi(t) \leq 2\beta(c,h)$ and $\psi(t-h) \leq 2\beta(c,h) < b$, then

$$r(\psi(t), \psi(t-h)) = (2\beta(c,h) - \psi(t))(1 - \psi(t-h)) + b\psi(t) =$$

$$2\beta(c,h)(1 - \psi(t-h)) + \psi(t)(b - 1 + \psi(t-h)) > \beta(c,h). \tag{16}$$

Next, we consider the non-delayed KPP-Fisher equation $u_t = u_{xx} + g(u)$. The profiles ϕ of the traveling fronts $u(x,t) = \phi(x + ct)$ for this equation satisfy

$$\phi''(t) - c\phi'(t) + g(\phi(t)) = 0,\ c \geq 2. \tag{17}$$

As before, $0 < \lambda \le \mu$ denote eigenvalues of equation (17) linearized around 0. Then $\chi(\lambda) = \chi(\mu) = 0$ where $\chi(z) := z^2 - cz + 1$. Recall also that $z_1 < 0 < z_2$ stand for the roots of the equation $z^2 - cz - b = 0$. In the sequel, $\phi_+(t)$ will denote the unique monotone front to (17) normalized by the condition

$$\phi_+(t) := (-t)^j e^{\lambda t}(1 + o(1)),\ t \to -\infty.$$

In fact, the latter asymptotic formula can be considerably improved since $\phi_+(t)$ for all t such that $\phi_+(t) < \beta(c,h)$ satisfies the linear differential equation

$$\phi''(t) - c\phi'(t) + \phi(t) = 0.$$

For example, if $c > 2$ then there exists (cf. [10, Theorem 6]) $K \ge 0$ such that

$$\phi_+(t) := e^{\lambda t} - Ke^{\mu t},\ t \le \phi_+^{-1}(\beta(c,h)). \tag{18}$$

Set $\epsilon' = z_2 - z_1$ and consider the following integral operator

$$(A_m\phi)(t) = \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{z_1(t-s)} r(\phi(s), \phi(s-h))ds + \int_{t}^{+\infty} e^{z_2(t-s)} r(\phi(s), \phi(s-h))ds\right\}.$$

Lemma 26 *Assume that $b > 2\beta(c,h) + 1$ and let $0 \le \phi(t) \le \phi_+(t)$, then ϕ_+ is an upper solution:*

$$0 \le (A_m\phi)(t) \le \phi_+(t).$$

PROOF. The lower estimate is obvious since $0 \le \phi(t) \le \phi_+(t) \le 2\beta(c,h)$ and therefore $r(\phi(t), \phi(t-h)) \ge 0$ in view of (15) and (16). Now, since $\phi(t) \le \phi_+(t)$ and $bu + g(u)$ is an increasing function, we find that

$$r(\phi(s), \phi(s-h)) \le b\phi(t) + g(\phi(t)) \le b\phi_+(t) + g(\phi_+(t)) =: R(\phi_+(t)).$$

Thus

$$(A_m\phi)(t) \le \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{z_1(t-s)} R(\phi_+(s))ds + \int_{t}^{+\infty} e^{z_2(t-s)} R(\phi_+(s))ds\right\} = \phi_+(t),$$

and the lemma is proved. □

Next, we need to find a lower solution for (14). Fortunately, for $c > 2$ we can use the following well known solution (e.g. see [24])

$$\phi_-(t) = \max\{0, e^{\lambda t}(1 - Me^{\epsilon t})\},$$

where $\epsilon \in (0, \lambda)$ and $M \gg 1$ is chosen in such a way that $-\chi(\lambda + \epsilon) > 1/M$, $\lambda + \epsilon < \mu$, and

$$0 < \phi_-(t) < \phi_+(t) < e^{\epsilon t} < 1, \quad t \le T_c,\ \text{where}\ \phi_-(T_c) = 0.$$

The above inequality $\phi_-(t) < \phi_+(t)$ is possible due to representation (18).

Lemma 27 *Let $b > 2\beta(c,h) + 2$ and $\phi_-(t) \leq \phi(t) \leq \phi_+(t),\ t \in \mathbb{R}$, then*

$$\phi_-(t) \leq (A_m\phi)(t) \leq \phi_+(t), \quad t \in \mathbb{R}. \tag{19}$$

PROOF. Due to Lemma 26, it suffices to prove the first inequality in (19) for $t \leq T_c$. Since $0 < \phi(t) < 1 < \beta(c,h),\ t \leq T_c$, we have, for $t \leq T_c$ that

$$(A_m\phi)(t) \geq \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{z_1(t-s)} r(\phi(s),\phi(s-h))ds + \int_{t}^{T_c} e^{z_2(t-s)} r(\phi(s),\phi(s-h))ds\right\} =$$

$$\frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{z_1(t-s)}\phi(s)(b+1-\phi(s-h))ds + \int_{t}^{T_c} e^{z_2(t-s)}\phi(s)(b+1-\phi(s-h))ds\right\} \geq$$

$$\frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{z_1(t-s)}\phi_-(s)(b+1-\phi_+(s-h))ds + \int_{t}^{T_c} e^{z_2(t-s)}(\dots)ds\right\} =$$

$$\frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{z_1(t-s)}\phi_-(s)(b+1-\phi_+(s-h))ds + \int_{t}^{+\infty} e^{z_2(t-s)}(\dots)ds\right\} =: Q(t),$$

where $(\dots)$ stands for $\phi_-(s)(b+1-\phi_+(s-h))$. In order to evaluate $Q(t)$, we consider the following chain of inequalities (for $t \leq T_c$)

$$\phi''_-(t) - c\phi'_-(t) - b\phi_-(t) + b\phi_-(t) + \phi_-(t)(1-\phi_+(t-h)) =$$

$$-\chi(\lambda+\epsilon)Me^{(\lambda+\epsilon)t} - \phi_+(t-h)e^{\lambda t}(1-Me^{\epsilon t}) \geq$$
$$-\chi(\lambda+\epsilon)Me^{(\lambda+\epsilon)t} - e^{\epsilon(t-h)}e^{\lambda t} > Me^{(\lambda+\epsilon)t}(-\chi(\lambda+\epsilon) - 1/M) > 0.$$

But then, rewriting the latter differential inequality in the equivalent integral form (e.g. see [16]) and using the fact that

$$\Delta\phi'_-|_{T_c} := \phi'_-(T_c+) - \phi'_-(T_c-) = -\phi'_-(T_c-) > 0,$$

we may conclude that $Q(t) \geq \phi_-(t),\ t \in \mathbb{R}$. Hence, $(A_m\phi)(t) \geq \phi_-(t),\ t \in \mathbb{R}$, and Lemma 27 is proved. □

Finally, it is clear that, in order to establish the existence of semi-wavefronts to equation (14), it suffices to prove that the equation $A_m\phi = \phi$ has at least one solution from the subset $K = \{x \in X : \phi_-(t) \leq x(t) \leq \phi_+(t),\ t \in \mathbb{R}\}$ of the Banach space $(X, \|\cdot\|)$, where

$$X = \{x \in C(\mathbb{R},\mathbb{R}) : \|x\| = \sup_{s \leq 0} e^{-\lambda s/2}|x(s)| + \sup_{s \geq 0} e^{-\rho s}|x(s)| < \infty\}$$

is defined with some fixed $\rho > 0$. Observe that the convergence $x_n \to x$ on K is equivalent to the uniform convergence $x_n \rightrightarrows x$ on compact subsets of $\mathbb{R}$.

Lemma 28 *Take $c > 2$. Then K is a closed, bounded, convex subset of X and $A_m : K \to K$ is completely continuous.*

PROOF. By the previous lemma, $A_m(K) \subset K$. It is also obvious that K is a closed, bounded, convex subset of X. Since

$$|x(t)| + |(A_m x)'(t)| \leq 2\beta(c,h)(1+\epsilon'), \text{ for all } x \in K, \tag{20}$$

due to the Ascoli-Arzelà theorem $A_m(K)$ is precompact in K . Next, by the Lebesgue's dominated convergence theorem, if $x_j \to x_0$ in K then $(A_m x_j)(t) \to (A_m x_0)(t)$ at every $t \in \mathbb{R}$. The precompactness of $\{A_m x_j\} \subset K$ assures that, in fact, $A_m x_j \to A_m x_0$ in K. Hence, the map $A_m : K \to K$ is completely continuous.

Theorem 29 *Assume that $c \geq 2$. Then the integral equation $A_m\phi = \phi$ has at least one positive bounded solution in K.*

PROOF. If $c > 2$ then, due to the previous lemma, we can apply the Schauder's fixed point theorem to $A_m : K \to K$. Let now $c = 2$ and consider $c_j := 2 + 1/j$ with $h_0 := 2\tau, h_j := c_j\tau$. By the first part of the theorem, we know that for each c_j there exists a semi-wavefront ϕ_j: we can normalize it by the condition $\phi_j(0) = 1/2,\ \phi_j'(s) > 0, s \leq 0$. It is clear from (20) that the set $\{\phi_j, j \geq 0\}$ is precompact in K and therefore we can also assume that $\phi_j \to \phi_0$ in K, where $\phi_0(0) = 1/2$ and ϕ_0 is monotone increasing on $(-\infty, 0]$. In addition, $R_j(s) := r(\phi_j(s), \phi_j(s - h_j)) \to R_0(s) := r(\phi_0(s), \phi_0(s - h_0))$ for each fixed $s \in \mathbb{R}$. The sequence $\{R_j(t)\}$ is also uniformly bounded on $\mathbb{R}$. All this allows us to apply the Lebesgue's dominated convergence theorem in

$$(A_{m,j}\phi_j)(t) := \frac{1}{\epsilon_j'}\left\{\int_{-\infty}^{t} e^{z_{1,j}(t-s)}R_j(s)ds + \int_{t}^{+\infty} e^{z_{2,j}(t-s)}R_j(s)ds\right\} = \phi_j(t),$$

where $z_{1,j} < 0 < z_{2,j}$ satisfy $z^2 - c_j z - b = 0$. In this way we obtain that $A_m\phi_0 = \phi_0$ with $c = 2$ and therefore ϕ_0 is a non-negative solution of equation (7) satisfying condition $\phi_0(0) = 1/2$ and monotone increasing on $(-\infty, 0]$. It is immediate to see that $\phi_0(-\infty) = 0$ and therefore ϕ_0 is a semi-wavefront. □

6 Admissible wavefront speeds

First, we observe that the necessity of the condition $c \geq 2$ for the existence of monotone wavefronts was already established in [10, Lemma 19]. Since the leading edge of each semi-wavefront is monotone, the proof of the mentioned lemma is also valid for the broader family of semi-wavefronts.

Consider now some semi-wavefront ϕ propagating at the velocity $c > c^\star$. We know that ϕ is slowly oscillating around the positive steady state. In this section, we show that these oscillations are non-decaying.

Arguing by contradiction, assume that $\phi(+\infty) = 1$. Then $w(t) = \phi(t) - 1$, $w(+\infty) = 0$, solves

$$w''(t) - cw'(t) - w(t-h)(1 + w(t)) = 0, \quad t \in \mathbb{R}.$$

Since $w(+\infty) = 0$, there exists a subsequence $\{t_n\}$, $\lim t_n = +\infty$, of the sequence $\{T_n\}$ defined in Lemma 22 such that $|w(t_n)| = \max_{s \geq t_n} |w(s)| > 0$, $w'(t_n) = 0$, $w''(t_n)w(t_n) < 0$, $w(t_n)w(t_n - h) < 0$. In fact, there is a unique $q_n \in (t_n - h, t_n)$ such that $w(q_n) = 0$. Without restricting the generality, we can suppose that $w(t_n) > 0$ and that $\{r_n\}$, $r_n := t_n - q_n \in (0, h)$, is monotonically converging to $r_* \in [0, h]$. Clearly, $w(s) < 0$ for $s \in [t_n - h, q_n)$ and $w(s) > 0$ for $s \in (q_n, t_n]$.

Now, each $y_n(t) := w(t + t_n)/w(t_n)$, $t \in \mathbb{R}$, satisfies

$$y''(t) - cy'(t) - (1 + w(t + t_n))y(t - h) = 0. \tag{21}$$

It is clear that $y_n(0) = 1$ and $|y_n(t)| \leq 1$, $t \geq -r_n$. In addition, $y_n(-r_n) = 0$, $y_n(-h) < 0$. We also can suppose that $|w(t + t_n)| \leq 0.1$ for all n and $t \geq 0$.

Next, we are going to estimate $|y_n'(t)|$, $t \geq 0$. Let $s \geq 0$ be the leftmost local extremum point for $y_n'(t)$. Then $y_n''(s) = 0, y_n'(s) < 0$, and therefore

$$0 > cy_n'(s) = -y_n(s - h)(1 + w(s + t_n)).$$

Thus $y_n(s - h) > 0$ that yields $s - h > -r_n$. Consequently, $\bar{s} - h > -r_n$ for each other critical point $\bar{s}$ of $y_n'(t)$. All this implies that $|y_n(\bar{s} - h)| \in [0, 1]$. Therefore $|y_n'(t)| \leq 1.1/c$ for $t \geq 0$, and, in particular, $y_n(t) \geq 0.45$ on $[0, c/2]$. Next, due to the Ascoli-Arzelà theorem, the sequence $y_n(t)$ has a subsequence which converges on $[0, +\infty)$, in the compact-open topology, to some continuous function $y_*(t)$. Evidently, $\max\{|y_*(s)|, s \geq 0\} = y_*(0) = 1$ and $y_*(t) \geq 0.45$ on $[0, c/2]$. Next, for some fixed positive b and all $t \in [h, +\infty)$, it holds that

$$g_n(t) := by_n(t) - (1 + w(t + t_n))y_n(t - h) \to g_*(t) := by_*(t) - y_*(t - h).$$

Obviously, $0 \leq |g_*(t)| \leq 1 + b$ for $t \geq h$.

In order to establish some further properties of $y_*(t)$, let us present the family of all solutions to (21) which are bounded at $+\infty$:

$$y(t) = Ae^{z_1 t} + \frac{1}{\epsilon'} \left\{ \int_h^t e^{z_1(t-s)} g_n(s) ds + \int_t^{+\infty} e^{z_2(t-s)} g_n(s) ds \right\}, \ t \geq h. \tag{22}$$

Here $\epsilon' = z_2 - z_1$ is defined in the same way as in Lemma 26. Replacing $y(t)$ with $y_n(t)$ in (22), we obtain that, for some A_n,

$$y_n(t) = A_n e^{z_1 t} + \frac{1}{\epsilon'}\left\{\int_h^t e^{z_1(t-s)} g_n(s)ds + \int_t^{+\infty} e^{z_2(t-s)} g_n(s)ds\right\},\ t \geq h.$$

The latter inequality implies that $A_n,\ n \in \mathbb{N}$, are uniformly bounded:

$$|A_n| = e^{-z_1 h}\left|y_n(h) - \frac{1}{\epsilon'}\int_h^{+\infty} e^{z_2(h-s)} g_n(s)ds\right| \leq e^{-z_1 h}\left(1 + \frac{1.1 + b}{\epsilon' z_2}\right).$$

Hence, taking limit as $n \to +\infty$ (through passing to a subsequence if necessary) we find that $y_*(t)$ satisfies

$$y_*(t) = A e^{z_1 t} + \frac{1}{\epsilon'}\left\{\int_h^t e^{z_1(t-s)} g_*(s)ds + \int_t^{+\infty} e^{z_2(t-s)} g_*(s)ds\right\},\ t \geq h, \quad (23)$$

with some finite A. Now, (23) implies that $y_*(t)$ is a solution of the equation

$$y''(t) - cy'(t) - y(t-h) = 0,\ t \geq h. \quad (24)$$

We claim that $y_*(t)$ is not a small solution. Indeed, on the contrary, let us suppose that $y_*(t)$ has superexponential decay. Since the characteristic function $z^2 - cz - e^{-zh}$ to (24) has the exponential type h, an application of [12, Theorem 3.1] assures that $y_*(t) = 0$ for all $t \geq 2h$. But then equation (24) implies that $y_*(t) = 0$ for all $t \geq h$ and, in consequence, $y_*(t) = 0$, for all $t \geq 0$. This contradicts the inequality $y_*(t) \geq 0.45$ on $[0, c/2]$ and therefore $y_*(t)$ is not a small solution.

Hence, by [18, Proposition 7.2], for every sufficiently large $\nu < 0$, we have that

$$y_*(t) = u(t) + O(\exp(\nu t)),\ t \to +\infty,$$

where u is a *non empty* finite sum of eigensolutions of (24) associated to the eigenvalues $\lambda_j \in F = \{\nu < \Re\lambda_j \leq 0\}$. Now, Lemmas 3 and 31 say that, for every $c > c^\star$,

$$F \cap (-\infty, 0] \times [-2\pi/h, 2\pi/h] = \emptyset.$$

In consequence, there exist $A > 0,\ \beta > 2\pi/h,\ \alpha \geq 0,\ \xi \in \mathbb{R}$, such that

$$y_*(t) = (A\cos(\beta t + \xi) + o(1))e^{-\alpha t},\ t \geq 0.$$

This implies the existence of an interval $(a, a+h)$, $a > 3h$, such that $y_*(t)$ changes its sign on $(a, a+h)$ exactly three times. Since $y_{n_j}(t) \to y_*(t)$ uniformly on $[a, a+h]$, we can conclude that $\mathrm{sc}(\bar{y}_{n_j, a+h}) \geq 3$ for all large j. However, this contradicts to the slowly oscillating behavior of $y_{n_j}(t)$. In consequence, the equality $\phi(+\infty) = 1$ can not hold for $c > c^\star$.

7 Appendix: Proof of Lemma 3

In this section, we study the zeros of $\psi(z,c) := z^2 - cz - e^{-zc\tau}$, $c \geq 2$, $\tau > 0$. It is straightforward to see that ψ always has a unique positive simple zero λ_{-1}. Since $\psi'''(z,c)$ is positive, ψ can have at most three (counting multiplicities) real zeros, one of them positive and the other two (when they exist) negative.

Fix some $\tau \geq 0$. We should prove that $\psi(z,c), c \geq 2$, has exactly one (counting multiplicity) zero in the open right half-plane $\Re z > 0$ if and only if $\tau \leq \tau_2$ and $c \leq c^\star(\tau)$. Aiming this objective, we first consider $\psi(z,c)$ without restriction $c \geq 2$. Then the next result follows from [10, Section 2]:

Lemma 30 *There exists function $C^* = C^*(\cdot) : [0,+\infty) \to [0,+\infty]$ such that $\psi(z,c),\ c > 0$, has exactly two (counting multiplicity) negative zeros (say, $\lambda_1 \leq \lambda_0$) in the half plane $\{\Re z < 0\}$ if and only if $c \in (0, C^*(\tau)]$. Moreover, $C^*(\tau) = +\infty$ if and only if $\tau \leq 1/e$ while on the interval $(1/e,+\infty)$ function $C^*(\tau)$ is decreasing and $C^*(+\infty) = 0$. Furthermore, $\Re\lambda_j < \lambda_1$ for every complex root of $\psi(\lambda_j, c) = 0$. Note also that $C^*(\tau) = c^*(\tau)$ for $\tau \in [0,\tau_1]$.*

On the other hand, Lemma 17 (2-3), Remarks 19,20 in [22] and the proof of Lemma 10 in [10] imply

Lemma 31 *Let $c > C^*(\tau)$. Then every root $\lambda_j(c)$ of $\psi(z,c) = 0$ is simple and depends smoothly on c. Moreover, each vertical half-line $\Re z = a, \Im z \geq 0$, contains at most one root λ_j and all roots $\lambda_j, \Im\lambda_j \geq 0$, can be ordered in such a way that $\ldots < \Re\lambda_{j+1}(c) < \Re\lambda_j(c) < \ldots < \Re\lambda_0(c) < \lambda_{-1}(c)$. Finally, if $\Re\lambda_j(c) \leq 0, \Im\lambda_j(c) \geq 0$, then $c\tau\Im\lambda_j(c) \in (2j\pi,(2j+1)\pi)$.*

Here we would like to stress the following important fact: if $\lambda(c_0) = iw, w > 0$, is purely imaginary zero of $\psi(z,c_0)$ then $\exp(-ic\tau w) = -w^2 - icw$ and thus

$$\Re\lambda'(c_0) = \frac{2w^2(1+\tau w^2)}{c^2(1+\tau w^2)^2 + w^2(c^2\tau - 2)^2} > 0.$$

Therefore the point $\lambda(c) \in \mathbb{C}$ must cross transversally the imaginary axis *only from the left to the right* and at the unique moment c_0. In view of the above lemmas, this means that $\psi(z,c)$ can have more that one zero in $\Re z \geq 0$ if and only if at some *uniquely determined* moment $c = C^\star(\tau) > C^*(\tau)$ the point $c\tau\lambda_0(c)$ crosses the vertical segment $i[0,\pi]$. Moreover, for each $c \leq C^\star(\tau)$ characteristic function $\psi(z,c)$ has only one zero (i.e. $\lambda_{-1}(c)$) in the open right half-plane, while for $c > C^\star(\tau)$ it has at least three zeros (i.e. $\lambda_{-1}(c),\ \lambda_0(c), \bar{\lambda}_0(c)$) in $\{\Re z > 0\}$. By the last lemma, the strip $\Pi := (-\infty, 0] \times [0, 2\pi/(c\tau)] \subset \mathbb{C}$ contains at most one complex zero (i.e. $\lambda_0(c)$) for $c > C^*(\tau)$ and therefore Π does not contain any zero of $\psi(z,c)$ for $c > C^\star(\tau)$.

Hence, $C^\star$ can be determined as a unique positive real number such that equation $\psi(z, C^\star) = 0$, or, equivalently,

$$\cos(C^\star \tau w) = -w^2, \quad \sin(C^\star \tau w) = C^\star w. \tag{25}$$

has a solution $z = iw$ with $C^\star \tau w \in [0, \pi]$. From the first equation of (25) we obtain that actually $C^\star \tau w \in (\pi/2, \pi]$. Therefore $1/\tau = \sin(C^\star \tau w)/(C^\star \tau w) < 2/\pi$. This means that $C^\star(\tau) = +\infty$ for all $\tau \in [0, \pi/2]$. On the other hand, if $\tau > \pi/2$, equation $1/\tau = \sin(c\tau w)/(c\tau w)$ has a unique root $c\tau w$ on $(\pi/2, \pi]$ and therefore w can be determined uniquely as $\sqrt{-\cos(c\tau w)}$. It is clear that also $w^4 + c^2 w^2 = 1$, from which $w^2(c) = 0.5(-c^2 + \sqrt{c^4 + 4})$. This proves the representation (6). Finally, it is easy to see that $\tau(c)$ strictly decreases on $(0, +\infty)$, with $\tau(+\infty) = \pi/2$. Therefore

$$\tau > \tau(2) = \frac{\arccos(2 - \sqrt{5})}{2\sqrt{\sqrt{5} - 2}} =: \tau_2$$

implies that $\psi(z, c)$ with $c \geq 2$ has at least three zeros on $\Re z \geq 0$.

Acknowledgments

This research was realized within the framework of the OPVK program, project CZ.1.07/2.300/20.0002. Sergei Trofimchuk was also partially supported by FONDECYT (Chile), project 1071053, and by CONICYT (Chile) through PBCT program ACT-56.

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