Complete Residue Systems: A Primer and an Application

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Abstract

Complete residue systems play an integral role in abstract algebra and number theory, and a description is typically found in any number theory textbook. This note provides a concise overview of complete residue systems, including a robust definition, several well-known results, a proof to the converse of a well-known theorem, ancillary results pertaining to an application arising from the study of the roots of nonnegative matrices, and extends our knowledge of complete residue systems in relation to complete sets of roots of unity.

1 Introduction

Complete residue systems play an integral role in abstract algebra and number theory, and a description is typically found in any number theory textbook (see, for instance, [4, 1, 3]). In Section 1, we provide a concise overview of complete residue systems, including a robust definition, several well-known results, and a proof to the converse of a well-known theorem. In Section 2, we derive ancillary results pertaining to an application arising from the study of the roots of nonnegative matrices, which extend, to a certain extent, Theorems 65 and 66 in [1].

2 Complete Residue Systems

Our discussion of complete residue systems begins with the following definition.

Definition 1. Let $R = \{a_0, a_1, \ldots, a_{h-1}\}$ be a set of integers, and for $h > 1$ an integer, let $R(h) = \{0, 1, \ldots, h-1\}$. Then R is said to be a *complete residue system* $(\mathrm{mod} h)$, denoted $\mathrm{CRS}(h)$, if the map $f : R \to R(h)$ defined by

$$a_i \to q_i := a_i \,(\mathrm{mod} h)$$

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is injective or, equivalently, surjective.

With the aforementioned definition, we readily glean the following equivalent properties if R is a CRS(h):

(P1) if $i \neq j$, then $q_i \neq q_j$;

(P2) if $i \neq j$, then $a_i \not\equiv a_j \,(\mathrm{mod} h)$; and

(P3) for $a \in \mathbb{Z}$, if $a \equiv a_i \,(\mathrm{mod} h)$, then $a \not\equiv a_j \,(\mathrm{mod} h)$ for all $j \neq i$.

The next two results are well-known number-theoretic results (see, for instance, [4, Theorem 3.3], [5, Theorem 4.6 and Corollary 4.7] or [1, Theorems 54 and 55]).

Theorem 2. *If* $\gcd(k, m) = d$, *then* $ka \equiv kb \,(\mathrm{mod} m)$ *if and only if* $a \equiv b \,(\mathrm{mod}(m/d))$.

In particular is the following useful case.

Corollary 3. *If* $\gcd(k, m) = 1$, *then* $ka \equiv kb \,(\mathrm{mod} m)$ *if and only if* $a \equiv b \,(\mathrm{mod} m)$

Theorem 4. *If* $\{a_0, a_1, \ldots, a_{h-1}\}$ *is a complete residue system* $(\mathrm{mod} h)$ *and* $p \in \mathbb{Z}$, *then* $\{pa_0, pa_1, \ldots, pa_{h-1}\}$ *is a complete residue system* $(\mathrm{mod} h)$ *if and only if* $\gcd(p, h) = 1$.

Remark 5. Quite frequently in the literature (see [1, Theorem 56], [4, Theorem 3.5], [5, Section 4.1, Exercise 10], [2, Theorem 20(i)], or [3, Theorem 62]), one finds only *sufficiency*, however, as the following proof shows, *necessity* holds as well.

Proof. If $\gcd(p, h) = 1$, then, following Corollary 3, $pa_i \equiv pa_j \,(\mathrm{mod} h)$ if and only if $a_i \equiv a_j \,(\mathrm{mod} h)$, which holds if and only if $i = j$.

Conversely, if $\gcd(p, h) = d > 1$, then $\{a_0, a_1, \ldots, a_{h-1}\}$ is not a CRS(h/d). Thus, there exist distinct integers i and j such that $a_i \equiv a_j \,(\mathrm{mod}(h/d))$, which holds, by Theorem 2, if and only if $pa_i \equiv pa_j \,(\mathrm{mod} h)$, i.e., $\{pa_0, pa_1, \ldots, pa_{h-1}\}$ is not a CRS(h). □

3 The Application

For $h > 1$, let $\omega := e^{\frac{2\pi i}{h}} \in \mathbb{C}$ and $\Omega_h := \left\{1, \omega, \ldots, \omega^{h-1}\right\} \subseteq \mathbb{C}$, i.e., $\Omega_h = \left\{z \in \mathbb{C} : z^h - 1 = 0\right\}$.

Theorem 6. *If* $\{a_0, a_1, \ldots, a_{h-1}\}$ *is a set of integers, then* $\{\omega^{a_0}, \omega^{a_1}, \ldots, \omega^{a_{h-1}}\} = \Omega_h$ *if and only if* $\{a_0, a_1, \ldots, a_{h-1}\}$ *is a complete residue system* $(\mathrm{mod} h)$.

Proof. If $\{a_0, a_1, \ldots, a_{h-1}\}$ is a CRS(h), then, for each $k \in \{0, 1, \ldots, h-1\}$, there exists one (and only one) $q_k \in \{0, 1, \ldots, h-1\}$ such that $a_k \equiv q_k \,(\mathrm{mod} h)$, i.e., there exists an integer l_k such that $a_k = hl_k + q_k$. For each such k,

$$\omega^{a_k} = \omega^{hl_k+q_k} = \left(e^{\frac{2\pi i}{h}}\right)^{hl_k+q_k} = e^{2\pi l_k i + \frac{2\pi q_k i}{h}} = e^{2\pi l_k i} \cdot e^{\frac{2\pi q_k i}{h}} = \left(e^{\frac{2\pi i}{h}}\right)^{q_k} = \omega^{q_k},$$

so that $\{\omega^{a_0}, \omega^{a_1}, \ldots, \omega^{a_{h-1}}\} = \{\omega^{q_0}, \omega^{q_1}, \ldots, \omega^{q_{h-1}}\} = \Omega_h$.

Conversely, suppose $\{a_0, a_1, \ldots, a_{h-1}\}$ is not a CRS(h), and for every k, let $q_k := a_k \,(\mathrm{mod} h)$. Then $\{q_0, q_1, \ldots, q_{h-1}\} \subset \{0, 1, \ldots, h-1\}$ so that $\{\omega^{a_0}, \omega^{a_1}, \ldots, \omega^{a_{h-1}}\} \neq \Omega_h$. □

Corollary 7. *For $p > 1$, let $\Omega_h^p := \left\{1, \omega^p, \ldots, \omega^{(h-1)p}\right\}$. Then $\Omega_h^p = \Omega_h$ if and only if $\gcd(p, h) = 1$.*

Lemma 8. *If a, b, c, and $h \in \mathbb{Z}$, then $a \equiv b \,(\mathrm{mod} h)$ if and only if $(a+c) \equiv (b+c) \,(\mathrm{mod} h)$.*

Proof. Note that

$$\begin{aligned} a \equiv b \,(\mathrm{mod} h) &\Leftrightarrow a - b = hk \qquad (k \in \mathbb{Z}) \\ &\Leftrightarrow (a+c) - (b+c) = hk \\ &\Leftrightarrow (a+c) \equiv (b+c) \,(\mathrm{mod} h)\,. \end{aligned}$$ □

Corollary 9. *If $\{a_0, a_1, \ldots, a_{h-1}\}$ is a complete residue system $(\mathrm{mod} h)$ and l is any integer, then $\{pa_0 + l, pa_1 + l, \ldots, pa_{h-1} + l\}$ is a complete residue system $(\mathrm{mod} h)$ if and only if $\gcd(p, h) = 1$.*

Lemma 10. *If a, b, c d and $h \in \mathbb{Z}$, then $(a + hc) \equiv (b + hd) \,(\mathrm{mod} h)$ if and only if $a \equiv b \,(\mathrm{mod} h)$.*

Proof. Note that

$$\begin{aligned} a \equiv b \,(\mathrm{mod} h) &\Leftrightarrow a - b = hk \qquad (k \in \mathbb{Z}) \\ &\Leftrightarrow (a + hc) - (b + hd) = h(k + c - d) \\ &\Leftrightarrow (a + c) \equiv (b + c) \,(\mathrm{mod} h)\,. \end{aligned}$$ □

Corollary 11. *If $\{a_0, a_1, \ldots, a_{h-1}\}$ and $\{l_0, l_1, \ldots, l_{h-1}\}$ are sets of integers, then $\{a_0, a_1, \ldots, a_{h-1}\}$ is a complete residue system $(\mathrm{mod} h)$ if and only if $\{a_0 + hl_0, a_1 + hl_1, \ldots, a_{h-1} + hl_{h-1}\}$ is a complete residue system $(\mathrm{mod} h)$.*

Proof. If $\{a_0 + hl_0, a_1 + hl_1, \ldots, a_{h-1} + hl_{h-1}\}$ is not a CRS(h), then there exist distinct integers i, $j \in \{0, 1, \ldots, h-1\}$ such that $(a_i + hl_i) \equiv (a_j + hl_j) \,(\mathrm{mod} h)$. Following Lemma 10, $(a_i + hl_i) \equiv (a_j + hl_j) \,(\mathrm{mod} h)$ if and only if $a_i \equiv a_j \,(\mathrm{mod} h)$; i.e., if and only if $\{a_0, a_1, \ldots, a_{h-1}\}$ is not a CRS(h). □

Definition 12. For $z = re^{i\theta} \in \mathbb{C}$, let $z^{\frac{1}{p}} := r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$ and, for $l \in \{0, 1, \ldots, p-1\}$, let $f_l(z) := z^{\frac{1}{p}} e^{\frac{2\pi l i}{p}} = r^{\frac{1}{p}} e^{\frac{(\theta + 2\pi l)i}{p}}$. Thus, f_l denotes the $(l+1)^{\text{th}}$ branch of the p^{th}-root function.

Theorem 13. *If* $L := \{l = (l_0, l_1, \ldots, l_{h-1}) : l_k \in \{0, 1, \ldots, p-1\}\}$, *and* $(\Omega_h)_l^{\frac{1}{p}} := \left\{f_{l_0}(1), f_{l_1}(\omega), \ldots, f_{l_{h-1}}(\omega^{h-1})\right\}$, *then there exists* $l \in L$ *such that* $(\Omega_h)_l^{\frac{1}{p}} = \Omega_h$ *if and only if* $\gcd(h, p) = 1$.

Remark 14. For any $k \in \{0, 1, \ldots, h-1\}$, note that

$$f_{l_k}(\omega^k) = \omega^{\frac{k}{p}} \cdot e^{\frac{2\pi l_k i}{p}} = e^{\frac{2\pi k i}{hp}} \cdot e^{\frac{2\pi l_k i}{p}} = e^{\frac{2\pi i(k + hl_k)}{hp}} = \left(e^{\frac{2\pi i}{h}}\right)^{\frac{k+hl_k}{p}} = \omega^{\frac{k+hl_k}{p}}$$

so that $(\Omega_h)_l^{\frac{1}{p}} = \left\{\omega^{\frac{hl_0}{p}}, \omega^{\frac{1+hl_1}{p}}, \ldots, \omega^{\frac{(h-1)+hl_{h-1}}{p}}\right\}$.

Proof. If $\gcd(p, h) = 1$, then, following Corollary 9, for each k, the set $\{k + hl\}_{l=0}^{p-1}$ is a CRS(p); in particular, there exists $l_k \in \{0, 1, \ldots, p-1\}$ such that $k + hl_k \equiv 0 \pmod{p}$, i.e., $k + hl_k$ is divisible by p. Moreover, the set

$$\left\{\frac{k + hl_k}{p}\right\}_{k=0}^{h-1}$$

is a CRS(h); indeed, following Corollary 3,

$$\left(\frac{i + hl_i}{p}\right) \equiv \left(\frac{j + hl_j}{p}\right) \pmod{h}$$

if and only if $(i + hl_i) \equiv (j + hl_j) \pmod{h}$, which holds, following Lemma 10, if and only if $i \equiv j \pmod{h}$. Since $\{0, 1, \ldots, h-1\}$ is a CRS(h), it follows that $i = j$, and the claim is established.

Conversely, suppose $\gcd(p, h) = d > 1$. We claim that for at least one k, there does not exist $l_k \in \{0, 1, \ldots, p-1\}$ such that $k + hl_k \equiv 0 \pmod{p}$; suppose the assertion is false so that for each k, there exists l_k such that $k + hl_k \equiv 0 \pmod{p}$, i.e., there exists an integer q_k such that $k + hl_k = pq_k$, or $hl_k = pq_k - k$ so that $pq_k \equiv k \pmod{h}$. Since $k \in \{0, 1, \ldots, h-1\}$, the set $\{pq_k\}_{k=0}^{h-1}$ is a CRS(h), establishing a contradiction since $\gcd(p, h) \neq 1$. Following Remark 14, at least one exponent in the set $(\Omega_h)_l^{\frac{1}{p}} := \left\{\omega^{\frac{k+hl_k}{p}}\right\}_{k=0}^{h-1}$ is not an integer so that $(\Omega_h)_l^{\frac{1}{p}} \neq \Omega_h$. □

Corollary 15. *If* $L := \{l = (l_0, l_1, \ldots, l_{h-1}) : l_k \in \{0, 1, \ldots, p-1\}\}$, *and* $(\Omega_h)_l^{\frac{q}{p}} := \left(\Omega_h^q\right)_l^{\frac{1}{p}}$ *or* $(\Omega_h)_l^{\frac{q}{p}} := \left((\Omega_h)_l^{\frac{1}{p}}\right)^q$, *then there exists* $l \in L$ *such that* $(\Omega_h)_l^{\frac{q}{p}} = \Omega_h$ *if and only if* $\gcd(h, p) = \gcd(h, q) = 1$.

References

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