ON THE POWER-BOUNDED OPERATORS OF CLASSES $C_{0\cdot}$ AND $C_{1\cdot}$

PATRYK PAGACZ

Abstract. By a bounded backward sequence of the operator T we mean a bounded sequence $\{x_n\}$ satisfying $Tx_{n+1} = x_n$. In [8] we have characterized contractions with strongly stable nonunitary part in terms of bounded backward sequences.

The main purpose of this work is to extend that result to power-bounded operators.

Aditionally, we show that a power-bounded operator is strongly stable ($C_{0\cdot}$) if and only if its adjoint does not have any nonzero bounded backward sequence. Similarly, a power-bounded operator is non-vanishing ($C_{1\cdot}$) if and only if its adjoint has a lot of bounded backward sequences.

[1]

1. Preliminaries

Let $\mathcal{H}$ be a complex, separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear transformations acting on $\mathcal{H}$. By a *contraction* we mean $T \in \mathcal{B}(\mathcal{H})$ such that $\|Tx\| \leq \|x\|$ for each $x \in \mathcal{H}$. By a *power-bounded* operator we mean $T \in \mathcal{B}(\mathcal{H})$ such that $\|T^n\|$ is uniformly bounded for all $n = 1, 2, 3, ...$

An operator T is said to be *completely nonunitary* (abbreviated *cnu*) if T restricted to every reducing subspace of $\mathcal{H}$ is nonunitary. As usual, by T^* we mean the adjoint of T.

We define as usually:

Definition 1.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be of class $C_{0\cdot}$ if*

$$\liminf_{n\to\infty} \|T^n x\| = 0$$

for each $x \in \mathcal{H}$.

Note that a power-bounded operator T is of class $C_{0\cdot}$ if and only if it is strongly stable ($T^n \to 0$, SOT).

Indeed, let T be $C_{0\cdot}$. If we fix $x \in \mathcal{H}$ then for each $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $\|T^k x\| < \epsilon$, so for all $m > k$ we have

$$\|T^m x\| = \|T^{m-k} T^k x\| \leq \|T^{m-k}\| \|T^k x\| \leq \epsilon \sup_{n\in\mathbb{N}} \|T^n\|.$$

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In general, $C_{0\cdot}$ operators can be extremely different from strongly stable operators. The following example shows a bounded operator of class $C_{0\cdot}$, which is not strongly stable at any (nonzero) point.

Example 1.2. Let $\{N_k\}_k$ be the sequence such that

$$\begin{cases} N_1 & = 1 \\ N_{k+1} & = 3N_k + 2N_k^2 \text{ , for } k = 1, 2, ... \end{cases}$$

Then let us define the operator S as the unilateral shift with weights $w_1, w_2, ...$, i.e., $S : l^2 \ni (x_1, x_2, ...) \mapsto (0, w_1x_1, w_2x_2, ...) \in l^2$, where

$$\begin{cases} w_1 & = 1 \\ w_i & = \frac{1}{2} \text{ , for } i = N_k + 1, N_k + 2, ..., 3N_k \\ w_i & = 2^{\frac{1}{N_k}} \text{ , for } i = 3N_k + 1, 3N_k + 2, ..., N_{k+1}. \end{cases}$$

For nonzero $x = (x_1, x_2, ...) \in l^2$ there is i_0 such that $x_{i_0} \neq 0$.
But by definition of S we have $S^{N_k}e_1 = e_{N_k+1}$ for all $k \in \mathbb{N}$, thus $\|S^{N_k-i_0}x\| \geq \|S^{N_k-i_0}x_{i_0}e_{i_0}\| = \frac{1}{w_1w_2\cdot\ldots\cdot w_{i_0}}|x_{i_0}|$.
So $S^nx \not\to 0$ for all nonzero $x \in l^2$.

Now we show that S is of class $C_{0\cdot}$.
Fix $x = (x_1, x_2, ...) \in l^2$ and $\epsilon > 0$. We can assume $\|x\| = 1$.
Since $\{N_k\}_k$ increases, then there is $N \in \{N_k | k = 1, 2, ...\}$ such that $\sum_{i=N+1}^{\infty} |x_i|^2 < \frac{\epsilon}{32}$ and $\left(\frac{1}{2^N}\right)^2 < \frac{\epsilon}{2}$. By that we obtain:

$$\|S^{2N}x\|^2 = \sum_{i=1}^{N} |x_i|^2\|S^{2N}e_i\|^2 + \sum_{j=N+1}^{\infty} |x_j|^2\|S^{2N}e_j\|^2 =$$

$$= \sum_{i=1}^{N} |x_i|^2\|S^N(w_iw_{i+1}...w_{i+N-1}e_{i+N})\|^2 + \sum_{j=N+1}^{\infty} |x_j|^2|w_jw_{j+1}...w_{j+2N-1}|^2 \leq$$

$$\leq \sum_{i=1}^{N} |x_i|^2\|w_iw_{i+1}\cdot\ldots\cdot w_N\cdot\left(\frac{1}{2}\right)^{i-1} S^Ne_{i+N}\|^2 +$$

$$+ \sum_{j=N+1}^{\infty} |x_j|^2|\underbrace{2^{\frac{1}{N}}2^{\frac{1}{N}}\cdot\ldots\cdot 2^{\frac{1}{N}}}_{2N}|^2 \leq \sum_{i=1}^{N} |x_i|^2\|S^Ne_{i+N}\|^2 + \sum_{j=N+1}^{\infty} |x_j|^24^2 =$$

$$= \sum_{i=1}^{N} |x_i|^2\|\left(\frac{1}{2}\right)^N e_{i+2N}\|^2 + \frac{\epsilon}{32}16 \leq \|x\|\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

In contrast to the above notion we have:

Definition 1.3. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be of class $C_{1\cdot}$ if*

$$\liminf_{n\to\infty} \|T^nx\| > 0$$

for each nonzero $x \in \mathcal{H}$.

Operators of class $C_{1\cdot}$ are also called *non-vanishing*.
We also say that T is of class $C_{\cdot 0}$ or $C_{\cdot 1}$ if its adjoint is of class $C_{0\cdot}$ or $C_{1\cdot}$, respectively.

Let us define $\mathcal{M}(T) := \{x \in \mathcal{H} | \ \exists\{x_n\}_{n\in\mathbb{N}} : x = x_0, Tx_{n+1} = x_n$ and $\{x_n\}_{n\in\mathbb{N}}$ is bounded $\}$. Naturally, such a sequence $\{x_n\}_{n\in\mathbb{N}}$ can be called as the *bounded backward sequence.*

2. Introduction

In the paper [8] we have presented the following theorem with some applications.

Theorem 2.1. *Let T be a contraction. The following conditions are equivalent:*

- *for any bounded backward sequence $\{x_n\}_{n\in\mathbb{N}}$ of T, the sequence of norms $\{\|x_n\|\}_{n\in\mathbb{N}}$ is constant,*
- *the nonunitary part of T is of class $C_{\cdot 0}$.*

We have been asked the natural question about a possible generalization to power-bounded operators. In this work we will try to answer this question.

The easy extension of the above theorem for power-bounded operators is not true. To see this, let us consider the following:

Example 2.2.

$$\text{Let } T : l^2 \ni (x_1, x_2, x_3, ...) \mapsto (0, x_1 + x_2, 0, x_3 + x_4, 0, ...) \in l^2.$$

It is clear that T is power-bounded, in fact $T = T^2$. Additionally, if $x = (a_1, a_2, a_3, ...) \in \mathcal{M}(T) \subset T(\mathcal{H})$, then $a_{2k+1} = 0$ for all $k \in \mathbb{N}$. Hence $T^{-1}(\{x\}) = \{x\}$. Thus, even any (not necessary bounded) backward sequence of T must be constant.

On the other hand, T has trivial unitary part and is not $C_{\cdot 0}$, since $T^* = T^{*2}$.

3. Characterization of $C_{1\cdot}$ and $C_{0\cdot}$ power-bounded operators

To introduce the next theorem, let us recall the construction of isometric asymptotes (see [7]).

Let us define a new semi-inner product on $\mathcal{H}$:

$$[x, y] := \text{glim}\{\langle T^{*n}x, T^{*n}y\rangle\}_{n\in\mathbb{N}},$$

where glim denote a Banach limit.
Thus, the factor space $\mathcal{H}/\mathcal{H}_0$, where $\mathcal{H}_0$ stands for the linear manifold $\mathcal{H}_0 := \{x \in \mathcal{H} | [x, x] = 0\}$, endowed with the inner product $[x + \mathcal{H}_0, y + \mathcal{H}_0] = [x, y]$, is an inner product space. Let $\mathcal{K}$ denote the resulting Hilbert space obtained by completion. Let X denote the natural embedding of Hilbert space $\mathcal{H}$ into $\mathcal{K}$ i.e. $X : \mathcal{H} \ni x \mapsto x + \mathcal{H}_0 \in \mathcal{K}$.

We can see that: $\|XT^*x\| = \|Xx\|$. So there is an isometry $V : \mathcal{K} \to \mathcal{K}$ such that $XT^* = VX$. The isometry V is called *isometric asymptote*.

Lemma 3.1. *For any power-bounded operator $T \in \mathcal{B}(\mathcal{H})$ the corresponding X from the construction above satisfies:*

$$X^*(\mathcal{K}) = \mathcal{M}(T).$$

Proof. By definition of V and X, we have $TX^* = X^*V^*$. Let $x_n := X^*V^n x$, then

$$Tx_{n+1} = TX^*V^{n+1}x = X^*V^*V^{n+1}x = x_n.$$

Moreover $\|x_n\| \le \|X^*\|\|x\|$, for all $n \in \mathbb{N}$. Thus $X^*x = x_0 \in \mathcal{M}(T)$, where $x \in \mathcal{H}$. Hence $X^*(\mathcal{H}) \subset \mathcal{M}(T)$.

To prove the converse, let us fix $x \in \mathcal{M}(T)$. By definition of $\mathcal{M}(T)$, there exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, a bounded backward sequence of x.
Let $y \in \mathcal{H}$, then

$$|\langle x, y\rangle| = |\langle T^n x_n, y\rangle| = |\langle x_n, T^{*n}y\rangle| \le \|x_n\|\|T^{*n}y\|. \tag{1}$$

So $|\langle x, y\rangle| \le \sup_{n\in\mathbb{N}} \|x_n\| \liminf_{n\to\infty} \|T^{*n}y\| \le \sup_{n\in\mathbb{N}} \|x_n\|\|Xy\|$. So by Theorem 1 in [9], we have $x \in X^*(\mathcal{K})$. □

At the begining, we have observed that if $\liminf_{n\to\infty} \|T^{*n}x\| = 0$, for some $x \in \mathcal{H}$, then $\lim_{n\to\infty} \|T^{*n}x\| = 0$. So we have:

$$\{x \in \mathcal{H} | T^{*n}x \to 0\} = \mathcal{N}(X).$$

Now by Lemma 3.1 we obtain:

Corollary 3.2. *Let T be a power-bounded operator, then*

$$\mathcal{H} = \{x \in \mathcal{H} | T^{*n}x \to 0\} \oplus \overline{\mathcal{M}(T)}.$$

We also have:

Theorem 3.3. *A power-bounded operator T is $C_{\cdot 1}$ if and only if $\overline{\mathcal{M}(T)} = \mathcal{H}$.*

It is trivial that $\mathcal{M}(T)$ is included in the set of origins of all backward sequences, that is, $T^\infty(\mathcal{H}) := \bigcap_{n\in\mathbb{N}} T^n(\mathcal{H})$. But in general, the converse inclusion does not hold, even for $C_{\cdot 1}$ contractions.

To see this let us consider the following example.

Example 3.4. Let $\mathcal{H} = l^2$. Then $\mathbb{H} := l^2(\mathcal{H}) = \{\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H} | \sum_{n\in\mathbb{N}} \|x_n\|^2 < \infty\}$ is a separable Hilbert space, with the norm $\|\{x_n\}_{n\in\mathbb{N}}\| := \sqrt{\sum_{n\in\mathbb{N}} \|x_n\|^2}$.
For the element $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \in \mathbb{H}$ sometimes we write $\bigoplus_{n\in\mathbb{N}} \mathbf{x}_n$.
Let S_w be the backward unilateral shift with weights $w = (w_1, w_2, ...)$,

i.e., $S_w : \mathcal{H} \ni (x_1, x_2, ...) \mapsto (w_1 x_2, w_2 x_3 ...) \in \mathcal{H}$. If we put $w_i^n = (\frac{1}{n})^{\frac{1}{i-1}-\frac{1}{i}}$ for all $n \in \mathbb{N}$ and $i > 2$, and $w_1^n = w_2^n = 1$ for all $n \in \mathbb{N}$, then $T = \bigoplus_{n\in\mathbb{N}} S_{w^n}$ is a $C_{\cdot 1}$ contraction.
Indeed, $T^m = \bigoplus_{n\in\mathbb{N}} S^m_{w^n}$, where $S^m_{w^n}(x_1, x_2, x_3, ...) = (w_1^n w_2^n \cdot ... \cdot w_m^n x_{m+1}, w_2^n w_3^n \cdot ... \cdot w_{m+1}^n x_{m+2}, ...)$ and $\lim_{m\to\infty} w_1^n w_2^n \cdot ... \cdot w_m^n = \lim_{m\to\infty} (\frac{1}{n})^{\frac{1}{2}-\frac{1}{m}} = \frac{1}{n^2} > 0$.
Now, let us consider $x = \bigoplus_{n\in\mathbb{N}} (\frac{1}{n}, 0, 0, ...) \in \mathbb{H}$.
For $m \in \mathbb{N}$ we have $x = T^m a_m$, where
$a_m = \bigoplus_{n\in\mathbb{N}} (\underbrace{0, 0, ..., 0}_{m}, (\frac{1}{n})^{\frac{1}{2}+\frac{1}{m}}, 0, 0, ...) \in \mathbb{H}$. Thus $x \in T^\infty(\mathcal{H})$.
Now let $\{b_m\}_{m\in\mathbb{N}} \subset \mathbb{H}$ be a backward sequence for x, then $x = T^m b_m$ and thus $b_m = \bigoplus_{n\in\mathbb{N}} (q_1, q_2, ..., q_m, (\frac{1}{n})^{\frac{1}{2}+\frac{1}{m}}, 0, 0, ...)$ for some complex $q_1, q_2, ..., q_m$. So $\|a_m\| \le \|b_m\|$, but
$\|a_m\|^2 = \sum_{n\in\mathbb{N}} (\frac{1}{n})^{2(\frac{1}{2}+\frac{1}{m})} \to \infty($ for $m \to \infty)$. Hence $x \notin \mathcal{M}(T)$.

One more consequence of Corollary 3.2 is the following:

Theorem 3.5. *A power-bounded operator T is $C_{\cdot 0}$ if and only if $\mathcal{M}(T) = \{0\}$.*

Another proof of this theorem (in the case of operators considered on Banach spaces) can be found in [10].

Corollary 3.6. *If T is power-bounded and invertible, then $\|T^{*n}x\| \to 0$ for all $x \in \mathcal{H}$ if and only if $\|T^{-n}x\| \to \infty$ for all $x \in \mathcal{H}$.*

Proof. If T is power-bounded, then T^* is power-bounded too.
By Theorem 3.5 we obtain that T^* is strongly stable if and only if each nontrivial sequence such that $Tx_{n+1} = x_n$ is unbounded.
But we have $x_n = T^{-n}x_0$. Thus the second condition means that $\sup_{n\in\mathbb{N}} \|T^{-n}x\| = \infty$ for each nonzero $x \in \mathcal{H}$.
Now, if for some $x \in \mathcal{H}$ there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\sup_{k\in\mathbb{N}} \|T^{-n_k}x\| < N$, then for each $n \in \mathbb{N}$ we have

$$\|T^{-n}x\| = \|T^{n_k-n}T^{-n_k}x\| \le \|T^{n_k-n}\|\|T^{-n_k}x\| \le N \sup_{n\in\mathbb{N}} \|T^n\|,$$

since $n_k > n$ for some $k \in \mathbb{N}$. □

Example 3.7. Let V be the classical integral Volterra operator defined, on the space $L^2[0, 1]$, by

$$(Vf)(x) := \int_0^x f(t)dt, \text{ for } f \in L^2[0, 1].$$

It is easy to calculate that $(V^*f)(x) = \int_x^1 f(t)dt$.
Hence $V + V^* = P$, where P is the one-dimensional projection on

subspace of constant functions. It is well-known that $\|(I+V)^{-1}\| = 1$ (see Problem 150 in [4]). The Allan-Pedersen relation (see [1])

$$S^{-1}(I-V)S = (I+V)^{-1},$$

where $Sf(t) = e^t f(t)$ show us that $I - V$ is similar to a contraction. So it is power-bounded.
Furthermore, Proposition 3.3 from [5] yields to

$$\lim_{n\to\infty} \sqrt{n}(I-V)^n V f = 0, \text{ for all } f \in L^2[0,1]. \tag{2}$$

But as we mentioned, $I - V$ is power-bounded. Moreover, V has dense range. Therefore $I - V$ is $C_{0\cdot}$. (To obtain this, instead of (2) we can use the Esterle-Katznelson-Tzafriri theorem (see [3], [6]), since $\sigma(I-V) = \{1\}$.)
Now, by Corollary 3.6 we obtain

$$\|(I+V-P)^{-n}f\| = \|((I-V)^*)^{-n}f\| \to \infty \text{ for all } f \in L^2[0,1]\backslash\{0\}.$$

Additionally, form (1) in the proof of Lemma 3.1 and (2) we have

$$\tfrac{1}{\sqrt{n}}\|(I+V-P)^{-n}f\| \to \infty \text{ for all } f \in V(L^2[0,1])\backslash\{0\}.$$

Remark. *To obtain the first part of this result we also can use Theorem 3.4 form* [2] *and observe that each local spectrum $\sigma_x(I+V-P)$ is equal $\{1\}$. (Because $\{1\} = \sigma(I-V) = \sigma((I+V-P)^*) = \sigma(I+V-P)$.)*

Example 3.8. According to the above example we see that the contraction $(I+V)^{-1}$ is of class $C_{0\cdot}$ and as before $\sigma(I+V) = \{1\}$.
So using Theorem 3.4 form [2] we obtain that $\|(I+V)^n f\| \to \infty$ for all nonzero $f \in L^2[0,1]$.
Now by Corollary 3.6 we have

$$(I-V+P)^{-n}f = (I+V)^{-*n}f \to 0 \text{ for all } f \in L^2[0,1].$$

So the contraction $(I+V)^{-1}$ is of class C_{00}.

4. Main result

To give a generalization of Theorem 2.1, we will need the following lemma(due to Kérchy, see [7]):

Lemma 4.1. *If T is power-bounded, then T can be represented by the matrix*

$$\begin{bmatrix} T_{11} & T_{21} \\ 0 & T_{22} \end{bmatrix}, \tag{3}$$

where T_{11}, T_{22} are power-bounded, T_{11} is of class $C_{0\cdot}$ and T_{22} is of class $C_{1\cdot}$.

Proof. Let (3) be the matrix of T with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$, where $\mathcal{N} := \{x \in \mathcal{H} | \ T^n x \to 0\}$. By definition $\mathcal{N}$ is invariant for T. So $T|_{\mathcal{N}} = T_{11}$, thus T_{11} is of class $C_{0\cdot}$ (and power-bounded). Moreover, we have:

$$T_{22} = P_{\mathcal{K}} T \in \mathcal{B}(\mathcal{K}), \text{ where } \mathcal{K} := \mathcal{N}^\perp \neq \{0\}.$$

The subspace $\mathcal{K}$ is invariant for T^*. So we obtain $T^* |_{\mathcal{K}} = T_{22}^*$, thus T_{22} is power-bounded.

Now, we will show that T_{22} is $C_{1\cdot}$.

To see this, let us assume that $T_{22}^n f \to 0$ for some $f \in \mathcal{K}$.
For an arbitrary $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|T_{22}^{n_0} f\| < \frac{\epsilon}{2M}$, where $M := \sup\limits_{n\in\mathbb{N}} \|T^n\|$.
Let us suppose for a while that $T_{22} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. By definition of T_{22} we have: $(T - T_{22})x = P_{\mathcal{N}} T x \in \mathcal{N}$ for each $x \in \mathcal{K}$.
Hence for each $k \in \{1, 2, 3, ..., n_0\}$ there exists $m_k \in \mathbb{N}$ such that $\|T^{m'+k-1}(T - T_{22})T_{22}^{n_0-k} f\| \leq \frac{\epsilon}{2n_0}$ for all $m' \geq m_k$.
Now, for $m := \max\{m_k |\ k = 1, 2, ..., n_0\}$ we have:

$$\|T^{m+n_0} f\| = \|T^m(T^{n_0} - T^{n_0-1}T_{22} + T^{n_0-1}T_{22} - T^{n_0-2}T_{22}^2 + ... +$$
$$+TT_{22}^{n_0-1} - T_{22}^{n_0} + T_{22}^{n_0})f\| = \|\sum_{k=1}^{n_0} T^{m+k-1}(T - T_{22})T_{22}^{n_0-k} f + T^m T_{22}^{n_0} f\| \leq$$
$$\leq \sum_{k=1}^{n_0} \|T^{m+k-1}(T - T_{22})T_{22}^{n_0-k} f\| + \|T^m T_{22}^{n_0} f\| \leq n_0 \frac{\epsilon}{2n_0} + M\frac{\epsilon}{2M} = \epsilon.$$

Thus $T^n f \to 0$, contrary to $f \in \mathcal{K}$.
So T_{22} is of class $C_{1\cdot}$. □

Now, we can give our generalization of Theorem 2.1:

Theorem 4.2. *Let T be a power-bounded operator. The following conditions are equivalent:*

- *for any bounded backward sequence $\{x_n\}_{n\in\mathbb{N}}$ of T, the sequence of norms $\{\|x_n\|\}_{n\in\mathbb{N}}$ is constant;*
- *T can be decomposed as $T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & U \end{bmatrix}$, where U is a unitary and T_{11} is of class $C_{\cdot 0}$.*

Proof. To the proof of the first implication, let $T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$ be the matrix form Lemma 4.1, where $T_{11} \in \mathcal{B}(\mathcal{H}_1)$ is $C_{\cdot 0}$ and $T_{22} \in \mathcal{B}(\mathcal{H}_2)$ is $C_{\cdot 1}$. Now, $\mathcal{H}_2$ is invariant for T, thus $T_{22} = T|_{\mathcal{H}_2}$. Hence, each bounded backward sequence of T_{22} is bounded backward sequence of T. So, by our assumption T_{22} is an isometry on $\overline{\mathcal{M}(T_{22})}$. But by Theorem 3.3 we have $\overline{\mathcal{M}(T_{22})} = \mathcal{H}_2$. So T_{22} is an isometry.

Finally, it can be decomposed as $T_{22} = U \oplus S_+$, where U is unitary and S_+ is the unilateral shift. But T_{22} is $C_{\cdot 1}$. So we have $T_{22} = U$.

To prove the converse implication, let us assume that $\{x_n\}_{n\in\mathbb{N}}$ is the bounded backward sequence of T. Let $x_n = a_n + b_n$, where $a_n \in \mathcal{H}_1$ and $b_n \in \mathcal{H}_2$. We have:

$$T_{11}a_{n+1} + (T_{21}a_{n+1} + Ub_{n+1}) = Ta_{n+1} + Tb_{n+1} = Tx_{n+1} = x_n = a_n + b_n.$$

So $T_{11}a_{n+1} = a_n$ and $\|a_n\| \le \|x_n\|$. It means that $\{a_n\}_{n\in\mathbb{N}}$ is a bounded backward sequence of T_{11}, but T_{11} is of class $C_{\cdot 0}$. So by Theorem 3.5 we obtain $a_n \equiv 0$. Thus $\|x_{n+1}\| = \|b_{n+1}\| = \|Ub_{n+1}\| = \|b_n\| = \|x_n\|$. □

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Instytut Matematyki, Uniwersytet Jagielloński,
Łojasiewicza 6, 30-348, Kraków, Poland
E-mail address: patryk.pagacz@im.uj.edu.pl