SINGULARITIES OF THE DIVERGENCE OF CONTINUOUS VECTOR FIELDS AND UNIFORM HAUSDORFF ESTIMATES

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ABSTRACT. We prove that every closed set which is not σ-finite with respect to the Hausdorff measure $\mathcal{H}^{N-1}$ carries singularities of continuous vector fields in $\mathbb{R}^N$ for the divergence operator. We also show that finite measures which do not charge sets of finite Hausdorff measure $\mathcal{H}^{N-1}$ can be written as an L^1 perturbation of the divergence of a continuous vector field. The main tool is a property of approximation of measures in terms of the Hausdorff content.

1. INTRODUCTION AND MAIN RESULTS

The original motivation of this paper is related to the following problem: to find a simple characterization of all closed sets S such that if $V : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous vector field such that

$$\operatorname{div} V = 0 \quad \text{in } \mathbb{R}^N \setminus S,$$

then

$$\operatorname{div} V = 0 \quad \text{in } \mathbb{R}^N.$$

Such sets S cannot carry singularities of continuous vector fields for the divergence operator; we say in this case that S is C^0 removable. We show that the answer to this question is given by the following

Theorem 1.1. *Let $S \subset \mathbb{R}^N$ be a closed set. Then, S is C^0 removable for the divergence operator if and only if S is σ-finite for the Hausdorff measure $\mathcal{H}^{N-1}$.*

The reverse implication "$\Leftarrow$" has been established by de Valeriola and Moonens [10, Theorem 12]. In Section 2, we provide the direct implication "$\Rightarrow$" by showing that if S is not σ-finite for the Hausdorff measure $\mathcal{H}^{N-1}$, then there exists a (Borel) positive measure μ supported on S such that the equation

$$\operatorname{div} V = \mu \quad \text{in } \mathbb{R}^N$$

has a continuous solution.

This result completes the picture concerning the removability of singularities for C^0 and L^∞ vector fields. Indeed, Theorem 1.1 has the following counterpart concerning *bounded* (not necessarily continuous) vector fields:

Theorem 1.2. *Let $S \subset \mathbb{R}^N$ be a closed set. Then, S is L^∞ removable for the divergence operator if and only if S has zero Hausdorff measure $\mathcal{H}^{N-1}$.*

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Theorem 1.2 has been proved independently by Moonens [15, Theorem 4.7] and by Phuc and Torres [17, Theorem 5.1]. We also refer the reader to [8, Theorem 6.3] in the case where S is purely unrectifiable.

The implication "⇒" in Theorem 1.1 which concerns us relies on the study of the existence of continuous vector fields $V : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\operatorname{div} V = \mu \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where μ is a function or a measure. The case where μ belongs to $L^N(\mathbb{R}^N)$ has been investigated by Bourgain and Brezis [4, Proposition 1 and Remark 1; 9, Proposition 2.9]. They have given an affirmative answer using the closed-range theorem. This solution cannot be obtained from the equation

$$\operatorname{div} \nabla u = \Delta u = \mu \quad \text{in } \mathbb{R}^N$$

in view of the lack of embedding of the Sobolev space $W^{1,N}$ into the space of continuous functions C^0.

A characterization of finite measures — and more generally distributions — in $\mathbb{R}^N$ for which equation (1.1) has a continuous solution has been obtained by De Pauw and Pfeffer [9, Theorem 4.8]. Stated in terms of *strong charges*, they have proved that equation (1.1) has a C^0 solution if and only if for every $\epsilon > 0$ and for every compact set $K \subset \mathbb{R}^N$, there exists $C > 0$ such that for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ supported in K,

$$\left| \int_{\mathbb{R}^N} \varphi \, \mathrm{d}\mu \right| \le C \|\varphi\|_{L^1(\mathbb{R}^N)} + \epsilon \|D\varphi\|_{L^1(\mathbb{R}^N)}.$$

Phuc and Torres [17, Theorem 4.5] have shown that for nonnegative measures the condition of De Pauw and Pfeffer is equivalent to asking that for every compact set $K \subset \mathbb{R}^N$,

$$\lim_{\delta \to 0} \sup_{\substack{x \in K \\ r \le \delta}} \frac{\mu(B(x;r))}{r^{N-1}} = 0. \tag{1.2}$$

We may get some insight about the meaning of assumption (1.2) using the Besicovitch covering theorem [14, Theorem 2.7]: if μ satisfies (1.2) for every compact set K, then for every Borel set $A \subset \mathbb{R}^N$,

$$\mathcal{H}^{N-1}(A) < +\infty \quad \text{implies} \quad \mu(A) = 0. \tag{1.3}$$

Actually, one does not need the full power of (1.2) to deduce this property, since the pointwise convergence with respect to x already suffices to obtain the same conclusion. We would like to understand in what sense a measure satisfying (1.3) misses the uniform limit in (1.2).

For this purpose, note that any finite measure (positive or not) in $\mathbb{R}^N$ can be approximated by smooth functions, for instance via convolution, but such approximation is rather weak and holds in the sense of distributions. The convergence cannot be strong with respect to the total mass norm

$$\|\mu\|_{\mathcal{M}(\mathbb{R}^N)} = |\mu|(\mathbb{R}^N),$$

unless μ is an L^1 function by completeness of $L^1(\mathbb{R}^N)$.

The main tool in this paper asserts that we can pass from condition (1.3) to (1.2) by strong convergence of measures:

Proposition 1.3. *For every finite nonnegative measure μ in $\mathbb{R}^N$ satisfying* (1.3), *there exists a nondecreasing sequence of finite nonnegative measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that*

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{M}(\mathbb{R}^N)} = 0,$$

and for every $n \in \mathbb{N}$,

$$\lim_{\delta\to 0} \sup_{\substack{x\in\mathbb{R}^N \\ r\leq\delta}} \frac{\mu_n(B(x;r))}{r^{N-1}} = 0.$$

Each measure μ_n is actually obtained from μ by restriction: although μ need not satisfy (1.2), it suffices to remove a small part of μ and the remaining of the measure verifies that property.

A similar approximation result still holds concerning any positive dimension s and the main tool is a uniform comparison principle between the Hausdorff measure $\mathcal{H}^s$ and the outer measures $\mathcal{H}^s_\delta$; see Proposition 3.1 and Proposition 3.2.

Returning to equation (1.1), we have seen that if μ satisfies condition (1.3), then we can extract a small part of μ and the remaining part yields a continuous solution of the equation. As a consequence, we prove in Section 4 that there exist continuous solutions except for an L^1 perturbation of μ:

Theorem 1.4. *For every finite measure μ in $\mathbb{R}^N$ satisfying* (1.3) *and for every $\epsilon > 0$, there exist $f \in L^1(\mathbb{R}^N)$ and $V \in C^0(\mathbb{R}^N;\mathbb{R}^N)$ such that*

$$\mu = f + \operatorname{div} V \quad \text{in } \mathbb{R}^N$$

and $\|f\|_{L^1(\mathbb{R}^N)} \leq \epsilon$.

There is a counterpart of this result for the Laplace operator, in which case the $W^{1,2}$ capacity plays the role of the Hausdorff measure $\mathcal{H}^{N-1}$; see [5, Theorem 4.3].

Theorem 1.4 implies that any finite measure satisfying (1.3) is a *charge* in the sense that for every $\epsilon > 0$ and for every compact set $K \subset \mathbb{R}^N$, there exists $C > 0$ such that for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ supported in K,

$$\Big|\int_{\mathbb{R}^N} \varphi \,\mathrm{d}\mu\Big| \leq C\|\varphi\|_{L^1(\mathbb{R}^N)} + \epsilon\big(\|D\varphi\|_{L^1(\mathbb{R}^N)} + \|\varphi\|_{L^\infty(\mathbb{R}^N)}\big).$$

The notions of charges and strong charges have connections with the Divergence theorem and have been investigated by De Pauw, Pfeffer and collaborators; see [9, 16] and the references therein.

Combining Theorem 1.4 with [9, Theorem 6.2], we characterize in Section 5 finite measures in $\mathbb{R}^N$ which are charges:

Corollary 1.5. *For every finite measure μ in $\mathbb{R}^N$, μ is a charge if and only if property* (1.3) *holds.*

By the Hahn decomposition theorem, property (1.3) is equivalent to asking that for every Borel set $A \subset \mathbb{R}^N$,

$$\mathcal{H}^{N-1}(A) < +\infty \quad \text{implies} \quad |\mu|(A) = 0.$$

We deduce from the corollary above that if a finite measure μ is a charge, then the positive and the negative parts of μ are also charges. The counterpart of this property for strong charges is false in dimension $N \geq 2$. For instance, given $\alpha > 0$, let $u_\alpha : \mathbb{R}^N \to \mathbb{R}$ be the function defined for $x \in \mathbb{R}$ by

$$u_\alpha(x) = \begin{cases} |x| \sin \frac{1}{|x|^\alpha} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then, u_α is continuous and belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ for $\alpha < N$. In particular, if $W : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth vector field with compact support, then the function

$$f_\alpha = \operatorname{div}(u_\alpha W)$$

belongs to $L^1(\mathbb{R}^N)$ and defines a strong charge since the vector field $u_\alpha W$ is continuous. However, if $\alpha \geq 1$ and $W(0) \neq 0$, then

$$\lim_{r \to 0} \frac{1}{r^{N-1}} \int\limits_{B(0;r)} f_\alpha^+ > 0.$$

In particular, condition (1.2) is not satisfied, whence f_α^+ does not define a strong charge.

2. Proof of Theorem 1.1

The proof of the direct implication of Theorem 1.1 relies on three main tools: the characterization of strong charges of Phuc and Torres given by condition (1.2), Frostman's lemma and a property of sets which are not σ-finite for the Hausdorff measure due to Besicovitch.

We first recall the definition of the Hausdorff measure with dimension given by a continuous function h. Henceforth, we assume that $h : [0, +\infty) \to \mathbb{R}$ is a continuous function such that

(a) h is nondecreasing,
(b) $h(0) = 0$ and for every $t > 0$, $h(t) > 0$.

Given a set $A \subset \mathbb{R}^N$, define for $0 < \delta \leq +\infty$ the Hausdorff outer measures of dimension h,

$$\Lambda^h_\delta(A) = \inf \left\{ \sum_{n=0}^{\infty} h(r_n) : A \subset \bigcup_{n=0}^{\infty} B(x_n; r_n) \text{ and } 0 \leq r_n \leq \delta \right\}.$$

The outer measure $\Lambda^h_\infty(A)$ is usually called the Hausdorff content of A.

The Hausdorff measure of A of dimension h is then defined as the limit

$$\Lambda^h(A) = \lim_{\delta \to 0} \Lambda^h_\delta(A).$$

If $s > 0$ and h is defined for $t > 0$ by

$$h(t) = \omega_s t^s,$$

where $\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$, then Λ^h is the Hausdorff measure $\mathcal{H}^s$.

We have adopted Hausdorff's original definition [13, Definition 1] of Hausdorff measures. Nowadays, this definition is usually referred to as the spherical Hausdorff measure since we are covering the set A using balls instead of arbitrary bounded sets.

An important property related to the Hausdorff measure Λ^h is given by Frostman's lemma [6, Chapter II, Theorem 1; 12, No. 47; 14, Theorem 8.17]:

Proposition 2.1. *Let $A \subset \mathbb{R}^N$ be a Borel set. If $\Lambda^h(A) > 0$, then there exists a positive measure μ supported in A such that for every $x \in \mathbb{R}^N$ and for every $r > 0$,*

$$\mu(B(x;r)) \leq h(r).$$

We also need the following result of Besicovitch [2, Theorem 7; 18, Theorem 1]:

Proposition 2.2. *Let $A \subset \mathbb{R}^N$ be a Borel set. If A is not σ-finite for the Hausdorff measure $\mathcal{H}^{N-1}$, then there exists a continuous function $h : [0, +\infty) \to \mathbb{R}$ such that*

(i) *h is increasing,*
(ii) *$h(0) = 0$ and for every $t > 0$, $0 < h(t) \leq t^{N-1}$,*
(iii) $\displaystyle\lim_{t\to 0} \frac{h(t)}{t^{N-1}} = 0$,
(iv) *A is not σ-finite with respect to the Hausdorff measure Λ^h.*

The result of Besicovitch actually concerns any Hausdorff measure $\Lambda^{h'}$ instead of $\mathcal{H}^{N-1}$. Although quite different in nature, this type of property is reminiscent of a result of de la Vallée Poussin which asserts that for any L^1 function f there exists a superlinear function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi \circ f$ is also an L^1 function [7, Remarque 23].

Proof of Theorem 1.1. The reverse implication "$\Leftarrow$" is established in [10, Theorem 12]. In order to prove the direct implication "$\Rightarrow$", let $S \subset \mathbb{R}^N$ be a closed set which is not σ-finite for the Hausdorff measure $\mathcal{H}^{N-1}$. By the result of Besicovitch (Proposition 2.2) applied to the set S, there exists an increasing continuous function h satisfying properties (i)–(iv) above. In particular, $\Lambda^h(S) > 0$. By Frostman's lemma (Proposition 2.1), there exists a finite positive measure μ such that for every $x \in \mathbb{R}^N$ and for every $r > 0$,

$$\mu(B(x;r)) \leq h(r).$$

By Proposition 2.2 (iii), the measure μ satisfies the condition of Phuc and Torres (1.2), hence μ can be written as the divergence of a continuous vector field V; see [17, Theorem 4.5]. We deduce that the C^0 removable singularity property is not satisfied by S. ☐

3. Uniform Hausforff estimates

The goal of this section is to obtain estimates of measures in terms of densities or equivalently in terms of Hausdorff outer measures Λ^h_δ. The type of result we have in mind is the following:

Proposition 3.1. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. If for every Borel set $A \subset \mathbb{R}^N$,*

$$\Lambda^h(A) = 0 \quad \textit{implies} \quad \mu(A) = 0,$$

then for every $\epsilon > 0$ there exist $c > 0$ and a Borel set $E \subset \mathbb{R}^N$ such that

(i) $\mu\lfloor_E \leq c\Lambda^h_\infty$,
(ii) $\mu(\mathbb{R}^N \setminus E) \leq \epsilon$.

The following analog of Proposition 3.1 will be used in the proof of Theorem 1.4:

Proposition 3.2. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. If for every Borel set $A \subset \mathbb{R}^N$,*

$$\Lambda^h(A) < +\infty \quad \textit{implies} \quad \mu(A) = 0,$$

then for every $\epsilon > 0$, there exists a Borel set $E \subset \mathbb{R}^N$ such that

(i) *for every $c > 0$ there exists $\delta > 0$ such that $\mu\lfloor_E \leq c\Lambda^h_\delta$,*
(ii) $\mu(\mathbb{R}^N \setminus E) \leq \epsilon$.

Although Proposition 3.1 and Proposition 3.2 look similar, the proof of the latter is more involved and requires an additional argument in order to compensate the lack of additivity of the Hausdorff outer measures Λ^h_δ. In both cases, we do not rely on covering arguments in $\mathbb{R}^N$ so the proofs are also valid in metric spaces; concerning Proposition 3.2, we need the inner regularity of the measure μ.

The results of this section are based on the following

Lemma 3.3. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. For every nonnegative outer measure T, there exists a Borel set $E \subset \mathbb{R}^N$ such that*

$$\mu\lfloor_E \leq T \quad \textit{and} \quad T(\mathbb{R}^N \setminus E) \leq \mu(\mathbb{R}^N \setminus E).$$

We recall the a nonnegative outer measure T is a set function with values into $[0, +\infty]$ such that

(a) $T(\emptyset) = 0$,
(b) if $A \subset B$, then $T(A) \leq T(B)$,
(c) for every sequence $(A_n)_{n\in\mathbb{N}}$, $T\big(\bigcup\limits_{k=0}^{\infty} A_k\big) \leq \sum\limits_{k=0}^{\infty} T(A_k)$.

This lemma is inspired from [1, Lemma 2]; the outer measure T we have in mind is Λ^h_δ. When T is a finite measure, this lemma follows from the classical Hahn decomposition theorem [11, Theorem 3.3] applied to the measure $\mu - T$, in which case the set E may be chosen so that

$$\mu\lfloor_E \leq T \quad \text{and} \quad T\lfloor_{\mathbb{R}^N\setminus E} \leq \mu.$$

The main idea of the proof is that if the inequality $\mu \leq T$ does not hold on every Borel set, then there exists some Borel set $F \subset \mathbb{R}^N$ such that $T(F) < \mu(F)$ and we try to choose F so that $\mu(F)$ is as large as possible. Since μ is a finite measure, we eventually exhaust the part of μ that prevents the inequality $\mu \leq T$ to hold.

Proof of Lemma 3.3. Let $0 < \theta < 1$. By induction, there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of disjoint Borel sets of $\mathbb{R}^N$ such that

(a) for every $n \in \mathbb{N}$, $T(F_n) \le \mu(F_n)$,
(b) for every $n \in \mathbb{N}_*$, $\mu(F_n) \ge \theta \epsilon_n$, where

$$\epsilon_n = \sup \Big\{ \mu(F) : F \subset \mathbb{R}^N \setminus \bigcup_{k=0}^{n-1} F_k \text{ and } T(F) \le \mu(F) \Big\}.$$

By subadditivity of T and by additivity of μ,

$$T\big(\bigcup_{k=0}^{\infty} F_k \big) \le \sum_{k=0}^{\infty} T(F_k) \le \sum_{k=0}^{\infty} \mu(F_k) = \mu\big(\bigcup_{k=0}^{\infty} F_k \big).$$

We claim that

$$\mu \lfloor_{\mathbb{R}^N \setminus \bigcup\limits_{k=0}^{\infty} F_k} \le T.$$

Assume by contradiction that this inequality is not true. Then, there exists a Borel set $C \subset \mathbb{R}^N$ such that

$$T(C) < \mu\big(C \setminus \bigcup_{k=0}^{\infty} F_k \big).$$

Let

$$D = C \setminus \bigcup_{k=0}^{\infty} F_k.$$

By monotonicity of T, we have

$$T(D) \le T(C) < \mu(D).$$

In particular, $\mu(D) > 0$. Since D is an admissible set in the definition of the numbers ϵ_n, for every $n \in \mathbb{N}$ we have

$$\mu(D) \le \epsilon_n.$$

This is not possible since

$$\theta \sum_{k=1}^{\infty} \epsilon_k \le \sum_{k=1}^{\infty} \mu(F_k) = \mu\big(\bigcup_{k=1}^{\infty} F_k \big) \le \mu(\mathbb{R}^N) < +\infty.$$

In particular, the sequence $(\epsilon_n)_{n \in \mathbb{N}}$ converges to 0, but this contradicts the fact that $(\epsilon_n)_{n \in \mathbb{N}}$ is bounded from below by $\mu(D)$.

We have the conclusion of the lemma by choosing

$$E = \mathbb{R}^N \setminus \bigcup_{k=0}^{\infty} F_k. \qquad \square$$

The last ingredient in the proof of Proposition 3.1 gives a quantitative information of the absolute continuity with respect to the Hausdorff measure Λ^h in terms of the Hausdorff content Λ^h_∞. The proof uses the same strategy as in the proof of the usual absolute continuity of measures. It relies on the fact that for any set $A \subset \mathbb{R}^N$, $\Lambda^h(A) = 0$ if and only if $\Lambda^h_\infty(A) = 0$.

Lemma 3.4. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. If for every Borel set $A \subset \mathbb{R}^N$,*

$$\Lambda^h(A) = 0 \quad \text{implies} \quad \mu(A) = 0,$$

then for every $\epsilon > 0$ there exists $\eta > 0$ such that for every Borel set $A \subset \mathbb{R}^N$,

$$\Lambda^h_\infty(A) \le \eta \quad \text{implies} \quad \mu(A) \le \epsilon.$$

Proof. Assume by contradiction that there exists $\epsilon > 0$ such that for every $\eta > 0$ there exists a Borel set $A \subset \mathbb{R}^N$ such that $\Lambda^h_\infty(A) \leq \eta$ and $\mu(A) > \epsilon$.

Given a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive numbers, for every $n \in \mathbb{N}$ we take a Borel set A_n such that

$$\Lambda^h_\infty(A_n) \leq \eta_n \quad \text{and} \quad \mu(A_n) > \epsilon.$$

By subadditivity of Λ^h_∞ we have for every $n \in \mathbb{N}$,

$$\Lambda^h_\infty\big(\bigcup_{k=n}^{\infty} A_k\big) \leq \sum_{k=n}^{\infty} \Lambda^h_\infty(A_k) \leq \sum_{k=n}^{\infty} \eta_k$$

and by monotonicity of μ,

$$\mu\big(\bigcup_{k=n}^{\infty} A_k\big) \geq \mu(A_n) > \epsilon.$$

Take

$$B = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Choosing the sequence $(\eta_n)_{n\in\mathbb{N}}$ such that the series $\sum\limits_{k=0}^{\infty} \eta_k$ converges, we deduce from the above that

$$\Lambda^h_\infty(B) = 0 \quad \text{and} \quad \mu(B) \geq \epsilon.$$

Since $\Lambda^h_\infty(B) = 0$ is equivalent to $\Lambda^h(B) = 0$, we get a contradiction with the assumption on the measure μ. □

Proof of Proposition 3.1. Given $c > 0$, let $E_c \subset \mathbb{R}^N$ be the set given by Lemma 3.3 with measure μ and outer measure $T = c\Lambda^h_\infty$. Thus,

$$\mu\lfloor_{E_c} \leq c\Lambda^h_\infty$$

and

$$c\Lambda^h_\infty(\mathbb{R}^N \setminus E_c) \leq \mu(\mathbb{R}^N \setminus E_c) \leq \mu(\mathbb{R}^N).$$

In particular,

$$\lim_{c\to+\infty} \Lambda^h_\infty(\mathbb{R}^N \setminus E_c) = 0.$$

By the property of absolute continuity of μ with respect to Λ^h_∞ (Lemma 3.4), the conclusion of the proposition follows. □

In the proof of Proposition 3.2, we bypass the lack of additivity of the outer Hausdorff measures Λ^h_δ using the following

Lemma 3.5. *Let $\nu \in \mathcal{M}(\mathbb{R}^N)$ be a finite nonnegative measure in $\mathbb{R}^N$, let $\delta > 0$ and let $F_1, \ldots, F_n$ be disjoint Borel subsets of $\mathbb{R}^N$. If for every $k \in \{1, \ldots, n\}$,*

$$\nu\lfloor_{F_k} \leq \Lambda^h_\delta,$$

then for every $\epsilon > 0$, there exist $0 < \underline{\delta} \leq \delta$ and a Borel set $F \subset \bigcup\limits_{k=1}^{n} F_k$ such that

$$\nu\lfloor_F \leq \Lambda^h_{\underline{\delta}} \quad \textit{and} \quad \nu\big(\bigcup_{k=1}^{n} F_k \setminus E\big) \leq \epsilon.$$

Proof. For each $i \in \{1, \dots, n\}$, let $K_i \subset F_i$ be a compact subset. For every Borel set $A \subset \mathbb{R}^N$,

$$\nu\lfloor_{\bigcup_{i=1}^n K_i}(A) = \sum_{i=1}^n \nu(A \cap K_i) \le \sum_{i=1}^n \Lambda_\delta^h(A \cap K_i).$$

Let $0 < \underline{\delta} \le \delta$ be such that for every $i, j \in \{1, \dots, n\}$, if $i \ne j$, then $d(K_i, K_j) \ge \underline{\delta}$. In particular,

$$d(A \cap K_i, A \cap K_j) \ge \underline{\delta}.$$

By the metric additivity of the outer Hausdorff measure Λ_δ^h [11, proof of Proposition 11.17; 19, Theorem 16],

$$\sum_{i=1}^n \Lambda_{\underline{\delta}}^h(A \cap K_i) = \Lambda_{\underline{\delta}}^h\big(\bigcup_{i=1}^n (A \cap K_i)\big).$$

We deduce that for every Borel set $A \subset \mathbb{R}^N$,

$$\begin{aligned}\nu\lfloor_{\bigcup_{i=1}^n K_i}(A) &\le \sum_{i=1}^n \Lambda_\delta^h(A \cap K_i)\\ &\le \sum_{i=1}^n \Lambda_{\underline{\delta}}^h(A \cap K_i) = \Lambda_{\underline{\delta}}^h\big(\bigcup_{i=1}^n (A \cap K_i)\big) \le \Lambda_{\underline{\delta}}^h(A).\end{aligned}$$

Thus, the set

$$F = \bigcup_{i=1}^n K_i$$

satisfies the first property of the statement.

We now show how to choose the compact sets K_i in order to have the second property. Since

$$\big(\bigcup_{i=1}^n F_i\big) \setminus \big(\bigcup_{i=1}^n K_i\big) = \bigcup_{i=1}^n (F_i \setminus K_i),$$

by additivity of the measure μ,

$$\mu\Big(\big(\bigcup_{i=1}^n F_i\big) \setminus \big(\bigcup_{i=1}^n K_i\big)\Big) = \sum_{i=1}^n \mu(F_i \setminus K_i).$$

By inner regularity of the measure μ, we may choose the compact set $K_i \subset F_i$ such that

$$\mu(F_i \setminus K_i) \le \frac{\epsilon}{n}.$$

Thus,

$$\mu\Big(\big(\bigcup_{i=1}^n F_i\big) \setminus \big(\bigcup_{i=1}^n K_i\big)\Big) \le n\frac{\epsilon}{n} = \epsilon.$$

This proves the proposition. □

Proof of Proposition 3.2. We begin by proving the existence of a Borel set $E \subset \mathbb{R}^N$ depending on $c > 0$:

Claim. For every $\epsilon > 0$ and for every $c > 0$, there exist a Borel set $E \subset \mathbb{R}^N$ and $\delta > 0$ satisfying properties (i)–(ii).

Proof of the claim. Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Given $c > 0$, we construct a sequence of Borel sets $(F_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

(a) $\mu\lfloor_{F_0} \leq c\Lambda^h_{\delta_0}$,

(b) for every $n \in \mathbb{N}_*$, $\mu\lfloor_{F_n \setminus \bigcup\limits_{k=0}^{n-1} F_k} \leq c\Lambda^h_{\delta_n}$,

(c) for every $n \in \mathbb{N}$, $c\Lambda^h_{\delta_n}(\mathbb{R}^N \setminus F_n) \leq \mu\big(\mathbb{R}^N \setminus \bigcup\limits_{k=0}^{n} F_k\big)$.

We proceed by induction on $n \in \mathbb{N}$. Let $F_0 \subset \mathbb{R}^N$ be a Borel set satisfying the conclusion of Lemma 3.3 with $T = \Lambda^h_{\delta_0}$. Given $n \in \mathbb{N}_*$ and Borel sets $F_0, \ldots, F_{n-1}$, applying Lemma 3.3 with measure $\mu\lfloor_{\mathbb{R}^N \setminus \bigcup\limits_{k=0}^{n-1} F_k}$ and outer measure $T = c\Lambda^h_{\delta_n}$, we obtain a Borel set $F_n \subset \mathbb{R}^N$ such that

$$\mu\lfloor_{F_n \setminus \bigcup\limits_{k=0}^{n-1} F_k} \leq c\Lambda^h_{\delta_n}$$

and

$$c\Lambda^h_{\delta_n}(\mathbb{R}^N \setminus F_n) \leq \mu\big(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} F_k\big).$$

Let

$$C = \mathbb{R}^N \setminus \bigcup_{k=0}^{\infty} F_k.$$

Then,

$$\lim_{n\to\infty} \mu\big(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} F_k\big) = \mu(C).$$

and, by property (c),

$$c\Lambda^h_{\delta_n}(C) \leq c\Lambda^h_{\delta_n}(\mathbb{R}^N \setminus F_n) \leq \mu\big(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} F_k\big).$$

Choosing a sequence $(\delta_n)_{n\in\mathbb{N}}$ converging to zero, as n tends to infinity we get

$$c\Lambda^h(C) \leq \mu(C).$$

In particular, the Hausdorff measure $c\Lambda^h(C)$ is finite and by assumption on the measure μ,

$$\mu(C) = 0.$$

Thus,

$$\lim_{n\to\infty} \mu\big(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} F_k\big) = \mu(C) = 0.$$

Given a Borel set $E \subset \mathbb{R}^N \setminus \bigcup\limits_{k=0}^{n} F_k$, by additivity of the measure μ,

$$\mu(\mathbb{R}^N \setminus E) = \mu\big(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} F_k\big) + \mu\big(\bigcup_{k=0}^{n} F_k \setminus E\big).$$

Given $\epsilon > 0$, by the limit above there exists $n \in \mathbb{N}$ such that

$$\mu\big(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} F_k\big) \leq \frac{\epsilon}{2}.$$

By the property of weak additivity of the Hausdorff outer measures Λ^h_δ (Lemma 3.5) applied to the sets $F_0, F_1 \setminus F_0, \ldots, F_n \setminus \bigcup\limits_{k=0}^{n-1} F_k$, there exist

$$0 < \underline{\delta} \le \min\{\delta_0, \ldots, \delta_n\} \quad \text{and} \quad E \subset \bigcup_{k=0}^{n} F_k$$

such that

$$\mu\lfloor_E \le \Lambda^h_{\underline{\delta}} \quad \text{and} \quad \mu\Big(\bigcup_{k=0}^{n} F_k \setminus E\Big) \le \frac{\epsilon}{2}.$$

Thus,

$$\mu(\mathbb{R}^N \setminus E) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This concludes the proof of the claim. □

Given two sequences $(\epsilon_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ of positive numbers, by the previous claim for every $n \in \mathbb{N}$ there exist a Borel set $E_n \subset \mathbb{R}^N$ and $\delta_n > 0$ such that

$$\mu\lfloor_{E_n} \le c_n \Lambda^h_{\delta_n} \quad \text{and} \quad \mu(\mathbb{R}^N \setminus E_n) \le \epsilon_n.$$

Let

$$E = \bigcap_{k=0}^{\infty} E_k.$$

By monotonicity of μ we have for every $n \in \mathbb{N}$,

$$\mu\lfloor_E \le \mu\lfloor_{E_n} \le c_n \Lambda^h_{\delta_n}.$$

Choosing a sequence $(c_n)_{n\in\mathbb{N}}$ converging to zero, then for every $c > 0$, there exists $n \in \mathbb{N}$ such that $c_n \le c$ and we have

$$\mu\lfloor_E \le c_n \Lambda^h_{\delta_n} \le c \Lambda^h_{\delta_n}.$$

Thus, property (i) holds with $\delta = \delta_n$.

By subadditivity of the measure μ,

$$\mu(\mathbb{R}^N \setminus E) \le \sum_{k=0}^{\infty} \mu(\mathbb{R}^N \setminus E_k) \le \sum_{k=0}^{\infty} \epsilon_k.$$

Choosing the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that the series $\sum\limits_{k=0}^{\infty} \epsilon_k$ converges and is bounded from above by ϵ, we deduce property (ii). The proof of the proposition is complete. □

The next corollary is the counterpart of Lemma 3.4 for measures which do not charge sets of finite Hausdorff measure Λ^h. We have no direct proof of it as in the case of Lemma 3.4. If we could prove directly Corollary 3.6, then we would have a simpler proof of Proposition 3.2 in the lines of the proof of Proposition 3.1.

Corollary 3.6. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. If for every Borel set $A \subset \mathbb{R}^N$,*

$$\Lambda^h(A) < +\infty \quad \textit{implies} \quad \mu(A) = 0,$$

then for every $M \geq 0$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that for every Borel set $A \subset \mathbb{R}^N$,

$$\Lambda^h_\delta(A) \leq M \quad \textit{implies} \quad \mu(A) \leq \epsilon.$$

Proof. Given a Borel set $E \subset \mathbb{R}^N$, for every Borel set $A \subset \mathbb{R}^N$,

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E) \leq \mu(A \cap E) + \mu(\mathbb{R}^N \setminus E).$$

Given $\epsilon > 0$, let $E \subset \mathbb{R}^N$ be a Borel set satisfying the conclusion of Proposition 3.2. Thus, for every $c > 0$ there exists $\delta > 0$ such that

$$\mu(A) \leq c\Lambda^h_\delta(A) + \epsilon.$$

Choosing $c > 0$ such that $cM \leq \epsilon$, it follows that if $\Lambda^h_\delta(A) \leq M$, then

$$\mu(A) \leq c\Lambda^h_\delta(A) + \epsilon \leq cM + \epsilon = 2\epsilon.$$ □

4. Proof of Theorem 1.4

We begin with the following tool which relates conditions (1.2) and (1.3):

Proposition 4.1. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. If for every Borel set $A \subset \mathbb{R}^N$,*

$$\Lambda^h(A) < +\infty \quad \textit{implies} \quad \mu(A) = 0,$$

then there exists a nondecreasing sequence of finite nonnegative measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that for every $n \in \mathbb{N}$,

$$\lim_{\delta\to 0} \sup_{\substack{x\in\mathbb{R}^N\\ r\leq\delta}} \frac{\mu_n(B(x;r))}{h(r)} = 0$$

and

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{M}(\mathbb{R}^N)} = 0.$$

The proposition above extends Proposition 1.3 and is a consequence of the following lemma by choosing for every $n \in \mathbb{N}$,

$$\mu_n = \mu\lfloor_{\bigcup\limits_{k=0}^{n} F_k} = \sum_{k=0}^{n} \mu\lfloor_{F_k}.$$

Lemma 4.2. *Let μ be a finite nonnegative measure in $\mathbb{R}^N$. If for every Borel set $A \subset \mathbb{R}^N$,*

$$\Lambda^h(A) < \infty \quad \textit{implies} \quad \mu(A) = 0,$$

then there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of disjoint Borel sets of $\mathbb{R}^N$ such that for every $n \in \mathbb{N}$,

$$\lim_{\delta\to 0} \sup_{\substack{x\in\mathbb{R}^N\\ r\leq\delta}} \frac{\mu(F_n \cap B(x;r))}{h(r)} = 0$$

and for every Borel set $A \subset \mathbb{R}^N$,

$$\mu(A) = \sum_{k=0}^{\infty} \mu(F_k \cap A).$$

Proof. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. By Proposition 3.2, there exists a Borel set $E_0 \subset \mathbb{R}^N$ such that

(a) for every $c > 0$, there exists $\delta > 0$ such that $\mu\lfloor_{E_0} \le c\Lambda^h_\delta$,
(b) $\mu(\mathbb{R}^N \setminus E_0) \le \epsilon_0$.
We proceed by induction. Given $n \in \mathbb{N}_*$ and Borel sets $E_0, \dots, E_{n-1} \subset \mathbb{R}^N$, we apply Proposition 3.2 with measure $\mu\lfloor_{\mathbb{R}^N \setminus \bigcup_{k=0}^{n-1} E_k}$ and parameter ϵ_n to obtain a Borel set $E_n \subset \mathbb{R}^N$ such that
(a') for every $c > 0$, there exists $\delta > 0$ such that $\mu\lfloor_{E_n \setminus \bigcup_{k=0}^{n-1} E_k} \le c\Lambda^h_\delta$,
(b') $\mu(\mathbb{R}^N \setminus \bigcup_{k=0}^{n} E_k) \le \epsilon_n$.

Let $F_0 = E_0$ and for $n \in \mathbb{N}_*$,

$$F_n = E_n \setminus \bigcup_{k=0}^{n-1} E_k.$$

By additivity of the measure μ, for every Borel set $A \subset \mathbb{R}^N$,

$$\mu(A) - \sum_{k=0}^{n} \mu(F_k \cap A) = \mu\big(A \setminus \bigcup_{k=0}^{n} E_k\big) \le \epsilon_n.$$

Choosing a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero, we deduce that the sequence $(F_n)_{n\in\mathbb{N}}$ has the required properties. □

We prove Theorem 1.4 using a strategy of Boccardo, Gallouët and Orsina [3, proof of Theorem 2.1] originally used to show that every finite measure which is diffuse with respect to the $W^{1,2}$ capacity belongs to $L^1 + (W^{1,2}_0)'$.

Proof of Theorem 1.4. Let μ be a finite measure in $\mathbb{R}^N$ such that for every Borel set $A \subset \mathbb{R}^N$, $\mathcal{H}^{N-1}(A) < +\infty$ implies $\mu(A) = 0$. By the Hahn decomposition theorem, the same property is still satisfied by the positive and the negative parts of μ. We may thus assume throughout the proof that μ is a nonnegative measure.

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of disjoint Borel sets given by the previous lemma. We decompose the measure μ as a series of finite measures

$$\mu = \sum_{k=0}^{\infty} \mu\lfloor_{F_k}.$$

By the characterization of nonnegative measures for which the equation (1.1) has a continuous solution [17, Theorem 4.5], for each $n \in \mathbb{N}$ there exists a continuous vector field $W_n : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\mu\lfloor_{F_n} = \operatorname{div} W_n \quad \text{in } \mathbb{R}^N.$$

Given a smooth mollifier ρ and $\delta > 0$, let $\rho_\delta : \mathbb{R}^N \to \mathbb{R}$ be the function defined for $x \in \mathbb{R}^N$ by $\rho_\delta(x) = \frac{1}{\delta^N}\rho(\frac{x}{\delta})$. We write

$$\begin{aligned}\mu\lfloor_{F_n} &= (\mu\lfloor_{F_n} - \rho_\delta * \mu\lfloor_{F_n}) + \rho_\delta * \mu\lfloor_{F_n}\\ &= \operatorname{div}(W_n - \rho_\delta * W_n) + \rho_\delta * \mu\lfloor_{F_n}.\end{aligned}$$

By Fubini's theorem,

$$\|\rho_{\delta_n} * \mu\lfloor_{F_n}\|_{L^1(\mathbb{R}^N)} = \|\mu\lfloor_{F_n}\|_{\mathcal{M}(\mathbb{R}^N)} = \mu(F_n).$$

Thus, the series

$$\sum_{k=0}^{\infty} \rho_{\delta_k} * \mu\lfloor_{F_k}$$

converges in $L^1(\mathbb{R}^N)$.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Since for each $n \in \mathbb{N}$, W_n is continuous in $\mathbb{R}^N$, the family $(\rho_\delta * W_n)_{\delta>0}$ converges to W_n uniformly to W_n on bounded subsets of $\mathbb{R}^N$. We take $\delta_n > 0$ so that

$$\|W_n - \rho_{\delta_n} * W_n\|_{L^\infty(B_n(0))} \leq \alpha_n.$$

Choosing $(\alpha_n)_{n\in\mathbb{N}}$ such that the series $\sum\limits_{k=0}^{\infty} \alpha_k$ converges, the series

$$\sum_{k=0}^{\infty} (W_k - \rho_{\delta_k} * W_k)$$

converges locally uniformly in $\mathbb{R}^N$.

For any $j \in \mathbb{N}$, we now write the measure μ as

$$\begin{aligned}\mu &= \sum_{k=0}^{j} \mu\lfloor_{F_k} + \sum_{k=j+1}^{\infty} \mu\lfloor_{F_k}\\ &= \operatorname{div}\Big(\sum_{k=0}^{j} W_k\Big) + \operatorname{div}\Big(\sum_{k=j+1}^{\infty} (W_k - \rho_{\delta_k} * W_k)\Big) + \sum_{k=j+1}^{\infty} \rho_{\delta_k} * \mu\lfloor_{F_k}.\end{aligned}$$

We thus have a decomposition of the form

$$\mu = \operatorname{div} V_j + f_j \quad \text{in } \mathbb{R}^N,$$

where the vector field $V_j : \mathbb{R}^N \to \mathbb{R}^N$ is continuous and $f_j \in L^1(\mathbb{R}^N)$ satisfies

$$\|f_j\|_{L^1(\mathbb{R}^N)} = \sum_{k=j+1}^{\infty} \|\rho_{\delta_k} * \mu\lfloor_{F_k}\|_{L^1(\mathbb{R}^N)} = \sum_{k=j+1}^{\infty} \mu(F_k).$$

Given $\epsilon > 0$, we have the conclusion by choosing $j \in \mathbb{N}$ such that

$$\sum_{k=j+1}^{\infty} \mu(F_k) \leq \epsilon.$$

The proof of the theorem is complete. □

5. Proof of Corollary 1.5

We first explain the main estimate we need based on a strategy from [9]. Let μ be a finite measure in $\mathbb{R}^N$ such that

$$\mu = f + \operatorname{div} V \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

where $f \in L^1_{\mathrm{loc}}(\Omega)$ and $V : \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Given $\varphi \in C_c^\infty(\mathbb{R}^N)$ and a smooth vector field $W : \mathbb{R}^N \to \mathbb{R}^N$, we write

$$\begin{aligned}\int_{\mathbb{R}^N} \varphi \,\mathrm{d}\mu &= \int_{\mathbb{R}^N} f\varphi - \int_{\mathbb{R}^N} V \cdot \nabla\varphi \\ &= \int_{\mathbb{R}^N} f\varphi - \int_{\mathbb{R}^N} (V - W) \cdot \nabla\varphi - \int_{\mathbb{R}^N} (\operatorname{div} W)\varphi.\end{aligned}$$

Thus,

(5.2)

$$\left|\int_{\mathbb{R}^N} \varphi \,\mathrm{d}\mu\right| \le \|f\|_{L^1(\operatorname{supp}\varphi)}\|\varphi\|_{L^\infty(\mathbb{R}^N)} + \|V - W\|_{L^\infty(\operatorname{supp}\varphi)}\|D\varphi\|_{L^1(\mathbb{R}^N)} + \|\operatorname{div} W\|_{L^\infty(\operatorname{supp}\varphi)}\|\varphi\|_{L^1(\mathbb{R}^N)}.$$

Proof of Corollary 1.5. If the measure μ satisfies property (1.3), then by Theorem 1.4 for every $\epsilon > 0$, μ can be written in the form (5.1) with

$$\|f\|_{L^1(\mathbb{R}^N)} \le \epsilon.$$

Let $K \subset \mathbb{R}^N$ be a compact set. Since the vector field V is continuous, by the Weierstrass approximation theorem there exists a smooth vector field $W : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\|V - W\|_{L^\infty(K)} \le \epsilon.$$

In view of estimate (5.2), for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ supported in K,

$$\left|\int_{\mathbb{R}^N} \varphi \,\mathrm{d}\mu\right| \le \epsilon\big(\|\varphi\|_{L^\infty(\mathbb{R}^N)} + \|D\varphi\|_{L^1(\mathbb{R}^N)}\big) + \|\operatorname{div} W\|_{L^\infty(K)}\|\varphi\|_{L^1(\mathbb{R}^N)}.$$

Thus, μ satisfies the definition of charge with constant $C = \|\operatorname{div} W\|_{L^\infty(K)}$.

Conversely, if μ is a charge, then by [9, Theorem 6.2] there exists $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a strong charge ν such that

$$\mu = f + \nu \quad \text{in } \mathbb{R}^N.$$

Note that f satisfies property (1.3) and ν is a locally finite measure in $\mathbb{R}^N$. We need to show that ν also satisfies property (1.3). By the characterization of charges by De Pauw and Pfeffer [9, Theorem 4.8], there exists a continuous vector field $V : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\nu = \operatorname{div} V \quad \text{in } \mathbb{R}^N.$$

Let $W : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth vector field. By estimate (5.2) with $f = 0$, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have

$$\left|\int_{\mathbb{R}^N} \varphi \,\mathrm{d}\nu\right| \le \|V - W\|_{L^\infty(\operatorname{supp}\varphi)}\|D\varphi\|_{L^1(\mathbb{R}^N)} + \|\operatorname{div} W\|_{L^\infty(\operatorname{supp}\varphi)}\|\varphi\|_{L^1(\mathbb{R}^N)}.$$

The rest of the proof is based on a standard covering argument. Indeed, given a compact set $K \subset \mathbb{R}^N$ such that $\mathcal{H}^{N-1}(K) < +\infty$, let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^N)$ such that

(a) $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^N)$,

(b) $(D\varphi_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}^N)$,
(c) $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwisely to 0 in $\mathbb{R}^N \setminus K$,
(d) for every $n \in \mathbb{N}$, $\varphi_n \geq 1$ in K,
(e) there exists $R > 0$ such that for every $n \in \mathbb{N}$, $\operatorname{supp}\varphi_n \subset B[0; R]$.

The construction of the sequence $(\varphi_n)_{n\in\mathbb{N}}$ can be found for instance in [8, Lemma 3.2]; the constant $C > 0$ such that for every $n \in \mathbb{N}$,

$$\|D\varphi_n\|_{L^1(\mathbb{R}^N)} \leq C$$

can be chosen to be of the order of $\mathcal{H}^{N-1}(K)$ if $\mathcal{H}^{N-1}(K) > 0$.

Denote by $(v_n)_{n\in\mathbb{N}}$ the sequence obtained by truncating each function φ_n at level 1. It follows by approximation of v_n by smooth functions that for every $n \in \mathbb{N}$,

$$\left|\int_{\mathbb{R}^N} v_n \, \mathrm{d}\nu\right| \leq \|V - W\|_{L^\infty(\operatorname{supp}\varphi)} \|Dv_n\|_{L^1(\mathbb{R}^N)} + \|\operatorname{div} W\|_{L^\infty(\operatorname{supp}\varphi)} \|v_n\|_{L^1(\mathbb{R}^N)}.$$

Thus,

$$\left|\int_{\mathbb{R}^N} v_n \, \mathrm{d}\nu\right| \leq C\|V - W\|_{L^\infty(B[0;R])} + \|\operatorname{div} W\|_{L^\infty(B[0;R])} \|\varphi_n\|_{L^1(\mathbb{R}^N)}.$$

As n tends to infinity, we get by the dominated convergence theorem,

$$|\nu(K)| \leq C\|V - W\|_{L^\infty(B[0;R])}.$$

By the Weierstrass approximation theorem, for every $\epsilon > 0$ we may choose W such that

$$C\|V - W\|_{L^\infty(B[0;R])} \leq \epsilon.$$

Thus,

$$|\nu(K)| \leq \epsilon.$$

Since ϵ is arbitrary, we deduce that $\nu(K) = 0$. By inner regularity of the measure ν, property (1.3) is satisfied by ν for every Borel set, not necessarily compact. □

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