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A TYPE (4) SPACE IN (FR)-CLASSIFICATION

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Abstract. We present a reflexive Banach space with an unconditional basis which is quasi-minimal and tight by range, i.e. of type (4) in Ferenczi-Rosendal list within the framework of Gowers' classification program of Banach spaces. The space is an unconditional variant of the Gowers Hereditarily Indecomposable space with asymptotically unconditional basis.

Introduction

In the celebrated papers [11, 12] W.T. Gowers started his classification program for Banach spaces. The goal is to identify classes of Banach spaces which are

- hereditary, i.e. if a space belongs to a given class, then all of its closed infinite dimensional subspaces belong to the same class as well as well,
- inevitable, i.e. any Banach space contains an infinite dimensional subspace in one of those classes,
- defined in terms of richness of family of bounded operators in the space.

The famous Gowers' dichotomy brought the first two classes: spaces with an unconditional basis and hereditary indecomposable spaces. Recall that a space is called *hereditarily indecomposable* (HI) if none of its infinite dimensional subspaces can be written as a direct sum of two closed infinite dimensional subspaces.

Further classes were defined in terms of the family of isomorphisms defined in a space. Recall that a Banach space is minimal if it embeds isomorphically into any of its closed infinite dimensional subspaces. Relaxing of this notion on obtains quasi-minimality, which asserts that any two subspaces of a given space contain further two isomorphic subspace. W.T. Gowers obtained a dichotomy between quasi-minimality and tightness by support in [12]. The latter notion, among other types of tightness, was explicity defined and studied in [6]. Recall that a subspace Y of a Banach space X with a basis (e_n) is *tight in* X iff there is a sequence of successive subsets $I_1 < I_2 < \dots$ of $\mathbb{N}$ such that the support of any isomorphic copy of Y in X intersects all but finitely many I_n's. X is called *tight* if any of its subspaces is tight in X. Adding requirements on the subsets (I_n) with respect to the given Y one obtains more specific notions, in particular in *tightness by support* mentioned above the subsets witnessing tightness of a subspace Y spanned by a block sequence (x_n) are chosen to be supports of (x_n) [12].

V.Ferenczi and C.Rosendal have presented in [6] further dichotomies refining Gowers list of classes: the "third dichotomy" contrasting tightness with minimality and the "forth dichotomy" between *tightness by range*, where the subsets witnessing the tightness of a subspace Y spanned by a block sequence (x_n) are chosen to

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be ranges of (x_n), with a stronger form of quasi-minimality, namely sequential minimality. A Banach space X is *sequentially minimal* if it is quasi-minimal and is block saturated with block sequences (x_n) with the following property: any subspace of X contains a sequence equivalent to a subsequence of (x_n).

The obvious observations relate some of the properties listed above to HI/unconditional dichotomy - in particular clearly any HI space is quasi-minimal and any tight basis is unconditional. V.Ferenczi and C.Rosendal in [7] studied the spaces already known identifying their properties with respect to the dichotomies mentioned above. Their study left open two particular cases. Namely, an HI and sequentially minimal space and also a quasi-minimal and tight by range space with an unconditional basis. The answer to the first question was provided by a version of Gowers-Maurey HI space, as it was proved by V.Ferenczi and Th.Schlumprecht recently [8]. We recall now the list of classes developed in [6] as stated in [8], mentioning also some already known examples.

Theorem 0.1 ((FR)-classification)**.** *Any infinite dimensional Banach space contains a subspace from one of the following classes:*

1. *HI, tight by range (Gowers space with asymptotically unconditional basis* [9, 7]*),*
2. *HI, tight, sequentially minimal (a version of Gowers-Maurey space,* [8]*),*
3. *tight by support (Gowers space with unconditional basis* [10, 6]*),*
4. *with unconditional basis, tight by range, quasi-minimal (?),*
5. *with unconditional basis, tight, sequentially minimal (Tsirelson space* [7]*),*
6. *with unconditional basis, minimal (ℓ_p, c_0, dual to Tsirelson space* [5]*, Schlumprecht space* [1]*)*

The aim of the present paper is to construct a reflexive space of type (4) in the above classification. Namely the following is proven.

Theorem 0.2. *There exists a reflexive space $\mathcal{X}_{(4)}$ with an unconditional basis which is quasi-minimal and tight by range.*

As we have mentioned the space $\mathcal{X}_{(4)}$ is the unconditional version of Gowers HI space which is asymptotically unconditional [10]. Banach spaces with an unconditional basis which are variants of HI spaces have occured with Gowers' solution of the hyperplane problem [9] and were followed by the most recent [2, 3]. Among the features of those spaces is the non homogeneous structure. For example in all [9, 2, 3] the spaces are tight by support. The new phenomenon in the present construction is that the space $\mathcal{X}_{(4)}$ is quasi-minimal. This is a consequence of the definition of the norming set W, which is slightly different from the initial Gowers definition, in the following manner. Starting with an appropriately chosen double sequence $(m_j, n_j)_j$ we consider the following norming sets W_1, W_2.

The set W_1 is the smallest subset of $c_{00}(\mathbb{N})$ satisfying

(i) W_1 contains $(e_n)_n$
(ii) For every $f \in W_1$ and $g \in c_{00}$ with $|g| = |f|$, then $g \in W_1$
(iii) It is closed in the projections on the subsets of $\mathbb{N}$
(iv) It is closed in the even operations $(\frac{1}{m_{2j}}, \mathcal{A}_{n_{2j}})$
(v) It is closed in the odd operations $(\frac{1}{m_{2j+1}}, \mathcal{A}_{n_{2j+1}})$ on special sequences $f_1, f_2, \ldots, f_{n_{2j+1}}$ (Here $f_1, f_2, \ldots, f_{n_{2j+1}}$ is a special sequence if the weight

if each f_i is even and for $1 < i$ the weight of f_i is uniquely determined by the sequence $|f_1|, |f_2|, \dots, |f_{i-1}|$)

Let $\|\cdot\|_1$ be the norm induced on $c_{00}(\mathbb{N})$ by the set W_1 and $\mathfrak{X}_1$ its completion. Then the space $\mathfrak{X}_1$ is reflexive with a 1-unconditional basis, tight by support (hence not quasi-minimal) and shares all the properties of Gowers space [9].

Consider next the norming set W_2 which satisfies properties (i), (ii), (iv), (v) of W_1 and the following

(iii)′ The set W_2 is closed in the projections of its elements on intervals of $\mathbb{N}$.

Denoting by $\|\cdot\|_2$ the norm induced by W_2 and $\mathfrak{X}_2$ the corresponding completion, the space $\mathfrak{X}_2$ is reflexive with a 1-unconditional basis and quasi-minimal.

This key difference between $\mathfrak{X}_1$ and $\mathfrak{X}_2$ permits the construction of the space $\mathcal{X}_{(4)}$. The norming set W of the space $\mathcal{X}_{(4)}$ is the smallest subset of c_{00} satisfying all the properties of the set W_2 and an additional one, called "Gowers operation" which is used to show that the space is tight by range.

It is unclear to us what the structure of the space of the operators $\mathcal{L}(\mathcal{X}_{(4)})$ is. We recall from [14] that every bounded linear operator on $\mathfrak{X}_1$ is of the form $D + S$ with D a diagonal operator and S a strictly singular one. Such a property seems to fail for the space $\mathfrak{X}_2$.

We describe now briefly the content of the paper. The first section is devoted to the definition of the norming set of the space $\mathcal{X}_4$. The second section contains the basic estimations, providing tools to be used in the last two sections in order to show quasi-minimality and tightness by range of the space $\mathcal{X}_{(4)}$.

1. The norming set W

Let us recall the usual basic notation. Let X be a Banach space with basis (e_i). The *support* of a vector $x = \sum_i x_i e_i$ is the set $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$, the *range* of x - the minimal interval containing $\operatorname{supp} x$. Given any $x = \sum_i a_i e_i$ and finite $E \subset \mathbb{N}$ put $Ex = \sum_{i \in E} a_i e_i$ and $|x| = \sum_i |a_i| e_i$. We write $x < y$ for vectors $x, y \in X$, if $\max \operatorname{supp} x < \min \operatorname{supp} y$. A *block sequence* is any sequence $(x_i) \subset X$ satisfying $x_1 < x_2 < \dots$, a *block subspace* of X - any closed subspace spanned by an infinite block sequence. Given a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ we say that a block sequence $(x_i)_{i=1}^d$ is $\mathcal{F}$-*admissible* if $(\operatorname{minsupp}(x_i))_{i=1}^d \in \mathcal{F}$. By the $(\theta, \mathcal{F})$-*operation*, for $\theta \in (0,1]$, we mean an operation which associates with any $\mathcal{F}$-admissible sequence $(x_1, \dots, x_d)$ the average $\theta(x_1 + \dots + x_d)$.

We define the space $\mathcal{X}_{(4)}$ to be the completion of $(c_{00}(\mathbb{N})$ under the norm $\|\cdot\|$ given by some set $W \subset c_{00}(\mathbb{N})$, described below, as the norming set. (i.e. $\|x\| = \sup\{f(x) : f \in W\}$ for $x \in c_{00}(\mathbb{N})$).

To define the set W we fix two sequences of natural numbers $(m_j)_j$ and $(n_j)_j$ defined recursively as follows.

We set $m_1 = 2$ and $m_{j+1} = m_j^5$ and $n_1 = 4$ and $n_{j+1} = (5n_j)^{s_j}$ where $s_j = \log_2(m_{j+1}^3)$, $j \geq 1$. We also fix a partition of $\mathbb{N}$ into two infinite sets N_1, N_2.

The set W is defined to be the smallest subset of $c_{00}(\mathbb{N})$ satisfying the following properties

α) It is unconditional (i.e. for $f \in W$, $g \in c_{00}(\mathbb{N})$ with $|g| = |f|$ we have $g \in W$).

β) It contains $(\pm e_n^*)_n$, where $(e_n^*)_n$ is the usual basis of $c_{00}(\mathbb{N})$.

γ) It is closed on the interval projections.

δ) It is closed under the $(\frac{1}{m_{2j}}, \mathcal{A}_{n_{2j}})$-operations (i.e. for every $f_1 < f_2 < \cdots < f_{n_{2j}}$ in W the functional $f = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i \in W$).

ε) It is closed under the $(\frac{1}{m_{2j+1}}, \mathcal{A}_{n_{2j+1}})$-operations on $(2j+1)$-special sequences.

A sequence $f_1 < f_2 < \cdots < f_{n_{2q+1}}$ in W is a $(2j+1)$-special sequence if the following are satisfied

(1) $n_{2j+1} < m_{2j_1} < \cdots < m_{2j_{2j+1}}$,
(2) $w(f_1) = m_{2j_1}$ for some $j_1 \in N_1$,
(3) $w(f_i) = m_{2\sigma(|f_1|,\ldots,|f_{i-1}|)}$ for any $1 < i \le n_{2j+1}$.
(4) For $1 < i \le 2q+1$ the sequence $(|f_1|, |f_2|, \ldots, |f_{i-1}|)$ is uniquely determined by $w(f_i)$

The special sequences can be defined in a similar manner as in [13],[4], with the use of a coding function σ.

A functional $f = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i$ with $(f_1, \ldots, f_{n_{2j+1}})$ an $(2j+1)$-special sequence is called a $(2j+1)$-special functional.

ζ) It is closed under the G-operation, defined as follows.
For any set $F = \{n_1 < \cdots < n_{2q}\} \subset \mathbb{N}$ which is Schreier (i.e. $2q \le n_1$) we set

$$S_F f = \chi_{\cup_{p=1}^{q}[n_{2p-1}, n_{2p})} f.$$

The G-operation associates with any $f \in c_{00}$ the vector $g = \frac{1}{2} S_F f$, for any F as above.

Remarks 1.1. (i) Clearly the natural basis $(e_n)_n$ is 1-unconditional in $\mathcal{X}_4$. Moreover, standard argument shows that $\mathcal{X}$ is a reflexive space.

(ii) The space $\mathcal{X}_{(4)}$ is an unconditional variant of W.T. Gowers, [10], HI space with an asymptotically unconditional basis. The key ingredient in Gowers construction beyond the standard ones is an operation similar to ζ). V.Ferenczi and C.Rosendal, [7], have shown that the Gowers space is tight by range.

(iii) It is worth pointing out that the quasi-minimal property of $\mathcal{X}_{(4)}$ is a result of the fact that the set W is not closed in rational convex combinations. Indeed if we include the rational convex combinations in the set W (even if we exclude property ζ)) we will get a space similar to Gowers space with an unconditional basis [9] which is tight by support and hence not quasi-minimal [7].

1.1. **The analysis of a norming functional.** As in the previous cases of norming sets defined to be closed under certain operations every functional $f \in W$ admits a tree-analysis which in the present case is described as follows.

Definition 1.2. Let $f \in W$. A family $(f_\alpha)_{\alpha \in \mathcal{A}}$ with $\mathcal{A}$ is a rooted finite tree of finite sequences of $\mathbb{N}$ is a tree-analysis of f if the following are satisfied

1) $f = f_0$ where 0 denotes the root of f.
2) If α is maximal element of $\mathcal{A}$ then $f_\alpha = \varepsilon e_n^*$ for some $\varepsilon = 1$ or -1 and $n \in \mathbb{N}$.
If $\alpha \in \mathcal{A}$ is not maximal, then one of the following conditions hold
3) $f_\alpha = \frac{1}{m_j} \sum_{\beta \in S_\alpha} f_\beta$ where $f_\alpha = E_\alpha \tilde{f}_\alpha$, E_α interval of $\mathbb{N}$, $\tilde{f}_\alpha = \frac{1}{m_j} \sum_{i=1}^{n_j} f_i$, $S_\alpha = \{(a, i) : E_\alpha f_i \neq 0\}$ and $f_\beta = E_\alpha f_i$ for $\beta = (\alpha, i)$. In this case we set the weight $w(f_\alpha)$ of f_α to be $w(f_\alpha) = m_j$.
4) $f_\alpha = \frac{1}{2} S_{F_\alpha} f_\beta$ with $\beta = (\alpha, 1)$, $S_\alpha = \{\beta\}$, F_α Schreier and range$(f_\alpha) =$ range(f_β).

Since the $(2j+1)$-special sequences $(f_i)_{i\leq n_{2j+1}}$ is determined by $(|f_i|)_{i\leq n_{2j+1}}$ it is easy to see the following

Lemma 1.3. *Let $f \in W$ with a tree-analysis $(f_\alpha)_{a\in\mathcal{A}}$ and $g \in W$ with $|g| = |f|$. Then g admits a tree analysis $(g_\alpha)_{\alpha\in\mathcal{A}}$ such that $|g_\alpha| = |f_\alpha|$ for all $\alpha \in \mathcal{A}$. In particular if f_α is a weighted functional then g_α is also weighted functional and $w(f_\alpha) = w(g_\alpha)$. If f_α is a G-functional, i.e. of the form $f_\alpha = \frac{1}{2}S_{F_\alpha} f_\beta$ then also $g_\alpha = \frac{1}{2}S_{F_\alpha} g_\beta$.*

The following follows easily.

Lemma 1.4. *Let $f \in W$ with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$. Let also $D \subset \mathcal{A}$ be a set of incomparable nodes of $\mathcal{A}$ and for every $\alpha \in D$ let $g_\alpha \in W$ such that $|g_\alpha| = |f_\alpha|$. Then there exists $g \in W$ satisfying*

i) $|g| = |f|$
ii) *g admits a tree analysis $(\tilde{g}_\alpha)_{\alpha\in\mathcal{A}}$ such that for every $\alpha \in \mathcal{A}$, $|\tilde{g}_\alpha| = |f_\alpha|$ and for every $\alpha \in D$, $\tilde{g}_\alpha = g_\alpha$.*

2. Basic estimations

In this section we shall give the definition of some special vectors as well as estimations of the functionals of the norming set on these special vectors. All the definitions and estimations have appeared in a series of papers, [15, 13, 4], so for the proofs we shall refer to a paper where the corresponding result has appeared.

2.1. Special vectors.

Definition 2.1. A $C - \ell_1^n$-average, $C \geq 1$, $n \in \mathbb{N}$, is a vector $x = \frac{x_1+\cdots+x_n}{n}$ where $\|x_i\| \leq C$, $\|x\| > 1$ and $n \leq x_1 < x_2 < \cdots < x_n$.

Lemma 2.2. *Let $(x_k)_k$ be a normalized block sequence. Then for every $n \in \mathbb{N}$ there exists $l(n) \in N$ such that for every finite subsequence $(x_n)_{n\in F}$ with $\#F \geq l(n)$ of $(x_k)_k$ there exists a block sequence $y_1 < y_2 < \cdots < y_n$ of $(x_k)_{k\in F}$ such that $\frac{y_1+\cdots+y_n}{n} = \sum_{n\in F} a_n x_n$ is an $2 - \ell_1^n$-average.*

The proof of the above lemma originates from [15, 13]. For a proof we refer to [4], Lemma II.22.

Definition 2.3. A block sequence $(x_k)_k$ is said to be a (C, ε) rapidly increasing sequence (RIS) if $\|x_k\| \leq C$ for each k and there exists a strictly increasing sequence (j_k) of positive integers such that

(1) $\max \text{range}(x_k) m_{j_{k+1}}^{-1} < \varepsilon$,
(2) for every $k = 1, 2, \ldots$ and every $f \in W$ with $w(f) = m_i < m_{j_k}$ the following holds, $|f(x_k)| \leq \frac{C}{m_i}$.

Definition 2.4. A pair (x, ϕ) with $x \in \mathcal{X}_{(4)}$ and $\phi \in W$ is said to be a $(C, 2j)$-exact pair, where $C \geq 1$, $j \in \mathbb{N}$, if the following conditions holds

(1) $1 \leq \|x\| \leq C$, for every $\psi \in W$ with $w(\psi) < m_{2j}$ we have $|\psi(x)| \leq \frac{3C}{w(\psi)}$, while for $\psi \in W$ with $w(\psi) > m_{2j}$, $|\psi(x)| \leq \frac{C}{m_{2j}^2}$,
(2) $w(\phi) = m_{2j}$,
(3) $\phi(x) = 1$ and $\text{range}(x) = \text{range}(\phi)$.

Definition 2.5. We shall call the sequence $(x_i, x_i^*)_{i=1}^{n_{2j+1}}$ a $(C, 2j+1)$-dependent sequence if

1. $j_1 \in N_1$ and $m_{2j_1} \geq n_{2j+1}$,
2. for every $i \leq n_{2j+2}$, (x_i, x_i^*) is an $(C, 2j_i)$-exact pair,
3. $(x_1^*, \ldots, x_{n_{2j+1}}^*)$-is a special sequence.

2.2. **Basic estimations.**

Lemma 2.6. *Let x be a $2-\ell_1^{n_j}$-average. Then for every $f \in W$ with $w(f) = m_i < m_{j_k}$ the following holds*

$$|f(x_k)| \leq 3\frac{1}{m_i}. \tag{2.1}$$

We refer to [15],[13] Lemma 5, [4] Lemma II.23 for the proof.

The following result follows from Lemmas 2.6 and 2.2

Proposition 2.7. *For every $\varepsilon > 0$ and every block subspace Z of $\mathcal{X}_{(4)}$ there exists a $(3, \varepsilon)$-RIS $(x_k)_k$ in Z.*

The following proposition will be the main tool for the estimations we shall need in the sequel. For the proof we refer to [4], Propositions II.14, II.19.

Proposition 2.8. *Let $(x_k)_{k=1}^{n_j}$ be a (C, ε)-RIS with $\varepsilon \leq m_j^{-2}$ and $f \in W$. Then*

$$|f(\frac{m_j}{n_j}\sum_{k=1}^{n_j} x_k)| \leq \begin{cases} 3Cw(f)^{-1} & \text{if } \ w(f) < m_j \\ C(w(f)^{-1}m_j + \frac{m_j}{n_j} + m_j\varepsilon) & \text{if } \ w(f) \geq m_j \end{cases} \tag{2.2}$$

In particular $\|\frac{m_j}{n_j}\sum_{k=1}^{n_j} x_k\| \leq 2C$.

Moreover if $(f_\alpha)_{\alpha \in \mathcal{A}}$ is a tree analysis of f and for every $\alpha \in \mathcal{A}$ with $w(f_\alpha) = m_j$ and every interval E of positive integers we have that

$$|f_\alpha(\sum_{k \in E} x_k)| \leq C(1 + \varepsilon \# E)$$

then $|f(\frac{m_j}{n_j}\sum_{k=1}^{n_j} y_k)| \leq \frac{4C}{m_j}$.

Proposition 2.8 yields the following

Proposition 2.9. *For every block subspace Z, every $\varepsilon > 0$ and $j \in \mathbb{N}$ there exists a $(6, 2j)$-exact pair (x, ϕ) with $x \in Z$.*

Proof. From Proposition 2.7 there exists $(x_k)_{k=1}^{n_{2j}}$ $(3, \varepsilon)$ RIS with $\varepsilon < 1/m_{2j}^5$. Choose $x_k^* \in W$ with $x_k^*(x_k) = 1$ and range(x_k^*) = range(x_k). Then Proposition 2.8 yields that

$$(\frac{m_j}{n_j}\sum_{k=1}^{n_{2j}} x_k, \frac{1}{m_{2j}}\sum_{i=1}^{n_{2j}} x_k^*)$$

is an $(6, 2j)$-exact pair. □

Corollary 2.10. *Let $(x_i, x_i^*)_{i=1}^{n_{2j+1}}$ be a $(6, 2j+1)$-dependent sequence and $f = \frac{1}{m_{2j+1}} E\sum_{r=1}^{n_{2k+1}} f_r$ a special functional such that $w(f_r) \neq w(x_i^*)$ for every $i, r \leq n_{2j+1}$. Then*

$$|f(\sum_{i=1}^{n_{2j+1}} x_i)| \leq \frac{1}{m_{2j+1}m_{2j+2}^2}. \tag{2.3}$$

Proof. For every $i \leq n_{2j+1}$ set

$$R_{i,1} = \{r \leq n_{2j+1} : \mathrm{range}(f_r) \cap \mathrm{range}(x_i) \neq \emptyset \,\mathrm{and}\, w(f_r) < m_{2j_i}\}$$

and

$$R_{i,2} = \{r \leq n_{2j+1} : \mathrm{range}(f_r) \cap \mathrm{range}(x_i) \neq \emptyset \,\mathrm{and}\, w(f_r) > m_{2j_i}\}.$$

Note that for every r there exists at most two i's such that $r \in R_{i,1}$ and $\mathrm{range}(x_i) \subsetneqq \mathrm{range}(f_r)$. From (2.2) we get

$$|\sum_{r \in R_{i,1}} f_r(x_i)| \leq \sum_{r \in R_{i,1}} \frac{18}{w(f_r)}. \tag{2.4}$$

and

$$|\sum_{r \in R_{i,2}} f_r(x_i)| \leq \frac{6}{m_{2j_i}^2} \# R_{i,2} \tag{2.5}$$

Using that $w(f_r) \geq m_{2j_1} \geq n_{2j+1}$, by (2.4),(2.5) we finish the proof. □

3. The quasi-minimality

We shall prove the quasi-minimality in two steps. In the first step we shall handle a special case. More precisely we shall consider block sequences $(y_k)_{k=1}^{n_{2j+1}}$, $(z_k)_{k=1}^{n_{2j+1}}$ such that $x_k = y_k + z_k$, $k \in \mathbb{N}$ for some dependent sequence $(x_k, f_k)_{k=1}^{n_{2j+1}}$. For a suitable splittings of (x_k) we show that for every $f \in W$ there exists $g \in W$ such that $\frac{1}{2} f(\frac{m_{2j+1}}{n_{2j+1}} \sum_k y_k) - 2\frac{31}{m_{2j+1}} \leq g(\frac{m_{2j+1}}{n_{2j+1}} \sum_k z_k))$.

In the second step we prove the quasi-minimality of $\mathcal{X}_4$ basing on the first step.

Let $(x_k, f_k)_{k=1}^{n_{2j+1}}$ be a $(6, 2j+1)$-dependent sequence such that each exact pair (x_k, f_k) is of the form as in the proof of Proposition 2.9. Split each x_k and f_k as follows.

$$x_k = y_k + z_k = \frac{m_{2j_k}}{n_{2j_k}} \sum_{i=1}^{n_{2j_k}/2} (y_{k,i} + z_{k,i}), \quad f_k = \frac{1}{m_{2j_k}} \sum_{i=1}^{n_{2j_k}/2} (y_{k,i}^* + z_{k,i}^*),$$

where for every i, $y_{k,i} < z_{k,i} < y_{k,i+1}$, $y_{k,i}^*(y_{k,i}) = 1 = z_{k,i}^*(z_{k,i})$ and $\mathrm{range}(y_{k,i}^*) = \mathrm{range}(y_{k,i})$, $\mathrm{range}(z_{k,i}^*) = \mathrm{range}(z_{k,i})$.

Set

$$y = \frac{m_{2j+1}}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} y_k = \frac{m_{2j+1}}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} \frac{m_{2j_k}}{n_{2j_k}} \sum_{i=1}^{n_{2j_k}/2} y_{k,i}$$

and

$$z = \frac{m_{2j+1}}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} z_k = \frac{m_{2j+1}}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} \frac{m_{2j_k}}{n_{2j_k}} \sum_{i=1}^{n_{2j_k}/2} z_{k,i}$$

Proposition 3.1. *Let y, z be as above. For all $f \in W$ there exists $g \in W$ such that $|f| = |g|$ and*

$$g(z) \geq \frac{1}{2} f(y) - 2\frac{31}{m_{2j+1}} \tag{3.1}$$

Proof. Let $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree analysis of f. Set I_1 to be the set of $k \in \{1, \ldots, n_{2j+1}\}$ such that there exists $\alpha_k \in \mathcal{A}$ with $w(f_{\alpha_k}) = m_{2j+1}^{-1}$, $f_{\alpha_k} = E_{\alpha_k} \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i^{\alpha_k}$ be a special functional satisfying

A) $\mathrm{range}(x_k) \subset E_{\alpha_k}$.
B) $|f_k^{\alpha_k}| = |f_k|$.

C) $\prod_{\beta\prec\alpha_k} w(f_\beta) < m_{2j+1}$.

We define I_2 as I_1 with the exception that C) is replaced by

C_1) $\prod_{\beta\prec\alpha_k} w(f_\beta) \geq m_{2j+1}$.

The complement of $I_1 \cup I_2$ is denoted by I_3. Set

$$w_i = \frac{m_{2j+1}}{n_{2j+1}} \sum_{k\in I_i} x_k \ \text{ for } i = 1, 2, 3.$$

We shall estimate f on w_1, w_2, w_3.

Lemma 3.2. *The following holds*

$$|f(w_3)| < \frac{24}{m_{2j+1}}. \tag{3.2}$$

Proof. By Proposition 2.8 it is enough to show that for every $\alpha \in \mathcal{A}$ with $w(f_\alpha) = m_{2j+1}$ and every interval E the following holds

$$|f_\alpha(\sum_{k\in I_3\cap E} x_k)| < 6(1 + \#E/m^2_{2j+1}). \tag{3.3}$$

Note that if $k \in I_3$, $\alpha \in \mathcal{A}$ with $f_\alpha = m^{-1}_{2j+1} E_\alpha \sum_{k=1}^{n_{2j+1}} f^\alpha_k$ and $\text{range}(f_\alpha) \cap \text{range}(x_k) \neq \emptyset$

$$\text{either } |f^\alpha_k| \neq |f_k| \text{ or } |f^\alpha_k| = |f_k| \text{ and } \text{range}(x_k) \not\subseteq E_\alpha.$$

Let $k_0 = \min\{k : \text{range}(f_\alpha) \cap \text{range}(x_k) \neq \emptyset\}$ and $k_1 = \min \Sigma$ where

$$\Sigma = \{k \leq n_{2j+1} : |f^\alpha_k| \neq |f_k| \text{ and } \text{range}(f_\alpha) \cap \text{range}(x_k) \neq \emptyset\}$$

If $\Sigma = \emptyset$ it follows that $\text{range}(f_\alpha)\cap\text{range}(x_k) \neq \emptyset$ for at most two k and hence (3.3) holds. For every $k > k_1$ we have that $w(f^\alpha_i) \neq m_{2j_k}$ for every i. Corollary 2.10 yields that

$$|f_\alpha(\sum_{k>k_1} x_k)| \leq \frac{1}{m^2_{2j+2} m_{2j+1}}. \tag{3.4}$$

For all $k_0 < k < k_1$, $k \in (I_1 \cup I_2)$ except maybe for k_0. Indeed assume that some $k_0 < k < k_1$ is not in $(I_1 \cup I_2)$. Then by the definition of k_1 it follows that $|f^\alpha_k| = |f_k|$ and $\text{range}(x_k) \not\subseteq \text{range}(E_\alpha)$, which yields a contradiction since $k \in I_3$ and $\text{range}(f^\alpha_k) = \text{range}(f_k) \subset \text{range}(x_k) \subset \text{range}(E_\alpha)$.

Thus for k_0 (or k_1) there exists at most one i with $w(f^\alpha_i) = m_{2j_{k_0}}$ (or $w(f^\alpha_i) = m_{2j_{k_1}}$), hence

$$|f_\alpha(x_{k_0} + x_{k_1})| \leq \frac{12}{m_{2j+1}} + \frac{1}{m_{2j+1} m^2_{2j+2}}. \tag{3.5}$$

To finish the proof note that (3.4) and (3.5) yield (3.3). ☐

Lemma 3.3. *The following holds*

$$|f(w_2)| = |f(\frac{m_{2j+1}}{n_{2j+1}} \sum_{k\in I_2} x_k)| \leq \frac{6}{m_{2j+1}}. \tag{3.6}$$

Proof. Note that for $k \in I_2$ it holds that $\text{range}(f_k) \subset \text{range}(x_k)$ and $|f^{\alpha_k}_k| = |f_k|$. It follows

$$|f_{\alpha_k}(\frac{m_{2j+1}}{n_{2j+1}} x_k)| = \frac{1}{m_{2j+1}} \frac{m_{2j+1}}{n_{2j+1}} |f_k(x_k)| \leq \frac{6}{n_{2j+1}}.$$

Property C_1) implies that $|f(\frac{m_{2j+1}}{n_{2j+1}}x_k)| \leq \frac{6}{n_{2j+1}m_{2j+1}}$. Summing over I_2 we obtain (3.6). □

It remains to estimate f on w_1. We shall consider a partition of w_1 into three vectors which is imposed by the G-functionals.

Let $\alpha \in \mathcal{A}$ such that $\alpha \prec \alpha_k$ for some $k \in I_1$ and $f_\alpha = \frac{1}{2}Sf_\beta$ is G-functional determined by the intervals $q_\alpha \leq n_1^\alpha < \cdots < n_{q_\alpha}^\alpha$. We set

$$L_1^\alpha = \{(k,i) : k \in I_1, \alpha \prec \alpha_k \,\text{and exists}\, d \leq q_\alpha \,\text{with}\, n_d^\alpha \in \text{range}(y_{k,i}+z_{k,i}) \,\text{for some}\, i\},$$
$$L_2^\alpha = \{(k,i) : k \in I_1, \alpha \prec \alpha_k \,\text{and exists}\, d \leq q_\alpha/2 \,\text{with range}(y_{k,i}+z_{k,i}) \subset (n_{2d}^\alpha, n_{2d+1}^\alpha)\},$$
$$L_3^\alpha = \{(k,i) : k \in I_1, \alpha \prec \alpha_k \text{ and } (k,i) \notin L_1^\alpha \cup L_2^\alpha\}.$$

We also set

$$\Gamma = \{\alpha \in \mathcal{A} : \alpha \prec \alpha_k \,\text{for some}\, k \in I_1 \text{ and } f_\alpha \text{ is } G\text{-functional}\},$$

and $L_i = \cup_{\alpha\in\Gamma} L_i^\alpha$ for $i = 1,2,3$. Without loss of generality we can assume that these sets define a partition of the whole vector w_1.

Remark 3.4. An easy inductive argument yields that for every $(k,i) \in L_3$ and every $\alpha \prec \alpha_k$ we have that $\text{supp}(y_{k,i}+z_{k,i}) \cap \text{supp}(f_\alpha) = \text{supp}(y_{k,i}+z_{k,i}) \cap \text{supp}(f_{\alpha_k})$.

Indeed, for a G-functional f_α the above follows from the definition of the set L_3^α, as f_α has one successor f_β and satisfies $f_\alpha(y_{k,i}+z_{k,i}) = \frac{1}{2}f_\beta(x_{k,i}+z_{k,i})$.

If f_α is weighted functional then there exists a unique $\beta \in S_\alpha$ such that $\text{supp}(y_{k,i}+z_{k,i}) \subset \text{supp}(f_\beta)$ and $f_\alpha(y_{k,i}+z_{k,i}) = \frac{1}{w(f_\alpha)}f_\beta(x_{k,i}+z_{k,i})$.

The following lemma give us an upper bound for the cardinality of the set Γ.

Lemma 3.5. *Let $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a tree analysis of a functional. Let*

$$B = \{\alpha \in \mathcal{A} : \prod_{\beta \precneqq \alpha} w(f_\beta) < m_{2j+1}\}$$

Then $\#B \leq (5n_{2j})^{\log_2(m_{2j+1})-1} \leq n_{2j+1}^{1/3}$.

For the proof we refer to the proof of Lemma II.9, [4].

The sets L_1^α, L_2^α and L_3^α implies the following partition of w_1.

$$u_i = \frac{m_{2j+1}}{n_{2j+1}} \sum_{k\in I_1} \frac{m_{2j_k}}{n_{2j_k}} \sum_{(k,i)\in L_i} (y_{k,i}+z_{k,i}),\ i = 1,2,3.$$

Lemma 3.6. *We have that $f(u_2) = 0$.*

Proof. Let $(k,i) \in L_2^\alpha$ for some α. From the definition of L_2^α it follows that $\text{supp}(f_\beta) \cap \text{supp}(x_{k,i}) = \emptyset$ for every $\beta \preceq \alpha$. It follows that $f(u_2) = 0$. □

Lemma 3.7. *It holds that*

$$|f(u_1)| = |f(\frac{m_{2j+1}}{n_{2j+1}} \sum_{k\in I_1} \frac{m_{2j_k}}{n_{2j_k}} \sum_{i:(k,i)\in L_1} (y_{k,i}+z_{k,i}))| \leq \frac{1}{m_{2j+1}^2}. \tag{3.7}$$

Proof. Let f_α be a G-functional determined by the intervals $q_\alpha \leq n_1^\alpha < \cdots < n_{q_\alpha}^\alpha$. Let k_0 be the smallest k such that $(k,i) \in L_1^\alpha$ for some $i \leq n_{2j_k}$. It follows that $q_\alpha \leq \text{maxsupp}\, x_{k_0}$. If $k_0 < n_{2j+1}$, as $(x_k, x_k^*)_k$ is a dependent sequence, we have that $q_\alpha \leq \text{maxsupp}\, x_{k_0} \leq m_{2j_{k_0+1}}$. Therefore

$$\#\{(k,i) \in L_1^\alpha : k > k_0\} \leq m_{2j_{k_0+1}}.$$

The above inequality implies the following.

$$|f_\alpha(\sum_{(k,i)\in L_1^\alpha}\frac{m_{2j_k}}{n_{2j_k}}(y_{k,i}+z_{k,i}))| = \sum_{k\in I_1}|f_\alpha(\frac{m_{2j_k}}{n_{2j_k}}\sum_{i:(k,i)\in L_1^\alpha}(y_{k,i}+z_{k,i}))|$$
$$\leq \|x_{k_0}\| + \sum_{k>k_0} 6m_{2j_{k_0+1}}\frac{m_{2j_k}}{n_{2j_k}} \leq 6+\frac{1}{n_{2j+2}} \leq 7.$$

Since for every $k \in I_1$ we have $\prod_{\beta\prec\alpha_k} w(f_\beta) < m_{2j+1}$, Lemma 3.5 yields that $\#\Gamma \leq (5n_{2j})^{\log_2(m_{2j+1})-1} \leq n_{2j+1}^{1/3}$. Therefore

$$|f(u_1)| \leq \frac{m_{2j+1}}{n_{2j+1}}\sum_{\alpha\in\Gamma}|f_\alpha(\sum_{(k,i)\in L_1^\alpha}\frac{m_{2j_k}}{n_{2j_k}}(y_{k,i}+z_{k,i}))|$$
$$\leq \frac{m_{2j+1}}{n_{2j+1}}7n_{2j+1}^{1/3} \leq \frac{1}{m_{2j+1}^2}.$$

□

It remains to estimate the action of f on u_3.

Lemma 3.8. *Let $y_3 = u_3|_{\operatorname{supp} y}$ and $z_3 = u_3|_{\operatorname{supp} z}$. There exist a functional $g \in W$ with $|g| = |f|$ satisfying*

$$g(z_3) \geq \frac{1}{2}f(y_3). \tag{3.8}$$

Recall that for every $k \in I_1$ it holds that $|f_k^{\alpha_k}| = |f_k|$ and $\operatorname{range}(x_k) \subset E_{\alpha_k}$. Also since $\prod_{\beta\prec\alpha_k} w(f_\beta) < m_{2j+1}$ it follows that the nodes α_k, α_l are incomparable for $k \neq l \in I_1$.

Let g be the functional defined by a tree-analysis we obtain by replacing each $f_k^{\alpha_k}$ by f_k for every $k \in I_1$. Lemma 1.4 yields that the resulting functional is a norming functional.

Setting $y_{3,k} = u_3|_{\operatorname{supp} y_k}$ and $z_{3,k} = u_3|_{\operatorname{supp} z_k}$ and using that $|f_k^{\alpha_k}| = |f_k|$ we have the following

$$|f_k^{\alpha_k}(y_{3,k})| \leq 2f_k(z_{3,k}) \tag{3.9}$$

Remark 3.4 yields that for every $\gamma \prec \alpha_k$ we have that

$$|f_\gamma(y_{k,3})| \leq \left(\prod_{\gamma\preceq\delta\preceq\alpha_k} w(f_\delta)^{-1}\right)|f_k^{\alpha_k}(y_{k,3})|.$$

Lemma 1.4 yields also that $g_\gamma(z_{3,k}) = (\prod_{\gamma\preceq\delta\preceq\alpha_k} w(f_\delta)^{-1})f_k(z_{3,k})$ for every $\gamma \prec \alpha_k$. Therefore

$$|f_\gamma(y_{3,k})| \leq \left(\prod_{\gamma\preceq\delta\preceq\alpha_k} w(f_\delta)^{-1}\right)|f_k^{\alpha_k}(y_{3,k})| \leq 2g_\gamma(z_{3,k}). \tag{3.10}$$

The above inequality proves (3.8). □

Combining (3.2), (3.6), (3.7) and (3.8) we obtain that

$$g(z) \geq \frac{1}{2}f(y) - 2\frac{31}{m_{2j+1}}.$$

Theorem 3.9. *The space $\mathcal{X}_{(4)}$ is quasi-minimal.*

Proof. The proof is based on the arguments we use in the proof of Proposition 3.1.

Let Y, Z be two block subspaces of $\mathcal{X}_{(4)}$. Inductively, by Proposition 2.9, we choose a a sequence $(x_l, x_l^*)_{l\in\mathbb{N}}$ such that (x_l, x_l^*) is a $(2j_l+1)$-dependent sequence, $3 < j_l \nearrow +\infty$, which splits as in the first step i.e.

$$x_l = y_l + z_l = \frac{m_{2j_l+1}}{n_{2j_l+1}} \sum_{k=1}^{n_{2j_l+1}/2} y_{l,k} + z_{l,k}$$

where $y_{l,k} + z_{l,k} = \dfrac{m_{2j_{k,l}}}{n_{2j_{l,k}}}(y_{l,k,1} + z_{l,k,1} + \cdots + y_{l,k,n_{2j_{l,k}}} + z_{l,k,n_{2j_{l,k}}})$ and additionally $y_{l,k,i} \in Y$, $z_{l,k,i} \in Z$.

We may also assume that the weights that appear in the choice of the dependent sequence (x_l, x_l^*) are bigger than the weights we use in (x_{l-1}, x_{l-1}^*).

Let $\|\sum_l a_l y_l\| = 1$ and let $f \in W$ with $f(\sum_l a_l y_l) > 1/2$. Let $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a tree-analysis of f.

We define for every $l \in \mathbb{N}$ the set $I_{l,1}, I_{l,2}, I_{l,3}$ as in Proposition 3.1. For each of the sets $I_{l,1}$, $I_{l,2}$ we get

$$f(\sum_{k\in I_{l,3}} (y_{l,k} + z_{l,k})) \leq \frac{24}{m_{2j_l+1}}$$

and

$$f(\sum_{k\in I_{l,2}} (y_{l,k} + z_{l,k})) \leq \frac{6}{m_{2j_l+1}}$$

For the sets $I_{l,1}$ as in Proposition 3.1 we define the sets $L_{l,1}, L_{l,2}$ and $L_{l,3}$ and vectors $u_{l,1}, u_{l,2}, u_{l,3}$. As before we obtain that $f(u_{l,2}) = 0$ and $f(u_{l,1})$ is dominated by $m_{2j_l+1}^{-1}$.

For the sets $L_{l,1}$ we work as in Lemma 3.8 substituting $f_{l,k}^{\alpha_k}$ (which corresponds to $f_k^{\alpha_k}$ of Lemma 3.8) by the functional $f_{l,k}$ (which corresponds to f_k of Lemma 3.8). In this way we get a functional g such that $|g| = |f|$ and

$$g(\sum_l a_l u_{l,3}|_{\operatorname{supp} z_l}) \geq \frac{1}{2} f(\sum_l a_l u_{l,3}|_{\operatorname{supp} z_l}).$$

The above yields that

$$g(\sum_l a_l z_l) \geq \frac{1}{2} f(\sum_l a_l y_l) - 2\sum_l |a_l| \frac{31}{m_{2j_l+1}} \geq \frac{1}{5}.$$

which ends the proof. □

4. Tightness by range

We show now, using G-operations, the following

Theorem 4.1. *The space $\mathcal{X}_4$ is tight by range.*

Proof. Let (x_i) be normalized block sequence. We show that there exists no bounded operator T such that $\operatorname{supp} T(x_i) \cap \operatorname{range}(x_i) = \emptyset$ and T can be extended to an isomorphism from $[(x_i)]$ to X. This will prove that $\mathcal{X}_{(4)}$ is tight by range.

Let T be an operator as above and assume without loss of generality that $\|T\| \leq 1$. By the reflexivity of the space and passing to a subsequence we may assume that $(T(x_i))_i$ is a block sequence and moreover

$$\operatorname{range}(x_i + Tx_i) < \operatorname{range}(x_{i+1} + Tx_{i+1})\ \forall i \in \mathbb{N}.$$

Let $j \in \mathbb{N}$. By Lemma 2.2 we can choose $w = \sum_{k=1}^{l(n_{2j})} a_k x_{r(j)+k}$ such that w is an $2 - \ell^{n_{2j}}$-average and $l(n_{2j}) \leq w$.

Let $f_0 \in W$ be a functional such that $f_0(w) \geq 1$. Without loss of generality we may assume that $\text{range}(f_0) = \text{range}(w)$. For $k \leq l(n_{2j})/2$ set

$$E_{2k-1} = \text{range}(x_{r(j)+2k-1}) \text{ and } E_{2k} = (\text{maxran}\, x_{r(j)+2k-1}, \text{minsupp}\, x_{r(j)+2k+1}),$$

It follows that $(E_k)_{k=1}^{l(n_{2j})}$ are consecutive intervals and $l(n_{2j}) \leq E_1 < \cdots < E_{n_{2j}}$. Then the functional $f = \frac{1}{2}\sum_{k=1}^{n_{2j}/2} E_{2k-1} f_0 \in W$ and without loss of generality we may assume $f(w) \geq 1/2$. Otherwise we can take the restriction of f to $\cup_k \text{range}(x_{r(j)+2k})$. It follows that f satisfies

$$f(w) \geq \frac{1}{2} \text{ and } \text{supp}(Tx_i) \cap \text{supp}(f) = \emptyset \ \forall i \in \mathbb{N}. \tag{4.1}$$

Definition 4.2. We shall call the pair (w, f) a $(2, n_{2j})$-special pair disjoint from (Tx_i) if w is an $2 - \ell_1^{n_{2j}}$-average and (4.1) holds.

Let $j \in \mathbb{N}$. Inductively, by Proposition 2.9, we choose a $(6, n_{2j+1})$-dependent sequence $((u_i, f_i))_{i \leq n_{2j+1}}$ such that

P1) (u_i, f_i) is a $(6, \varepsilon_i)$-exact pair of the form $u_i = \frac{m_{2j_i}}{n_{2j_i}} \sum_{k=1}^{n_{2j_i}} u_{i,k}$ and $f_i = \frac{1}{m_{2j_i}} \sum_{k=1}^{n_{2j_i}} f_{i,k}$ for any i,

P2) $(u_{i,k}, f_{i,k})$ is a $(2, n_{2j_{i,k}})$-special pair disjoint from $(Tx_i)_i$ for every i, k,

P3) $(f_1, \ldots, f_{n_{2j+1}})$ is a $(2j+1)$-special sequence.

Note that

$$\left\|\frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} u_i\right\| \geq \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i\Big(\frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} u_i\Big) \geq \frac{1}{2}. \tag{4.2}$$

Moreover $\text{supp}(f) \cap \text{supp}(Tx_j) = \emptyset$ for all $f = \frac{1}{m_{2j+1}} E \sum_{i=1}^{n_{2j+1}} h_i$ and $|h_i| = |f_i|$ for all i.

Lemma 4.3. *The following holds*

$$\left\|\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} Tu_i\right\| \leq 25 m_{2j+1}^{-2} \tag{4.3}$$

Proof of Lemma 4.3. Let $u_{i,k} = \sum_{j \in D_{i,k}} a_j x_j = \frac{1}{n_{2j_{i,k}}}(\bar{x}_{i,k,1} + \ldots \bar{x}_{i,k,n_{2j_{i,k}}})$ be a $2 - \ell_1^{n_{2j_{i,k}}}$ average. We set $y_{i,k} = \sum_{j \in D^*_{i,k}} a_j T x_j$ where $D^*_{i,k} = D_{i,k} \setminus \{\max D_{i,k}\}$.

Note that $\text{maxsupp}\, y_{i,k} < \text{maxsupp}\, u_{i,k}$,

$$y_{i,k} = \frac{1}{n_{2j_{i,k}}}(\bar{y}_{i,k,1} + \cdots + \bar{y}_{i,k,n_{2j_{i,k}}})$$

and $\|\bar{y}_{i,k,j}\| \leq \|T\| \|\bar{x}_{i,k,j}\| \leq 2$ for all j.

Set $w_i = \frac{m_{2j_i}}{n_{2j_i}} \sum_{k=1}^{n_{2j_i}} y_{i,k}$. Since

$$\|Tu_i - w_i\| \leq \frac{m_{2j_i}}{n_{2j_i}} \sum_{k=1}^{n_{2j_i}} \|Tu_{i,k} - y_{i,k}\| \leq \frac{m_{2j_i}}{n_{2j_i}} \sum_{k=1}^{n_{2j_i}} \frac{1}{n_{2j_{i,k}}} \leq \frac{1}{m_{2j_i}^2}$$

it is enough to show that

$$\left\|\frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} w_i\right\| \leq 24 m_{2j+1}^{-2}$$

To get the above inequality we shall use Proposition 2.8. We show that $(y_{i,k})_{k=1}^{n_{2j_i}}$ is a $(3,\varepsilon_i)$-RIS.

Indeed as in Lemma 2.6 we obtain that for all $f \in W$ with $w(f) = m_p < m_{2j_{i,k}}$ it holds that

$$|f(y_{i,k})| \leq \frac{3}{m_p}.$$

Also since $m_{2j_{i,k+1}}^{-1} \max\operatorname{supp} u_{i,k} \leq \varepsilon_i$ and $\operatorname{maxsupp} y_{i,k} < \operatorname{maxsupp} u_{i,k}$ we get that $(y_{i,k})_{k \leq n_{2j_i}}$ is a $(3,\varepsilon_i)$-RIS. By Proposition 2.8 and P3) we have that $(w_i)_{i=1}^{n_{2j+1}}$ is a $(6, n_{2j+1}^{-1})$-RIS.

We will show now that for every $f \in W$ with $w(f) = m_{2j+1}^{-1}$ and every interval E we have that

$$|f(\sum_{i \in E} w_i)| \leq 6(1 + \frac{\#E}{m_{2j+1}^2})$$

Let $f = E\frac{1}{m_{2j+1}}\sum_{r=1}^{n_{2j+1}} h_r$ be a special functional. Let $i_0 = \min\{i : |h_i| \neq |f_i|\}$. If $i_0 > 1$ it follows that for every $i < i_0$ $f(w_i) = 0$, since by P2) $f(Tu_{i,k}) = 0$ for every (i,k) with $i < i_0$. For every $i > i_0$ the assumptions of Corollary 2.10 hold, hence

$$|f(\sum_{i > i_0} w_i)| \leq \frac{1}{m_{2j+2}^2} \tag{4.4}$$

For the w_{i_0}, $w(h_{i_0}) = w(f_{i_0})$ and hence using Corollary 2.10 we get

$$|f(w_{i_0})| \leq 6 + \frac{1}{m_{2j+2}^2} \tag{4.5}$$

It follows

$$|f(\sum_{i \in E} w_i)| \leq 6(1 + \frac{\#E}{m_{2j+1}^2})$$

Proposition 2.8 yields

$$\|\frac{m_{2j+1}}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}} w_i\| \leq \frac{24}{m_{2j+1}},$$

which ends the proof of the lemma. □

Notice that combining (4.2) and (4.3) we get that T is not an isomorphism, which ends the proof of the theorem. □

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