RESTRICTED INVERTIBILITY AND THE BANACH-MAZUR DISTANCE TO THE CUBE

PIERRE YOUSSEF

ABSTRACT. We prove a normalized version of the restricted invertibility principle obtained by Spielman-Srivastava in [12]. Applying this result, we get a new proof of the proportional Dvoretzky-Rogers factorization theorem recovering the best current estimate. As a consequence, we also recover the best known estimate for the Banach-Mazur distance to the cube: the distance of every n-dimensional normed space from ℓ_∞^n is at most $(2n)^{\frac{5}{6}}$. Finally, using tools from the work of Batson-Spielman-Srivastava in [2], we give a new proof for a theorem of Kashin-Tzafriri on the norm of restricted matrices.

1. INTRODUCTION

Given an $n \times m$ matrix U, viewed as an operator from ℓ_2^m to ℓ_2^n, the restricted invertibility problem asks if we can extract a large number of linearly independent columns of U and provide an estimate for the norm of the restricted inverse. If we write U_σ for the restriction of U to the columns Ue_i, $i \in \sigma \subset \{1, \ldots, m\}$, we want to find a subset σ, of cardinality k as large as possible, such that $\|U_\sigma x\|_2 \geqslant c\|x\|_2$ for all $x \in \mathbb{R}^\sigma$ and to estimate the constant c (which will depend on the operator U). This question was studied by Bourgain-Tzafriri (see [4]) who obtained a result for square matrices:

Given an $n \times n$ matrix T (viewed as an operator on ℓ_2^n) whose columns are of norm one, there exists $\sigma \subset \{1, \ldots, n\}$ with $|\sigma| \geqslant d\frac{n}{\|T\|_2^2}$ such that $\|T_\sigma x\|_2 \geqslant c\|x\|_2$ for all $x \in \mathbb{R}^\sigma$, where $d, c > 0$ are absolute constants.

Here and in the rest of the paper, $\|\cdot\|_2$ denotes the Euclidean norm when applied to a vector and the operator norm when applied to a matrix (seen as an operator on l_2). For a matrix A, $\|A\|_{\text{HS}}$ denotes the Hilbert-Schmidt norm, i.e.

$$\|A\|_{\text{HS}} = \sqrt{Tr(A \cdot A^*)} = \left(\sum_i \|C_i\|_2^2\right)^{1/2}$$

where C_i are the columns of A.

Since the identity operator can be decomposed in the form $Id = \sum_j e_j e_j^t$ where (e_j) is the canonical basis of $\mathbb{R}^n$, the previous result states that one can find a large part of this basis (of cardinality greater than $d\frac{n}{\|T\|_2^2}$) on the span of which the operator T is invertible and the norm of its inverse is controlled by an absolute constant.

Vershynin generalized this result for any decomposition of the identity (see [16]) and improved the estimate for the size of the subset. Using a technical iteration scheme based on the previous result of Bourgain-Tzafriri, combined with a theorem of Kashin-Tzafriri which we will discuss in the last section, he obtained the following :

Let $Id = \sum_{j \leqslant m} x_j x_j^t$ and let T be a linear operator on ℓ_2^n. For any $\varepsilon \in (0, 1)$ one can find $\sigma \subset \{1, \ldots, m\}$ with

$$|\sigma| \geqslant (1 - \varepsilon)\frac{\|T\|_{\text{HS}}^2}{\|T\|_2^2}$$

such that

$$\left\|\sum_{j\in\sigma} a_j \frac{Tx_j}{\|Tx_j\|_2}\right\|_2 \geqslant c(\varepsilon)\left(\sum_{j\in\sigma} a_j^2\right)^{\frac{1}{2}}$$

for all scalars (a_j).

One can easily check that, in the case of the canonical decomposition, this is a generalization of the Bourgain-Tzafriri theorem, which was previously only proved for a fixed value of ε. The constant $c(\varepsilon)$ plays a crucial role in applications and we will be able to improve its value significantly (see Proposition 2.1 for the precise statement).

Back to the original restricted invertibility problem, a recent work of Spielman-Srivastava (see [12]) provides the best known estimate for the norm of the inverse matrix. Their proof uses a new deterministic method based on linear algebra, while the previous works on the subject employed probabilistic, combinatorial and functional-analytic arguments.

More precisely, Spielman-Srivastava proved the following:

Theorem 1.1. *Let $x_1,\dots x_m \in \mathbb{R}^n$ such that $Id = \sum_i x_i x_i^t$ and let $0<\varepsilon<1$. For every linear operator $T:\ell_2^n \to \ell_2^n$ there exists a subset $\sigma \subset \{1,\dots,m\}$ of size $|\sigma| \geqslant \left\lfloor (1-\varepsilon)^2 \frac{\|T\|_{\mathrm{HS}}^2}{\|T\|_2^2}\right\rfloor$ for which $\{Tx_i\}_{i\in\sigma}$ is linearly independent and*

$$\lambda_{\min}\left(\sum_{i\in\sigma}(Tx_i)(Tx_i)^t\right) > \frac{\varepsilon^2\|T\|_{\mathrm{HS}}^2}{m},$$

where $\lambda_{\min}$ is computed on $\mathrm{span}\{Tx_i\}_{i\in\sigma}$ or simply here $\lambda_{\min}$ denotes the smallest nonzero eigenvalue of the corresponding operator. .

One can view the previous result as an invertibility theorem for rectangular matrices. Given, as above, a decomposition of the identity and a linear operator T, we can associate to these an $n\times m$ matrix U whose columns are the vectors $(Tx_j)_{j\leqslant m}$. Since $Id=\sum_j x_j x_j^t$, one can easily check that

$$U\cdot U^t = T\cdot T^t = \sum_j (Tx_j)\cdot(Tx_j)^t.$$

Hence,

$$\|U\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}} \quad \text{and} \quad \|U\|_2 = \|T\|_2,$$

and thus the previous result can be written in terms of the rectangular matrix U.

In the applications, one might need to extract multiples of the columns of the matrix. Adapting the proof of Spielman-Srivastava, we will generalize the restricted invertibility theorem for any rectangular matrix and, under some conditions, for any choice of multiples.

If D is an $m\times m$ diagonal matrix with diagonal entries $(\alpha_j)_{j\leqslant m}$, we set $I_D := \{j\leqslant m \mid \alpha_j \neq 0\}$ and write D_σ^{-1} for the restricted inverse of D i.e the diagonal matrix whose diagonal entries are the inverses of the respective entries of D for indices in σ and zero elsewhere. The main result of this paper is the following:

Theorem 1.2. *Given an $n\times m$ matrix U and a diagonal $m\times m$ matrix D with $(\alpha_j)_{j\leqslant m}$ on its diagonal, with the property that $\mathrm{Ker}(D)\subset \mathrm{Ker}(U)$, then for any $\varepsilon\in(0,1)$ there exists $\sigma\subset\{1,\dots,m\}$ with*

$$|\sigma| \geqslant (1-\varepsilon)^2 \frac{\|U\|_{\mathrm{HS}}^2}{\|U\|_2^2}$$

such that

$$s_{\min}\left(U_\sigma D_\sigma^{-1}\right) > \frac{\varepsilon\|U\|_{\mathrm{HS}}}{\|D\|_{\mathrm{HS}}},$$

where $s_{\min}$ denotes the smallest singular value.

Note that if we apply this fact to the matrix U which we associated with a linear operator T and a decomposition of the identity, and we take D to be the identity operator, we recover the restricted invertibility theorem of Spielman-Srivastava.

In Section 2, we discuss some applications of our result: an improved version of Vershynin's normalized invertibility theorem and an alternative proof of the best known estimates for the proportional Dvoretzky-Rogers factorization and the Banach-Mazur distance to the cube. In Section 3, we prove our main result. Finally, in Section 4 we give a new proof of a theorem due to Kashin-Tzafriri which deals with the norm of coordinate projections of a matrix.

2. Applications

2.1. **Normalized restricted invertibility.** Our first application is an improved version of a normalized restricted invertibility principle for an arbitrary operator on l_2^n and any decomposition of the identity on $\mathbb{R}^n$. The formulation below is essentially due to Vershynin, where the constant depending on ε satisfied $c(\varepsilon) \sim \varepsilon^{-\log \varepsilon}$. We show that one can have a similar result with $c(\varepsilon) = \varepsilon$. In [16], Vershynin established a non trivial upper bound as well; he applied this to get a Dvoretzky-Rogers type lemma and to deduce information about embeddings of l_∞^k into finite dimensional spaces.

Proposition 2.1. *Let $Id = \sum_{j \leqslant m} x_j x_j^t$ be a decomposition of the identity on $\mathbb{R}^n$ and let T be a linear operator on l_2^n. For any $\varepsilon > 0$ there exists $\sigma \subset \{1, \ldots, m\}$ with*

$$|\sigma| \geqslant (1-\varepsilon)^2 \frac{\|T\|_{\mathrm{HS}}^2}{\|T\|_2^2}$$

such that, for any choice of scalars $(a_j)_{j\in\sigma}$,

$$\left\| \sum_{j\in\sigma} a_j \frac{Tx_j}{\|Tx_j\|_2} \right\|_2 \geqslant \varepsilon \left(\sum_{j\in\sigma} a_j^2 \right)^{1/2}.$$

Proof. Let U be the $n \times m$ matrix whose columns are the vectors $(Tx_j)_{j\leqslant m}$. As mentioned before, we have $\|U\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$ and $\|U\|_2 = \|T\|_2$. Let D be the matrix with diagonal entries $\|Tx_j\|_2$. Then,

$$\|D\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}.$$

Now, we apply Proposition 2.1 with these U and D to find σ of the desired cardinality such that

$$s_{\min}(U_\sigma D_\sigma^{-1}) \geqslant \varepsilon.$$

Observing that, for any $a = (a_j) j \in \sigma$,

$$U_\sigma D_\sigma^{-1} a = \sum_{j\in\sigma} a_j \frac{Tx_j}{\|Tx_j\|_2},$$

we conclude the proof. □

2.2. **Dvoretzky-Rogers factorization.** We say that an n-dimensional normed space X is in John's position if the Euclidean unit ball B_2^n is the ellipsoid of maximal volume inscribed in the unit ball B_X of X. In [9] (see also [1]) John characterized this position as follows: If X is in John's position, then we have a decomposition $Id = \sum_j x_j x_j^t$ of the identity, where $\frac{x_j}{\|x_j\|_2}$ are contact points of B_X and B_2^n.

The proportional Dvoretzky-Rogers factorization is given by the following statement.

Proposition 2.2. *Let X be a n-dimensional normed space. For any $\varepsilon > 0$, there exists $k \geqslant (1-\varepsilon)^2 n$ such that the identity $i_{2,\infty} : l_2^k \longrightarrow l_\infty^k$ can be written as $i_{2,\infty} = \alpha \circ \beta$, where $\beta : l_2^k \longrightarrow X$, $\alpha : X \longrightarrow l_\infty^k$ and $\|\alpha\| \cdot \|\beta\| \leqslant \frac{1}{\varepsilon}$.*

Proof. By changing the Euclidean structure in $\mathbb{R}^n$, one may assume that X is in John's position. Hence, $\|.\|_{X^*} \geqslant \|.\|_2$ and by John's theorem one can write $Id = \sum_{j\leqslant m} x_j x_j^t$ where x_j are multiples of contact points of X, i.e. $\|x_j\|_X = \|x_j\|_{X^*} = \|x_j\|_2$.

Now, we apply Proposition 2.1 with $T = Id$ to find $\sigma \subset \{1,\dots,m\}$ with $|\sigma| = k \geqslant (1-\varepsilon)^2 n$ such that:

$$\left\| \sum_{j\in\sigma} a_j \frac{x_j}{\|x_j\|_2} \right\|_2 \geqslant \varepsilon \left(\sum_{j\in\sigma} a_j^2 \right)^{1/2}. \tag{1}$$

Obviously, the vectors $\{x_j\}_{j\in\sigma}$ are linearly independent. Next, we define $\alpha' : l_1^k \longrightarrow X^*$ by setting $\alpha'(e_j) = \frac{x_j}{\|x_j\|_2}$, and $\beta' : X^* \longrightarrow l_2^k$ by setting $\beta'(\frac{x_j}{\|x_j\|_2}) = e_j$. Clearly, $i_{1,2} = \beta' \circ \alpha'$.

- Since $\|x_j\|_2 = \|x_j\|_{X^*}$, we have $\|\alpha'\|_{l_1^k \to X^*} \leqslant 1$.
- By (1), we have $\|\beta'\|_{X^* \to l_2^k} \leqslant \frac{1}{\varepsilon}$.

Then we get the result by duality. □

Remark 2.3. This form of factorization was first proven by Bourgain-Szarek (see [3]) with a weaker dependence on ε. After that, came the work of Szarek-Talagrand (see [14]) and of Giannopoulos (see [6]) that improved the dependence on ε to ε^{-2} and $\varepsilon^{-\frac{3}{2}}$ respectively. In these two works, the factorization for $i_{1,1}$ was established by combining an isomorphic variant of Sauer-Shelah's lemma with a factorization theorem of Grothendieck. Finally, Giannopoulos proved in [7] that the statement holds true with $c(\varepsilon) = \frac{c}{\varepsilon}$, which is the best known dependence on ε. Our Proposition recovers this result (with $c = 1$) using a completely different method.

2.3. **Banach-Mazur distance to the cube.** Let $\mathbb{BM}_n$ denote the space of all n-dimensional normed spaces X, known as the Banach-Mazur compactum. If X, Y are in $\mathbb{BM}_n$, we define the Banach-Mazur distance between X and Y as follows:

$$\begin{aligned} d(X,Y) &= \inf\{\|T\| \cdot \|T^{-1}\| \mid \ T \text{ is an isomorphism between X and Y } \} \\ &= \inf\{\alpha/\beta \mid \ \beta B_Y \subset T(B_X) \subset \alpha B_Y\} \\ &= \inf\{\alpha/\beta \mid \ \beta\|x\|_X \leqslant \|T(x)\|_Y \leqslant \alpha\|x\|_X\}. \end{aligned}$$

More precisely, $\log(d)$ defines a distance (in the classical sense) in the quotient space of $\mathbb{BM}_n$, which is derived after we identify isometric spaces. By duality we have $d(X,Y) = d(X^*, Y^*)$.

Obtaining sharp estimates for Banach-Mazur distances is a classical topic in the asymptotic theory of finite dimensional normed spaces. The first result in this direction was a theorem of John (see [1],[9]) which provided a sharp upper bound for the distance between any n-dimensional normed space and the Euclidean space ℓ_2^n. If we set $R_2^n = \max\{d(X, l_2^n) \mid \ X \in \mathbb{BM}_n\}$, John proved that $R_2^n = \sqrt{n}$. A theorem of Gluskin (see [8]) established the existence of spaces of dimension n which are at distance proportional to n from each other; it follows that l_2^n is at the center of $\mathbb{BM}_n$ and that $\mathbb{BM}_n$ is contained in a ball of radius $\sqrt{n}$ with respect to the Banach-Mazur distance centered at ℓ_2^n. One natural question is to ask for an analogous estimate when l_2^n is replaced by some other l_p^n-space, and in particular by l_∞^n. This problem was first studied by Bourgain-Szarek (see [3]), then Szarek-Talagrand (see [14]) and finally Giannopoulos (see [6]) who improved the work of Szarek-Talagrand and proved that $R_\infty^n \leqslant cn^{\frac{5}{6}}$. This is the best known result and it is probably not optimal. The Dvoretzky-Rogers factorization theorem, stated above, plays a crucial role in estimating this distance, since Proposition 2.3 allows us to find a "large subspace" of our space whose distance from l_2 and l_1 can be bounded well. For the remaining part, which will be a "small subspace", we use a trivial estimate and combine it with the previous one.

Proposition 2.4. $R_1^n = R_\infty^n \leqslant (2n)^{\frac{5}{6}}$.

Proof. Let $X \in \mathbb{BM}_n$ and assume that X^* is in John's position (by changing the Euclidean structure in $\mathbb{R}^n$ if necessary), i.e. B_2^n is the ellipsoid of minimal volume containing B_X.
By John's theorem, one can find $x_1, \ldots, x_m$ which are multiples of contact points ($\|x_j\|_X = \|x_j\|_{X^*} = \|x_j\|_2$) such that $Id = \sum_{j \leqslant m} x_j x_j^t$. Note that $B_2^n \subset B_{X^*} \subset \sqrt{n} B_2^n$, and hence $\|x\|_2 \leqslant \|x\|_X \leqslant \sqrt{n}\|x\|_2$.

Applying Proposition 2.1, one can find $\sigma \subset \{1, \ldots, m\}$ with $|\sigma| = k \geqslant (1 - 2\varepsilon)n$ such that, for any choice of scalars (a_j),

$$\left\| \sum_{j \in \sigma} a_j \frac{x_j}{\|x_j\|_2} \right\|_2 \geqslant \varepsilon \left(\sum_{j \in \sigma} a_j^2 \right)^{1/2}.$$

In order to simplify notation, we assume that $\sigma = \{1, \ldots, k\}$. Let $y_{k+1}, \ldots, y_n$ be an orthogonal basis of $\mathrm{span}(x_1, \ldots, x_k)^\perp$ normalized so that $\|y_j\|_X \leqslant 1$. To do this, we require that $\|y_j\|_2 = \frac{1}{\sqrt{n}}$.

Define $T : l_1^n \longrightarrow X$ by setting $T(e_j) = \frac{x_j}{\|x_j\|_2}$ if $j \leqslant k$ and $T(e_j) = y_j$ for $j > k$. Let $a = (a_j)_{j \leqslant n} \in \mathbb{R}^n$ and

$$Ta = \sum_{j=1}^{k} a_j \frac{x_j}{\|x_j\|_2} + \sum_{j=k+1}^{n} a_j y_j.$$

Then,

$$\|a\|_1 = \sum_{j \leqslant k} |a_j| + \sum_{j > k} |a_j| \geqslant \left\| \sum_{j \leqslant k} a_j \frac{x_j}{\|x_j\|_2} + \sum_{j > k} a_j y_j \right\|_X \geqslant \left\| \sum_{j \leqslant k} a_j \frac{x_j}{\|x_j\|_2} + \sum_{j > k} a_j y_j \right\|_2.$$

We write

$$\begin{aligned}
\|Ta\|_2 &\geqslant \left[\left\| \sum_{j \leqslant k} a_j \frac{x_j}{\|x_j\|_2} \right\|_2^2 + \left\| \sum_{j > k} a_j y_j \right\|_2^2 \right]^{\frac{1}{2}} \quad \text{by orthogonality} \\
&\geqslant \left[\varepsilon^2 \sum_{j \leqslant k} a_j^2 + \sum_{j > k} a_j^2 \|y_j\|_2^2 \right]^{\frac{1}{2}} \\
&\geqslant \left[\frac{\varepsilon^2}{n} \left(\sum_{j \leqslant k} |a_j| \right)^2 + \frac{1}{n(n-k)} \left(\sum_{j > k} |a_j| \right)^2 \right]^{\frac{1}{2}} \quad \text{by Cauchy-Shwarz} \\
&\geqslant \left[\frac{\varepsilon^2}{n} \left(\sum_{j \leqslant k} |a_j| \right)^2 + \frac{1}{2\varepsilon n^2} \left(\sum_{j > k} |a_j| \right)^2 \right]^{\frac{1}{2}} \\
&\geqslant \frac{1}{\sqrt{2}} \left[\frac{\varepsilon}{\sqrt{n}} \sum_{j \leqslant k} |a_j| + \frac{1}{n\sqrt{2\varepsilon}} \sum_{j > k} |a_j| \right] \\
&\geqslant \frac{1}{(2n)^{\frac{5}{6}}} \sum_{j=1}^{n} |a_j| \quad \text{taking } \varepsilon = (2n)^{-\frac{1}{3}}.
\end{aligned}$$

It follows that $(2n)^{-\frac{5}{6}} \|a\|_1 \leqslant \|Ta\|_X \leqslant \|a\|_1$ and therefore $d(X, l_1^n) \leqslant (2n)^{\frac{5}{6}}$ for all $X \in \mathbb{BM}_n$. □

Remark 2.5. Here we are interested in high dimensional results; this is why the constant is not that important for us. If we want an estimate for "small" dimensions, then the value of the constant becomes important. Note that a trivial estimate would be $R_\infty^n \leqslant n$. In [6], Giannopoulos proved that $R_\infty^n \leqslant cn^{\frac{5}{6}}$ with $c = \frac{2^{\frac{7}{6}}}{(\sqrt{2}-1)^{\frac{1}{3}}} \sim 3,0116$, and thus his result becomes nontrivial when the dimension is larger than 746. On the other hand, our result becomes nontrivial whenever the dimension is bigger than 32. Moreover, if we are interested in small dimensions, we can obtain a better result by choosing ε in the last inequality in a different way: in fact we have chosen $(2n)^{-\frac{1}{3}}$ in the asymptotic regime, otherwise one just needs to optimize ε so that it satisfies $\frac{\varepsilon}{\sqrt{(1-\varepsilon)^2 n}} = \frac{1}{n\sqrt{1-(1-\varepsilon)^2}}$; then our result becomes nontrivial when the dimension is larger than 15. In [15], Taschuk has also obtained an estimate for the Banach-Mazur distance to the cube of "small"-dimensional spaces. One can check that our result improves on that whenever the dimension is larger than 15.

3. Proof of Theorem 1.2

Since the rank and the eigenvalues of $(U_\sigma D_\sigma^{-1})^t \cdot (U_\sigma D_\sigma^{-1})$ and $(U_\sigma D_\sigma^{-1}) \cdot (U_\sigma D_\sigma^{-1})^t$ are the same, it suffices to prove that $(U_\sigma D_\sigma^{-1}) \cdot (U_\sigma D_\sigma^{-1})^t$ has rank equal to $k = |\sigma|$ and its smallest positive eigenvalue is greater than $\varepsilon^2 \frac{\|U\|_{\mathrm{HS}}^2}{\|D\|_{\mathrm{HS}}^2}$. Note that

$$(U_\sigma D_\sigma^{-1}) \cdot (U_\sigma D_\sigma^{-1})^t = \sum_{j\in\sigma} \left(UD_\sigma^{-1}e_j\right) \cdot \left(UD_\sigma^{-1}e_j\right)^t = \sum_{j\in\sigma} \left(\frac{Ue_j}{\alpha_j}\right) \cdot \left(\frac{Ue_j}{\alpha_j}\right)^t$$

We are going to construct the matrix $A_k = \sum_{j\in\sigma} \left(UD_\sigma^{-1}e_j\right) \cdot \left(UD_\sigma^{-1}e_j\right)^t$ by iteration. We begin by setting $A_0 = 0$ and at each step we will be adding a rank one matrix $(\left(\frac{Ue_j}{\alpha_j}\right) \cdot \left(\frac{Ue_j}{\alpha_j}\right)^t$ for a suitable j) which will give a new positive eigenvalue. This will guarantee that the vector $UD_\sigma^{-1}e_j$ chosen in each step is linearly independent from the previous ones.

If A and B are symmetric matrices, we write $A \preceq B$ if $B - A$ is a positive semidefinite matrix. Recall the Sherman-Morrison Formula which will be needed in the proof. For any invertible matrix A and any vector v we have

$$(A + v\cdot v^t)^{-1} = A^{-1} - \frac{A^{-1}v \cdot v^t A^{-1}}{1 + v^t A^{-1} v}.$$

We will also apply the following lemma which appears as Lemma 6.3 in [13]:

Lemma 3.1. *Suppose that $A \succeq 0$ has q nonzero eigenvalues, all greater than $b' > 0$. If $v \neq 0$ and*

$$v^t(A - b'I)^{-1}v < -1, \tag{2}$$

then $A + vv^t$ has $q+1$ nonzero eigenvalues, all greater than b'.

The proof of the lemma is simple and makes use of the Sherman-Morrison formula.

For any symmetric matrix A and any $b > 0$, we define

$$\phi_b(A) = \mathrm{Tr}\left(U^t(A - bI)^{-1}U\right)$$

as the potential corresponding to the barrier b.
At each step l, the matrix already constructed is denoted by A_l and the barrier by b_l and to simplify the notations, we will use ϕ_l for the potential ϕ_{b_l}. A_l has l nonzero eigenvalues all greater than b_l. As mentioned before, we will try to construct A_{l+1} by adding a rank one matrix $v \cdot v^t$ to A_l so that A_{l+1} has $l+1$ nonzero eigenvalues all greater than $b_{l+1} = b_l - \delta$ and

$\phi_{l+1}(A_{l+1}) \leqslant \phi_l(A_l)$. Note that

$$\begin{aligned}\phi_{l+1}(A_{l+1}) &= \mathrm{Tr}\left(U^t(A_l+vv^t-b_{l+1}I)^{-1}U\right)\\ &= \mathrm{Tr}\left(U^t(A_l-b_{l+1}I)^{-1}U\right) - \mathrm{Tr}\left(\frac{U^t(A_l-b_{l+1}I)^{-1}vv^t(A_l-b_{l+1}I)^{-1}U}{1+v^t(A_l-b_{l+1}I)^{-1}v}\right)\\ &= \phi_{l+1}(A_l) - \frac{v^t(A_l-b_{l+1}I)^{-1}UU^t(A_l-b_{l+1}I)^{-1}v}{1+v^t(A_l-b_{l+1}I)^{-1}v}.\end{aligned}$$

So, in order to have $\phi_{l+1}(A_{l+1}) \leqslant \phi_l(A_l)$, we must choose a vector v verifying

$$-\frac{v^t(A_l-b_{l+1}I)^{-1}UU^t(A_l-b_{l+1}I)^{-1}v}{1+v^t(A_l-b_{l+1}I)^{-1}v} \leqslant \phi_l(A_l) - \phi_{l+1}(A_l). \tag{3}$$

Since $v^t(A_l-b_{l+1}I)^{-1}UU^t(A_l-b_{l+1}I)^{-1}v$ and $\phi_l(A_l)-\phi_{l+1}(A_l))$ are positive, choosing v verifying conditions (2) and (3) is equivalent to choosing v which satisfies the following:

$$v^t(A_l-b_{l+1}I)^{-1}UU^t(A_l-b_{l+1}I)^{-1}v \leqslant (\phi_l(A_l)-\phi_{l+1}(A_l))\left(-1-v^t(A_l-b_{l+1}I)^{-1}v\right). \tag{4}$$

Since $UU^t \preceq \|U\|_2^2 Id$ and $(A_l-b_{l+1}I)^{-1}$ is symmetric, it is sufficient to choose v so that

$$v^t(A_l-b_{l+1}I)^{-2}v \leqslant \frac{1}{\|U\|_2^2}(\phi_l(A_l)-\phi_{l+1}(A_l))\left(-1-v^t(A_l-b_{l+1}I)^{-1}v\right). \tag{5}$$

Recall the notation $I_D := \{j \leqslant m \mid \alpha_j \neq 0\}$ where $(\alpha_j)_{j\leqslant m}$ are the diagonal entries of D. Since we have assumed that $\mathrm{Ker}(D) \subset \mathrm{Ker}(U)$, we have

$$\|U\|_{\mathrm{HS}}^2 = \sum_{j\leqslant m}\|Ue_j\|_2^2 = \sum_{j\in I_D}\|Ue_j\|_2^2 \leqslant |I_D|\cdot\|U\|_2^2,$$

and thus $|I_D| \geqslant \frac{\|U\|_{\mathrm{HS}}^2}{\|U\|_2^2}$. At each step, we will select a vector v satisfying (5) among $(\frac{Ue_j}{\alpha_j})_{j\in I_D}$. Our task therefore is to find $j\in I_D$ such that

$$(Ue_j)^t(A_l-b_{l+1}I)^{-2}Ue_j \leqslant \frac{\phi_l(A_l)-\phi_{l+1}(A_l)}{\|U\|_2^2}\left(-\alpha_j^2-(Ue_j)^t(A_l-b_{l+1}I)^{-1}Ue_j\right). \tag{6}$$

The existence of such a $j\in I_D$ is guaranteed by the fact that condition (6) holds true if we take the sum over all $(\frac{Ue_j}{\alpha_j})_{j\in D}$. The hypothesis $\mathrm{Ker}(D)\subset\mathrm{Ker}(U)$ implies that:

- $\sum_{j\in I_D}(Ue_j)^t(A_l-b_{l+1}I)^{-2}Ue_j = \mathrm{Tr}\left(U^t(A_l-b_{l+1}I)^{-2}U\right)$,
- $\sum_{j\in I_D}(Ue_j)^t(A_l-b_{l+1}I)^{-1}Ue_j = \mathrm{Tr}\left(U^t(A_l-b_{l+1}I)^{-1}U\right)$.

Therefore it is enough to prove that, at each step, one has

$$\mathrm{Tr}(U^t(A_l-b_{l+1}I)^{-2}U) \leqslant \frac{\phi_l(A_l)-\phi_{l+1}(A_l)}{\|U\|_2^2}\left(-\|D\|_{\mathrm{HS}}^2-\phi_{l+1}(A_l)\right). \tag{7}$$

The rest of the proof is similar to the one in [13]. One just needs to replace m by $\|D\|_{\mathrm{HS}}^2$. For the sake of completeness, we include the proof. The next lemma will determine the conditions required at each step in order to prove (7).

Lemma 3.2. *Suppose that A_l has l nonzero eigenvalues all greater than b_l, and write Z for the orthogonal projection onto the kernel of A_l. If*

$$\phi_l(A_l) \leqslant -\|D\|_{\mathrm{HS}}^2 - \frac{\|U\|_2^2}{\delta} \tag{8}$$

and

$$0<\delta<b_l\leqslant\delta\frac{\|ZU\|_{\mathrm{HS}}^2}{\|U\|_2^2}, \tag{9}$$

then there exists $i \in I_D$ *such that* $A_{l+1} := A_l + \left(\frac{Ue_i}{\alpha_i}\right) \cdot \left(\frac{Ue_i}{\alpha_i}\right)^t$ *has* $l+1$ *nonzero eigenvalues all greater than* $b_{l+1} := b_l - \delta$ *and* $\phi_{l+1}(A_{l+1}) \leqslant \phi_l(A_l)$.

Proof. As mentioned before, it is enough to prove inequality (7). We set $\Delta_l := \phi_l(A_l) - \phi_{l+1}(A_{l+1})$. By (8), we get

$$\phi_{l+1}(A_l) \leqslant -\|D\|_{\mathrm{HS}}^2 - \frac{\|U\|_2^2}{\delta} - \Delta_l.$$

Inserting this in (7), we see that it is sufficient to prove the following inequality:

$$\operatorname{Tr}\left(U^t(A_l - b_{l+1}I)^{-2}U\right) \leqslant \Delta_l \left(\frac{\Delta_l}{\|U\|_2^2} + \frac{1}{\delta}\right). \tag{10}$$

Now, denote by P the orthogonal projection onto the image of A_l. We set

$$\phi_l^P(A_l) := \operatorname{Tr}\left(U^tP(A_l - b_lI)^{-1}PU\right) \quad \text{and} \quad \Delta_l^P := \phi_l^P(A_l) - \phi_{l+1}^P(A_l)$$

and use similar notation for Z. Since P, Z and A_l commute, one can write

$$\Delta_l = \Delta_l^P + \Delta_l^Z \quad \text{and} \quad \phi_l(A_l) = \phi_l^P(A_l) + \phi_l^Z(A_l).$$

Note that:

$$\begin{aligned}(A_l - b_lI)^{-1} - (A_l - b_{l+1}I)^{-1} &= (A_l - b_lI)^{-1}(b_lI - A_l + A_l - b_{l+1}I)(A_l - b_{l+1}I)^{-1} \\ &= \delta(A_l - b_lI)^{-1}(A_l - b_{l+1}I)^{-1}\end{aligned}$$

and since $P(A_l - b_lI)^{-1}P$ and $P(A_l - b_{l+1}I)^{-1}P$ are positive semidefinite, we have:

$$U^tP(A_l - b_lI)^{-1}PU - U^tP(A_l - b_{l+1}I)^{-1}PU \succeq \delta U^tP(A_l - b_{l+1}I)^{-2}PU.$$

Inserting this in (10), it is enough to prove that:

$$\operatorname{Tr}\left(U^tZ(A_l - b_{l+1}I)^{-2}ZU\right) \leqslant \Delta_l \left(\frac{\Delta_l}{\|U\|_2^2} + \frac{1}{\delta}\right) - \frac{\Delta_l^P}{\delta}.$$

Since $A_lZ = 0$, we have:

- $\operatorname{Tr}(U^tZ(A_l - b_{l+1}I)^{-2}ZU) = \frac{\|ZU\|_{\mathrm{HS}}^2}{b_{l+1}^2}$ and
- $\Delta_l^Z = -\frac{\|ZU\|_{\mathrm{HS}}^2}{b_l} + \frac{\|ZU\|_{\mathrm{HS}}^2}{b_{l+1}} = \delta\frac{\|ZU\|_{\mathrm{HS}}^2}{b_lb_{l+1}}$,

so taking into account the fact that $\Delta_l \geqslant \Delta_l^Z \geqslant 0$, it remains to prove the following:

$$\frac{\|ZU\|_{\mathrm{HS}}^2}{b_{l+1}^2} \leqslant \delta^2\frac{\|ZU\|_{\mathrm{HS}}^4}{\|U\|_2^2b_l^2b_{l+1}^2} + \frac{\|ZU\|_{\mathrm{HS}}^2}{b_lb_{l+1}}. \tag{11}$$

By Hypothesis (9), this last inequality follows by

$$\frac{\|ZU\|_{\mathrm{HS}}^2}{b_{l+1}^2} \leqslant \delta\frac{\|ZU\|_{\mathrm{HS}}^2}{b_lb_{l+1}^2} + \frac{\|ZU\|_{\mathrm{HS}}^2}{b_lb_{l+1}}, \tag{12}$$

which is trivially true since $b_{l+1} = b_l - \delta$. □

We are now able to complete the proof of Theorem 1.2. To this end, we must verify that conditions (8) and (9) hold at each step. At the beginning we have $A_0 = 0$ and $Z = Id$, so we must choose a barrier b_0 such that:

$$-\frac{\|U\|_{\mathrm{HS}}^2}{b_0} \leqslant -\|D\|_{\mathrm{HS}}^2 - \frac{\|U\|_2^2}{\delta} \tag{13}$$

and

$$b_0 \leqslant \delta\frac{\|U\|_{\mathrm{HS}}^2}{\|U\|_2^2}. \tag{14}$$

We choose

$$b_0 := \varepsilon \frac{\|U\|_{\mathrm{HS}}^2}{\|D\|_{\mathrm{HS}}^2} \quad \text{and} \quad \delta := \frac{\varepsilon}{1-\varepsilon} \frac{\|U\|_2^2}{\|D\|_{\mathrm{HS}}^2},$$

and we note that (13) and (14) are verified. Also, at each step (8) holds because $\phi_{l+1}(A_{l+1}) \leqslant \phi_l(A_l)$. Since $\|ZU\|_{\mathrm{HS}}^2$ decreases at each step by at most $\|U\|_2^2$, the right-hand side of (9) decreases by at most δ, and therefore (9) holds once we replace b_l by $b_l - \delta$.

Finally note that, after $k = (1-\varepsilon)^2 \frac{\|U\|_{\mathrm{HS}}^2}{\|U\|_2^2}$ steps, the barrier will be

$$b_k = b_0 - k\delta = \varepsilon^2 \frac{\|U\|_{\mathrm{HS}}^2}{\|D\|_{\mathrm{HS}}^2}.$$

This completes the proof.

4. Projection on coordinate subspaces

Given an $n \times m$ rectangular matrix, a theorem of Kashin-Tzafriri (see [10]) allows us to find a coordinate projection of arbitrary size (say λm with $\lambda < 1$) and to estimate its norm in terms of the norm of the matrix and λ. The precise formulation of the result (see [16] for another formulation and proof) is the following.

Theorem 4.1 (Kashin-Tzafriri)**.** *Let U be an $n \times m$ matrix. Fix λ with $1/m \leqslant \lambda \leqslant \frac{1}{4}$. Then, there exists a subset ν of $\{1, \ldots, m\}$ of cardinality $|\nu| \geqslant \lambda m$ such that*

$$\|U_\nu\| \leqslant c \left(\sqrt{\lambda} \|U\|_2 + \frac{\|U\|_{\mathrm{HS}}}{\sqrt{m}} \right),$$

where $U_\nu = UP_\nu$ and P_ν denotes the coordinate projection onto $\mathbb{R}^\nu$.

The proof of this theorem uses a standard selection argument combined with a factorization theorem of Grothendieck. We will give an alternative proof of this result which exploits the method introduced by Batson-Spielman-Srivastava in [2]. This allows us to slightly improve Theorem 4.1 by obtaining an estimate for all possible values of the rank of the projection with the constant c replaced by $\frac{\sqrt{2}}{\sqrt{1-\lambda}}$.

Theorem 4.2. *Let U be an $n \times m$ matrix and let $0 < \lambda \leqslant \eta < 1$. Then, there exists $\sigma \subset \{1, \ldots, m\}$ with $|\sigma| = k \geqslant \lambda m$ such that*

$$\|U_\sigma\|_2 \leqslant \frac{1}{\sqrt{1-\lambda}} \left(\sqrt{\lambda + \eta} \|U\|_2 + \sqrt{1 + \frac{\lambda}{\eta}} \frac{\|U\|_{\mathrm{HS}}}{\sqrt{m}} \right),$$

In particular,

$$\|U_\sigma\|_2 \leqslant \frac{\sqrt{2}}{\sqrt{1-\lambda}} \left(\sqrt{\lambda} \|U\|_2 + \frac{\|U\|_{\mathrm{HS}}}{\sqrt{m}} \right),$$

where U_σ denotes the selection of the columns of U with indices in σ.

Proof. We denote by $(e_j)_{j \leqslant m}$ the canonical basis of $\mathbb{R}^m$. Since

$$U_\sigma \cdot U_\sigma^t = \sum_{j \leqslant \sigma} (Ue_j) \cdot (Ue_j)^t,$$

our problem reduces to the question of estimating the largest eigenvalue of this sum of rank one matrices. We will follow the same procedure as in the proof of the restricted invertibility theorem: at each step, we would like to add a column of the original matrix and then study the evolution of the largest eigenvalue. However, it will be convenient for us to add suitable multiples of the columns of U in order to construct the k-th matrix; for each k we will choose a subset σ_k of cardinality $|\sigma_k| = k$ and consider the matrix $A_k = \sum_{j \in \sigma_k} s_j (Ue_j) \cdot (Ue_j)^t$ where

$(s_j)_{j\in\sigma}$ will be positive numbers which will be suitably chosen. At the step l, the barrier will be denoted by b_l, namely the eigenvalues of A_l will be all smaller than b_l. The corresponding potential is $\phi_l(A_l) := \mathrm{Tr}\,(U^t(b_lI - A_l)^{-1}U)$. We set $A_0 = 0$, while b_0 will be determined later.

As we did before, at each step the value of the potential $\phi_l(A_l)$ will decrease so that we can continue the iteration, while the value of the barrier will increase by a constant δ, i.e. $b_{l+1} = b_l + \delta$. We will use a lemma which appears as Lemma 3.4 in [13]. We state it here in the notation introduced above.

Lemma 4.3. *Assume that* $\lambda_{\max}(A_l) \leqslant b_l$. *Let* v *be a vector in* $\mathbb{R}^n$ *satisfying*

$$\mathbb{V}_l(v) := \frac{v^t(b_{l+1}I - A_l)^{-2}v}{\phi_l(A_l) - \phi_{l+1}(A_l)}\|U\|_2 + v^t(b_{l+1}I - A_l)^{-1}v \leqslant \frac{1}{s}.$$

Then, if we define $A_{l+1} = A_l + svv^t$ *we have*

$$\lambda_{\max}(A_{l+1}) \leqslant b_{l+1} \quad and \quad \phi_{l+1}(A_{l+1}) \leqslant \phi_l(A_l).$$

We write α for the initial potential, i.e. $\alpha = \frac{\|U\|_{\mathrm{HS}}^2}{b_0}$. Suppose that $A_l = \sum_{j\in\sigma_l} s_j\,(Ue_j)\cdot(Ue_j)^t$ is constructed so that $\phi_l(A_l) \leqslant \phi_{l-1}(A_{l-1}) \leqslant \alpha$ and $\lambda_{\max}(A_l) \leqslant b_l$. We will now use Lemma 4.3 in order to construct A_{l+1}. To this end, we must find a vector Ue_j not chosen before and a scalar s_{l+1} so that $\mathbb{V}_l(Ue_j) \leqslant \frac{1}{s_{l+1}}$, and then use the lemma. We choose j by calculating the sum of $\mathbb{V}_l(Ue_j)$ over all $j \notin \sigma_l$. Since $(b_lI - A_l)^{-1}$ and $(b_{l+1}I - A_l)^{-1}$ are positive semidefinite, one can easily check that

$$(b_lI - A_l)^{-1} - (b_{l+1}I - A_l)^{-1} \succeq \delta(b_{l+1}I - A_l)^{-2}.$$

Therefore,

$$\mathrm{Tr}\left(U^t(b_{l+1}I - A_l)^{-2}U\right) \leqslant \frac{1}{\delta}\left(\phi_l(A_l) - \phi_{l+1}(A_l)\right).$$

It follows that

$$\begin{aligned}\sum_{j\notin\sigma_l}\mathbb{V}_l(Ue_j) \leqslant \sum_{j\leqslant m}\mathbb{V}_l(Ue_j) &= \frac{Tr\,(U^t(b_{l+1}I - A_l)^{-2}U)}{\phi_l(A_l) - \phi_{l+1}(A_l)}\|U\|_2^2 + \phi_{l+1}(A_l)\\ &\leqslant \frac{\|U\|_2^2}{\delta} + \alpha,\end{aligned}$$

and therefore one can find $i \notin \sigma_l$ such that

$$\mathbb{V}_l(Ue_i) \leqslant \frac{1}{|\sigma_l^c|}\left(\frac{\|U\|_2^2}{\delta} + \alpha\right) \leqslant \frac{1}{|\sigma_k^c|}\left(\frac{\|U\|_2^2}{\delta} + \alpha\right),$$

where k is the maximum number of steps (which is in our case λm).
We are going to choose all s_j equal to $s := \frac{(1-\lambda)m}{\alpha + \frac{\|U\|_2^2}{\delta}}$. By the previous lemma, it is sufficient to take $A_{l+1} = A_l + s\,(Ue_i)\cdot(Ue_i)^t$. After $k = \lambda m$ steps, we get $\sigma = \sigma_k$ such that

$$\begin{aligned}\lambda_{\max}\left(\sum_{j\in\sigma_k}(Ue_j)\cdot(Ue_j)^t\right) &\leqslant \frac{1}{s}(b_0 + k\delta) = \frac{\alpha + \frac{\|U\|_2^2}{\delta}}{(1-\lambda)m}(b_0 + k\delta)\\ &= \frac{1}{1-\lambda}\left[\frac{\|U\|_{\mathrm{HS}}^2}{m} + \lambda\|U\|_2^2 + \lambda\|U\|_{\mathrm{HS}}^2\frac{\delta}{b_0} + \frac{\|U\|_2^2}{m}\frac{b_0}{\delta}\right]\end{aligned}$$

The result follows by taking $b_0 = \eta m\delta$. □

Aknowledgement. I am grateful to my PhD advisor Olivier Guédon for many helpful discussions. I would also like to thank the doctoral school of Paris-Est for giving me the opportunity to visit the University of Athens and Apostolos Giannopoulos for his hospitality and help with the final version of this paper.

References

[1] K. Ball. Ellipsoids of maximal volume in convex bodies. *Geom. Dedicata*, 41(2):241–250, 1992.
[2] J. D. Batson, D. A. Spielman, and N. Srivastava. Twice-Ramanujan sparsifiers. In *STOC'09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 255–262. ACM, New York, 2009.
[3] J. Bourgain and S. J. Szarek. The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization. *Israel J. Math.*, 62(2):169–180, 1988.
[4] J. Bourgain and L. Tzafriri. Invertibility of large submatrices with applications to the geometry of banach spaces and harmonic analysis. *Israel J. Math.*, 57:137–224, 1987.
[5] J. Diestel. *Absolutely Summing Operators*. Cambridge University Press, 1995.
[6] A. A. Giannopoulos. A note on the Banach-Mazur distance to the cube. In *Geometric aspects of functional analysis (Israel, 1992–1994)*, volume 77 of *Oper. Theory Adv. Appl.*, pages 67–73. Birkhäuser, Basel, 1995.
[7] A. A. Giannopoulos. A proportional Dvoretzky-Rogers factorization result. *Proc. Amer. Math. Soc.*, 124(1):233–241, 1996.
[8] E. D. Gluskin. The diameter of the Minkowski compactum is roughly equal to n. *Funktsional. Anal. i Prilozhen.*, 15(1):72–73, 1981.
[9] F. John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.
[10] B. Kashin and L. Tzafriri. Some remarks on the restriction of operators to coordinate subspaces. preprint.
[11] M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Springer, Berlin, 1991.
[12] D. Spielman and N. Srivastava. An elementary proof of the restricted invertibility theorem. to appear in Israel J. Math.
[13] N. Srivastava. *Spectral Sparsification and Restricted Invertibility*. PhD thesis, Yale University, Mar. 2010.
[14] S. J. Szarek and M. Talagrand. An "isomorphic" version of the Sauer-Shelah lemma and the Banach-Mazur distance to the cube. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 105–112. Springer, Berlin, 1989.
[15] S. Taschuk. The banach-mazur distance to the cube in low dimensions. *Discrete Computational Geometry*, 46(2):175–183, 2011.
[16] R. Vershynin. John's decompositions: selecting a large part. *Israel J. Math.*, 122:253–277, 2001.

Pierre Youssef,
Université Paris-Est Marne-La-Vallée
Laboratoire d'Analyse et Mathématiques Appliquées,
5, boulevard Descartes, Champs sur Marne,
77454 Marne-la-Vallée, Cedex 2, France
e-mail: pierre.youssef@univ-mlv.fr