A Family of Exact, Analytical Time Dependent Wave Packet Solutions to a Nonlinear Schrödinger Equation

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We obtain time dependent q-Gaussian wave-packet solutions to a non linear Schrödinger equation recently advanced by Nobre, Rego-Montero and Tsallis (NRT) [Phys. Rev. Lett. **106** (2011) 10601]. The NRT non-linear equation admits plane wave-like solutions (q-plane waves) compatible with the celebrated de Broglie relations connecting wave number and frequency, respectively, with energy and momentum. The NRT equation, inspired in the q-generalized thermostatistical formalism, is characterized by a parameter q, and in the limit $q \to 1$ reduces to the standard, linear Schrödinger equation. The q-Gaussian solutions to the NRT equation investigated here admit as a particular instance the previously known q-plane wave solutions. The present work thus extends the range of possible processes yielded by the NRT dynamics that admit an analytical, exact treatment. In the $q \to 1$ limit the q-Gaussian solutions correspond to the Gaussian wave packet solutions to the free particle linear Schrödinger equation. In the present work we also show that there are other families of nonlinear Schrödinger-like equations, besides the NRT one, exhibiting a dynamics compatible with the de Broglie relations. Remarkably, however, the existence of time dependent Gaussian-like wave packet solutions is a unique feature of the NRT equation not shared by the aforementioned, more general, families of nonlinear evolution equations.

I. INTRODUCTION

A nonlinear Schrödinger equation has been recently advanced by Nobre, Rego-Monteiro and Tsallis [1, 2]. The NRT proposal constitutes an intriguing contribution to a line of enquiry that has been the focus of continuous research activity for several years: the exploration of non-linear versions of some of the fundamental equations of physics [3, 4]. The NRT equation is inspired in the thermostatistical formalism based upon the Tsallis S_q non-additive, power-law entropic functional, whose applications to the study of diverse physical system and processes have attracted considerable attention in recent years (see, for instance, [5–9] and references therein). In particular, the S_q entropy constitutes a useful tool for the analysis of diverse problems in quantum physics [10–17].

The NRT nonlinear Schrödinger equation governing the field $\Phi(x,t)$ ("wave function") corresponding to a particle of mass m reads [1, 2],

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\frac{\Phi(x,t)}{\Phi_0}\right]^{2-q}, \tag{1}$$

where the scaling constant Φ_0 guarantees an appropriate physical normalization for the different terms appearing in the equation, i is the imaginary constant, $\hbar$ is Planck's constant, and q is a real parameter formally associated with Tsallis' entropic index in non-extensive thermostatistics [6]. It was shown in [1] that the wave equation (1) admits time dependent solutions having the "q-plane wave" form,

$$\Phi(x,t) = \Phi_0 \left[1 + (1-q)i(kx - wt)\right]^{\frac{1}{1-q}}, \tag{2}$$

with k and w real parameters having, respectively, dimensions of inverse length and inverse time (that is, k can be regarded as a wave number and w as a frequency). In the limit $q \to 1$, the q-plane waves (2) reduce to the plane wave solutions $\Phi_0 \exp(-i(kx - wt))$ of the standard, linear Schrödinger equation describing a free particle of mass m.

The q-plane wave solutions (2) propagate at a constant velocity $c = w/k$ without changing shape, thus exhibiting a soliton-like behaviour. Moreover, and in contrast to the $q \to 1$ case yielding standard plane waves, the solutions corresponding to $q \neq 1$ don't have a spatially constant modulus. In fact (defining $\psi = \Phi/\Phi_0$) we have,

$$|\psi(x,t)|^2 = \left[1 + (1-q)^2(kx - wt)^2\right]^{\frac{1}{1-q}}, \tag{3}$$

which corresponds, for $1 < q < 3$ to a normalizable q-Gaussian centered at $x = wt/k$. Therefore, in this case the q-plane wave solution describes a phenomenon characterized by a certain degree of spatial localization. A field-theoretical approach to the NRT equation was developed in [2], where it was shown that this equation can be derived from a variational principle. The nonlinear NRT equation is formally related to the nonlinear Fokker-Planck equation (NLFP) with a diffusion term depending on a power of the density. These kind of evolution equations, and their relations with the nonextensive thermostatitical formalism, have been the focus of an intensive research recently [18–27]. In spite of the formal resemblance between the NRT Schrödinger equation and the nonlinear Fokker-Planck, there are profound differences between these two types of equations. For instance, the nonlinear Fokker-Planck equation does not admit q-plane wave solutions of the form (2), that propagate without changing their shape.

A property of the solutions (2) that was highlighted by NRT [1] is that they are consistent with the celebrated de Broglie relations [28],

$$\begin{aligned} E &= \hbar w, \\ p &= \hbar k, \end{aligned} \tag{4}$$

connecting, respectively, energy with frequency and momentum with wave number. Indeed, it can be verified that the q-plane wave (2) satisfies the equation (1) if and only if the parameters w and k comply with the relation,

$$w = \frac{\hbar k^2}{2m}, \tag{5}$$

which, combined with (4), lead to the standard relation between linear momentum and kinetic energy,

$$E = \frac{p^2}{2m}. \tag{6}$$

This suggests that it is conceivable that the q-plane wave (2) represents a particle of mass m with kinetic energy $\hbar w$ and momentum $\hbar k$ [1].

Wave packets (in particular Gaussian wave packets) are of paramount importance in quantum mechanics, both from the conceptual and practical points of view [28, 29]. Wave packets also played a distinguished role in the historical development of quantum physics [30]. It is interesting to explore the existence of time dependent wave packet solutions of the NRT nonlinear Schrödinger equation. The original presentation of the NRT equation [1] was strongly focused upon the q-plane wave solutions. However, as a first step towards elucidating the meaning of the NRT equation, it is imperative to explore more general solutions. The aim of the present effort is to investigate a family of exact analytical time dependent solutions to the NRT equation exhibiting the form of q-Gaussian wave packets and corresponding, in the limit $q \to 1$, to the celebrated Gaussian wave packets solutions of the linear Schrödinger equation.

II. TIME DEPENDENT WAVE PACKET SOLUTIONS

In this work we are going to investigate solutions to the NRT equation based upon the q-Gaussian wave packet ansatz,

$$\psi(x,t) = \frac{\Phi(x,t)}{\Phi_0} = \left[1-(1-q)(a(t)x^2+b(t)x+c(t))\right]^{\frac{1}{1-q}}, \tag{7}$$

where a, b, and c are appropriate (complex) time dependent coefficients. Notice that ψ depends on time only through these three parameters. Inserting the ansatz (7) into the left and the right hand sides of the NRT equation (1) one obtains,

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar(\dot{a}(t)x^2+\dot{b}(t)x+\dot{c}(t))\psi^q, \tag{8}$$

and,

$$-\frac{1}{2-q}\frac{\partial^2\psi^{2-q}}{\partial x^2} = \Big[-2(3-q)a(t)^2x^2-2(3-q)a(t)b(t)x+2a(t)-2(1-q)a(t)c(t)-b(t)^2\Big]\psi^q. \tag{9}$$

Combining now equations (1), (8), and (9) one sees that the ansatz (7) constitutes a solution of the NRT equation provided that the coefficients a, b, and c comply with the set of coupled ordinary differential equations,

$$i\dot{a}(t) = \frac{\hbar}{m}(3-q)a(t)^2 \tag{10}$$

$$i\dot{b}(t) = \frac{\hbar}{m}(3-q)a(t)b(t) \tag{11}$$

$$i\dot{c}(t) = \frac{\hbar}{m}\left((1-q)a(t)c(t)-a(t)+\frac{b(t)^2}{2}\right). \tag{12}$$

The above set of differential equations admits the general solution,

$$a(t) = \frac{1}{\frac{(3-q)i\hbar t}{m} + \alpha} \tag{13}$$

$$b(t) = \frac{\beta}{\frac{(3-q)i\hbar t}{m} + \alpha} \tag{14}$$

$$c(t) = \left(\frac{(3-q)i\hbar}{m}t + \alpha\right)^{-\frac{1-q}{3-q}} \left[\frac{\left(\frac{(3-q)i\hbar}{m}t + \alpha\right)^{\frac{1-q}{3-q}}}{1-q} + \frac{\beta^2}{4}\left(\frac{(3-q)i\hbar}{m}t + \alpha\right)^{\frac{1-q}{3-q}-1} + \gamma - \frac{1}{1-q}\right], \tag{15}$$

where α, β, and γ are integrations constants determined by the initial conditions $a(0)$, $b(0)$, and $c(0)$,

$$\alpha = \frac{1}{a(0)}, \tag{16}$$

$$\beta = \frac{b(0)}{a(0)}, \tag{17}$$

$$\gamma = a(0)^{\frac{q-1}{3-q}} \left(c(0) - \frac{1}{1-q} - \frac{1}{4}\frac{b(0)^2}{a(0)}\right) + \frac{1}{1-q}. \tag{18}$$

A. Limit Case $q \to 1$

Let us now briefly consider the limit $q \to 1$ of the evolving q-Gaussian wave packet. In this case the time dependent parameters $a(t)$, $b(t)$, and $c(t)$ are,

$$a(t) = \frac{1}{\frac{2i\hbar t}{m} + \alpha} \tag{19}$$

$$b(t) = \frac{\beta}{\frac{2i\hbar t}{m} + \alpha} \tag{20}$$

$$c(t) = \frac{1}{2}\ln\left(\frac{2i\hbar t}{m} + \alpha\right) + \frac{1}{4}\left(\frac{\beta^2}{\frac{2i\hbar t}{m} + \alpha}\right) + \gamma, \tag{21}$$

with α, β and γ integration constants. The general solution can be written as

$$\begin{aligned} \psi &= \lim_{q\to 1}\left[1 - (1-q)(a(t)x^2 + b(t)x + c(t))\right]^{\frac{1}{1-q}} \\ &= \exp\left(-\frac{1}{2}\ln\left(\frac{2i\hbar t}{m} + \alpha\right) - \frac{(\beta/2 + x)^2}{\frac{2i\hbar t}{m} + \alpha} + \gamma\right). \end{aligned} \tag{22}$$

Taking now $\exp(4\gamma) = \frac{2\alpha}{\pi}\exp(-\alpha)$ and defining $k_0 = i\beta/\alpha$, and $\tan 2\theta = \frac{2\hbar t}{m\alpha}$, (22) can be cast under the guise,

$$\psi = \left(\frac{2\alpha/\pi}{\frac{4\hbar^2 t^2}{m^2} + \alpha^2}\right)^{1/4} \exp\left(-i\,\left(\theta + \hbar k_0^2 t/2m\right)\right)\exp(ik_0 x)\exp\left(-\frac{(x - \hbar k_0 t/m)^2}{\frac{2i\hbar t}{m} + \alpha}\right), \tag{23}$$

recovering the well known Gaussian wave packet solution of the standard linear Schrödinger equation.

B. Illustrative Example of Wave Packet Evolution for $q = 2$.

The evolution of the time dependent q-Gaussian solution is illustrated in Figure 1, where the square modulus $|\psi|^2$ of an initially localized solution is depicted against the (nondimensional) time variable $\bar{t} = \left(\frac{\hbar a_0}{m}\right) t$ and spatial coordinate $\bar{x} = \frac{1}{\sqrt{a_0}} x$, for $q = 2$ (here $a_0 = |a(0)|$ stands for the modulus of the intial value of the parameter a). It is interesting that in the nonlinear ($q = 2$) case, as time progresses, $|\psi(x,t)|^2$ develops two peaks that depart from each other. This behaviour exhibits some qualitative similarity with the evolution of an initially localized particle in the tight binding model [31].

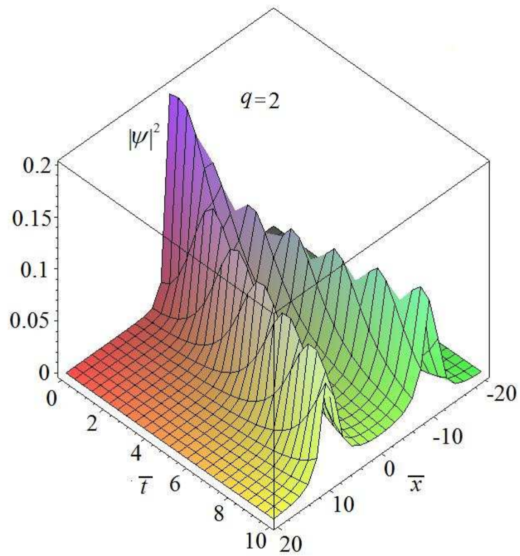

FIG. 1: Behaviour of $|\psi(x,t)|^2$ determined by the NRT nonlinear Schrödinger equation with $q = 2$ and initial conditions given by $\alpha = 1$, $\beta = 1$, and $\gamma = 1$. All depiced quantities are dimensionless.

C. The Case $q = 3$ and "Frozen" Solutions

When $q = 3$ the set of coupled differential goberning the evolution of the parameters a, b, and c, admit the solution,

$$\begin{aligned} a &= a_c, \\ b &= b_c, \\ c &= \frac{b_c^2 - 2a_c}{4a_c} + c_1 \exp\left(2i\frac{\hbar}{m}a_c t\right), \end{aligned} \tag{24}$$

where a_c, b_c, and c_1, are time independent constants. The concomitant solution to the NRT equation reads,

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left[a_c x^2 + b_c x + \frac{b_c^2}{4a_c} + c_1 \exp\left(2i\frac{\hbar}{m}a_c t\right)\right]^{-\frac{1}{2}}. \tag{25}$$

It is interesting that the solution (25) is periodic (with period, in the dimensionless time variable $\bar{t} = \frac{\hbar}{m} a_c t$, equal to π) even though the particle is not subjected to an external confining potential. Indeed, this solution describes a quasi-stationary scenario where the shape of $|\psi(x,t)|^2$ "pulsates" with the abovementioned period. In the extreme case given by $c_1 = 0$, the amplitude of the "pulsations" vanishes and we obtain the stationary solution,

$$\psi(x) \;=\; \frac{1}{\sqrt{2}}\left[a_c x^2 + b_c x + \frac{b_c^2}{4a_c}\right]^{-\frac{1}{2}}. \tag{26}$$

The norm $N = \int |\psi|^2 dx$ of this last solution is finite provided that $a_c \neq 0$ and b_c/a_c is not a real number.

It is interesting that there are "frozen" solutions like (26) also for instances of the NRT equation characterized by other values of q. Indeed, it can be verified that the NRT equation admits stationary solutions of the form

$$\psi(x) \;=\; [bx + ic]^{\frac{1}{2-q}}, \tag{27}$$

with $b \neq 0$ and $c \neq 0$ real constants. The square modulus $|\psi|^2$ of the "frozen" solutions (27) has a q-Gaussian profile,

$$|\psi(x)|^2 \;=\; \left[b^2x^2 + c^2\right]^{\frac{1}{2-q}} \tag{28}$$

and a finite norm for $2 < q < 4$. Notice that the solutions (27) can be cast under the guise of a q-exponential, but with a q-value given by $\tilde{q} = q - 1$, which is different from the q-value associated with the concomitant nonlinear NRT evolution equation. Therefore, strictly speaking, the solutions (27) are not comprised within those corresponding to the ansatz (7).

D. q-Plane Waves and Related Particular Solutions

The case $a = 0$ leads to a family of particular solution of the set of equations (10), given by,

$$\begin{aligned} a &= 0, \\ b &= b_c, \\ c &= -i\frac{\hbar b_c^2}{2m}\,t + c_0 \end{aligned} \tag{29}$$

where b_c and c_0 are complex constants. A simple but important case is obtained when the constant b_c is purely imaginary and $c_0 = 0$,

$$\begin{aligned} a &= 0, \\ b &= -ik, \quad k \in \mathbf{R}, \\ c &= i\frac{\hbar k^2}{2m}\,t, \end{aligned} \tag{30}$$

which, after setting $w = \hbar k^2/2m$, is clearly seen to correspond to the q-plane wave solutions considered in [1]. Therefore, the exact solutions to the NRT equation studied in [1] constitute a particular case of the time dependent q-Gaussian wave packet investigated here.

Another interesting particular instance of the solutions associated with (29) corresponds to the case where the constant b_c is a real number and $c_0 = -i\frac{\hbar b_c^2}{2m}t_0$, with $t_0 > 0$ a real constant with dimensions of time, leading to,

$$\psi(x,t) \;=\; \left[1 + (1-q)\left(-b_c x + \frac{i\hbar}{2m}b_c^2(t+t_0)\right)\right]^{\frac{1}{1-q}}. \tag{31}$$

The squared modulus of the above soution has a q-Gaussian form,

$$|\psi(x,t)|^2 \;=\; \left[\frac{(1-q)^2\hbar^2}{4m^2}b_c^4(t+t_0)^2\right]^{\frac{1}{1-q}}\left\{1 + \left[\frac{2m(1-(1-q)b_c x)}{(1-q)\hbar b_c^2(t+t_0)}\right]^2\right\}^{\frac{1}{1-q}}, \tag{32}$$

leading to a finite norm, $N = \int |\psi|^2 dx$, for $t > -t_0$ and $1 < q < 3$ (see next subsection). The solution (31) exhibits a finite-time singularity in the past: indeed, its norm diverges at the finite time $t = -t_0$. In the limit $q \to 1$ the solutions (31) go to

$$\psi(x,t) \;=\; \exp(-b_c x) \exp\left[\frac{i\hbar}{2m} b_c^2 (t+t_0)\right], \tag{33}$$

which are formal solutions of the standard linear Schrödinger equation but are, evidently, physically unacceptable because they are not normalizable (and the wave function's square modulus $|\psi|^2$ itself diverges when $xb_c \to -\infty$). It is nevertheless interesting that the non-linearity associated with $q > 1$ not only regularizes (in the sense of leading to a finite norm N) the plane wave solutions of the Schrödinger equation (as stressed by NRT in [1]) but it also regularizes the exponential solutions (33) (at least for all times $t > -t_0$).

E. (Non-)Preservation of the Norm

It is known that, in general, time dependent solutions to the NRT nonlinear Schrödinger equation do not preserve the norm [2]. The q-plane wave solutions studied in [2] constitute a remarkable exception: they do preserve the norm. Up to now the q-plane wave solutions where the only known time dependent solutions to the NRT equation. This means that no norm non-preserving explicit solution was known before our present work. Therefore, it is of some interest to explore whether the solutions to the NRT equation investigated here preserve the norm or not. The norm N of the q-Gaussian wave packet is given by,

$$N = \int_{-\infty}^{\infty} dx |\psi|^2, \tag{34}$$

where

$$|\psi|^2 = \left[1 - 2(1-q)\Re(a(t)x^2 + b(t)x + c(t)) + (1-q)^2 |a(t)x^2 + b(t)x + c(t)|^2\right]^{\frac{1}{1-q}}. \tag{35}$$

When $a{\neq}0$ the q-Gaussian wave packet is normalizable (that is, $N < \infty$) provided that $1 < q < 5$ and the polynomial $P(z) = 1 - (1-q)(a(t)z^2 + b(t)z + c(t))$ doesn't have real roots. Notice that for $a \neq 0$ the range of q-values admitting normalizable q-Gaussian wave functions of the form (7) is larger than the range of q-values leading to normalizable q-plane wave functions (which is $1 < q < 3$ [1]).

Let us now consider a concrete instance of a solution to the NRT equation that does not preserve the norm. In the case of the solution (31) the norm is,

$$\begin{aligned} N &= \frac{1}{|1-q|b_c} \left[\frac{(1-q)^2\hbar^2}{4m^2} b_c^4 (t+t_0)^2\right]^{\frac{1}{2}+\frac{1}{1-q}} \int_{-\infty}^{+\infty} du \left[1+u^2\right]^{\frac{1}{1-q}} \\ &= \frac{1}{|1-q|b_c} \left[\frac{(1-q)^2\hbar^2}{4m^2} b_c^4 (t+t_0)^2\right]^{\frac{1}{2}+\frac{1}{1-q}} \sqrt{\pi}\, \frac{\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\Gamma\left(\frac{1}{q-1}\right)}. \end{aligned} \tag{36}$$

The norm is finite for $t > -t_0$ and $1 < q < 3$ and proportional to $(t+t_0)^{1+\frac{2}{1-q}}$. Therefore, the norm is not conserved: it vanishes in the limit $t \to +\infty$, it is a finite and monotonously decreasing function of time for all finite times $t > t_0$, and it diverges at $t = -t_0$. Consequently, (31) constitutes an explicit example of a norm non-preserving solution to the NRT equation.

As a final comment, let us mention that the non conservation of the norm suggests that, when considering pairs of time dependent solutions ψ_1 and ψ_2 to the NRT equation, the overlap $\int \psi_1^* \psi_2 dx$ is not conserved either. However, there may still be some conserved measure of the "distance" or "fidelity" between pairs of time dependent solutions. The search for such a measure constitutes a line of enquiry that may shed new light on the nature of the NRT dynamics. A possible direction to explore in this regard would be the one suggested by Yamano and Iguchi in [32], where non-Csiszar f-divergence measures where considered in connection with nonlinear Liouville-type equations.

III. WAVE PACKET SOLUTIONS IN THE PRESENCE OF AN HARMONIC CONFINING POTENTIAL

Now we are going to use the q-Gaussian ansatz (7) to investigate time dependent solutions of nonlinear Schrödinger equation given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2\psi^{2-q}}{\partial x^2} + V(x)\psi^q, \tag{37}$$

where $V(x) = \frac{1}{2}Kx^2$. In the limit $q \to 1$ the above equation reduces to the time dependent Schrödinger equation of the quantum harmonic oscillator. Inserting the ansatz (7) in the right hand side of equation (37) we get

$$\begin{aligned} -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2\psi^{2-q}}{\partial x^2} + V(x)\psi^q &= \frac{\hbar^2}{2m}\left[2a(t)(1-(1-q)(a(t)x^2+b(t)x+c(t))) - (2a(t)x+b(t))^2\right]\psi^q + \frac{1}{2}Kx^2\psi^q \\ &= \frac{\hbar^2}{2m}\left[\left(\frac{m}{\hbar^2}K - 2(3-q)a(t)^2\right)x^2 - 2(3-q)a(t)b(t)x + 2a(t) - 2(1-q)a(t)c(t) - b(t)^2\right]\psi^q \end{aligned}$$

Comparing Eq.(8) with (9) we obtain the following set of coupled differential equations for the evolution of the parameters appearing in the ansatz

$$\begin{aligned} i\dot{a}(t) &= \frac{\hbar}{m}(3-q)a(t)^2 - \frac{K}{2\hbar} \\ i\dot{b}(t) &= \frac{\hbar}{m}(3-q)a(t)b(t) \\ i\dot{c}(t) &= \frac{\hbar}{m}\left((1-q)a(t)c(t) - a(t) + \frac{b(t)^2}{2}\right). \end{aligned} \tag{38}$$

Integrating the first of these equation one obtains,

$$\frac{1}{\frac{\hbar}{m}(3-q)}\int\frac{da(t)}{\frac{mK}{2\hbar^2(3-q)} - a(t)^2} = \frac{1}{\frac{\hbar}{m}(3-q)}\frac{1}{2\sqrt{\frac{mK}{2\hbar^2(3-q)}}}\log\left|\frac{\sqrt{\frac{mK}{2\hbar^2(3-q)}} + a(t)}{\sqrt{\frac{mK}{2\hbar^2(3-q)}} - a(t)}\right| = it + \delta, \tag{39}$$

where δ is an integration constant.

A. Quasi-Stationary Solutions

A particularly interesting solution of the set of equations (38) is given by,

$$\begin{aligned} a &= a_c = \frac{1}{\hbar}\sqrt{\frac{mK}{2(3-q)}}, \\ b &= 0, \\ c &= \frac{1}{1-q}\left[1 - \exp\left(-i(1-q)\frac{\hbar a_c t}{m}\right)\right], \end{aligned} \tag{40}$$

leading, in turn, to the following solution for the non-linear Schrödinger equation,

$$\psi = \left[\exp\left(-i(1-q)\frac{\hbar a_c t}{m}\right) - (1-q)a_c x^2\right]^{\frac{1}{1-q}}. \tag{41}$$

It is interesting to consider now the $q \to 1$ limit of the above solution,

$$\lim_{q\to1}\psi = \exp\left(-i\frac{\hbar a_c t}{m}\right)\lim_{q\to1}\left[1-(1-q)a_c\exp\left(i(1-q)\frac{\hbar a_c t}{m}\right)x^2\right]^{\frac{1}{1-q}}$$
$$= \exp\left(-i\frac{\hbar a_c t}{m}\right)\exp\left(-a_c x^2\right)$$
$$= \exp\left(-i\frac{\omega t}{2}\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right), \tag{42}$$

which is the (unnormalized) wave function associated with the ground state of a standard harmonic oscillator with natural frequency $\omega = \sqrt{\frac{K}{m}}$ and zero point energy $E_0 = \frac{1}{2}\hbar\omega$.

The norm of the solution (41) is finite for $1 < q < 5$ and $-t_c < t < t_c$ with $t_c = \frac{\pi m}{(q-1)\hbar a_c}$, and it has finite-time sigularities at $t = \pm t_c$. The time derivative of the norm of the solution (41) is,

$$\frac{dN}{dt} = 2(1-q)\frac{\hbar a_c^2}{m}\sin\left[(1-q)\frac{\hbar a_c t}{m}\right]\int_{-\infty}^{\infty}x^2\left[1-2a_c(1-q)\cos\left[(1-q)\frac{\hbar a_c t}{m}\right]x^2+(1-q)^2a_c^2x^4\right]^{\frac{q}{1-q}}dx. \tag{43}$$

The expression between square brackets appearing in the integrand in the right hand side of the above equation is in general larger than zero and, consequently, the time derivative of the norm is different from zero. Therefore, the time dependent solution (41) constitutes another explicit example of a solution that does not preserve the norm. In the limit $q \to 1$ we get $|t_c| \to \infty$ and, of course, $dN/dt \to 0$.

IV. GENERALIZATIONS OF THE NRT APPROACH AND UNIQUE FEATURES OF THE NRT EQUATION

The existence of q-plane wave solutions consistent with the de Broglie relations connecting frequency and wave number respectively with energy and momentum was one of the features of NRT equation discussed in [1]. It is worth noticing that there are other possible nonlinear Schrödinger-like equations exhibiting similar properties. That is, the NRT approach can be substantially generalized. Let us consider a pair of one-variable functions $L(u)$ and $F(u)$ satisfying the functional relation,

$$\frac{d^2}{du^2}\left[L(F(u))\right] = \frac{dF}{du}. \tag{44}$$

It can then be verified after some algebra that the (in general non linear) Schrödinger -like equation

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[L\left(\frac{\Phi(x,t)}{\Phi_0}\right)\right] \tag{45}$$

admits exact time dependent plane wave-like solutions of the form,

$$\Phi(x,t) = \Phi_0 F[i(kx-wt)], \tag{46}$$

with $\hbar w = \hbar^2k^2/2m$. Indeed, inserting the ansatz (46) into the right and left hand sides (45) and setting $u = i(kx-wt)$ we get,

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = \left(\frac{dF}{du}\right)\left(\frac{\partial u}{\partial t}\right) = \hbar w\,\frac{dF}{du},$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[L\left(\frac{\Phi(x,t)}{\Phi_0}\right)\right] = -\frac{\hbar^2}{2m}\frac{d^2}{du^2}\left[L\left(u\right)\right]\left(\frac{\partial u}{\partial x}\right)^2 = \frac{\hbar^2k^2}{2m}\,\frac{d^2}{du^2}\left[L(F(u))\right], \tag{47}$$

where, in the last equation, the fact that $\partial^2u/\partial x^2 = 0$ was used. It is plain that the functional relation (44) implies (45), provided that $\hbar w = \hbar^2k^2/2m$. This in turn leads, via the de Broglie relations, to the kinetic energy-momentum relation $E = p^2/2m$. Similarly to what happens within the NRT scenario, the solutions (46) to the non linear evolution equation (45) propagate without changing shape and with constant velocity $v = w/k$. Furthermore, as we have seen, they comply with a relation between w and k that is consistent with the de Broglie connection between frequency, wave number, energy, and momentum.

One procedure to generate pairs of functions satisfying (44) is the following. One starts with a function $G(u)$ such that its derivative dG/du admits an inverse. Then we define,

$$\begin{aligned} F(u) &= \frac{dG}{du}, \\ L(u) &= G\left[F^{(-1)}(u)\right], \end{aligned} \tag{48}$$

where $F^{(-1)}(u)$ is the inverse function of $F(u)$, satisfing $F^{(-1)}(F(u)) = u$. It can be verified that the functions defined by (48) comply with the required relation (44).

In the case of the NRT equation we have,

$$\begin{aligned} G(u) &= \frac{1}{2-q}[1+(1-q)u]^{\frac{2-q}{1-q}}, \\ F(u) &= [1+(1-q)u]^{\frac{1}{1-q}}, \\ L(u) &= \frac{u^{2-q}}{2-q}. \end{aligned} \tag{49}$$

An example (different from the NRT one) of a nonlinear Schrödinger -like equation of the form (45) admitting the plane wave-like solutions (46) is given by $F(u) = \sinh(u)$ and $L(u) = \sqrt{1+u^2}$. This case corresponds to the nonlinear evolution equation,

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left\{1+\left[\frac{\Phi(x,t)}{\Phi_0}\right]^2\right\}^{\frac{1}{2}}, \tag{50}$$

having plane wave-like solutions

$$\Phi(x,t) = \Phi_0 \sinh[i(kx - wt)], \tag{51}$$

with $\hbar w = \hbar^2 k^2/2m$.

As mentioned in the Introduction, the NRT equation was inspired by the nonextensive generalized thermostatistical formalism based on the constrained optimization of Tsallis' entropy [1]. Indeed, there are formal connections between the NRT scenario and the alluded thermostatistical formalism. An intriguing question, which is beyond the scope of the present work but certainly deserves to be explored, is the possible existence of connections between (some of) the nonlinear Schrödinger equations (45), on the one hand, and other formalisms based on non-standard entropic functionals different from the Tsallis one [9, 33], on the other hand.

We have seen that the NRT equation is not the only nonlinear equation of the form (45) admitting plane wave-like solutions consistent with the de Broglie relations. It is then natural to ask which of the equations (45) also admit solutions of the form

$$\Phi(x,t) = \Phi_0 F[d_2(t)x^2 + d_1(t)x + d_0(t)], \tag{52}$$

with $d_1(t), d_2(t), d_0(t)$ appropriate time dependent coefficients. The solutions (52) would constitute generalizations of the q-Gaussian solutions to the NRT equation previously discussed in the present work. It turns out that the NRT equation is the only member of the family (45) admitting both solutions of the form (46) and of the form (52). If one inserts in the left and rigth hand sides of the evolution equation (45) the expression for $\psi = \frac{\Phi}{\Phi_0}$ given by the ansatz ansatz (52) one gets,

$$i\hbar\frac{\partial \psi}{\partial t} = i\hbar(\dot{d}_2 x^2 + \dot{d}_1 x + \dot{d}_0)F'(u), \tag{53}$$

and,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Big[L(F(u))\Big] \;=\; -\frac{\hbar^2}{2m}\Big\{(2d_2x+d_1)^2\frac{d^2}{du^2}\Big[L(F(u))\Big]+2d_2\frac{d}{du}\Big[L(F(u))\Big]\Big\}, \tag{54}$$

where $u = d_2(t)x^2 + d_1(t)x + d_0(t)$. Then, to have solutions of the form (46) (plane wave-like), equation (44) must hold. On the other hand, to have solutions of the form (52) one further condition is required,

$$\frac{d}{du}\Big[L(F(u))\Big] \;=\; (r_1 + r_2u)F'(u), \tag{55}$$

with r_1 and r_2 appropriate constants. It follows from (55) that,

$$L''(F(u))F'(u) \;=\; r_2. \tag{56}$$

Combining now equations (44), (55) and (56) one obtains,

$$r_2F'(u) + (r_1 + r_2u)F''(u) \;=\; F'(u), \tag{57}$$

which leads to,

$$F'(u) \;=\; F_0(r_1 + r_2u)^{\frac{1-r_2}{r_2}}, \tag{58}$$

where F_0 is an integration constant. Finally, we have,

$$F(u) \;=\; F_0(r_1 + r_2u)^{\frac{1}{r_2}} + F_1, \tag{59}$$

involving one more integration constant F_1. Making now the identification $r_2 = 1-q$ one sees that (up to multiplicative and additive constants) the form of the solution (59) coincides with q-Gaussian wave packet.

V. CONCLUSIONS

We have obtained a new family of exact, analytical time dependent wave packet solutions to the nonlinear Schrödinger equation recently proposed by Nobre, Rego-Montero and Tsallis [1, 2]. Our solutions have the form of a q-exponential evaluated upon a quadratic function of the spatial coordinate x with time dependent coefficients. Therefore, these solutions have a q-Gaussian form. They extend and generalize the previously known solutions to the NRT equation. The solutions investigated here by us correspond, in the limit $q \to 1$ of the parameter q, to the Gaussian wave packet solutions to the standard linear Schrödinger equation. Our present wave packet solutions admit as a special particular case the q-plane wave solutions studied in [1]. In the present work we also discuss other families of nonlinear Schrödinger-like equations, besides the NRT one, leading to a dynamics compatible with the de Broglie relations. In this regard, it is remarkable that the existence of the Gaussian-like time dependent solutions investigated in this work is a unique feature of the NRT equation not shared by the abovementioned, more general, families of nonlinear evolution equations.

We also obtained q-Gaussian wave packet solutions for the case of a harmonic potential. In the limit $q \to 1$ these latter solutions reduce to Schrödinger's celebrated Gaussian wave packet solutions to the harmonic oscillator. As a particular case of the time dependent q-Gaussian wave packets associated with the harmonic potential we found a quasi-stationary solution yielding in the $q \to 1$ limit the wave function corresponding to the ground state of the quantum harmonic oscillator.

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