A Note on the Voting Problem

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Abstract

Let $v(n)$ be the minimum number of voters with transitive preferences which are needed to generate any strong preference pattern (ties not allowed) on n candidates. Let $k = \lfloor \log_2 n \rfloor$. We show that $v(n) \leq n - k$ if n and k have different parity, and $v(n) \leq n - k + 1$ otherwise.

1 Introduction

Let us consider a set of n candidates or options $A = \{a, b, c, \ldots\}$ which are ordered by order of preference by each individual of a set U of voters. Thus, each $\boldsymbol{\alpha} \in U$ can be identified with a permutation $\boldsymbol{\alpha} = x_1 x_2 \cdots x_n$ of the elements of A, where x_i is preferred over x_j (denoted $x_i \to x_j$) if and only if $i < j$. The set of voters determine what is called a *preference pattern* which summarizes the majority opinion about each pair of options.

In this note only *strong* preference patterns are considered, that is, it is assumed that there are no ties. So, each preference pattern on n options is fully represented by a tournament T_n on n vertices where the arc (a, b) means $a \to b$, that is, a is preferred over b by a majority of voters. Conversely, given any pattern T_n we may be interested in finding a minimum set of voters, denoted $U(T_n)$, which generates T_n. Let $v(T_n) = |U(T_n)|$ and let $v(n) = \max\{v(T_n)\}$ computed over all tournaments with n vertices. In [2] McGarvey showed that $v(n)$ is well defined, that is, for any T_n there always exist a set $U(T_n)$ and $v(n) \leq 2\binom{n}{2}$. Sterns [3] showed that $v(n) \leq n + 2$ if n is even and $v(n) \leq n + 1$ if n is odd. Finally, Erdös and Moser [1] were able to prove that $v(n)$ is of the order $O(n/\log_2 n)$. In fact all the above results were given for preference patterns which are not necessarily strong (in this case a tie between a and b can be represented either by an absence of arcs between a and b or by an edge $\{a, b\}$). It is worth noting that, contrarily to the method of Erdös and Moser, the approaches of McGarvey and Sterns give explicit constructions of a set of voters which generate any desired pattern. In the case of strong patterns we improve the results of the latter authors by giving and inductive method to obtain a suitable set of voters.

2 Strong preference patterns

Let us begin with a very simple but useful result, which is a direct consequence of the fact that in our preference patterns there are no ties.

Lemma 2.1. *Let $v(n)$ be defined as above. Then, $v(n)$ is odd.*

Proof. By contradiction, suppose that, for a given strong pattern T_n, $v(n)$ is even. Then, for any two options a, b we have that either $a \to b$ or $b \to a$ with at least two votes of difference. Consequently, the removing of a voter does not change the preference pattern. □

Notice that, from this lemma, Sterns' result particularized for strong patterns are $v(n) \leq n+1$ for n even and $v(n) \leq n$ for n odd.

Our results are based on the following theorem.

Theorem 2.2. *Let T_{n+2} be a strong pattern containing two options, say a and b. Let $T_n = T_{n+2} \setminus \{a, b\}$. Then, $v(T_{n+2}) \leq v(T_n) + 2$.*

Proof. Let $U(T_n) = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_r\}$ be a minimum set of $r = v(T_n)$ voters generating T_n. By Lemma 2.1, r is odd. Besides, suppose without loss of generality that $a \to b$, and consider the sets $A_1 = \{x \neq a \,|\, x \to b\}$ and $A_2 = \{x \neq b \,|\, a \to x\}$. Assuming $A_1 \cap A_2 \neq \emptyset, A_1, A_2$ (any other case follows trivially from this one), we can write $A_1 = \{y_1, y_2, \ldots, y_s, \ldots, y_t\}$ and $A_2 = \{y_s, y_{s+1}, \ldots, y_t, \ldots, y_m\}$, $1 < s \leq t < m$. Now, let us define the sequences $\boldsymbol{\gamma} = y_1 y_2 \cdots y_{s-1}$, $\boldsymbol{\delta} = y_s y_{s+1} \cdots y_t$, $\boldsymbol{\sigma} = y_{t+1} y_{t+2} \cdots y_m$ and $\boldsymbol{\mu} = y_{m+1} y_{m+2} \cdots y_n$, and consider the following set of $r+2$ voters:

$$
\begin{aligned}
\boldsymbol{\beta}_i &= b\boldsymbol{\alpha}_i a, \qquad 1 \leq i \leq (r+1)/2, \\
\boldsymbol{\beta}_j &= a\boldsymbol{\alpha}_j b, \qquad (r+3)/2 \leq j \leq r, \\
\boldsymbol{\beta}_{r+1} &= \boldsymbol{\gamma} a \boldsymbol{\delta} b \boldsymbol{\sigma} \boldsymbol{\mu}, \\
\boldsymbol{\beta}_{r+2} &= \overline{\boldsymbol{\mu}} a \overline{\boldsymbol{\sigma}} \overline{\boldsymbol{\delta}} \overline{\boldsymbol{\gamma}} b,
\end{aligned}
$$

where $\overline{\boldsymbol{\gamma}} = y_{s-1} \cdots y_2 y_1$, $\overline{\boldsymbol{\delta}} = y_t \cdots y_{s+1} y_s$, etc. Now it is routine to verify that these voters generate the pattern T_{n+2} and, hence, $v(T_{n+2}) \leq r + 2 = v(T_n) + 2$. □

A tournament or strong preference pattern T is called *transitive* if $a \to b$ and $b \to c$ implies $a \to c$. In this case it is clear that $v(T) = 1$. The proof of the following result can be found in [1].

Theorem 2.3 ([1])**.** *Let $f(n)$ be the maximum number such that every tournament on n vertices has a transitive subtournament on $f(n)$ vertices. Then,*

$$\lfloor \log_2 n \rfloor + 1 \leq f(n) \leq 2\lfloor \log_2 n \rfloor + 1.$$

The proof of the lower bound, due to Sterns, gives a very simple algorithm to find a subtournament which attains such a bound, see again [1].

From Theorems 2.2 and 2.3 we get the following corollary.

Corollary 2.4. *Given* $n \geq 2$*, set* $k = \lfloor \log_2 n \rfloor$*. Then* $v(n) \leq n - k$ *if* n *and* k *have different parity, and* $v(n) \leq n - k + 1$ *otherwise.*

Proof. Let T_n be any tournament on n vertices. First, use Theorem 2.3 to find a transitive subtournament T on $k+1$ vertices. If n and k have different parity, then $n-k-1$ is even. So, starting from T, we can apply Theorem 2.2 repeatedly, $(n-k-1)/2$ times, to obtain a set of $n-k$ voters which generates T_n. Otherwise, we consider a subtournament of T on k vertices and proceed as above with the remaining $n-k$ vertices. □

References

[1] P. Erdös and L. Moser, On the representation of directed graphs as unions of orderings, *Publ. Math. Inst. Hung. Acad. Sci.* **9** (1964), 125–132.

[2] D.C. McGarvey, A theorem on the construction of voting paradoxes, *Econometria* **21** (1953), 608–610.

[3] R. Stearns, The voting problem, *Amer. Math. Monthly* **66** (1959), 761–763.