Generalized T-Q relations and the open spin-s XXZ chain with nondiagonal boundary terms

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Abstract

We consider the open spin-s XXZ quantum spin chain with nondiagonal boundary terms. By exploiting certain functional relations at roots of unity, we derive a generalized form of T-Q relation involving more than one independent $Q(u)$, which we use to propose the Bethe-ansatz-type expressions for the eigenvalues of the transfer matrix. At most two of the boundary parameters are set to be arbitrary and the bulk anisotropy parameter has values $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \ldots$. We also provide numerical evidence for the completeness of the Bethe-ansatz-type solutions derived, using $s = 1$ case as an example.

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1 Introduction

Considerable effort has been put into solving integrable quantum spin chains for many years. In particular, integrable open quantum spin chains have attracted much interest over the years. In this regard, open XXX and XXZ quantum spin chains have been extensively investigated due to their growing applications in fields of physics such as statistical mechanics, string theory and condensed matter physics. Much progress has been made on the topic up to this point. Numerous successes in the past [1]-[8] (also refer to [9]-[32] and references therein, for other related work on the subject) have motivated further investigations of these models. In addition, in a series of publication, Bethe ansatz solutions have been derived for open spin-1/2 XXZ quantum spin chain where the boundary parameters obey a certain constraint. Readers are urged to refer to [33]-[40] for related work on the subject. Two sets of Bethe ansatz equations are needed there to obtain all 2^N eigenvalues, where N is the number of sites. A special case of the above solution was generalized to open XXZ quantum spin chain with alternating spins by Doikou [41] using the functional relation approach proposed by Nepomechie in [35] to solve the spin-1/2 case. In [42], related work was carried out using the method in [33]. In [43], the spin-1/2 XXZ Bethe ansatz solution (for boundary parameters obeying certain constraint) is generalized to the spin-s case by utilizing an approach based on the Q-operator and the T-Q equation [44] (see below), which was developed earlier for the spin-1/2 XXZ chain in [39] and subsequently applied to the spin-1/2 XYZ chain in [45]. Two sets of Bethe ansatz equations are also needed there to produce all $(2s+1)^N$ eigenvalues, where again N represents the number of sites. This was later followed by another work for spin-s with such constraint removed, but limiting the Bethe ansatz solutions for cases with at most two arbitrary boundary parameters for some special values of the bulk anisotropy parameter [46].

In a number of works cited above, the well known Baxter $T-Q$ relation [44], with the following schematic form

$$t(u)Q(u) = Q(v) + Q(w) \tag{1.1}$$

has provided a way to obtain the Bethe ansatz equations for the eigenvalues of the transfer matrix $t(u)$. In [47, 48], a generalization of this relation involving two $Q(u)$ of the following form for the open spin-1/2 XXZ quantum spin chain was given:

$$\begin{aligned} t(u)Q_1(u) &= Q_2(v) + Q_2(w) \\ t(u)Q_2(u) &= Q_1(v') + Q_1(w') \end{aligned} \tag{1.2}$$

Motivated by this solution, in this paper, we obtain the corresponding solution for the open spin-s XXZ quantum spin chain. In addition, our work is motivated by the fact that such $T-Q$ relations for an open spin-s XXZ quantum spin chain are novel structures and therefore merit further studies and investigation. We remark that a more general form of $T-Q$ relations were found in [49] involving multiple $Q(u)$ functions. Moreover, the relation of $s=1$ case to the supersymmetric sine-Gordon (SSG) model [50]-[55], especially the boundary SSG model [56]-[60], has inspired us to consider the problem. As in [46], we note that these results hold for cases with at most two arbitrary boundary parameters at roots of unity, namely when the bulk anisotropy parameter has vales $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \ldots$. We follow similar approach as given in [35]-[38] and [47] that was used to solve the $s=1/2$ case, which is based on functional relations obeyed by transfer matrix at roots of unity. This yields Bethe-ansatz-type solutions which give completely all the $(2s+1)^N$ eigenvalues. As in [43], we rely on fusion [5], [61]-[67], the truncation of the fusion hierarchy at roots of unity [68]-[70] and the Bazhanov-Reshetikin [71] solution of the RSOS models.

The outline of the paper is as follows: In Sec. 2, we review the construction of the fused R [61]-[65], [72]-[76] and $K^{\mp}$ [5], [66, 67] matrices from the corresponding spin-1/2 matrices. One can refer to [77, 78] for some original work on spin-1/2 $K^{\mp}$ matrices. We then review the construction of commuting transfer matrices from these fused matrices (using Sklyanin's work [4], which relies on Cherednik's previous results [81]), together with some of their properties. Fusion hierachy and functional relations obeyed by transfer matrices are also reviewed. In Sec. 3, the generalized $T-Q$ relations are given along with some arguments behind their structure. This is done by exploiting the reviewed functional relations obeyed by the transfer matrices. From this, we derive the Bethe-ansatz-type equations for cases with at most two arbitrary boundary parameters at roots of unity, e.g. $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \ldots$. We then present numerical results in Sec. 4 to illustrate the completeness of our solution, using $s=1$ as an example. Here, the Bethe roots and energy eigenvalues derived from the Bethe-ansatz-type equations (for some values of p and N) are given. We remark that these eigenvalues coincide with the ones obtained from direct diagonalization of the open spin-1 XXZ chain Hamiltonian. Finally, we conclude the paper with discussion of the results and potential future works in Sec. 5.

2 Transfer matrices, fusion hierachy and functional relations at roots of unity

In this section, we review some crucial concepts on the construction of commuting transfer matrices for N-site open spin-s XXZ quantum spin chain. Materials reviewed here on fused R, $K^{\mp}$ and higher spin transfer matrices are mainly reproduced from [43]. Like the commuting transfer matrix for $s=1/2$, constructed in [4], which we denote (following notations adopted in [43]) by $t^{(\frac{1}{2},\frac{1}{2})}(u)$, whose auxiliary space as well as each of its N quantum spaces are two-dimensional, one can similarly construct a transfer matrix $t^{(j,s)}(u)$ whose auxiliary space is spin-j ($(2j+1)$-dimensional) and each of its N quantum spaces are spin-s ($(2s+1)$-dimensional), for any $j, s \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ using the fused R [61]-[65], [72]-[76] and $K^{\mp}$ [5], [66, 67] matrices. These R and $K^{\mp}$ matrices serve as building blocks in the construction of the commuting transfer matrices for higher spins. We list them below along with some of their properties. The fused-R matrices can be constructed as given below,

$$R^{(j,s)}_{\{a\}\{b\}}(u) = P^+_{\{a\}} P^+_{\{b\}} \prod_{k=1}^{2j} \prod_{l=1}^{2s} R^{(\frac{1}{2},\frac{1}{2})}_{a_k b_l}(u + (k + l - j - s - 1)\eta)\, P^+_{\{a\}} P^+_{\{b\}}\,, \tag{2.1}$$

where $\{a\} = \{a_1, \ldots, a_{2j}\}$, $\{b\} = \{b_1, \ldots, b_{2s}\}$, and $P^+_{\{a\}}$ is the symmetric projector given by

$$P^+_{\{a\}} = \frac{1}{(2j)!} \prod_{k=1}^{2j} \left(\sum_{l=1}^{k} \mathcal{P}_{a_l, a_k} \right)\,, \tag{2.2}$$

$\mathcal{P}$ is the permutation operator, with $\mathcal{P}_{a_k,a_k} \equiv 1$; Similar definition also holds for $P^+_{\{b\}}$. $R^{(\frac{1}{2},\frac{1}{2})}(u)$ is given by

$$R^{(\frac{1}{2},\frac{1}{2})}(u) = \begin{pmatrix} \mathrm{sh}(u+\eta) & 0 & 0 & 0 \\ 0 & \mathrm{sh}\, u & \mathrm{sh}\, \eta & 0 \\ 0 & \mathrm{sh}\, \eta & \mathrm{sh}\, u & 0 \\ 0 & 0 & 0 & \mathrm{sh}(u+\eta) \end{pmatrix}\,, \tag{2.3}$$

where η is the bulk anisotropy parameter. The fundamental R matrix satisfies the following unitarity relation

$$R^{(\frac{1}{2},\frac{1}{2})}(u) R^{(\frac{1}{2},\frac{1}{2})}(-u) = -\xi(u) 1\,, \qquad \xi(u) = \mathrm{sh}(u+\eta)\, \mathrm{sh}(u-\eta)\,. \tag{2.4}$$

The R matrices in the product (2.1) are ordered in the order of increasing k and l. The fused R matrices satisfy the Yang-Baxter equations [79, 80]

$$R^{(j,k)}_{\{a\}\{b\}}(u-v)\, R^{(j,s)}_{\{a\}\{c\}}(u)\, R^{(k,s)}_{\{b\}\{c\}}(v) = R^{(k,s)}_{\{b\}\{c\}}(v)\, R^{(j,s)}_{\{a\}\{c\}}(u)\, R^{(j,k)}_{\{a\}\{b\}}(u-v)\,. \tag{2.5}$$

Having defined fused-R matrices, the construction of the fused K^- matrices readily follows [5], [66, 67],

$$\begin{aligned} K^{-(j)}_{\{a\}}(u) \;&=\; P^+_{\{a\}} \prod_{k=1}^{2j} \Bigg\{ \left[\prod_{l=1}^{k-1} R^{(\frac{1}{2},\frac{1}{2})}_{a_l a_k}(2u + (k + l - 2j - 1)\eta) \right] \\ &\times\; K^{-(\frac{1}{2})}_{a_k}(u + (k - j - \frac{1}{2})\eta) \Bigg\} P^+_{\{a\}}\,, \end{aligned} \tag{2.6}$$

where $K^{-(\frac{1}{2})}(u)$ is the 2×2 matrix whose components are given by [77, 78]

$$
\begin{aligned}
K^-_{11}(u) &= 2\,(\operatorname{sh}\alpha_- \operatorname{ch}\beta_- \operatorname{ch}u + \operatorname{ch}\alpha_- \operatorname{sh}\beta_- \operatorname{sh}u) \\
K^-_{22}(u) &= 2\,(\operatorname{sh}\alpha_- \operatorname{ch}\beta_- \operatorname{ch}u - \operatorname{ch}\alpha_- \operatorname{sh}\beta_- \operatorname{sh}u) \\
K^-_{12}(u) &= e^{\theta_-} \operatorname{sh}2u\,, \qquad K^-_{21}(u) = e^{-\theta_-} \operatorname{sh}2u\,,
\end{aligned}
\tag{2.7}
$$

where $\alpha_-\,, \beta_-\,, \theta_-$ are the boundary parameters. The products of braces $\{\ldots\}$ in (2.6) are ordered in the order of increasing k. The fused K^- matrices satisfy the boundary Yang-Baxter equations [81]

$$
\begin{aligned}
& R^{(j,s)}_{\{a\}\{b\}}(u-v)\, K^{-(j)}_{\{a\}}(u)\, R^{(j,s)}_{\{a\}\{b\}}(u+v)\, K^{-(j)}_{\{b\}}(v) \\
&\quad = K^{-(j)}_{\{b\}}(v)\, R^{(j,s)}_{\{a\}\{b\}}(u+v)\, K^{-(j)}_{\{a\}}(u)\, R^{(j,s)}_{\{a\}\{b\}}(u-v)\,.
\end{aligned}
\tag{2.8}
$$

The fused K^+ matrices are given by

$$
K^{+(j)}_{\{a\}}(u) = \frac{1}{f^{(j)}(u)}\, K^{-(j)}_{\{a\}}(-u-\eta)\Big|_{(\alpha_-,\beta_-,\theta_-)\to(-\alpha_+,-\beta_+,\theta_+)}\,,
\tag{2.9}
$$

where the normalization factor is,

$$
f^{(j)}(u) = \prod_{l=1}^{2j-1} \prod_{k=1}^{l} [-\xi(2u + (l+k+1-2j)\eta)]
\tag{2.10}
$$

Using the above results, one can construct the transfer matrix $t^{(j,s)}(u)$,

$$
t^{(j,s)}(u) = \operatorname{tr}_{\{a\}} K^{+(j)}_{\{a\}}(u)\, T^{(j,s)}_{\{a\}}(u)\, K^{-(j)}_{\{a\}}(u)\, \hat{T}^{(j,s)}_{\{a\}}(u)\,,
\tag{2.11}
$$

where the monodromy matrices are given by products of N fused R matrices,

$$
\begin{aligned}
T^{(j,s)}_{\{a\}}(u) &= R^{(j,s)}_{\{a\},\{b^{[N]}\}}(u) \ldots R^{(j,s)}_{\{a\},\{b^{[1]}\}}(u)\,, \\
\hat{T}^{(j,s)}_{\{a\}}(u) &= R^{(j,s)}_{\{a\},\{b^{[1]}\}}(u) \ldots R^{(j,s)}_{\{a\},\{b^{[N]}\}}(u)\,.
\end{aligned}
\tag{2.12}
$$

These transfer matrices commute for different values of spectral parameter for any $j, j' \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and any $s \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$,

$$
\left[t^{(j,s)}(u)\,, t^{(j',s)}(u')\right] = 0\,.
\tag{2.13}
$$

Furthermore, they also obey the fusion hierarchy [5, 43, 66, 67][1]

$$
t^{(j-\frac{1}{2},s)}(u - j\eta)\, t^{(\frac{1}{2},s)}(u) = t^{(j,s)}(u - (j-\frac{1}{2})\eta) + \delta^{(s)}(u-\eta)\, t^{(j-1,s)}(u - (j+\frac{1}{2})\eta)\,,
\tag{2.14}
$$

[1] See the appendix in [43] for more details on the fusion hierachy.

$j = 1, \frac{3}{2}, \ldots$, where $t^{(0,s)} = 1$, and $\delta^{(s)}(u)$ is given by

$$\delta^{(s)}(u) = \delta_0^{(s)}(u)\delta_1^{(s)}(u) \tag{2.15}$$

where

$$\begin{aligned}
\delta_0^{(s)}(u) &= \left[\prod_{k=0}^{2s-1} \xi(u + (s - k + \frac{1}{2})\eta)\right]^{2N} \frac{\mathrm{sh}(2u)\,\mathrm{sh}(2u + 4\eta)}{\mathrm{sh}(2u + \eta)\,\mathrm{sh}(2u + 3\eta)} \\
\delta_1^{(s)}(u) &= 2^4\,\mathrm{sh}(u + \alpha_- + \eta)\,\mathrm{sh}(u - \alpha_- + \eta)\,\mathrm{ch}(u + \beta_- + \eta)\,\mathrm{ch}(u - \beta_- + \eta) \\
&\times\ \mathrm{sh}(u + \alpha_+ + \eta)\,\mathrm{sh}(u - \alpha_+ + \eta)\,\mathrm{ch}(u + \beta_+ + \eta)\,\mathrm{ch}(u - \beta_+ + \eta)\,.
\end{aligned} \tag{2.16}$$

To avoid confusion, we emphasize that the $\delta^{(s)}(u)$ in [43] differs from the one given here merely by a shift in η.

Next, we list a few important properties of the rescaled "fundamental" transfer matrix $\tilde{t}^{(\frac{1}{2},s)}(u)$ (defined below), which are useful in determining its eigenvalues. Following the definition of $\tilde{t}^{(\frac{1}{2},s)}(u)$ as in [43], we have

$$\tilde{t}^{(\frac{1}{2},s)}(u) = \frac{1}{g^{(\frac{1}{2},s)}(u)^{2N}} t^{(\frac{1}{2},s)}(u)\,, \tag{2.17}$$

where

$$g^{(\frac{1}{2},s)}(u) = \prod_{k=1}^{2s-1} \mathrm{sh}(u + (s - k + \frac{1}{2})\eta) \tag{2.18}$$

This transfer matrix has the following useful properties:

$$\tilde{t}^{(\frac{1}{2},s)}(u + i\pi) = \tilde{t}^{(\frac{1}{2},s)}(u) \qquad (i\pi \text{ - periodicity}) \tag{2.19}$$

$$\tilde{t}^{(\frac{1}{2},s)}(-u - \eta) = \tilde{t}^{(\frac{1}{2},s)}(u) \qquad (\text{crossing}) \tag{2.20}$$

$$\tilde{t}^{(\frac{1}{2},s)}(0) = -2^3\,\mathrm{sh}^{2N}((s + \frac{1}{2})\eta)\,\mathrm{ch}\,\eta\,\mathrm{sh}\,\alpha_-\,\mathrm{ch}\,\beta_-\,\mathrm{sh}\,\alpha_+\,\mathrm{ch}\,\beta_+\mathbb{I} \quad (\text{initial condition}) \tag{2.21}$$

$$\begin{aligned}
\tilde{t}^{(\frac{1}{2},s)}(u)\Big|_{\eta=0} &= 2^3\,\mathrm{sh}^{2N} u\Big[-\mathrm{sh}\,\alpha_-\,\mathrm{ch}\,\beta_-\,\mathrm{sh}\,\alpha_+\,\mathrm{ch}\,\beta_+\,\mathrm{ch}^2 u \\
&+\ \mathrm{ch}\,\alpha_-\,\mathrm{sh}\,\beta_-\,\mathrm{ch}\,\alpha_+\,\mathrm{sh}\,\beta_+\,\mathrm{sh}^2 u \\
&-\ \mathrm{ch}(\theta_- - \theta_+)\,\mathrm{sh}^2 u\,\mathrm{ch}^2 u\Big]\mathbb{I} \quad (\text{semi-classical limit})
\end{aligned} \tag{2.22}$$

where $\mathbb{I}$ is the identity matrix.

Due to the commutativity property (2.13), the corresponding simultaneous eigenvectors are independent of the spectral parameter. Hence, (2.19) - (2.22) hold for the corresponding eigenvalues as well. In addition to the above mentioned properties, for bulk anisotropy parameter values $\eta = \frac{i\pi}{p+1}$, with $p = 1\,, 2\,, \ldots$, the "fundamental" transfer matrix, $t^{(\frac{1}{2},s)}(u)$ (and hence each of the corresponding eigenvalues, $\Lambda^{(\frac{1}{2},s)}(u)$) obeys functional relations of order $p+1$ [35]-[37],

$$\begin{aligned}
& t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+\eta)\ldots t^{(\frac{1}{2},s)}(u+p\eta) \\
- \quad & \delta^{(s)}(u-\eta)t^{(\frac{1}{2},s)}(u+\eta)t^{(\frac{1}{2},s)}(u+2\eta)\ldots t^{(\frac{1}{2},s)}(u+(p-1)\eta) \\
- \quad & \delta^{(s)}(u)t^{(\frac{1}{2},s)}(u+2\eta)t^{(\frac{1}{2},s)}(u+3\eta)\ldots t^{(\frac{1}{2},s)}(u+p\eta) \\
- \quad & \delta^{(s)}(u+\eta)t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+3\eta)t^{(\frac{1}{2},s)}(u+4\eta)\ldots t^{(\frac{1}{2},s)}(u+p\eta) \\
- \quad & \delta^{(s)}(u+2\eta)t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+\eta)t^{(\frac{1}{2},s)}(u+4\eta)\ldots t^{(\frac{1}{2},s)}(u+p\eta) - \ldots \\
- \quad & \delta^{(s)}(u+(p-1)\eta)t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+\eta)\ldots t^{(\frac{1}{2},s)}(u+(p-2)\eta) \\
+ \quad & \ldots = f(u)\,. \qquad (2.23)
\end{aligned}$$

For example, for $p = 3$, the above functional relation reduces to

$$\begin{aligned}
& t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+\eta)t^{(\frac{1}{2},s)}(u+2\eta)t^{(\frac{1}{2},s)}(u+3\eta) - \delta^{(s)}(u)t^{(\frac{1}{2},s)}(u+2\eta)t^{(\frac{1}{2},s)}(u+3\eta) \\
& -\delta^{(s)}(u+\eta)t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+3\eta) - \delta^{(s)}(u+2\eta)t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u+\eta) \\
& -\delta^{(s)}(u+3\eta)t^{(\frac{1}{2},s)}(u+\eta)t^{(\frac{1}{2},s)}(u+2\eta) + \delta^{(s)}(u)\delta^{(s)}(u+2\eta) \\
& +\delta^{(s)}(u+\eta)\delta^{(s)}(u+3\eta) = f(u)\,. \qquad (2.24)
\end{aligned}$$

The scalar function $f(u)$ (which can be expressed as $f(u) = f_0(u)f_1(u)$) is given in terms of the boundary parameters $\alpha_\mp\,, \beta_\mp\,, \theta_\mp$ (for odd p) by

$$f_0(u) = \begin{cases} (-1)^{N+1}2^{-4spN}\,\mathrm{sh}^{4sN}\left((p+1)u\right)\mathrm{th}^2\left((p+1)u\right)\,, \\ \qquad s = \frac{1}{2}\,, \frac{3}{2}\,, \frac{5}{2}\,, \ldots \\ (-1)^{N+1}2^{-4spN}\,\mathrm{ch}^{4sN}\left((p+1)u\right)\mathrm{th}^2\left((p+1)u\right)\,, \\ \qquad s = 1\,, 2\,, 3\,, \ldots \end{cases} \qquad (2.25)$$

and

$$\begin{aligned}
f_1(u) \quad = \quad & -2^{3-2p}\Big(\\
& \mathrm{ch}\left((p+1)\alpha_-\right)\mathrm{ch}\left((p+1)\beta_-\right)\mathrm{ch}\left((p+1)\alpha_+\right)\mathrm{ch}\left((p+1)\beta_+\right)\mathrm{sh}^2\left((p+1)u\right) \\
- \quad & \mathrm{sh}\left((p+1)\alpha_-\right)\mathrm{sh}\left((p+1)\beta_-\right)\mathrm{sh}\left((p+1)\alpha_+\right)\mathrm{sh}\left((p+1)\beta_+\right)\mathrm{ch}^2\left((p+1)u\right) \\
+ \quad & (-1)^N\,\mathrm{ch}\left((p+1)(\theta_- - \theta_+)\right)\mathrm{sh}^2\left((p+1)u\right)\mathrm{ch}^2\left((p+1)u\right)\Big)\,. \qquad (2.26)
\end{aligned}$$

for $s=\frac{1}{2}\,,\frac{3}{2}\,,\frac{5}{2}\ldots$ and

$$\begin{aligned} f_1(u) &= (-1)^{N+1}2^{3-2p}\Big(\\ & \mathrm{ch}\left((p+1)\alpha_-\right)\mathrm{ch}\left((p+1)\beta_-\right)\mathrm{ch}\left((p+1)\alpha_+\right)\mathrm{ch}\left((p+1)\beta_+\right)\mathrm{sh}^2\left((p+1)u\right) \\ &- \mathrm{sh}\left((p+1)\alpha_-\right)\mathrm{sh}\left((p+1)\beta_-\right)\mathrm{sh}\left((p+1)\alpha_+\right)\mathrm{sh}\left((p+1)\beta_+\right)\mathrm{ch}^2\left((p+1)u\right) \\ &+ \mathrm{ch}\left((p+1)(\theta_- - \theta_+)\right)\mathrm{sh}^2\left((p+1)u\right)\mathrm{ch}^2\left((p+1)u\right)\Big)\,. \end{aligned} \tag{2.27}$$

for $s=1\,,2\,,3\ldots$, Note that $f(u)$ satisfies

$$f(u+\eta)=f(u)\,,\qquad f(-u)=f(u)\,. \tag{2.28}$$

and

$$f_0(u)^2=\prod_{j=0}^{p}\delta_0^{(s)}(u+j\eta)\,. \tag{2.29}$$

where $\delta_0^{(s)}(u)$ is given by (2.16).

3 Generalized T-Q relations and Bethe ansatz

In this section, we give the main results of this paper. We derive the generalized $T-Q$ relations for the transfer matrix eigenvalues and obtain the Bethe-ansatz-type equations, for cases where at most two of the boundary parameters $\{\alpha_-,\alpha_+,\beta_-,\beta_+\}$ are arbitrary, by adopting the steps given in [47] while setting $\theta_-=\theta_+=\theta$, where θ is also arbitrary. More on this is given below.

3.1 $T-Q$ relations

The transfer matrix $t^{(\frac{1}{2},s)}(u)$ and its eigenvalues ($\Lambda^{(\frac{1}{2},s)}(u)$) obey the functional relations (2.23). We exploit this fact to obtain the $T-Q$ relations. Following [71], one could recast the functional relations as the condition that the determinant of a certain matrix vanishes, namely

$$\det\mathcal{M}(u)=0\,, \tag{3.1}$$

where $\mathcal{M}(u)$ is given by the $(p+1)\times(p+1)$ matrix

$$\mathcal{M}(u)=\begin{pmatrix} \Lambda^{(\frac{1}{2},s)}(u) & -\frac{\delta^{(s)}(u)}{h^{(1)}(u)} & 0 & \ldots & 0 & -\frac{\delta^{(s)}(u-\eta)}{h^{(2)}(u-\eta)} \\ -h^{(1)}(u) & \Lambda^{(\frac{1}{2},s)}(u+\eta) & -h^{(2)}(u+\eta) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h^{(2)}(u-\eta) & 0 & 0 & \ldots & -h^{(1)}(u+(p-1)\eta) & \Lambda^{(\frac{1}{2},s)}(u+p\eta) \end{pmatrix} \tag{3.2}$$

where $h^{(1)}(u)$ and $h^{(2)}(u)$ are functions which are $i\pi$-periodic, but otherwise not yet specified. We note that the above matrix has the following symmetry,

$$T\,\mathcal{M}(u)T^{-1} = \mathcal{M}(u+2\eta)\,, \qquad T \equiv S^2\,, \tag{3.3}$$

where S is given by,

$$S = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}\,. \tag{3.4}$$

Assuming that

$$\det \mathcal{M}(u) = 0\,, \tag{3.5}$$

then $\mathcal{M}(u)$ has a null eigenvector $v(u)$,

$$\mathcal{M}(u)\; v(u) = 0\,. \tag{3.6}$$

The symmetry (3.3) is consistent with

$$T\;\, v(u) = v(u+2\eta)\,, \tag{3.7}$$

which implies that $v(u)$ has the form

$$v(u) = \left(Q_1(u)\,, Q_2(u+\eta)\,, \ldots\,, Q_1(u-2\eta)\,, Q_2(u-\eta)\right)\,, \tag{3.8}$$

with

$$Q_1(u) = Q_1(u+i\pi)\,, \qquad Q_2(u) = Q_2(u+i\pi)\,. \tag{3.9}$$

That is, the components of $v(u)$ are determined by *two* independent functions, $Q_1(u)$ and $Q_2(u)$.[2]

The null eigenvector condition (3.6) together with the explicit forms of $\mathcal{M}(u)$ and $v(u)$, given by (3.2) and (3.8) respectively, now lead to the following $T-Q$ relations,

$$\Lambda^{(\frac{1}{2},s)}(u) = \frac{\delta^{(s)}(u)}{h^{(1)}(u)}\frac{Q_2(u+\eta)}{Q_1(u)} + \frac{\delta^{(s)}(u-\eta)}{h^{(2)}(u-\eta)}\frac{Q_2(u-\eta)}{Q_1(u)}\,, \tag{3.10}$$

$$= h^{(1)}(u-\eta)\frac{Q_1(u-\eta)}{Q_2(u)} + h^{(2)}(u)\frac{Q_1(u+\eta)}{Q_2(u)}\,. \tag{3.11}$$

[2]In [46], all of the matrices $\mathcal{M}(u)$ possess a stronger symmetry, $S\,\mathcal{M}(u)S^{-1} = \mathcal{M}(u+\eta)$, implying the null eigenvector with single $Q(u)$. We refer the reader to Sec. 3 of [47] for more detail discussion on this.

Due to the crossing symmetry (2.20) and

$$\delta^{(s)}(u) = \delta^{(s)}(-u - 2\eta) \tag{3.12}$$

which is the crossing property for $\delta^{(s)}(u)$, it is then natural to have the two terms in (3.10) transform into each other under crossing. Hence, we set

$$h^{(2)}(u) = h^{(1)}(-u - 2\eta)\,, \tag{3.13}$$

and we make the following ansatz

$$Q_j(u) \;=\; \prod_{k=1}^{M_j} \sinh(u - u_k^{(j)}) \sinh(u + u_k^{(j)} + \eta)\,, \tag{3.14}$$

which is consistent with the required periodicity (3.9) and crossing properties

$$Q_j(u) \;=\; Q_j(-u - \eta)\,. \tag{3.15}$$

where $j = 1\,, 2$. In (3.14), $\{u_k^{(j)}\}$ represents the Bethe roots (or zeros of $Q_j(u)$) and there are M_j of these roots. Further, one can verify that the condition $\det \mathcal{M}(u) = 0$ indeed implies the functional relations (2.23), if $w(u)$ satisfies

$$f(u) = w(u) \prod_{j=0,2,\ldots}^{p-1} \delta^{(s)}(u + j\eta) + \frac{1}{w(u)} \prod_{j=1,3,\ldots}^{p} \delta^{(s)}(u + j\eta)\,, \tag{3.16}$$

where

$$w(u) \equiv \frac{\prod_{j=1,3,\ldots}^{p} h^{(2)}(u + j\eta)}{\prod_{j=0,2,\ldots}^{p-1} h^{(1)}(u + j\eta)}\,. \tag{3.17}$$

It follows from (3.16) that the process of finding $w(u)$ reduces to solving a quadratic equation, which when used together with (3.13) and (3.17), yields the explicit form for the function $h^{(1)}(u)$. Here, we consider even number of sites, N. Below, we give the solutions of (3.17) for $h^{(1)}(u)$ for two cases:

I. β_- and β_+ arbitrary while setting $\alpha_\pm = 0$, $\theta_- = \theta_+ = \theta =$ arbitrary.

$$h^{(1)}(u) = 4 \left[\prod_{k=0}^{2s-1} \mathrm{sh}(u + (s - k + \frac{3}{2})\eta) \right]^{2N} \frac{\mathrm{sh}^2(u + \eta)\, \mathrm{sh}(2u + 4\eta)}{\mathrm{sh}(2u + 3\eta)}\,,$$

$$M_1 = sN + \frac{1}{2}(p + 1)\,, \quad M_2 = M_1 - 1\,. \tag{3.18}$$

II. α_- and α_+ arbitrary while setting $\beta_\pm = 0$, $\theta_- = \theta_+ = \theta =$ arbitrary.

$$h^{(1)}(u) = 4\left[\prod_{k=0}^{2s-1} \mathrm{sh}(u+(s-k+\frac{3}{2})\eta)\right]^{2N} \frac{\mathrm{ch}^2(u+\eta)\,\mathrm{sh}(2u+4\eta)}{\mathrm{sh}(2u+3\eta)}\,,$$

$$M_1 = sN + \frac{1}{2}(p+1)\,, \quad M_2 = M_1 - 1\,. \tag{3.19}$$

Now, using the analyticity of $\Lambda^{(\frac{1}{2},s)}(u)$, given by (3.10) and (3.11), one can write down the Bethe-ansatz-type equations for the zeros $\{u_j^{(1)}\,, u_j^{(2)}\}$ of $Q_1(u)$ and $Q_2(u)$,

$$\frac{\delta^{(s)}(u_j^{(1)})\ h^{(2)}(u_j^{(1)}-\eta)}{\delta^{(s)}(u_j^{(1)}-\eta)\ h^{(1)}(u_j^{(1)})} = -\frac{Q_2(u_j^{(1)}-\eta)}{Q_2(u_j^{(1)}+\eta)}\,, \qquad j = 1\,,2\,,\ldots\,,M_1\,, \tag{3.20}$$

$$\frac{h^{(1)}(u_j^{(2)}-\eta)}{h^{(2)}(u_j^{(2)})} = -\frac{Q_1(u_j^{(2)}+\eta)}{Q_1(u_j^{(2)}-\eta)}\,, \qquad j = 1\,,2\,,\ldots\,,M_2\,. \tag{3.21}$$

We remark here that for each case, there are more than one solutions for $h^{(1)}(u)$ that correspond to the above expression for $w(u)$. The solutions found are largely by trial and error, verifying numerically for small values of N that the eigenvalues can indeed be expressed as (3.10), (3.11) with $Q(u)$'s of the form (3.14).

To summarize, we have proposed that for the case where the bulk anisotropy parameter, $\eta = \frac{i\pi}{p+1}$ with p being odd integers and that at most two of the boundary parameters $\{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ are arbitrary, the eigenvalues $\Lambda^{(\frac{1}{2},s)}(u)$ of the transfer matrix $t^{(\frac{1}{2},s)}(u)$ for two cases (I and II) are given by a generalized form of $T-Q$ relations (3.10), (3.11), with $Q_1(u)$ and $Q_2(u)$ given by (3.14) and $h^{(2)}(u)$ given by (3.13). The $h^{(1)}(u)$ is given by (3.18) and (3.19) respectively, for the two cases considered. The zeros $\{u_j^{(1)}\,, u_j^{(2)}\}$ of $Q_1(u)$ and $Q_2(u)$ are indeed the solutions of the Bethe-ansatz-type equations, (3.20) and (3.21). These equations reproduce results in [47, 48] for $s = \frac{1}{2}$. In the following section, we shall use these results (specifically $\tilde{\Lambda}^{(\frac{1}{2},s)}(u)$ which represents the eigenvalues of the rescaled "fundamental" transfer matrix given by (2.17)) to derive expressions for energy eigenvalues for the case $s = 1$.

4 Energy eigenvalues and Bethe roots

In this section, we provide numerical evidence for the completeness of the Bethe-ansatz-type solutions derived in Sec. 3 using the case $s = 1$ as an example. We derive an expression for the energy eigenvalues for the open spin-1 XXZ quantum spin chain and compute the complete energy levels for this case from the Bethe roots given by (3.20) and (3.21). We do this for both cases, I and II.

4.1 Open spin-1 XXZ quantum spin chain: The Hamiltonian

In this section, we review the integrable Hamiltonian for the open spin-1 XXZ quantum spin chain (adopting notations used in [43]). The Hamiltonian is given by

$$\mathcal{H} = \sum_{n=1}^{N-1} H_{n,n+1} + H_b \,, \tag{4.1}$$

where $H_{n,n+1}$ represents the bulk terms. Explicitly, these terms are given by [84],

$$\begin{aligned} H_{n,n+1} &= \sigma_n - (\sigma_n)^2 + 2\,\mathrm{sh}^2\,\eta \left[\sigma_n^z + (S_n^z)^2 + (S_{n+1}^z)^2 - (\sigma_n^z)^2\right] \\ &- 4\,\mathrm{sh}^2(\frac{\eta}{2}) \left(\sigma_n^\perp \sigma_n^z + \sigma_n^z \sigma_n^\perp\right) \,, \end{aligned} \tag{4.2}$$

where

$$\sigma_n = \vec{S}_n \cdot \vec{S}_{n+1} \,, \quad \sigma_n^\perp = S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \,, \quad \sigma_n^z = S_n^z S_{n+1}^z \,, \tag{4.3}$$

and $\vec{S}$ are the $su(2)$ spin-1 generators. H_b represents the boundary terms with the following form (see e.g., [43, 85])

$$\begin{aligned} H_b &= a_1 (S_1^z)^2 + a_2 S_1^z + a_3 (S_1^+)^2 + a_4 (S_1^-)^2 + a_5 S_1^+ \, S_1^z + a_6 S_1^z \, S_1^- \\ &+ a_7 S_1^z \, S_1^+ + a_8 S_1^- \, S_1^z + (a_j \leftrightarrow b_j \text{ and } 1 \leftrightarrow N) \,, \end{aligned} \tag{4.4}$$

where $S^\pm = S^x \pm i S^y$. The coefficients $\{a_i\}$ of the boundary terms at site 1 are functions of the boundary parameters $(\alpha_-, \beta_-, \theta_-)$ and the bulk anisotropy parameter η. They are given by,

$$\begin{aligned} a_1 &= \frac{1}{4} a_0 \left(\mathrm{ch}\, 2\alpha_- - \mathrm{ch}\, 2\beta_- + \mathrm{ch}\, \eta\right) \mathrm{sh}\, 2\eta\, \mathrm{sh}\, \eta \,, \\ a_2 &= \frac{1}{4} a_0\, \mathrm{sh}\, 2\alpha_-\, \mathrm{sh}\, 2\beta_-\, \mathrm{sh}\, 2\eta \,, \\ a_3 &= -\frac{1}{8} a_0 e^{2\theta_-}\, \mathrm{sh}\, 2\eta\, \mathrm{sh}\, \eta \,, \\ a_4 &= -\frac{1}{8} a_0 e^{-2\theta_-}\, \mathrm{sh}\, 2\eta\, \mathrm{sh}\, \eta \,, \\ a_5 &= a_0 e^{\theta_-} \left(\mathrm{ch}\, \beta_-\, \mathrm{sh}\, \alpha_-\, \mathrm{ch}\, \frac{\eta}{2} + \mathrm{ch}\, \alpha_-\, \mathrm{sh}\, \beta_-\, \mathrm{sh}\, \frac{\eta}{2}\right) \mathrm{sh}\, \eta\, \mathrm{ch}^{\frac{3}{2}}\, \eta \,, \\ a_6 &= a_0 e^{-\theta_-} \left(\mathrm{ch}\, \beta_-\, \mathrm{sh}\, \alpha_-\, \mathrm{ch}\, \frac{\eta}{2} + \mathrm{ch}\, \alpha_-\, \mathrm{sh}\, \beta_-\, \mathrm{sh}\, \frac{\eta}{2}\right) \mathrm{sh}\, \eta\, \mathrm{ch}^{\frac{3}{2}}\, \eta \,, \\ a_7 &= -a_0 e^{\theta_-} \left(\mathrm{ch}\, \beta_-\, \mathrm{sh}\, \alpha_-\, \mathrm{ch}\, \frac{\eta}{2} - \mathrm{ch}\, \alpha_-\, \mathrm{sh}\, \beta_-\, \mathrm{sh}\, \frac{\eta}{2}\right) \mathrm{sh}\, \eta\, \mathrm{ch}^{\frac{3}{2}}\, \eta \,, \\ a_8 &= -a_0 e^{-\theta_-} \left(\mathrm{ch}\, \beta_-\, \mathrm{sh}\, \alpha_-\, \mathrm{ch}\, \frac{\eta}{2} - \mathrm{ch}\, \alpha_-\, \mathrm{sh}\, \beta_-\, \mathrm{sh}\, \frac{\eta}{2}\right) \mathrm{sh}\, \eta\, \mathrm{ch}^{\frac{3}{2}}\, \eta \,, \end{aligned} \tag{4.5}$$

where

$$a_0 = \Big[\mathrm{sh}(\alpha_- - \frac{\eta}{2})\,\mathrm{sh}(\alpha_- + \frac{\eta}{2})\,\mathrm{ch}(\beta_- - \frac{\eta}{2})\,\mathrm{ch}(\beta_- + \frac{\eta}{2})\Big]^{-1} . \tag{4.6}$$

Similarly, the coefficients $\{b_i\}$ of the boundary terms at site N which are functions of the boundary parameters $(\alpha_+, \beta_+, \theta_+)$ and η, are given by the following correspondence,

$$b_i = a_i\Big|_{\alpha_- \to \alpha_+, \beta_- \to -\beta_+, \theta_- \to \theta_+} . \tag{4.7}$$

The Hamiltonian $\mathcal{H}$ (4.1), according to [4], is related to the first derivative of the spin-1 transfer matrix, namely $t^{(1,1)}(u)$, which one constructs from $t^{(\frac{1}{2},1)}(u)$ by using the fusion hierarchy formula (2.14),

$$t^{(1,1)}(u) = t^{(\frac{1}{2},1)}(u - \frac{\eta}{2})t^{(\frac{1}{2},1)}(u + \frac{\eta}{2}) - \delta^{(1)}(u - \frac{\eta}{2}) \tag{4.8}$$

where $\delta^{(1)}(u)$ is given by (2.15)-(2.16) with $s = 1$. Following [43], we work with the rescaled transfer matrix given by

$$\tilde{t}^{(1,1)\ gt}(u) = \frac{\mathrm{sh}(2u)\,\mathrm{sh}(2u + 2\eta)}{[\mathrm{sh}\,u\,\mathrm{sh}(u + \eta)]^{2N}}\, t^{(1,1)\ gt}(u)\,, \tag{4.9}$$

where $t^{(1,1)\ gt}(u)$ is the transfer matrix constructed from "gauge"-transformed $R^{(1,1)}(u)$ and $K^{\mp(1)}(u)$ matrices [3]. We note here that the rescaled transfer matrix does not vanish at $u = 0$.

The Hamiltonian $\mathcal{H}$ (4.1), can now be expressed in terms of the first derivative of $\tilde{t}^{(1,1)\ gt}(u)$,

$$\mathcal{H} = c_1^{(1)} \frac{d}{du}\tilde{t}^{(1,1)\ gt}(u)\Big|_{u=0} + c_2^{(1)}\mathbb{I}\,, \tag{4.10}$$

where

$$\begin{aligned} c_1^{(1)} &= \mathrm{ch}\,\eta\Big\{16[\mathrm{sh}\,2\eta\,\mathrm{sh}\,\eta]^{2N}\,\mathrm{sh}\,3\eta\,\mathrm{sh}(\alpha_- - \frac{\eta}{2})\,\mathrm{sh}(\alpha_- + \frac{\eta}{2})\,\mathrm{ch}(\beta_- - \frac{\eta}{2})\,\mathrm{ch}(\beta_- + \frac{\eta}{2}) \\ &\times\ \mathrm{sh}(\alpha_+ - \frac{\eta}{2})\,\mathrm{sh}(\alpha_+ + \frac{\eta}{2})\,\mathrm{ch}(\beta_+ - \frac{\eta}{2})\,\mathrm{ch}(\beta_+ + \frac{\eta}{2})\Big\}^{-1} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} c_2^{(1)} &= -\frac{a_0}{4}b\,\mathrm{ch}\,\eta - (N-1)(4 + \mathrm{ch}\,2\eta) + 2N\,\mathrm{ch}^2\,\eta \\ &- \frac{\mathrm{sh}\,\eta}{2d}\Big\{ -2\,\mathrm{ch}\,2\alpha_+\Big(\mathrm{ch}\,\eta(3 + 7\,\mathrm{ch}\,2\eta + \mathrm{ch}\,4\eta) + \mathrm{ch}\,2\beta_+(4 + 5\,\mathrm{ch}\,2\eta + 2\,\mathrm{ch}\,4\eta)\Big) \\ &+ 2\,\mathrm{ch}\,\eta\Big(\mathrm{ch}\,2\beta_+(3 + 7\,\mathrm{ch}\,2\eta + \mathrm{ch}\,4\eta) + \mathrm{ch}\,\eta(5 + 3\,\mathrm{ch}\,2\eta + 3\,\mathrm{ch}\,4\eta)\Big)\Big\} \\ &- \frac{\mathrm{sh}\,2\eta}{2d}\Big\{\mathrm{ch}\,2\beta_+(2 + 4\,\mathrm{ch}\,\eta\,\mathrm{ch}\,3\eta) + \mathrm{ch}\,\eta(5\,\mathrm{ch}\,2\eta + \mathrm{ch}\,4\eta) - 2\,\mathrm{ch}\,2\alpha_+\Big(1 + \mathrm{ch}\,2\eta \\ &+ \mathrm{ch}\,2\beta_+(\mathrm{ch}\,\eta + 2\,\mathrm{ch}\,3\eta) + \mathrm{ch}\,4\eta\Big)\Big\}\,. \end{aligned} \tag{4.12}$$

[3]Such a transformation results in a more symmetric form of these matrices. For a detailed discussion on this, refer to Sec. 4 of [43].

In (4.12), b and d are given by

$$b = 2(-\operatorname{ch} 2\beta_- - \operatorname{ch}^3 \eta + \operatorname{ch} 2\alpha_-(1 + \operatorname{ch} 2\beta_- \operatorname{ch} \eta)) \tag{4.13}$$

and

$$d = -4 \operatorname{sh} 3\eta \operatorname{sh}(\alpha_+ + \frac{\eta}{2}) \operatorname{sh}(\alpha_+ - \frac{\eta}{2}) \operatorname{ch}(\beta_+ + \frac{\eta}{2}) \operatorname{ch}(\beta_+ - \frac{\eta}{2}) \tag{4.14}$$

4.2 Open spin-1 XXZ quantum spin chain: Energy eigenvalues

Next, we proceed to the eigenvalues of the Hamiltonian (4.10). Note that (4.10) implies the following result for the corresponding eigenvalues,

$$E = c_1^{(1)} \frac{d}{du} \tilde{\Lambda}^{(1,1)\ gt}(u)\Big|_{u=0} + c_2^{(1)}, \tag{4.15}$$

where $\tilde{\Lambda}^{(1,1)\ gt}(u)$ represents the transfer matrix eigenvalues which assume the following form after using (2.17), (4.8) and (4.9),

$$\begin{aligned} \tilde{\Lambda}^{(1,1)\ gt}(u) &= \frac{\operatorname{sh}(2u)\operatorname{sh}(2u+2\eta)}{[\operatorname{sh} u \operatorname{sh}(u+\eta)]^{2N}} \Big\{ [g^{(\frac{1}{2},s)}(u-\frac{\eta}{2}) g^{(\frac{1}{2},s)}(u+\frac{\eta}{2})]^{2N} \tilde{\Lambda}^{(\frac{1}{2},1)}(u-\frac{\eta}{2}) \tilde{\Lambda}^{(\frac{1}{2},1)}(u+\frac{\eta}{2}) \\ &- \delta^{(1)}(u-\frac{\eta}{2}) \Big\} \end{aligned} \tag{4.16}$$

where $\delta^{(1)}(u)$ is given by (2.15)-(2.16) with $s = 1$. Furthermore, we have also used the fact that $\Lambda^{(1,1)\ gt}(u) = \Lambda^{(1,1)}(u)$. Finally, from (3.11), (4.8) and (4.9), we obtain the energy in terms of Bethe roots $\{u_k^{(j)}\}$ for cases I and II. Below, we present the analytic forms of the energy eigenvalues for these two cases:

Case I. β_- and β_+ arbitrary while setting $\alpha_\pm = 0$, $\theta_- = \theta_+ = \theta =$ arbitrary :

$$\begin{aligned} E &= \operatorname{sh}^2(2\eta) \sum_{k=1}^{M_1} \frac{1}{\operatorname{sh}(u_k^{(1)} + \frac{3\eta}{2}) \operatorname{sh}(u_k^{(1)} - \frac{\eta}{2})} + 2 \operatorname{sh} 2\eta[(N+1) \operatorname{cth} \eta - \operatorname{cth} \frac{\eta}{2}] \\ &+ c_1^{(1)} C'(0) + c_2^{(1)} \end{aligned} \tag{4.17}$$

Case II. α_- and α_+ arbitrary while setting $\beta_\pm = 0$, $\theta_- = \theta_+ = \theta =$ arbitrary :

$$\begin{aligned} E &= \operatorname{sh}^2(2\eta) \sum_{k=1}^{M_1} \frac{1}{\operatorname{sh}(u_k^{(1)} + \frac{3\eta}{2}) \operatorname{sh}(u_k^{(1)} - \frac{\eta}{2})} + 2 \operatorname{sh} 2\eta[(N+1) \operatorname{cth} \eta - \operatorname{th} \frac{\eta}{2}] \\ &+ c_1^{(1)} C'(0) + c_2^{(1)} \end{aligned} \tag{4.18}$$

where in (4.17) and (4.18),

$$C(u) = -\frac{\operatorname{sh} 2u \operatorname{sh}(2u+2\eta)}{[\operatorname{sh} u \operatorname{sh}(u+\eta)]^{2N}} \delta^{(1)}(u - \frac{\eta}{2}). \tag{4.19}$$

We recall that $M_1 = N + \frac{1}{2}(p+1)$ and $M_2 = M_1 - 1$. Note that in (4.17) and (4.18), the contribution to E comes only from $\{u_k^{(1)}\}$ and not from both $\{u_k^{(1)}\}$ and $\{u_k^{(2)}\}$ as one may initially expect. Perhaps, if one uses (3.10) instead of (3.11) in the derivation of E, an equivalent expression involving only $\{u_k^{(2)}\}$ or both $\{u_k^{(1)}\}$ and $\{u_k^{(2)}\}$ may result. This however will not affect the numerical value of the energy eigenvalues tabulated in the next section.

4.3 Numerical results

In this section, we tabulate the energies computed using (4.17) and (4.18) for some values of $N\,, p$ (therefore η) and the boundary parameters $\{\alpha_\pm\,, \beta_\pm\,, \theta_\pm\}$ with the Bethe roots, $\{u_k^{(1)}\}$, in Tables 1 and 2 for cases I and II respectively. These Bethe roots are obtained using McCoy's method [82, 83]. These numerical results demonstrate the completeness of the Bethe-ansatz-type equations, (3.20) and (3.21). We have verified that the energies given in Tables 1 and 2 coincide with those obtained from direct diagonalization of (4.1).

5 Discussion

By using a method that relies on certain functional relations that the "fundamental" transfer matrices, $t^{(\frac{1}{2},s)}(\mathrm{u})$, obey at roots of unity and the truncation of fusion hierarchy, we set up a generalized form of the $T-Q$ relation, (3.10) and (3.11), for the open spin-s XXZ quantum spin chain with nondiagonal boundary terms. From these relations, we have determined Bethe-ansatz-type solutions of the model, (3.20) and (3.21). These solutions hold only for $\eta = \frac{i\pi}{2}\,, \frac{i\pi}{4}\,, \ldots$. The solutions found here hold for arbitrary values of boundary parameters (at most two). We have checked these solutions for chains of length up to $N = 4$, and have verified that indeed they give the complete set of $(2s+1)^N$ eigenvalues. Moreover, we also presented numerical evidence for the completeness of the Bethe-ansatz-type solutions found (using $s = 1$ as examples) in Tables 1 and 2. The completeness of the solutions found here for spin-s can perhaps be readily concluded from completeness of the corresponding Bethe-ansatz-type equations for spin-1/2 case and the fusion hierarchy (2.14) which is used in the construction of higher spin-s transfer matrices.

A number of problems remain that are worth investigating. It would be very interesting to see the relation of $s = 1$ case to the supersymmetric sine-Gordon (SSG) model, along the lines of [59] and [60], but now for spin-1 chain with nondiagonal boundary terms described by the generalized $T-Q$ relations instead of the conventional $T-Q$ relation. One could also try to generalize the solutions presented in [49] for the spin-1/2 case, where all six boundary parameters are completely arbitrary, to any spin s, and analyze the $s = 1$ case for this general solution in relation to the SSG model. In this regard, one can study the continuum limit of their Nonlinear Integral Equations (NLIEs), thus investigating the infrared (IR) and ultraviolet (UV) limits of the NLIEs. One could also investigate the boundary bound states of SSG models corresponding to all these cases such as reported recently in [86].

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E	Bethe roots, $\{u_k^{(1)}\}$
-5.6483	0.426847 + 2.19193 i, 0.719676 + 1.1781 i, 0.109151 i, 0.426847 + 0.164266 i
-4.67715	0.106242 + 2.28424 i, 0.379199 + 1.1781 i, 1.05101 + 1.1781 i, 0.106242 + 0.071957 i
-2.75841	0.387014 + 2.748893 i, 1.277532 i, 0.932369 + 1.1781 i, 0.0609966 i
-1.98286	0.185547 + 2.748893 i, 1.701637 i, 0.915819 + 1.1781 i, 0.138044 i
-1.54571	0.171807 + 3.046499 i, 0.171807 + 2.451287 i, 1.566925 i, 0.916569 + 1.1781 i
-0.489791	0.781754 + 1.921787i, 1.599981 i, 0.0312436 i, 0.781754 + 0.434407 i
-0.392189	3.109568 i, 0.779636 + 1.920991 i, 1.554992 i, 0.779636 + 0.435203 i
0.572634	0.810472 i, 0.624212 + 1.1781 i, 0.010646 i, 1.227343 + 1.1781 i
0.808501	3.130312 i, 0.791507 i, 0.618753 + 1.1781 i, 1.221033 + 1.1781 i

Table 1: The 9 energies and corresponding Bethe roots for $N = 2\,, s = 1\,, p = 3\,, \eta = i\pi/4\,, \alpha_- = 0\,, \beta_- = 0.767\,, \theta_- = 0.573\,, \alpha_+ = 0\,, \beta_+ = 0.598\,, \theta_+ = 0.573$

E	Bethe roots, $\{u_k^{(1)}\}$
-6.07709	0.0471453 + 3.1415 i, 0.0471453 + 2.61809 i, 1.74867 i, 0.74532 + 1.309 i, 0.48742 i
-4.65604	2.65564 i, 0.107433 + 2.35618 i, 0.321204 + 1.309 i, 0.557414 i, 0.107433 + 0.261819 i
-4.3506	0.00657235 + 3.07819 i, 0.00657235 + 2.6814 i, 2.07693 i, 0.12098 + 1.93837 i, 0.12098 + 0.679624 i
-2.55991	0.272597 + 3.13706 i, 0.272597 + 2.62253 i, 2.13098 i, 0.672718 + 1.309 i, 0.862768 i
-1.63092	0.326829 + 2.87979 i, 0.308315 + 2.35663 i, 2.13093 i, 0.890835 i, 0.308315 + 0.261367 i
0.0925845	0.248529 + 2.87979 i, 1.76311 i, 0.373083 + 1.309 i, 1.20497 + 1.309 i, 0.487 i
0.0971716	0.548694 + 2.59187 i, 2.13099 i, 0.518481 + 1.309 i, 0.856853 i, 0.548694 + 0.0261235 i
1.6757	0.70468 + 2.87979 i, 0.338436 + 1.309 i, 0.854426 i, 1.08306 + 1.309 i, 0.487 i
2.99332	1.7639 i, 0.273003 + 1.309 i, 0.720682 + 1.309 i, 0.487 i, 1.54847 + 1.309 i

Table 2: The 9 energies and corresponding Bethe roots for $N = 2\,, s = 1\,, p = 5\,, \eta = i\pi/6\,, \alpha_- = 0.854i\,, \beta_- = 0\,, \theta_- = 0.482\,, \alpha_+ = 0.487i\,, \beta_+ = 0\,, \theta_+ = 0.482$