ON REDUCTIVE AUTOMORPHISM GROUPS OF REGULAR EMBEDDINGS

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ABSTRACT. Let G be a connected reductive complex algebraic group acting on a smooth complete complex algebraic variety X. We assume that X under the action of G is a *regular embedding*, a condition satisfied in particular by smooth toric varieties and flag varieties. For any set $\mathcal{D}$ of G-stable prime divisors, we study the action on X of the group $\mathrm{Aut}^\circ(X, \mathcal{D})$, the connected automorphism group of X stabilizing $\mathcal{D}$. We determine a Levi subgroup A of $\mathrm{Aut}^\circ(X, \mathcal{D})$ and we compute relevant invariants of X as a spherical A-variety. As a byproduct, we obtain a description of the open A-orbit on X and the inclusion relation between A-orbit closures.

1. Introduction

In the 1970's Demazure described the connected automorphism groups of two distinguished classes of algebraic varieties equipped with the action of a connected reductive group G: the complete homogeneous spaces G/P for P a parabolic subgroup (see [De77]), and the smooth complete toric varieties, with G abelian (see [De70]). In the case of $X = G/P$, the group G goes surjectively onto the connected automorphism group $\mathrm{Aut}^\circ(X)$ except for three particular cases (with G a simple group) and products $(G_1 \times G_2)/(P_1 \times P_2)$ where $P_1 \subseteq G_1$, $P_2 \subseteq G_2$, and G_1/P_1 is one of these three exceptions. In the case where X is a toric G-variety, the image of G in $\mathrm{Aut}^\circ(X)$ is a maximal torus of the latter, and the corresponding root datum of $\mathrm{Aut}^\circ(X)$ is completely determined by the spaces of global sections $H^0(X, \mathcal{O}_X(Y))$, with Y varying in the set of G-stable prime divisors of X.

These two classes of G-varieties admit a common generalization: the *regular embeddings*, here also called G-regular embeddings or G-regular varieties, defined independently in [BDP90] and [Gi89]. With the additional assumption of completeness, Bien and Brion showed that these varieties correspond to a relevant class of spherical varieties, namely the smooth, complete, and *toroidal* ones (see [BB96]).

The spaces $H^0(X, \mathcal{O}_X(Y))$ play again an important role, especially for the case where X is *wonderful* in the sense of [Lu01] (see [Br07]), although they do not yield a direct description of $\mathrm{Aut}^\circ(X)$ if X is not toric. Also, the group $\mathrm{Aut}^\circ(X)$ may be non-reductive. Nevertheless, X is a

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spherical variety under the action of A, where A is any reductive subgroup $\mathrm{Aut}^\circ(X)$ containing the image of G, therefore it is natural to study the relationship between invariants of X as a spherical G-variety, the structure of A, and invariants of X with respect to the A-action. The results of [AG10] are also related to this problem, and classify those toric varieties that are homogeneous under the action of a semisimple group.

In this paper we provide a complete description of the action of A on X if A is a Levi subgroup of $\mathrm{Aut}^\circ(X, \mathcal{D})$. Here $\mathcal{D}$ is any subset of the set ∂X of G-stable prime divisors of X, and $\mathrm{Aut}^\circ(X, \mathcal{D})$ is the connected component of the group of automorphisms of X stabilizing each element of $\mathcal{D}$.

Our approach is based on the analysis of the following filtration:

$$\theta(G) \subseteq \mathrm{Aut}^\circ(X, \partial X) \subseteq \mathrm{Aut}^\circ(X, \mathcal{D} \cup (\partial X)^\ell) \subseteq \mathrm{Aut}^\circ(X, \mathcal{D}),$$

where $\theta(G)$ is the image of G in $\mathrm{Aut}^\circ(X)$ and $(\partial X)^\ell$ is a certain subset of ∂X (see Definition 2.5). The main motivation is the fact that the groups $\mathrm{Aut}^\circ(X, \partial X)$ and $\mathrm{Aut}^\circ(X, \mathcal{D} \cup (\partial X)^\ell)$ are reductive and X is regular under their actions, whereas both statements may fail for $\mathrm{Aut}^\circ(X, \mathcal{D})$.

For the group $\mathrm{Aut}^\circ(X, \partial X)$, we show in §4 that it is completely determined by results of [Br07] and [Pe09]. Then we consider $\mathcal{D}' = \mathcal{D} \cup (\partial X)^\ell$ and show in §7 that $\mathrm{Aut}^\circ(X, \mathcal{D}')$ is reductive, and that it can be studied using a certain G-equivariant map $X \to \mathbb{X}$, where $\mathbb{X}$ is a wonderful G-variety canonically associated with X. Namely, the group $\mathrm{Aut}^\circ(X, \mathcal{D}')$ (up to a central torus) is obtained lifting to X the action of the universal cover of a certain semisimple subgroup of $\mathrm{Aut}^\circ(\mathbb{X})$. The latter is known thanks to the results of [Pe09], which are somewhat similar to Demazure's theorem on flag varieties: the image of G is the whole $\mathrm{Aut}^\circ(\mathbb{X})$, up to some exceptions that can be explicitly described.

It is worth noticing that $\mathbb{X}$ is obtained from X using a procedure called *wonderful closure*, which is closely related to the well-known construction of the *spherical closure* of a spherical subgroup of G.

For the group $\mathrm{Aut}^\circ(X, \mathcal{D})$, we show that it is enough to deal with the case where $\mathcal{D}$ contains $(\partial X) \setminus (\partial X)^\ell$ (see the discussion at the end of §6). Under this assumption we show in §10 how to recover a Levi subgroup A of $\mathrm{Aut}^\circ(X, \mathcal{D})$ by an analysis of the fibers of the map $X \to \mathbb{X}$, which are finite unions of toric varieties.

We also give an explicit combinatorial description of all the invariants commonly associated to X as a spherical A-variety, invariants which uniquely determine X up to A-equivariant isomorphisms thanks to the classification of spherical varieties.

In particular, we describe both the invariants associated to the open A-orbit on X, the so-called *Luna invariants*, and the invariants associated to X considered as an embedding of its open A-orbit, according to the Luna-Vust theory of embeddings of spherical homogeneous spaces. Thanks to this theory, this accounts for a complete description of the structure of the A-orbits on X.

We also discuss explicitly in §8 and §9 the two special cases of G semisimple and G abelian.

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Notations. Through this paper G is a connected reductive linear algebraic group over the field of complex numbers $\mathbb{C}$. We assume that $G = G' \times C$ where C is an algebraic torus and G' is semisimple and simply connected. We denote by $\mathbb{G}_m$ the multiplicative algebraic group of non-zero complex numbers.

We fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We denote by B^- the Borel subgroup of G such that $B \cap B^- = T$. If H is any algebraic group then we denote by $\mathrm{Z}(H)$ ist center, by H° its connected subgroup containing the unit element e_H, and by $\mathcal{X}(H)$ the set of its characters, i.e. algebraic group homomorphisms $H \to \mathbb{G}_m$. If V is an H-module, then we denote by $V^{(H)}$ the set of *non-zero* H-semiinvariants of V, and for any $\chi \in \mathcal{X}(H)$ we set

$$V^{(H)}_\chi = \{v \in V \setminus \{0\} \mid hv = \chi(h)v \ \forall h \in H\}\,.$$

If $H \subseteq K$ are subgroups of G, then we denote by $\pi^{H,K} \colon G/H \to G/K$ the natural map sending $gH \in G/H$ to $gK \in G/K$.

For any subset R of a $\mathbb{Z}$-module Λ, we denote by $R^\vee$ (resp. $R^\perp$) the subset of $\mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Q})$ of all elements that are ≥ 0 (resp. $= 0$) on R. We define in the same way *mutatis mutandis* the subsets $R^\vee, R^\perp \subseteq \Lambda$ for $R \subseteq \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Q})$.

The term *algebraic variety* (or simply *variety*) stands here for separated, reduced and irreducible scheme of finite type over the field $\mathbb{C}$, and all actions of algebraic groups on varieties are be assumed to be algebraic. If X is a variety, the connected component containing id_X of its automorphism group is denoted by $\mathrm{Aut}^\circ(X)$. If a connected algebraic group H acts on X, we denote by

$$\theta_{H,X} \colon H \to \mathrm{Aut}^\circ(X)$$

the corresponding homomorphism.

If X is a G-variety, we denote by $\mathrm{Pic}^G(X)$ the group of isomorphism classes of G-linearized invertible sheaves. If X is normal and Y is a Cartier divisor, then the invertible sheaf $\mathcal{O}_X(Y)$ admits a (non unique) G-linearization (see [KKLV89, Remark after Proposition 2.4]). If in addition X is complete and Y is a G-stable prime divisor, we will always assume that the G-linearization is chosen in such a way that the induced G-action on $H^0(X, \mathcal{O}_X(Y))$ is equal to the action inherited via the usual inclusion $H^0(X, \mathcal{O}_X(Y)) \subset \mathbb{C}(X)$.

2. Complete regular embeddings

Definition 2.1. Suppose that an irreducible G-variety X has an open G-orbit. Then X is *G-regular* (or a *G-regular embedding*) if for any $x \in X$:

(1) the closure $\overline{Gx}$ of its orbit is smooth, and it is the transversal intersection of the G-stable prime divisors containing it;
(2) the stabilizer G_x has a dense orbit on the normal space in X to the orbit Gx in the point x.

As an immediate consequence of the definition, a G-regular embedding is smooth and has only a finite number of G-orbits. Examples of G-regular embeddings are the G-homogeneous spaces for any G, and if G is an algebraic torus then any smooth toric G-variety. Other examples come from the family of *spherical varieties*, which are by definition irreducible normal G-varieties with a dense B-orbit.

More precisely, suppose that a G-variety X is smooth and complete. Then X is G-regular if and only if it is spherical and *toroidal*, i.e. any B-stable prime divisor containing a G-orbit is also G-stable (see [BB96, Proposition 2.2.1]).

We review some relevant invariants associated to any spherical G-variety X. They are actually invariants under birational G-equivariant maps, therefore they only depend on the open G-orbit of X. If x_0 is a point on this orbit, then we also denote the orbit Gx_0 simply by G/H, where $H = G_{x_0}$ is called a *generic stabilizer* of X. In this case, H is also called a *spherical subgroup*, and (X, x_0) (or simply X) is called an *embedding* of G/H. A *morphism* between two embeddings (X, x_0) and (X', x'_0) is a G-equivariant map $X \to X'$ sending x_0 to x'_0.

We will always assume that x_0 is chosen in such a way that Bx_0 is dense in X. Then H is also called a *B-spherical subgroup*.

Definition 2.2. Let X be a spherical G-variety with open G-orbit G/H.

(1) We define[1] the lattice
$$\Lambda_G(X) = \left\{ \chi \in \mathcal{X}(B) \,\middle|\, \mathbb{C}(X)^{(B)}_\chi \neq \varnothing \right\},$$
whose rank is by definition the *rank* of X.
(2) We define
$$\mathrm{N}_G(X) = \mathrm{Hom}_\mathbb{Z}(\Lambda_G(X), \mathbb{Q}).$$
(3) We define $\Delta_G(X)$ to be the set of *colors* of X, i.e. the B-stable prime divisors of X having non-empty intersection with the open G-orbit G/H of X.

[1] We ignore the dependence on B of all the invariants we define. This is justified by the fact that for any reductive group under consideration the choice of a Borel subgroup will be either unique (when the group is abelian) or always explicitly fixed.

(4) For any discrete valuation $\nu\colon \mathbb{C}(X)\setminus\{0\}\to\mathbb{Q}$ we define an element $\rho_{G,X}(\nu)\in \mathrm{N}_G(X)$ with the formula

$$\langle\rho_{G,X}(\nu),\chi\rangle = \nu(f_\chi),$$

where $f_\chi\in\mathbb{C}(X)^{(B)}_\chi$. If D is a prime divisor of X and ν_D is the associated discrete valuation, then we will also write $\rho_{G,X}(D)$ for $\rho_{G,X}(\nu_D)$.

(5) We define

$$\mathrm{V}_G(X) = \{\rho_{G,X}(\nu)\mid \nu \text{ is } G\text{-invariant}\},$$

which is a polyhedral convex cone of maximal dimension in $\mathrm{N}_G(X)$; we denote its linear part by $\mathrm{V}^\ell_G(X)$.

(6) We define the *boundary* of X, denoted by $\partial_G X$, to be the set of the irreducible components of $X\setminus(G/H)$.

For the above, and for all the invariants defined later, we will drop the indices G and X whenever it is clear which group and which variety are considered. In loose terms the colors of X can also be considered as invariants under G-equivariant birational maps, since they are the closures in X of the colors of G/H.

The Luna-Vust theory of embeddings of homogeneous spaces specializes for spherical toroidal varieties in the following way (for details and proofs see [Kn96]).

Definition 2.3. Let X be a G-regular embedding, and Y an irreducible G-stable locally closed subvariety. Then we define $c_{X,Y}\subseteq \mathrm{N}(X)$ to be the polyhedral convex cone generated by $\rho(D_1),\ldots,\rho(D_n)$, where $D_1,\ldots,D_n$ are the the B-stable prime divisors containing Y. The *fan* of X is defined as

$$\mathcal{F}_G(X) = \{c_{X,Y}\mid Y \text{ a } G\text{-orbit of } X\}.$$

Notice that since X is toroidal then the divisors $D_1,\ldots,D_n$ above are also G-stable for any Y. The collection of convex cones $\mathcal{F}(X)$ satisfies the following properties:

(1) each cone of $\mathcal{F}(X)$ is contained in $\mathrm{V}(G/H)$, it is strictly convex, and all its faces belong to $\mathcal{F}(X)$,
(2) any element of $\mathrm{V}(G/H)$ belongs to the relative interior of at most one cone of $\mathcal{F}(X)$.

The map $X\mapsto\mathcal{F}(X)$ induces a bijection between toroidal embeddings of G/H (up to isomorphism of embeddings) and *fans*, i.e. collections of strictly convex polyhedral convex cones satisfying (1) and (2).

The *support* of a fan $\mathcal{F}$ is defined as

$$\operatorname{supp}\mathcal{F} = \bigcup_{c\in\mathcal{F}} c.$$

The embedding X is complete if and only if $\operatorname{supp}\mathcal{F}(X) = \mathrm{V}(X)$, and it is smooth if and only if for each $c \in \mathcal{F}(X)$ there exists a basis $\gamma_1 \ldots, \gamma_r$ of $\Lambda(X)$ and an integer k between 1 and r such that

$$c = \{\gamma_1, \ldots, \gamma_k\}^\vee .$$

For later reference, we recall that if a spherical embedding X is not toroidal, then it is also described by a similar datum, called a fan of *colored convex cones*. Here, the convex cone associated to a G-orbit $Y \subseteq X$ is replaced by the pair $(c_{X,Y}, d_{X,Y})$ where $d_{X,Y}$ is the set of colors containing Y, and $c_{X,Y}$ is defined as above.

In general, the set $\mathrm{V}(X)$ is also a polyhedral convex cone, of maximal dimension, and its linear part $\mathrm{V}^\ell(X)$ has the same dimension (as a $\mathbb{Q}$-vector space) of $\mathrm{N}_G H/H$ (as a complex algebraic group). The equations defining the maximal proper faces of $\mathrm{V}(X)$ are linearly independent (see [Br90, Corollaire 3.3]). In other words, there always exist $\sigma_1, \ldots, \sigma_k \in \Lambda(X)$ that are indivisible, linearly independent, and such that

$$\mathrm{V}(X) = \{-\sigma_1, \ldots, -\sigma_k\}^\vee .$$

Definition 2.4. The elements $\sigma_1, \ldots, \sigma_k$ above are uniquely determined by G/H and called the *spherical roots* of X; their set is denoted as

$$\Sigma_G(X) = \{\sigma_1, \ldots, \sigma_k\} .$$

The map $Y \mapsto c_{X,Y}$ sends a G-orbit of codimension d in X to a cone of dimension d, and this restricts to a bijection between the boundary ∂X and the set of 1-dimensional cones in $\mathcal{F}(X)$.

Definition 2.5. For a subset $\mathcal{D} \subseteq \partial X$, we define the subsets

$$\mathcal{D}^\ell = \left\{ Y \in \mathcal{D} \,\middle|\, c_{X,Y} \subset \mathrm{V}^\ell(X) \right\}$$

and

$$\mathcal{D}^{n\ell} = \mathcal{D} \setminus \mathcal{D}^\ell.$$

3. Spherical and wonderful closure

In this section we recall the notion, introduced in [Lu01], of the spherical closure $\overline{H}$ of a spherical subgroup $H \subseteq G$. We also define another subgroup containing H, called its wonderful closure. This is essentially already known, but not yet found in the literature. We gather at first some results from [Lu01, §6].

An element n of the normalizer N of H induces a G-equivariant isomorphism $G/H \to G/H$ given by $gH \mapsto gnH$. This induces an action of N on the set of colors $\Delta(G/H)$: the *spherical closure* $\overline{H}$ of H is defined as the kernel of this action.

If $\overline{H} = H$ then we say that H is *spherically closed*, and for any spherical subgroup H the spherical closure $\overline{H}$ is itself spherically closed. This is well known, but for lack of a detailed reference we provide a proof, also because $\mathrm{N}_G\overline{H}$ may well be strictly bigger than $\mathrm{N}_G H$.

Proposition 3.1. *For any spherical subgroup $H \subseteq G$, the spherical closure $\overline{H}$ is spherically closed.*

Proof. Since $\overline{H}$ is contained in $\mathrm{N}_G H$ the quotient $\overline{H}/H$ is diagonalizable (see [Kn94, Theorem 6.1]), and thus H is defined inside $\overline{H}$ as intersection of kernels of some characters. The colors of $G/\overline{H}$ generate $\mathrm{Pic}^G(G/\overline{H})$ (see [Br89, Proposition 2.2]) and the latter is isomorphic to $\mathcal{X}(\overline{H})$ (see [KKV89, §3.1]), therefore $\overline{\overline{H}}$ acts trivially on $\mathcal{X}(\overline{H})$.

This implies that $\overline{\overline{H}}$ normalizes H. By definition, it fixes all colors of $G/\overline{H}$, but these correspond to the colors of G/H via the natural map $\pi^{H,\overline{H}}\colon G/H \to G/\overline{H}$. Hence $\overline{\overline{H}} \subseteq \overline{H}$. □

For later convenience we report the following auxiliary result. Recall that whenever $H \subseteq K$ are spherical subgroups of G, the lattice $\Lambda(G/K)$ is contained in the lattice $\Lambda(G/K)$, since B-semiinvariant functions can be lifted from G/K to G/H via the map $\pi^{H,K}\colon G/H \to G/K$. We sometimes denote this inclusion as a map $(\pi^{H,K})^*\colon \Lambda(G/K) \to \Lambda(G/H)$, which induces a surjection $\pi^{H,K}_*\colon \mathrm{N}(G/H) \to \mathrm{N}(G/K)$. Moreover, we have $\pi^{H,K}_*(\mathrm{V}(G/H)) = \mathrm{V}(G/K)$ and $\ker \pi^{H,\overline{H}}_* = \mathrm{V}^\ell(G/H)$ (see [Kn96, Theorem 4.4 and Theorem 6.1]).

Lemma 3.2. *Let $H \subseteq K \subseteq \overline{H}$ be spherical subgroups. Then $(\pi^{H,K}_*)^{-1}(\mathrm{V}(G/H)) = \mathrm{V}(G/K)$.*

Proof. The claim stems from $\pi^{H,K}_*(\mathrm{V}(G/H)) = \mathrm{V}(G/K)$, together with

$$\ker\left(\pi^{H,K}_*\right) \subseteq \mathrm{V}^\ell(G/H).$$

This inclusion follows from the fact that $\pi^{K,\overline{H}}_* \circ \pi^{H,K}_* = \pi^{H,\overline{H}}_*$, and that the latter has kernel $\mathrm{V}^\ell(G/H)$. □

A class of subgroups slightly broader then the spherically closed ones is the following.

Definition 3.3. Suppose that $\Sigma(G/H)$ is a basis of $\Lambda(G/H)$. Then we say that H is a *wonderful* subgroup of G. In this case there exists a fan $\mathcal{F}$ having only one maximal cone equal to $\mathrm{V}(G/K)$; the associated toroidal embedding is denoted by $\mathbb{X}(G/H)$.

If H is wonderful then the embedding $\mathbb{X}(G/H)$ is smooth, has a unique closed G-orbit and it is *wonderful* in the sense of [Lu01]. A fundamental theorem of Knop (see [Kn96, Corollary 7.6]) states that a spherically closed subgroup is wonderful.

Example 3.4. The converse of the above statement is false: for example, if $G = \mathrm{SO}(2n+1)$ with $n \geq 2$, then $H = \mathrm{SO}(2n)$ is a wonderful subgroup, with $\overline{H} = \mathrm{N}_{\mathrm{SO}(2n+1)}\mathrm{SO}(2n) \neq H$ (see [Wa96, cases 7B, 8B of Table 1]).

It is possible to define canonically a minimal wonderful subgroup $\widehat{H}$ between H and $\overline{H}$. As a byproduct, the automorphism groups of regular embeddings of G/H are more directly related to the automorphism group of $\mathbb{X}(G/\widehat{H})$ than to that of $\mathbb{X}(G/\overline{H})$.

Definition 3.5. Let H and I be spherical subgroups of G. Then I is a *wonderful closure* of H if it is wonderful, satisfies $H \subseteq I \subseteq \overline{H}$, and is minimal with respect to these properties.

We will show that a wonderful closure always exists and is unique; for this we need to describe combinatorially all spherical subgroups having spherical closure equal to $\overline{H}$.

Let us fix a spherically closed subgroup K, and consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda(G/K) & \xrightarrow{\overline{\rho}} & \mathrm{Pic}^G(\mathbb{X}(G/K)) & \xrightarrow{\tau} & \mathrm{Pic}^G(G/K) & \longrightarrow & 0 \\ & & & & \big\downarrow \sigma & & & & \\ & & & & \mathrm{Pic}^G(G/B) & & & & \end{array}$$

where the row is exact (see also [Br07, Proposition 2.2.1].

The map τ is the pullback along the inclusion $G/K \to \mathbb{X}(G/K)$. For σ, observe that $\mathbb{X}(G/K)$ has a unique closed G-orbit Z, which is projective and therefore comes with a natural projection map $G/B \to Z$. The map σ is then the pullback along the composition $G/B \to Z \to \mathbb{X}(G/K)$.

The map $\overline{\rho}$ is defined in the following way: for any $\chi \in \Lambda(G/K)$ we take a function $f_\chi \in \mathbb{C}(G/K)^{(B)}_\chi$ and consider the G-stable part $D = \mathrm{div}(f_\chi)^G$ of $\mathrm{div}(f_\chi)$. Then we set $\overline{\rho}(\chi) = \mathcal{O}_{\mathbb{X}}(-D)$, which admits a unique G-linearization such that C acts trivially on the total space of the bundle.

These maps admit also a combinatorial definition, using the fact that $G = C \times G'$ and $K \supseteq C$, that $\Delta(G/K)$ is a basis of $\mathrm{Pic}(\mathbb{X}(G/K))$ (see [Br89, Proposition 2.2]), and the isomorphisms $\mathrm{Pic}^G(G/K) \cong \mathcal{X}(K)$, $\mathrm{Pic}^G(G/B) \cong \mathcal{X}(B)$. The resulting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda(G/K) & \xrightarrow{\overline{\rho}} & \mathcal{X}(C) \times \mathbb{Z}^\Delta & \xrightarrow{\tau} & \mathcal{X}(K) & \longrightarrow & 0 \\ & & & & \big\downarrow \sigma & & & & \\ & & & & \mathcal{X}(B) & & & & \end{array} \tag{3.1}$$

where $\Delta = \Delta(G/K)$, is also described in details in [Lu01, §6.3]. The map $\overline{\rho}$ is defined as:

$$\overline{\rho}(\chi) = (\chi|_C, \langle \rho_{G,G/K}(\cdot), \chi \rangle),$$

and $\sigma \circ \overline{\rho}$ is the identity on $\Lambda(G/K)$ (see loc.cit.).

Lemma 3.6. [Lu01, Lemme 6.3.1, Lemme 6.3.3] *Let $K \subseteq G$ be a spherically closed subgroup. The application*

$$H \to \tau^{-1}\left(\mathcal{X}(K)^H\right)$$

is an inclusion-reversing bijection between the set of normal subgroups H of K such that K/H is diagonalizable, and the set of subgroups of $\mathcal{X}(C)\times\mathbb{Z}^\Delta$ containing $\overline{\rho}(\Lambda(G/K))$. If the restriction of σ to $\tau^{-1}\left(\mathcal{X}(K)^H\right)$ is injective then H is spherical.

Lemma 3.7. *For any spherical subgroup $H\subseteq G$ contained and normal in K and all $D\in\Delta(G/H)$ we have $\pi^{H,K}(D)\in\Delta(G/K)$, and*

$$\pi^{H,K}_*(\rho_{G,G/H}(D))=\rho_{G,G/K}(\pi^{H,K}(D)).$$

Proof. Since H is normal in K, then K stabilizes the open set $BH\subseteq G$ acting by right multiplication on G (see also [BP87, First part of the proof of Proposition 5.1]). The complement $G\setminus BH$ is the union of $\pi^{\{e_G\},H}(E)$ for E varying in $\Delta(G/H)$, whence the first statement.

For the second statement, it is enough to show that a local equation of D on G/H can be chosen to be the pull-back of a function on G/K along $\pi^{H,K}$. Let $E_1,\dots,E_n$ be all the distinct B-stable prime divisors of G such that $\pi^{\{e_G\},K}(E_i)=\pi^{H,K}(D)$. Since G is factorial we can choose a global equation $f_i\in\mathbb{C}[G]$ for each E_i, and consider the product $f=f_1\cdot\ldots\cdot f_n$.

The divisor $\mathrm{div}(f)$ on G is B-stable under the left translation action of G on itself, but none of its components is G-stable therefore there exists an element $g\in G$ such that the function $f_0\colon x\mapsto f(gx)$ doesn't vanish on any divisor E_i. On the other hand $\mathrm{div}(f)$ is K-stable under the right translation action of G on itself, thus f is K-semiinvariant under this action. The function f_0 is then also K-semiinvariant, with same K-eigenvalue. It follows that

$$F=\frac{f}{f_0}$$

is K-invariant with respect to the right translation action. In other words $F=(\pi^{\{e_G\},K})^*(\widetilde{F})$ for some $\widetilde{F}\in\mathbb{C}(G/K)$.

Now for some i_0 the divisor E_{i_0} satisfies $\pi^{\{e_G\},H}(E_{i_0})=D$. The function F is equal to the pull-back of $(\pi^{H,K})^*(\widetilde{F})$ along $\pi^{\{e_G\},H}$ and is a local equation of E_{i_0} on G, hence $(\pi^{H,K})^*(\widetilde{F})$ is a local equation of $\pi^{\{e_G\},H}(E_{i_0})=D$ on G/H: the lemma follows. ☐

Thanks to Lemma 3.7, we can extend the map $\overline{\rho}$ to $\Lambda(G/H)$ in the following way.

Definition 3.8. We denote again by $\overline{\rho}$ the extension of the above map $\overline{\rho}\colon\Lambda(G/K)\to\mathcal{X}(C)\times\mathbb{Z}^\Delta$ to $\Lambda(G/H)$ given by the following formula:

$$\overline{\rho}(\chi)=(\chi|_C,\langle\rho_{G,G/H}(\cdot),\chi\rangle).$$

Lemma 3.9. *For any spherical subgroup $H\subseteq G$ contained and normal in K we have*

$$\overline{\rho}(\Lambda_G(G/H))=\tau^{-1}\left(\mathcal{X}(K)^H\right). \tag{3.2}$$

Proof. The equality stems from the description of the map τ given in [Lu01, §6.3], see in particular [Lu01, Proof of Proposition 6.3]. Indeed, for any $\chi \in \Lambda(G/H)$ the image $\tau(\overline{\rho}(\chi))$ is the K-eigenvalue of a rational function f on G such that f is the pull-back of a rational function on G/H. Hence its K-eigenvalue is trivial on H.

For the other inclusion, Lemma 3.7 implies that

$$\bigcup_{D\in\Delta(G/H)} \left(\pi^{\{e_G\},H}\right)^{-1}(D) = \bigcup_{D\in\Delta(G/K)} \left(\pi^{\{e_G\},K}\right)^{-1}(D)$$

is the union of all prime divisors of G that are B-stable under left translation and H-stable (or equivalently K-stable) under right translation. As a consequence, if $\tau(\gamma,(n_D)_{D\in\Delta})$ is a K-character that is trivial on H, then $\chi = \sigma(\tau(\gamma,(n_D)_{D\in\Delta}))$ is the B-eigenvalue of a B-semiinvariant rational function on G/H. Since $\overline{\rho}(\chi) = (\gamma,(n_D)_{D\in\Delta})$, the proof is complete. □

Proposition 3.10. *Let $H \subseteq G$ be a spherical subgroup, set $K = \overline{H}$ and $\Xi = \mathrm{span}_{\mathbb{Z}}\Sigma(G/H)$. Then*

$$\Lambda(G/K) \subseteq \Xi \subseteq \Lambda(G/H).$$

The normal subgroup $\widehat{H} \subseteq K$ associated to $\overline{\rho}(\Xi)$ via the map of Lemma 3.6 is the unique wonderful closure of H. It has the same dimension of $\overline{H}$, and it is the unique wonderful subgroup between H and $\overline{H}$ that satisfies $\Sigma_G(G/H) = \Sigma(G/\widehat{H})$. Moreover, the spherical closure of $\widehat{H}$ is $\overline{H}$.

Proof. The inclusion $\Xi \subseteq \Lambda(G/H)$ is obvious. The map $\pi_*^{H,K}\colon \mathrm{N}(G/H) \to \mathrm{N}(G/K)$ has kernel $\mathrm{V}^\ell(G/H)$, and satisfies $\pi_*(\mathrm{V}(G/H)) = \mathrm{V}(G/K)$. The other inclusion $\Lambda(G/K) \subseteq \Xi$ follows. Hence the subgroup $\widehat{H}$ contains H.

The lattice $\Lambda(G/\widehat{H}) = \Xi$ has basis $\Sigma(G/H)$ since the spherical roots are aways linearly independent. Since $\Lambda(G/\widehat{H})$ has finite index inside $\mathrm{V}^\ell(G/H)^\perp$ and $\pi^{H,\widehat{H}}(\mathrm{V}(G/H)) = \mathrm{V}(G/\widehat{H})$ we deduce that $\Sigma(G/H) = \Sigma(G/\widehat{H})$.

If $\widetilde{H}$ is another wonderful subgroup such that $H \subseteq \widetilde{H} \subseteq \overline{H}$, then $\Lambda(G/\widetilde{H})$ has also finite index in $\mathrm{V}^\ell(G/H)^\perp$, and $\Sigma(G/\widetilde{H})$ is equal to $\Sigma(G/H)$ up to taking (positive) multiples of the elements of the latter. The dimension, minimality and uniqueness properties of $\widehat{H}$ follow, since $\Lambda(G/\widetilde{H}) \subseteq \Lambda(G/\widehat{H})$ implies $\widetilde{H} \supseteq \widehat{H}$.

The last assertion follows from the last assertion of [Lu01, Lemme 6.3.3]: indeed the lattice $\sigma(\Phi')$ of loc.cit. is denoted here by Ξ, and the set S° of loc.cit. is here a subset of $\frac{1}{2}\Sigma(G/H)$. □

4. Automorphisms stabilizing all G-orbits

From now on, X denotes a complete G-regular variety, with open G-orbit G/H.

Definition 4.1. For any subset $\mathcal{D} \subseteq \partial_G X$ of G-stable prime divisors we define

$$\mathrm{Aut}^\circ(X,\mathcal{D}) = \{\phi \in \mathrm{Aut}^\circ(X) \mid \phi(D) = D,\ \forall D \in \mathcal{D}\}.$$

Since X is G-regular, the group $\mathrm{Aut}^\circ(X, \partial_G X)$ is also the connected group of automorphisms of X stabilizing each G-orbit.

We recall now some results from [BB96] (see also [Br07]). The group $\mathrm{Aut}^\circ(X)$ is a linear algebraic group, with Lie algebra

$$\mathrm{Lie}\,\mathrm{Aut}^\circ(X) = H^0(X, \mathcal{T}_X)$$

where $\mathcal{T}_X$ is the sheaf of sections of the tangent bundle of X. The structure of G-module on $\mathrm{Lie}\,\mathrm{Aut}^\circ(X)$, induced by the adjoint action of $\theta_{G,X}(G) \subseteq \mathrm{Aut}^\circ(X)$, is given in [BB96, Proposition 4.1.1] in terms of global sections of the line bundles $\mathcal{O}_X(D)$ where $D \in \partial_G X$.

Namely, there exists an exact sequence of G-modules

$$0 \to \mathrm{Lie}\,\mathrm{Aut}^\circ(X, \partial_G X) \to \mathrm{Lie}\,\mathrm{Aut}^\circ(X) \to \bigoplus_{D \in \partial_G X} \frac{H^0(X, \mathcal{O}_X(D))}{\mathbb{C}} \to 0. \tag{4.1}$$

Moreover, for any $\mathcal{D} \in \partial_G X$ the Lie algebra of the subgroup $\mathrm{Aut}^\circ(X, \mathcal{D})$ is the inverse image of the sum

$$\bigoplus_{D \in (\partial_G X) \setminus \mathcal{D}} \frac{H^0(X, \mathcal{O}_X(D))}{\mathbb{C}}.$$

Definition 4.2. Let $0 \neq \gamma \in \mathcal{X}(B)$. If it exists, we denote by $X(\gamma)$ the uniquely determined element of $\partial_G X$ such that $H^0(X, \mathcal{O}_X(X(\gamma)))^{(B)}_\gamma \neq \varnothing$.

A particular case of $\mathrm{Aut}^\circ(X)$ has been studied in [Pe09], where X is a wonderful variety. Recall that C acts trivially on any wonderful G-variety, hence we can consider G'-varieties without loss of generality. Moreover, in this case $\mathrm{Aut}^\circ(X, \mathcal{D})$ is always semisimple and X is wonderful under its action (see [Br07, Theorem 2.4.2]). It is possible to summarize the results of [Pe09] as follows.

Theorem 4.3. [Pe09] *Let $\mathbb{X}$ be a wonderful G'-variety and $\mathcal{D} \subseteq \partial_{G'}\mathbb{X}$. Decompose G' and $\mathbb{X}$ into products*

$$G' = G'_1 \times \ldots \times G'_n, \quad \mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n,$$

with a maximal number of factors in such a way that G'_i acts non-trivially only on $\mathbb{X}_i$ for all $i = 1, \ldots, n$. Then

$$\mathrm{Aut}^\circ(\mathbb{X}, \mathcal{D}) = \mathrm{Aut}^\circ(\mathbb{X}_1, \mathcal{D}_1) \times \ldots \times \mathrm{Aut}^\circ(\mathbb{X}_n, \mathcal{D}_n),$$

where $\mathcal{D}_i = \{D \cap \mathbb{X}_i \,|\, D \in \mathcal{D}\} \subseteq \partial_{G'_i}\mathbb{X}_i$. Moreover, if the image of G'_i in $\mathrm{Aut}^\circ(\mathbb{X}_i, \mathcal{D}_i)$ is a proper subgroup, then $(G'_i, \mathbb{X}_i)$ appears in the lists of "exceptions" of [Pe09, §§3.2 – 3.6]. *If $\mathcal{D} = \partial_{G'}\mathbb{X}$ then all such exceptional factors have rank* 0 *or* 1.

Let now $\mathbb{X} = \mathbb{X}(G/\overline{H})$: the group $\mathrm{Aut}^\circ(X, \partial_G X)$ is easily recovered from $\mathrm{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X})$. Indeed, thanks to [Br07, Theorem 4.4.1], there exists a split exact sequence of Lie algebras

$$0 \to \frac{\mathrm{Lie}\,\overline{H}}{\mathrm{Lie}\,H} \to \mathrm{Lie}\,\mathrm{Aut}^\circ(X, \partial_G X) \to \mathrm{Lie}\,\mathrm{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X}) \to 0. \tag{4.2}$$

It follows that $\mathrm{Aut}^\circ(X, \partial_G X)$ is reductive, its connected center is $(\overline{H}/H)^\circ = (\widehat{H}/H)^\circ$, and its semisimple part can be computed using Theorem 4.3 and the lists of [Pe09].

We point out that in the above exact sequence we may as well use the variety $\mathbb{X} = \mathbb{X}(G/\widehat{H})$. Indeed, the results of [Br07, §4.4] hold (with same proofs) if we replace the spherical closure of H with its wonderful closure.

5. Relating $\mathrm{Aut}^\circ(X)$ to $\mathrm{Aut}^\circ(\mathbb{X})$

From now on, $\mathbb{X} = \mathbb{X}(G/\widehat{H})$ denotes the wonderful embedding of $G/\widehat{H}$. As a consequence of the last section, we may suppose from now on that $\theta_{G,X}(G) = \mathrm{Aut}^\circ(X, \partial_G X)$ and that $\theta_{G,\mathbb{X}}(G) = \mathrm{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X})$. Indeed, if this is not the case we may first apply Theorem 4.3 to $\mathbb{X}$, replace G'_i with the universal cover of $\mathrm{Aut}^\circ(\mathbb{X}_i, \partial_{G_i} \mathbb{X}_i)$ for all i such that these groups are different, and then replace C with $C \times (\widehat{H}/H)^\circ$.

Thanks to [Kn96, Theorem 4.1], the natural surjection $\pi^{H,\widehat{H}}\colon G/H \to G/\widehat{H}$ extends to a surjective G-equivariant map

$$\pi\colon X \to \mathbb{X} = \mathbb{X}(G/\widehat{H}).$$

Definition 5.1. We denote by

$$X \xrightarrow{\ \psi\ } X' \xrightarrow{\ f\ } \mathbb{X}$$

the Stein factorization of the map $\pi\colon X \to \mathbb{X}$.

In [Br07, §4.4] it is shown that $\mathrm{Aut}^\circ(X)$ acts on X' in such a way that ψ is equivariant; we denote the corresponding homomorphism as follows:

$$\psi_*\colon \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(X').$$

Its kernel is the subgroup of automorphisms of X stabilizing each fiber of ψ.

Proposition 5.2. *The inclusions* $\mathrm{Z}(\theta_{G,X}(G))^\circ \subseteq \ker\psi_* \cap \theta_{G,X}(G) \subseteq \mathrm{Z}(\theta_{G,X}(G))$ *between subgroups of* $\mathrm{Aut}^\circ(X)$ *hold. Moreover, there is a local isomorphism*

$$\mathrm{Aut}^\circ\Big(X, (\partial_G X)^{n\ell}\Big) \cong \theta_{G,X}(G') \ltimes (\ker\psi_*)^\circ \tag{5.1}$$

induced by the inclusion of both factors of the right hand side in $\mathrm{Aut}^\circ(X)$.

Proof. The first inclusion stems from the fact that $C = \mathrm{Z}(G)^\circ$ acts trivially on $\mathbb{X}$, hence also on X'. On the other hand, if $g \in G$ stabilizes all fibers of ψ, then it acts trivially on X' and also on $\mathbb{X}$. Therefore, to show the second inclusion, we only have to check that no simple factor of G acts trivially on $\mathbb{X}$ but not on X. This is true because $\widehat{H}/H$ is abelian.

Let us prove the last statement. Both groups on the right hand side of (5.1) are subgroups of $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$: this is obvious for $\theta_{G,X}(G')$, so we only have to check it for $(\ker \psi_*)^\circ$. Notice that ψ maps any element D of $(\partial_G X)^{n\ell}$ onto a proper G-stable closed subset of X'. It follows that D is an irreducible component of $\psi^{-1}(\psi(D))$, hence it is stable under the action of $(\ker \psi_*)^\circ$. It also follows that ψ_* maps $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$ into $\mathrm{Aut}^\circ(X', \partial_G X')$.

The intersection $(\ker \psi_*)^\circ \cap \theta_{G,X}(G')$ is finite thanks to the first part of the proof, and $(\ker \phi_*)^\circ$ is a normal subgroup of $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$. It only remains to prove that $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$ is generated by $\theta_{G,X}(G')$ and $(\ker \psi_*)^\circ$.

By [Br07, Theorem 4.4.1], we know that $\mathrm{Aut}^\circ(X', \partial X')$ and $\mathrm{Aut}^\circ(\mathbb{X}, \partial \mathbb{X})$ are both semisimple and locally isomorphic. It follows that the universal cover of $\mathrm{Aut}^\circ(X', \partial X')$ acts on $\mathbb{X}$ in such a way that f is equivariant. On the other hand no element of this universal cover could act trivially on X' and non-trivially on $\mathbb{X}$, hence $\mathrm{Aut}^\circ(X', \partial X')$ itself acts on $\mathbb{X}$, preserving all G-orbits. This produces a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\theta_{G,X}} & \mathrm{Aut}^\circ\left(X, (\partial_G X)^{n\ell}\right) \\ \Big\downarrow{\scriptstyle \theta_{G,\mathbb{X}}} & & \Big\downarrow{\scriptstyle \psi_*} \\ \mathrm{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X}) & \xleftarrow{f_*} & \mathrm{Aut}^\circ(X', \partial_G X') \end{array}$$

where $\theta_{G,\mathbb{X}}$ is surjective by our assumptions. Therefore $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$ is generated by $\theta_{G,X}(G)$ and $\ker(f_* \circ \psi_*)$. Notice that f_* has finite kernel, that the kernel of ψ_* contains $\theta_{G,X}(C)$, and that $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$ is connected: we deduce that $\mathrm{Aut}^\circ(X, (\partial_G X)^{n\ell})$ is indeed generated by $\theta_{G,X}(G')$ and $(\ker \psi_*)^\circ$, and the proof is complete. ☐

If we denote by

$$d\psi_* \colon \mathrm{Lie}\,\mathrm{Aut}^\circ(X) \to \mathrm{Lie}\,\mathrm{Aut}^\circ(X')$$

the corresponding homomorphism of Lie algebras, then the following corollary is an immediate consequence of the above proposition.

Corollary 5.3. *The subspace* $\ker d\psi_* \subseteq \mathrm{Lie}\,\mathrm{Aut}^\circ(X)$ *is G-stable, and its intersection with* $\mathrm{Lie}\,\theta_{G,X}(G)$ *is equal to* $\mathrm{Lie}\,\theta_{G,X}(C)$*. There exists a G-equivariant splitting of the exact sequence (4.1) such that*

$$\ker d\psi_* = \mathrm{Lie}\,\theta_{G,X}(C) \oplus \bigoplus_{D \in (\partial_G X)^\ell} \frac{H^0(X, \mathcal{O}_X(D))}{\mathbb{C}}.$$

6. Restricting automorphisms of X to fibers of ψ

We study now the automorphisms of a generic fiber of ψ induced by automorphisms of X belonging to $\ker\psi_*$. For this it is convenient to exploit the *local structure* of spherical varieties.

Theorem 6.1. [Kn94, Theorem 2.3 and Proposition 2.4] *Let Y be a spherical G-variety. Let $P_{G,Y} \supseteq B$ be the stabilizer in G of the open B-orbit of Y, let $L_{G,Y}$ be the Levi subgroup of $P_{G,Y}$ containing T, and consider the following open subset of Y:*

$$Y_0 = Y \setminus \bigcup_{D\in\Delta_G(Y)} D.$$

Then there exists a closed $L_{G,Y}$-stable and $L_{G,Y}$-spherical subvariety $\mathcal{Z}_{G,Y}$ of Y_0 such that the map

$$\begin{array}{ccc} P^u_{G,Y} \times \mathcal{Z}_{G,Y} & \to & Y_0 \\ (p,z) & \mapsto & pz \end{array}$$

is a $P_{G,Y}$-equivariant isomorphism, where $L_{G,Y}$ acts on $P^u_{G,Y} \times \mathcal{Z}_{G,Y}$ by $l\cdot(p,z) = (lpl^{-1}, lz)$. The commutator subgroup $(L_{G,Y}, L_{G,Y})$ acts trivially on $\mathcal{Z}_{G,Y}$, and if Y is toroidal then every G-orbit meets $\mathcal{Z}_{G,Y}$ in an $L_{G,Y}$-orbit.

Definition 6.2. We define $T_{G,Y}$ to be the quotient of $L^r_{G,Y}$ by the kernel of its action on $\mathcal{Z}_{G,Y}$.

We get back to our complete G-regular variety X. The torus $T_{G,X}$ is a subquotient of T, and $\mathcal{Z}_{G,X}$ is a spherical (toric) $T_{G,X}$-variety, with lattice $\Lambda_{T_{G,X}}(\mathcal{Z}_{G,X}) = \mathcal{X}(T_{G,X}) = \Lambda_G(G/H)$ and fan of convex cones equal to $\mathcal{F}_G(X)$.

Definition 6.3. For any x' in the open G-orbit of X' we denote by $\kappa_{x'}$ the restriction map

$$\kappa_{x'}\colon (\ker\psi_*)^\circ \to \mathrm{Aut}^\circ(X_{x'})$$

where $X_{x'} = \psi^{-1}(x')$.

Recall that H is chosen in such a way that BH is open in G, and $x_0 = eH \in G/H \subseteq X$. Let us consider $x'_0 = \psi(x_0)$: the fiber $X_{x'_0}$ is smooth and complete, and it is a toric variety under the action of the torus $S = (\widehat{H}/H)^\circ = H'/H$, where H' is the stabilizer of x'_0.

Moreover, S acts naturally on G/H by G-equivariant automorphisms, and since S is connected this S-action extends to X, stabilizing all colors of X and all fibers of ψ. We may fix $\mathcal{Z}_{G,X'} \subset X'$ containing x'_0, and choose $\mathcal{Z}_{G,X}$ so that

$$\mathcal{Z}_{G,X} = \psi^{-1}(\mathcal{Z}_{G,X'}) \cap X_0,$$

which implies that $\mathcal{Z}_{G,X}$ contains x_0 and is stable under the action of S.

The same action of S on $\mathcal{Z}_{G,X}$ can be realized sending S injectively into $T_{G,X}$, and then letting it act on $\mathcal{Z}_{G,X}$ via the restriction of the usual action of G on X. Indeed, if $nH \in S$

and $f \in \mathbb{C}(G/H)^{(B)}_\chi$, then $gH \mapsto f(gnH)$ also belongs to $\mathbb{C}(G/H)^{(B)}_\chi$, therefore there is a homomorphism (depending only on χ) $\iota_\chi\colon S \to \mathbb{C}^*$ such that $f(gnH) = \iota_\chi(n^{-1}H)f(gH)$ for all $g \in G$. This induces a homomorphism

$$\iota\colon S \to \mathrm{Hom}(\Lambda_G(G/H), \mathbb{C}^*) \cong T_{G,X},$$

which can be shown to be injective, with image equal to the subtorus of $T_{G,X}$ corresponding to the subspace $\mathrm{V}^\ell_G(G/H) \subseteq \mathrm{N}_G(G/H)$ (see [Br97, Proof of Theorem 4.3]). Let us check that restricting to $\iota(S)$ the usual $T_{G,X}$-action on $\mathcal{Z}_{G,X}$ yields the action described above. The intersection $\mathcal{Z}_{G,X} \cap G/H$ is dense in $\mathcal{Z}_{G,X}$, and $\mathcal{Z}_{G,X}$ is a toric $T_{G,X}$-variety with lattice equal to $\Lambda_G(G/H)$: it follows that $\iota(nH)gH = gnH$, because

$$f(\iota(nH)gH) = \chi(\iota(n^{-1}H))f(gH) = \iota_\chi(n^{-1}H)f(gH) = f(gnH)$$

for all $nH \in S$, $gH \in \mathcal{Z}_{G,X} \cap G/H$, $\chi \in \Lambda_G(G/H)$ and $f \in \mathbb{C}(G/H)^{(B)}_\chi$.

The fiber $X_{x'_0}$ is also the fiber over x'_0 of the S-equivariant map $\mathcal{Z}_{G,X} \to \mathcal{Z}_{G,X'}$, which implies that its fan of convex cones is

$$\mathcal{F}_S(X_{x'_0}) = \left\{ c \;\middle|\; c \in \mathcal{F}_G(X),\ c \subset \mathrm{V}^\ell_G(G/H) \right\}. \tag{6.1}$$

Since the S-boundary of $X_{x'_0}$ is given intersecting $X_{x'_0}$ with the elements of $(\partial_G X)^\ell$, there is an exact sequence of S-modules

$$0 \to \mathrm{Lie}\, S \to \mathrm{Lie}\,\mathrm{Aut}^\circ(X_{x'_0}) \to \bigoplus_{D \in (\partial_G X)^\ell} \frac{H^0(X_{x'_0}, \mathcal{O}_X(D \cap X_{x'_0}))}{\mathbb{C}} \to 0. \tag{6.2}$$

Lemma 6.4. *If $V \subseteq \ker d\psi_*$ is a simple G-submodule and x' is in the open B-orbit of X', then $d\kappa_{x'}(V) = d\kappa_{x'}(\mathbb{C}v)$, where $v \in V$ is a highest weight vector.*

Proof. We may assume that $x' = x'_0$ and that $v = [s] \in H^0(X, \mathcal{O}_X(D))/\mathbb{C}$ for some $D \in (\partial X)^\ell$, in view of Corollary 5.3. From the expression in local coordinates of [BB96, Remark after Proposition 2.3.2] and the proof of [BB96, Proposition 4.1.1], we see that $d\kappa_{x'_0}(v)$ is sent by the surjective map of (6.2) to $[s|_{X_{x'_0}}]$ where $s|_{X_{x'_0}}$ is a section of $\mathcal{O}_{X_{x'_0}}(D \cap X_{x'})$.

If s is a B-eigenvector then its zeros are B-stable. On the other hand, since Bx'_0 is open in X', the only zeros of s intersecting $X_{x'_0}$ are G-stable. It also follows that $(gs)|_{X_{x'_0}}$ and $s|_{X_{x'_0}}$ have the same zeros (hence are linearly dependent) for any $g \in G$ such that gx_0 doesn't lie on any color of G/H. This is true for g lying in a dense subset U of G, and since V is generated as a vector space by elements of the form $[gs]$ for $g \in U$, the lemma follows. □

Lemma 6.5. *Let $i = 1, 2$ and $0 \neq \gamma_i \in \Lambda_G(X)$ be such that $X(\gamma_i)$ exists, with $X(\gamma_i) \in (\partial X)^\ell$. Suppose that $\langle m, \gamma_1 \rangle = -\langle m, \gamma_2 \rangle$ for all $m \in \mathrm{V}^\ell_G(X)$. Then*

$$\langle \rho_{G,X}(D), \gamma_i \rangle = 0$$

for all $i = 1, 2$, for all $D \in (\partial_G X)^{n\ell}$ and for all $D \in \Delta_G(X)$.

Proof. Consider the wonderful variety $\mathbb{X}$. Both sets $\rho_{G,\mathbb{X}}(\Delta_G(\mathbb{X}))$ and $\rho_{G,\mathbb{X}}(\partial_G\mathbb{X})$ generate $\mathrm{N}_G(\mathbb{X})$ as a vector space, and the convex cone generated by $\rho_{G,\mathbb{X}}(\Delta_G(\mathbb{X}))$ contains $-\rho_{G,\mathbb{X}}(\partial_G\mathbb{X})$ (see [Br07, Lemma 2.1.2]). On the other hand, the set $\pi_*(\rho_{G,X}((\partial_G X)^{n\ell})) \subset \mathrm{N}_G(\mathbb{X})$ generates the same convex cone $\mathrm{V}_G(\mathbb{X})$ generated by $\rho_{G,\mathbb{X}}(\partial_G\mathbb{X})$, and $\pi_*(\rho_{G,X}(\Delta_G(X))) = \rho_{G,\mathbb{X}}(\Delta_G(\mathbb{X}))$. It follows that there exists a linear combination

$$v = \sum_{Y\in(\partial_G X)^{n\ell}} n_Y \rho_{G,X}(Y) + \sum_{Z\in\Delta_G(X)} n_Z \rho_{G,X}(Z) \in \mathrm{V}^{\ell}_G(X)$$

where all the coefficients n_Y and n_Z are positive. From the assumptions on the characters γ_i, all the elements $\rho_{G,X}(Y)$ and $\rho_{G,X}(Z)$ above are non-negative on both γ_1 and γ_2: we deduce that $\langle v, \gamma_i\rangle \geq 0$, which yields $\langle v, \gamma_i\rangle = 0$. The lemma follows. □

In the next sections we will investigate $\mathrm{Aut}^\circ(X, \mathcal{D})$ for any subset $\mathcal{D} \subseteq \partial_G X$, using the results above. It is harmless to assume that each $E \in \mathcal{E} = \partial_G X \setminus \mathcal{D}$ is not stable under $\mathrm{Aut}^\circ(X, \mathcal{D})$, and it is convenient to treat separately the two subsets $\mathcal{E}^{n\ell}$, $\mathcal{E}^{\ell}$ of $\mathcal{E}$.

More precisely, we first consider in §7 the special case where $\mathcal{E}^\ell = \varnothing$, i.e. $\mathcal{D} \supseteq (\partial_G X)^\ell$. We determine the group $\mathrm{Aut}^\circ(X, \mathcal{D})$: it is obtained lifting from $\mathbb{X}$ to X the action of a certain subgroup of $\mathrm{Aut}^\circ(\mathbb{X})$, it is reductive and under its action X is G-regular, with boundary $\mathcal{D}$. Finally, we compute the related fan of convex cones.

For a general $\mathcal{D}$, we apply the above results to X where G is replaced by $\widetilde{G} = \mathrm{Aut}^\circ(X, \mathcal{D}\cup\mathcal{E}^\ell)$. It turns out (see Corollary 7.19) that the elements of $\mathcal{E}^\ell$ lie on the linear part of the valuation cone both with respect to the G-action and to the $\widetilde{G}$-action.

Therefore we may finally replace G with the group $\widetilde{G}$, and develop further analysis on $\mathrm{Aut}^\circ(X, \mathcal{D})$ where now X is a complete $\widetilde{G}$-regular variety satisfying $\mathcal{D} \supseteq (\partial_{\widetilde{G}} X)^{n\ell}$. This will be done in §10, after discussing the special cases of G abelian (§8) and G semisimple (§9).

7. G-stable prime divisors not on the linear part of the valuation cone

In this section we study $\mathrm{Aut}^\circ(X, \mathcal{D})$ under the assumption that $\mathcal{D} \supseteq (\partial_G X)^\ell$. We also suppose that $\mathcal{D}$ contains all the G-stable prime divisors D that satisfy $H^0(X, \mathcal{O}_X(D)) = \mathbb{C}$, since these prime divisors do not move under the action of $\mathrm{Aut}^\circ(X)$ anyway.

Before stating the main result of this section, Theorem 7.8, we need to establish a correspondence between the divisors in $\partial_G X \setminus \mathcal{D}$ and certain boundary divisors of $\mathbb{X}$.

Recall that since $\mathbb{X}$ is wonderful the set $-\rho_{G,\mathbb{X}}(\partial_X\mathbb{X})$ is a basis of $\mathrm{N}_G(\mathbb{X})$, dual to $\Sigma_G(\mathbb{X})$.

Definition 7.1. For an element $D \in \partial_G\mathbb{X}$, we denote by σ_D the spherical root of $\mathbb{X}$ dual to $-\rho_{G,\mathbb{X}}(D)$.

Since $\Lambda_G(\mathbb{X})$ is a sublattice of $\Lambda_G(X)$, we consider σ_D also as an element of the latter.

Also recall that, thanks to [Br07, Theorem 2.2.3], if $D \in \partial_G \mathbb{X}$ satisfies $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(D)) \neq \mathbb{C}$ then $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(D))/\mathbb{C}$ is irreducible with highest weight σ_D.

Lemma 7.2. *Let $E \in \mathcal{E} = \partial_G X \setminus \mathcal{D}$. Then:*

(1) *the image $\pi(E)$ is an element of $\partial_G \mathbb{X}$, with $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(\pi(E))) \neq \mathbb{C}$, and E is the only element of $\partial_G X$ whose image is $\pi(E)$;*

(2) *we have*

$$\pi_*(\rho_{G,X}(E)) = \rho_{G,\mathbb{X}}(\pi(E)),$$

and

$$\forall c \in \mathcal{F}_G(X) \setminus \{c_{X,E}\}, c \text{ 1-dimensional}: \quad c \subset \sigma_{\pi(E)}^\perp; \tag{7.1}$$

(3) *the G-modules $H^0(X, \mathcal{O}_X(E))$ and $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(\pi(E)))$ are isomorphic.*

Proof. Let $\gamma \neq 0$ be such that $H^0(X, \mathcal{O}_X(E))^{(B)}_\gamma \neq \varnothing$. Since $E \in (\partial_G X)^{n\ell}$, the character γ is non-negative on $\rho_{G,X}((\partial_G X)^\ell)$, which generates the whole $\mathrm{V}_G^\ell(X)$ as a convex cone, because X is complete. It follows that $\gamma \in \mathrm{V}_G^\ell(X)$, which implies that some positive integral multiple of γ, say $n\gamma$, lies in $\Lambda_G(\mathbb{X})$. Let us also assume that it is indecomposable in $\Lambda_G(\mathbb{X})$, i.e. that n is minimal satisfying $n > 0$ and $n\gamma \in \Lambda_G(X)$.

Consider $\pi(E)$: if it is not a G-stable prime divisor of $\mathbb{X}$, then $\pi_*(\rho_{G,X}(E))$ is in $\mathrm{V}_G(\mathbb{X})$ but doesn't lie on any 1-dimensional face of $\mathrm{V}_G(\mathbb{X})$. On the other hand, each element of $\partial_G \mathbb{X}$ is the image $\pi(D)$ of some G-stable prime divisor D of X, with $\pi_*(\rho_{G,X}(D))$ equal to a positive rational multiple of $\rho_{G,\mathbb{X}}(\pi(D))$. This implies that $n\gamma \in \Lambda_G(\mathbb{X})$ is non-negative on $\rho_{G,\mathbb{X}}(\partial_G \mathbb{X})$ and negative on $\pi_*(\rho_{G,X}(E))$, which is absurd because $\rho_{G,\mathbb{X}}(\partial_G \mathbb{X})$ generates $\mathrm{V}_G(\mathbb{X})$ as a convex cone.

We conclude that $\pi(E) \in \partial_G \mathbb{X}$, and that E is the unique element of $\partial_G X$ whose image is $\pi(E)$, because $n\gamma$ is non-negative on $\rho_{G,X}(E')$ for any $E' \in \partial_G X$ different from E. Let $0 > -m = \langle \rho_{G,\mathbb{X}}(\pi(E)), n\gamma \rangle$. Then $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(m\pi(E))) \neq \mathbb{C}$.

From [Br07, Theorem 2.2.3] it follows that $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(\pi(E))) \neq \mathbb{C}$, that $H^0(\mathbb{X}, \mathcal{O}_\mathbb{X}(\pi(E)))/\mathbb{C}$ is irreducible with highest weight $\sigma_{\pi(E)}$, and that any $\chi \in \Lambda_G(\mathbb{X})$ satisfying

$$\langle \rho_{G,\mathbb{X}}(D), \chi \rangle \geq 0 \quad \forall D \in (\partial_G \mathbb{X} \setminus \{\pi(E)\} \cup \Delta_G(\mathbb{X}), \qquad \langle \rho_{G,\mathbb{X}}(\pi(E)), \chi \rangle < 0 \tag{7.2}$$

is a positive multiple of $\sigma_{\pi(E)}$. We have then shown (1). It also follows that $n\gamma$ is a positive multiple of $\sigma_{\pi(E)}$, whence γ is non-positive on $\mathrm{V}_G(X)$ and so it is zero on $\rho_{G,X}(D)$ for all $D \in \partial_G X$ different from E. This shows (7.1).

Now recall that $n\gamma$ is indecomposable in $\Lambda_G(\mathbb{X})$. Since it is a positive multiple of $\sigma_{\pi(E)}$, it is equal to $\sigma_{\pi(E)}$. On the other hand $\Sigma_G(X) = \Sigma_G(\mathbb{X})$ and $\sigma_{\pi(E)}$ is also indecomposable in $\Lambda_G(X)$. Therefore $n = 1$, and we have

$$\langle \rho_{G,X}(E), \gamma \rangle = \langle \pi_*(\rho_{G,X}(E)), \gamma \rangle = -1 = \langle \rho_{G,\mathbb{X}}(\pi(E)), \sigma_{\pi(E)} \rangle,$$

whence $\pi_*(\rho_{G,X}(E)) = \rho_{G,\mathbb{X}}(\pi(E))$. The proof of part (2) is complete.

Since γ is the highest weight of an arbitrary non-trivial G-submodule of $H^0(X, \mathcal{O}_X(E))$, and the latter is multiplicity-free since X is spherical, the proof of (3) is also complete. $\square$

Definition 7.3. We denote by

$$\Lambda_G(X, \mathcal{E}) \subseteq \Lambda_G(X)$$

the sublattice generated by the elements $\sigma_{\pi(E)}$ for all $E \in \mathcal{E}$.

Corollary 7.4.

$$\Lambda_G(X) = \rho_{G,X}(\mathcal{E})^\perp \oplus \Lambda_G(X, \mathcal{E}).$$

Proof. From Lemma 7.2 we deduce that for all $E \in \mathcal{E}$ the element $\rho_{G,X}(E)$ is -1 on the spherical root $\sigma_{\pi(E)}$ of X, and zero on all other spherical roots of X. The corollary follows. $\square$

Remark 7.5. In the proof of Lemma 7.2 we used the crucial fact that X and $\mathbb{X}(G/\widehat{H})$ have the same spherical roots. The decomposition of $\Lambda(G/H)$ into the above direct sum would indeed be false in general, if we had used $\mathbb{X}(G/\overline{H})$ instead of $\mathbb{X}(G/\widehat{H})$.

Definition 7.6. Define

$$\mathbb{E} = \{\pi(E) \mid E \in \mathcal{E}\},$$

and

$$\mathbb{D} = \partial\mathbb{X} \setminus \mathbb{E}.$$

Definition 7.7. Let $A' = A'(X, \mathcal{D})$ be the universal cover of $\mathrm{Aut}^\circ(\mathbb{X}, \mathbb{D})$, and $A = A(X, \mathcal{D}) = A' \times C$. We denote by

$$\vartheta' \colon G' \to A'$$

the lift of $\theta_{G',\mathbb{X}} \colon G' \to \mathrm{Aut}^\circ(\mathbb{X}, \mathbb{D})$ to A', and we set

$$\vartheta = \vartheta' \times \mathrm{id}_C \colon G \to A.$$

We also choose a Borel subgroup B_A of A such that $B_A \supseteq \vartheta(B)$.

Now we are ready to state the main result of this section.

Theorem 7.8. *The action of $A(X, \mathcal{D})$ lifts from $\mathbb{X}$ to X, and the image of $A(X, \mathcal{D})$ inside $\mathrm{Aut}^\circ(X)$ is equal to $\mathrm{Aut}^\circ(X, \mathcal{D})$. As an $A = A(X, \mathcal{D})$-variety, X is G-regular with boundary $\mathcal{D}$. The vector space $\mathrm{N}_A(X)$ is naturally identified with $\Lambda_G(X, \mathcal{E})^\perp \subseteq \mathrm{N}_G(X)$. The fan $\mathcal{F}_A(X)$ of X as an A-variety is given by intersecting all cones of $\mathcal{F}_G(X)$ with $\Lambda_G(X, \mathcal{E})^\perp$.*

The proof occupies the rest of the section: the theorem follows from Lemma 7.13, Theorem 7.18, and Corollary 7.20.

Example 7.9. It is necessary to define A' as the universal cover of $\mathrm{Aut}^\circ(\mathbb{X},\mathbb{D})$. Consider for example $G=\mathrm{SL}(n{+}1)$ (with $n\geq 1$) acting linearly and diagonally on $\mathbb{P}^{n+1}\times(\mathbb{P}^n)^*$, where on the first factor it acts only on the first $n{+}1$ homogeneous coordinates. Then $X=\mathrm{Bl}_p(\mathbb{P}^{n+1})\times(\mathbb{P}^n)^*$, with $p=[0,\dots,0,1]$, is a G-regular variety with three G-stable prime divisors, of which only one lies in $(\partial_G X)^{n\ell}$. We have $\mathbb{X}=\mathbb{P}^n\times(\mathbb{P}^n)^*$, and if $\mathcal{D}=(\partial X)^\ell$ then $\mathbb{D}=\varnothing$. The action of $\mathrm{Aut}^\circ(\mathbb{X},\mathbb{D})=\mathrm{Aut}^\circ(\mathbb{X})=\mathrm{PGL}(n+1)\times\mathrm{PGL}(n+1)$ doesn't lift to X, whereas the action of its universal cover does.

In view of proving Theorem 7.8, we start finding a candidate for a generic stabilizer of the A-action on X. Let $\widehat{H}_A\subseteq A$ be[2] the stabilizer of the point $e\widehat{H}\in G/\widehat{H}\subseteq\mathbb{X}$. The colors of $\mathbb{X}$ as a G-variety and as an A-variety coincide, thanks to [Br07, Theorem 2.4.2], and we have $\vartheta(\widehat{H})=\widehat{H}_A\cap\vartheta(G)$.

We also notice that thanks to our general assumptions any G-linearization of an invertible sheaf $\mathbb{X}$ can be uniquely extended to an A-linearization, inducing an identification of the two groups $\mathrm{Pic}^G(\mathbb{X})$ and $\mathrm{Pic}^A(\mathbb{X})$.

Lemma 7.10. (1) *The pull-back of characters of B_A along $\vartheta|_B$ induces an injective map $r\colon \Lambda_A(\mathbb{X})\to\Lambda_G(\mathbb{X})$. It maps $\Sigma_A(\mathbb{X})$ to the set of spherical roots $\{\sigma_D\mid D\in\mathbb{D}\}$.*

(2) *The dual map $r^*\colon \mathrm{N}_G(\mathbb{X})\to\mathrm{N}_A(\mathbb{X})$ satisfies*

$$r^*(V_G(\mathbb{X}))=V_A(\mathbb{X}).$$

(3) *We have that $\partial_A\mathbb{X}=\mathcal{D}$, and*

$$A/\widehat{H}_A=\mathbb{X}\setminus\bigcup_{D\in\mathbb{D}}D.$$

(4) *The pull-back of characters of $\widehat{H}_A$ along $\vartheta|_{\widehat{H}}$ is a surjective homomorphism $r'\colon\mathcal{X}(\widehat{H}_A)\to\mathcal{X}(\widehat{H})$ with free kernel of rank $|\mathbb{E}|$.*

Proof. The injectivity of r is obvious, since it corresponds to taking a B_A-semiinvariant $f\in\mathbb{C}(\mathbb{X})$ and considering it as a B-semiinvariant. The rest of part (1) follows from the results of [Pe09], and it can also be shown directly using the following fact: the spherical roots of $\mathbb{X}$ are the T-weights appearing in the quotient of tangent spaces

$$\frac{\mathrm{T}_z\mathbb{X}}{\mathrm{T}_z(G\,z)},$$

where $z\in\mathbb{X}$ is the unique fixed point of B^-. Let us choose a maximal torus T_A of A containing $\vartheta(T)$: if $B_A^-\subseteq A$ is the Borel subgroup satisfying $B_A\cap B_A^-=T_A$ then B_A^- contains $\vartheta(B^-)$.

[2] Our notation is consistent thanks to Corollary 7.14.

Hence z is also the unique B_A^--fixed point, therefore the spherical roots of $\mathbb{X}$ as an A-variety are the T_A-weights appearing in the quotient of tangent spaces

$$\frac{\mathrm{T}_z\mathbb{X}}{\mathrm{T}_z(A\,z)},$$

form the set $\Sigma_A(\mathbb{X})$. This implies part (1), and part (2) is an immediate consequence.

The first statement of part (3) stems from the fact that each $E \in \mathbb{E}$ is not stable under the action of A, and the second follows from the first because $\mathbb{X}$ is wonderful under the action of A.

For part (4), we notice that r' can be identified with the natural map

$$\frac{\mathcal{X}(C) \times \mathbb{Z}^\Delta}{\overline{\rho}_{A,\mathbb{X}}(\Lambda_A(\mathbb{X}))} \to \frac{\mathcal{X}(C) \times \mathbb{Z}^\Delta}{\overline{\rho}_{G,\mathbb{X}}(\Lambda_G(\mathbb{X}))}$$

(see diagram (3.1)). The kernel of r' is then $\Lambda_G(\mathbb{X})/r(\Lambda_A(\mathbb{X}))$ which is free, generated by the spherical roots σ_E for all $E \in \mathbb{E}$ by part (1). $\square$

Let us put together two copies of the diagram (3.1), one for the G- and one for the A-action, also adding the extensions of $\overline{\rho}_G$ and $\overline{\rho}_A$ resp. to $\Lambda_G(G/H)$ and $\Lambda_A(A/\widehat{H}_A)$, as in §3. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
\Lambda_A(A/K_A) & \hookrightarrow & \Lambda_A(A/\widehat{H}_A) & \xhookrightarrow{\quad\overline{\rho}_A\quad} & & \mathcal{X}(C)\times\mathbb{Z}^\Delta & \xrightarrow{\tau_A} & \mathcal{X}(K_A) & \twoheadrightarrow \mathcal{X}(\widehat{H}_A) \\
 & & \big\downarrow r & & & \big\Vert & \searrow^{\sigma_A} \; \mathcal{X}(B_A) & & \big\downarrow r' \\
 & & & & & & \quad\downarrow (\vartheta|_B)^* & & \\
 & & & & & & \nearrow^{\sigma_G} \; \mathcal{X}(B) & & \\
\Lambda_G(G/K) & \hookrightarrow & \Lambda_G(G/\widehat{H}) & \hookrightarrow & \Lambda_G(G/H) & \xhookrightarrow{\overline{\rho}_G} \mathcal{X}(C)\times\mathbb{Z}^\Delta & \xrightarrow{\tau_G} & \mathcal{X}(K) & \twoheadrightarrow \mathcal{X}(\widehat{H})
\end{array}$$

where K_A is the spherical closure of $\widehat{H}_A$, and K is the spherical closure of $\widehat{H}$ (and of H). The last arrow of the first row is the restriction map, which can be seen as the quotient

$$\mathcal{X}(K_A) \to \mathcal{X}(K_A)/\mathcal{X}(K_A)^{\widehat{H}_A} \cong \mathcal{X}(\widehat{H}_A).$$

The same remark holds for the last map of the second row and the groups $K, \widehat{H}$.

In order to determine a generic stabilizer in A for X, we start defining a lattice $\Lambda \subseteq \Lambda_G(G/H)$. A posteriori, it will be the lattice of B-eigenvalues χ_f of B_A-eigenvectors $f \in \mathbb{C}(X)^{(B_A)}$. Such a function f cannot have zeros nor poles on the divisors in $\mathcal{E}$, since these are not A-stable, nor are A-colors of X. This suggests the definition of Λ given in the following.

Definition 7.11. Let Λ be the lattice

$$\Lambda = \rho_{G,X}(\mathcal{E})^\perp \subseteq \Lambda_G(G/H).$$

Proposition 7.12. *The following inclusion holds:*

$$\overline{\rho}_G(\Lambda) \supseteq \overline{\rho}_A(\Lambda_A(A/K_A)).$$

The subgroup H_A of K_A corresponding to the lattice $\overline{\rho}_G(\Lambda)$ is a spherical subgroup of A, and we have $\vartheta(H) = H_A \cap \vartheta(G)$. This induces a G-equivariant identification of G/H with an open subset of A/H_A.

Proof. Let $\chi \in \Lambda_A(A/K_A) \subseteq \Lambda_A(A/\widehat{H}_A)$. If $f \in \mathbb{C}(A/\widehat{H}_A)^{(B_A)}_\chi$, then consider its pull-back on X, denoted by $\widetilde{f}$. It is also a B-eigenvector with B-eigenvalue $\widetilde{\chi} = r(\chi)$.

We know that the divisor $\operatorname{div}(\widetilde{f})$ on X is B_A-stable, so in general it is a linear combination of colors and A-stable prime divisors. In any case, its components do not belong to $\mathcal{E}$, because the latter consists of prime divisors moved by A. It follows that all discrete valuations in $\mathrm{V}_G(G/H)$ coming from these elements of $\mathcal{E}$ must take the value 0 on $\widetilde{\chi}$.

Therefore $\widetilde{\chi} \in \Lambda$, and the first assertion is proved. In order to verify that H_A is spherical we have to show that σ_A restricted to $\overline{\rho}_G(\Lambda) = \tau_A^{-1}\left(\mathcal{X}(K_A)^{H_A}\right)$ is injective. But we already know that the restriction of σ_G on $\overline{\rho}_G(\Lambda_G(G/H))$ is injective, and that $\Lambda \subseteq \Lambda_G(G/H)$: this proves the second assertion.

Next, we claim that r' induces an isomorphism between $\mathcal{X}(\widehat{H}_A)^{H_A}$ and $\mathcal{X}(\widehat{H})^H$. This shows that $\widehat{H}_A/H_A \cong \widehat{H}/H$, and the rest of the lemma follows. To prove the claim, it is enough to notice that

$$\begin{aligned}
\mathcal{X}(\widehat{H}_A)^{H_A} &\cong \frac{\overline{\rho}_G(\Lambda)}{\ker \tau_A} \\
&= \frac{\overline{\rho}_G(\Lambda)}{\overline{\rho}_G(r(\Lambda_A(A/\widehat{H}_A)))} \\
&\cong \frac{\overline{\rho}_G(\Lambda) \oplus \overline{\rho}_G(\Lambda_G(X,\mathcal{E}))}{\overline{\rho}_G(\Lambda_G(G/\widehat{H}))} \\
&= \frac{\overline{\rho}_G(\Lambda \oplus \Lambda_G(X,\mathcal{E}))}{\overline{\rho}_G(\Lambda_G(G/\widehat{H}))} \\
&= \frac{\overline{\rho}_G(\Lambda_G(G/H))}{\overline{\rho}_G(\Lambda_G(G/\widehat{H}))} \\
&\cong \mathcal{X}(\widehat{H})^H,
\end{aligned}$$

and that the resulting isomorphism $\mathcal{X}(\widehat{H}_A)^{H_A} \cong \mathcal{X}(\widehat{H})^H$ is indeed induced by r'. □

We build *ex novo* an embedding X_A of A/H_A, and then prove that we actually obtain X.

Lemma 7.13. *The pull-back of characters of B_A to B along $\vartheta|_B$ induces an injective map $s\colon \Lambda_A(A/H_A) \to \Lambda_G(G/H)$ whose image is Λ. The dual map $s^*\colon \mathrm{N}_G(G/H) \to \mathrm{N}_A(A/H_A)$ satisfies*

$$s^*(\mathrm{V}_G(G/H)) = \mathrm{V}_A(A/H_A),$$

and induces an isomorphism

$$s^*|_{\mathrm{V}^\ell_G(G/H)}\colon \mathrm{V}^\ell_G(G/H) \to \mathrm{V}^\ell_A(A/H_A).$$

Proof. Let $\gamma \in \Lambda_A(A/H_A)$: it is the B_A-eigenvalue of a B_A-eigenvector $f \in \mathbb{C}(A/H_A)^{(B_A)}$. But f is a B-eigenvector too and the character $\chi = s(\gamma)$ is its B-eigenvalue. Both the B- and the B_A-eigenvalue determine f up to a multiplicative constant, hence s is injective.

Consider the commutative diagram

$$\begin{array}{ccc}
\Lambda_A(A/H_A) & \xhookrightarrow{\overline{\rho}_A} & \mathrm{Pic}^A(\mathbb{X}) \\
\Big\downarrow{\scriptstyle s} & & \Big\| \\
\Lambda_G(G/H) & \xhookrightarrow{\overline{\rho}_G} & \mathrm{Pic}^G(\mathbb{X})
\end{array}$$

From the definition of H_A we have $\overline{\rho}_A(\Lambda_A(A/H_A)) = \overline{\rho}_G(\Lambda)$, therefore we obtain $s(\Lambda_A(A/H_A)) = \Lambda$.

Let $v \in \mathrm{V}_A(A/H_A)$. It corresponds to an A-invariant valuation, which is *a fortiori* G-invariant too: in other words we can compute v also on $\Lambda_G(G/H)$ obtaining an element of $\mathrm{V}_G(G/H)$. This shows that $s^*(\mathrm{V}_G(G/H)) \supseteq \mathrm{V}_A(A/H_A)$.

Then we notice that s extends the map r of Lemma 7.10. This gives the commutative diagram

$$\begin{array}{ccc}
\mathrm{N}_G(G/H) & \xrightarrow{\pi_*^{H,\widehat{H}}} & \mathrm{N}_G(G/\widehat{H}) \\
\Big\downarrow{\scriptstyle s^*} & & \Big\downarrow{\scriptstyle r^*} \\
\mathrm{N}_A(A/H_A) & \xrightarrow{\pi_*^{H_A,\widehat{H}_A}} & \mathrm{N}_A(A/\widehat{H}_A)
\end{array}$$

where $V_A(A/H_A)$ (resp. $V_G(G/H)$) is the inverse image of $V_A(A/\widehat{H}_A)$ (resp. $V_G(G/\widehat{H})$) thanks to Lemma 3.2.

This, together with Lemma 7.10, part (2), proves $s^*(\mathrm{V}_G(G/H)) = \mathrm{V}_A(A/H_A)$. The image of $\mathrm{V}^\ell_G(G/H)$ is contained in $\mathrm{V}^\ell_A(A/H_A)$, and we conclude the proof observing that the dimensions of $\mathrm{V}^\ell_G(G/H)$ and $\mathrm{V}^\ell_G(A/H_A)$ are both equal to the dimension of $\widehat{H}/H \cong \widehat{H}_A/H_A$. □

Corollary 7.14. *The wonderful closure of H_A is $\widehat{H}_A$.*

Proof. By construction $H_A \subseteq \widehat{H}_A \subseteq K_A = \overline{H_A}$. From Lemma 7.13 we deduce that A/H_A and $A/\widehat{H}_A$ have the same spherical roots: the corollary follows then from Proposition 3.10. □

We shall now define the fan of convex cones of X_A, using that of X. First, we collect some consequences on $\mathcal{F}(X)$ of the analysis we have developed so far.

Definition 7.15. Let $\mathcal{F}$ be a fan of convex cones, consider a subset $\mathcal{F}' \subset \mathcal{F}$ and let $c \in \mathcal{F} \setminus \mathcal{F}'$ be 1-dimensional. Then $\mathcal{F}$ is the *join* of $\mathcal{F}'$ and c if each element of $\mathcal{F} \setminus \mathcal{F}'$ is the convex cone generated by c and an element of $\mathcal{F}'$.

Corollary 7.16. (1) *Let $E \in \mathcal{E}$, and let $\mathcal{F}_G^{\sigma_{\pi(E)}}(X)$ be the fan of convex cones obtained intersecting each element of $\mathcal{F}_G(X)$ with $\sigma_{\pi(E)}^{\perp}$. Then $\mathcal{F}_G(X)$ is the join of $\mathcal{F}_G^{\sigma_{\pi(E)}}(X)$ and $c_{X,E}$.*

(2) *Let $\mathcal{F}_G^{\Lambda}(X)$ be the fan of convex cones obtained intersecting each element of $\mathcal{F}_G(X)$ with $\Lambda_G(X,\mathcal{E})^{\perp}$. Then the restriction of s^* to $\operatorname{supp}\mathcal{F}_G^{\Lambda}(X)$ is injective, and $s^*(\operatorname{supp}\mathcal{F}_G^{\Lambda}(X)) = \mathrm{V}_A(A/H_A)$.*

(3) *The set*

$$\left\{ s^*(c) \,\middle|\, c \in \mathcal{F}_G^{\Lambda}(X) \right\}$$

is a fan of polyhedral convex cones in $\mathrm{N}_A(A/H_A)$. The associated embedding of A/H_A is smooth and complete.

Proof. Part (1) follows from Lemma 7.2, part (2). Part (2) follows from part (1) applied to all $E \in \mathcal{E}$, together with Corollary 7.4 and Lemma 7.13. We turn to part (3). Completeness of this embedding is an immediate consequence of part (2). For smoothness, we observe that a maximal cone c of $\mathcal{F}_G(X)$ can be written as

$$c = (\{-\sigma_E \,|\, E \in \mathcal{E}\} \cup \Psi)^{\vee}$$

where Ψ is a basis of $\Lambda = \rho_{G,X}(\mathcal{E})^{\perp}$, thanks to the smoothness of X together with part (1) applied to all $E \in \mathcal{E}$ and Corollary 7.4. Therefore

$$s^*\left(c \cap \left(\Lambda_G(X,\mathcal{E})^{\perp}\right)\right) = \left(s^{-1}\left(\Psi\right)\right)^{\vee}.$$

The smoothness characterization recalled in §2 is verified, since $s^{-1}(\Psi)$ is a basis of $\Lambda_A(A/H_A)$, and the proof is complete. □

Definition 7.17. We define

$$\mathcal{F}_A = \left\{ s^*(c) \,\middle|\, c \in \mathcal{F}_G^{\Lambda}(X) \right\},$$

and we denote by X_A the corresponding embedding of A/H_A.

Theorem 7.18. *The inclusion $G/H \subseteq A/H_A$ extends to an A-equivariant isomorphism between X and X_A.*

Proof. The group G acts on X_A via the map θ, and it is enough to show X_A is a toroidal embedding of G/H with fan $\mathcal{F}_G(X)$. Let us first prove the theorem with the assumption that $|\mathcal{E}| = 1$, say $\mathcal{E} = \{E\}$.

In addition to the G-equivariant map $\pi\colon X \to \mathbb{X}$ we also have by construction an A-equivariant map $\pi_A\colon X_A \to \mathbb{X}$ extending the projection $\pi^{H_A,\widehat{H}_A}\colon A/H_A \to A/\widehat{H}_A$. The A-colors and the G-colors of $\mathbb{X}$ coincide, and this implies the same for X_A: indeed any A-color (resp. G-color) of X_A is of the form $\pi_A^{-1}(D)$ for an A-color (resp. G-color) D of $\mathbb{X}$.

If $D \subset \mathbb{X}$ is a color such that $\pi_A^{-1}(D)$ contains a G-orbit $Y \subset X_A$, then D contains the G-orbit $\pi_A(Y)$: this is absurd because $\mathbb{X}$ is a toroidal G-variety. In other words X_A is a toroidal G-variety.

Next, we claim that A/H_A is a G-embedding of G/H whose fan contains $c_{X,E}$ as its unique non-trivial cone. Part (3) of Lemma 7.10 implies that $A/\widehat{H}_A$ is an elementary embedding of $G/\widehat{H}$, with orbits $G/\widehat{H}$, $\pi(E) \cap A/\widehat{H}_A$, and fan containing $c_{\mathbb{X},\pi(E)}$ as its unique non-trivial cone. The open subset $G/H \subset A/H_A (\subseteq X_A)$ is equal to $\pi_A^{-1}(G/\widehat{H})$, and the G-stable closed subset $E' = (A/H_A) \setminus (G/H)$ is equal to $\pi_A^{-1}(\pi(E)) \cap A/H_A$.

Consider the G-invariant prime divisors contained in E': they are neither colors nor A-stable prime divisors. We claim that there is only one of them, with associated convex cone $c_{X,E}$. Then E' itself is a G-stable prime divisor, because we already proved that A/H_A is a toroidal embedding of G/H.

For this, consider $f \in \mathbb{C}(G/H)^{(B)}_\lambda$ with $\lambda \in \Lambda$. By Lemma 7.13 we have that f is also a B_A-eigenvector, therefore its divisor $\operatorname{div}(f)$ on A/H_A has components which are either colors or A-stable prime divisors. It follows that $\rho_{G,A/H_A}(F) \in \lambda^\perp$ for all $\lambda \in \Lambda$ and all G-stable prime divisor $F \subseteq E'$. Since $c_{X,E} = \Lambda^\perp \cap \mathrm{V}_G(G/H)$, we deduce that there is only one such F and it satisfies $\rho_{G,A/H_A}(F) \in c_{X,E}$: the claim above follows.

Now Lemma 7.2, Lemma 7.13 and Corollary 7.16 part (1) hold also if we replace X with X_A and $\mathcal{D}$ with the set $(\partial_G X_A) \setminus \{E'\}$. From Corollary 7.16 part (1) we deduce that $\mathcal{F}_G(X_A)$ is the join of $\mathcal{F}_G^{\sigma_{\pi(E)}}(X_A)$ and $c_{X,E}$. From Lemma 7.13 we deduce that every G-stable prime divisor D of X_A such that $\rho_{G,X_A}(D) \in \sigma_{\pi(E)}^\perp$ is also A-stable, hence each G-orbit $Y \subseteq X_A$ such that $c_{X_A,Y} \subset \sigma_{\pi(E)}^\perp$ is also an A-orbit.

In other words $\mathcal{F}_G^{\sigma_{\pi(E)}}(X_A)$ and $\mathcal{F}_G^{\sigma_{\pi(E)}}(X)$ have the same image under s_*, which implies that they are equal. The theorem in the case $|E| = 1$ follows.

If $|\mathcal{E}| > 1$, we consider the chain of groups

$$\theta_{G,X}(G) \subseteq \mathrm{Aut}^\circ(X, \partial_G X \setminus \{E_1\}) \subseteq \mathrm{Aut}^\circ(X, \partial_G X \setminus \{E_1, E_2\}) \subseteq \ldots \subseteq \mathrm{Aut}^\circ(X, \mathcal{D}),$$

where $\mathcal{E} = \{E_1, E_2, \ldots\}$, and proceed by induction on $|\mathcal{E}|$. Let $A_i \subseteq A_{i+1}$ be two consecutive groups of this chain: we may apply the first part of the proof, together with Corollary 7.19 below (whose proof in the case $|\mathcal{E}| = 1$ only depends on the case $|\mathcal{E}| = 1$ of this theorem) to the A_i-variety X. We obtain the construction of an A_{i+1}-variety $X_{A_{i+1}}$, which is A_i-equivariantly isomorphic to X. □

Corollary 7.19. *We have $\partial_A X = \mathcal{D}$ and $(\partial_A X)^\ell = (\partial_G X)^\ell$.*

Proof. This is obvious from the definition of $\mathcal{F}_A$. □

Corollary 7.20. *The image of A in $\mathrm{Aut}^\circ(X)$ is equal to $\mathrm{Aut}^\circ(X, \mathcal{D})$.*

Proof. By construction A moves each element of $\mathbb{E}$ on $\mathbb{X}$ and stabilizes all elements of $\mathbb{D}$, hence $\mathbb{D} = \partial_A \mathbb{X}$.

Moreover $\widehat{H}_A$ is the wonderful closure of H_A, hence we can apply the exact sequence (4.2) to X as an A-variety, mapping onto the wonderful A-variety $\mathbb{X}$. Since the image of A contains by construction both the universal cover of $\mathrm{Aut}^\circ(\mathbb{X}, \mathbb{D}) = \mathrm{Aut}^\circ(\mathbb{X}, \partial_A \mathbb{X})$ and $(\widehat{H}_A/H_A)^\circ \cong (\widehat{H}/H)^\circ \subseteq C$, it follows that the image of A contains $\mathrm{Aut}^\circ(X, \partial_A X) = \mathrm{Aut}^\circ(X, \mathcal{D})$. □

8. Abelian case

In this section we will assume that $G = C$ is an algebraic torus, X as usual a complete G-regular variety, and $\mathcal{D} \subseteq \partial_G X$ any subset. Hence X is a toric variety under the acton of a quotient of G. Since G is equal to its own Borel subgroups, X has no G-color.

We recall now the desctiption of $\mathrm{Aut}^\circ(X)$ given in [Oda88]. In this setting the study of $\mathrm{Aut}^\circ(X)$ is simplified by the fact that, for all $D \in \partial_G X$, the G-module $H^0(X, \mathcal{O}_X(D))$ splits into the sum of 1-dimensional G-submodules.

Definition 8.1. Suppose that for some non-zero $\alpha \in \Lambda_G(X)$ the divisor $X(\alpha)$ exists, i.e. that there exist $X(\alpha) \in \partial_G X$ and an element $f_\alpha \in H^0(X, \mathcal{O}_X(X(\alpha)))^{(B)}_\alpha$. Then we denote by $u_\alpha \colon \mathbb{C} \to \mathrm{Aut}^\circ(X)$ the unipotent 1-PSG corresponding to α defined in [Oda88, Proposition 3.14], and such that $X(\alpha)$ is the unique G-stable prime divisor not stable under $U_\alpha = u_\alpha(\mathbb{C})$.

We recall that [Oda88, Proposition 3.14] gives explicit formulae for u_α, and that this 1-PSG can also be defined in the following way. The element $\alpha \in \Lambda_G(X)$ naturally corresponds to a semisimple 1-PSG of $\mathrm{Aut}^\circ(X)$ through the action of G on X. Denote by δ_α its derivative, which is a tangent vector field on X. Then the tangent vector field du_α is equal to $f_\alpha \delta_\alpha$.

Remark 8.2. If $X(\alpha)$ exists for some α, then $\langle \rho_{G,X}(X(\alpha)), \alpha \rangle = -1$ and $\langle \rho_{G,X}(D), \alpha \rangle \geq 0$ for all $D \in \partial_G X$ different from $X(\alpha)$. However, the difference in signs from our discussion and [Oda88, §3.4] is only apparent: a character $\lambda \in \mathcal{X}(\theta_{G,X}(G))$ is indeed a rational function on X and a G-eigenvector, but of G-eigenvalue $-\lambda$.

Notice that the assignment $\alpha \mapsto X(\alpha)$ might be not injective. Also, if both $X(\alpha)$ and $X(-\alpha)$ exist, then $\rho_{G,X}(X(\alpha))$ is not necessarily $-\rho_{G,X}(X(-\alpha))$. However, $X(\alpha)$ and $X(-\alpha)$ are the only G-stable prime divisors whose images through $\rho_{G,X}$ are non-zero on α.

Definition 8.3. Let $\mathcal{D} \subseteq \partial_G X$ any subset, and define $\Phi = \Phi(X, \mathcal{D})$ to be the maximal set of roots of X such that:

(1) if $\alpha \in \Phi(X, \mathcal{D})$ then also $-\alpha \in \Phi(X, \mathcal{D})$;
(2) if $\alpha \in \Phi(X, \mathcal{D})$ then $X(\alpha) \in \mathcal{E} = \partial X \setminus \mathcal{D}$.

The following result is an immediate consequence of [Oda88, Demazure's Structure Theorem, §3.4].

Theorem 8.4. *The subgroup of* $\mathrm{Aut}^\circ(X)$ *generated by* $\theta_{G,X}(G)$ *and* U_α *for all* $\alpha \in \Phi(X,\mathcal{D})$ *has* $\Phi(X,\mathcal{D})$ *as root system with respect to its maximal torus* $\theta_{G,X}(G)$*, and is a Levi subgroup of* $\mathrm{Aut}^\circ(X,\mathcal{D})$*.*

Definition 8.5. Define $A = A(X,\mathcal{D})$ the subgroup of $\mathrm{Aut}^\circ(X)$ generated by $\theta_{G,X}(G)$ and U_α for all $\alpha \in \Phi(X,\mathcal{D})$. Let us also choose a Borel subgroup $B_A \subseteq A$ containing G and, consequently, a subdivision of Φ into positive and negative roots, resp. denoted by $\Phi_+ = \Phi_+(X,\mathcal{D})$ and $\Phi_- = \Phi_-(X,\mathcal{D})$, and denote by $\Psi = \Psi(X,\mathcal{D})$ the basis of positive roots.

Since B_A is generated by $\theta_{G,X}(G)$ together with the subgroups U_α for all $\alpha \in \Psi$, we have that any G-stable prime divisor which doesn't appear as $X(\alpha)$ for some $\alpha \in \Psi$ is B_A-stable. In other words

$$\{X(\alpha) \mid \alpha \in \Phi_+\} = \{X(\alpha) \mid \alpha \in \Psi\}\,, \tag{8.1}$$

and for the same reason (replacing Ψ with $-\Psi$)

$$\{X(\alpha) \mid \alpha \in \Phi_-\} = \{X(\alpha) \mid \alpha \in (-\Psi)\}\,. \tag{8.2}$$

Lemma 8.6. *Let* $\alpha, \beta \in \Phi$*, and suppose that* $X(\alpha) = X(\beta)$*. Then* $\gamma = \alpha - \beta$ *and* $-\gamma$ *are also in* Φ*, with* $X(\gamma) = X(-\beta)$ *and* $X(-\gamma) = X(-\alpha)$*.*

Proof. Suppose that $X(-\alpha) = X(-\beta)$. Then $\alpha-\beta$ is zero on $\rho_{G,X}(X(\pm\alpha))$ and on $\rho_{G,X}(X(\pm\beta))$. On the other hand, if a G-stable prime divisor $D \subset X$ is not of the form $X(\pm\alpha)$ nor $X(\pm\beta)$, then both α and β are zero on $\rho_{G,X}(D)$. It follows that $\operatorname{supp}\mathcal{F}_G(X)$ is contained in the hyperplane $(\alpha-\beta)^\perp$ of $\mathrm{N}_G(X)$, which contradicts the completeness of X. Therefore $X(-\alpha) \neq X(-\beta)$, i.e. $X(\alpha)$, $X(-\alpha)$ and $X(-\beta)$ are three different prime divisors. The statement of the lemma is now obvious. □

Lemma 8.7. *The matrix*

$$(\langle \rho_{G,X}(X(\alpha)), \alpha\rangle)_{\alpha\in\Psi} \tag{8.3}$$

is non-degenerate. In particular, the elements $\rho_{G,X}(X(\alpha))$*, for* α *varying in* Ψ*, are linearly independent.*

Proof. Thanks to Lemma 8.6, the elements $\rho_{G,X}(X(\alpha))$ for $\alpha \in \Psi$ are pairwise distinct. If the matrix (8.3) is degenerate, there exists a linear combination

$$\sum_{\alpha\in\Psi'} a_\alpha \rho_{G,X}(X(\alpha)) \in \Psi^\perp \tag{8.4}$$

where $\varnothing \neq \Psi' \subseteq \Psi$ and $a_\alpha \neq 0$ for all $\alpha \in \Psi'$. Applying $\langle -, \alpha\rangle$ for a fixed $\alpha \in \Psi'$ to the linear combination (8.4), we see that both $\rho_{G,X}(X(\alpha))$ and $\rho_{G,X}(X(-\alpha))$ must appear in the sum. Indeed, the former appears, and the latter is the only other possible summand that is nonzero on α. The elements $\rho_{G,X}(X(-\alpha))$ for $\alpha \in \Psi$ are distinct, thanks to the first part of the proof applied to the set of simple roots $-\Psi$.

Hence each summand in (8.4) can also be rewritten as $a_\alpha \rho_{G,X}(X(-\tau(\alpha)))$ where $\tau\colon \Psi' \to \Psi'$ is a bijection. We also know that $\rho_{G,X}(X(\alpha)) \neq \rho_{G,X}(X(-\alpha))$, therefore τ has no fixed points. Now consider

$$\gamma = \sum_{\alpha \in \Psi'} \alpha.$$

Its value on $\rho_{G,X}(D)$ is zero, if $D \subset X$ is a G-stable prime divisor not of the form $X(\pm\alpha)$ for some $\alpha \in \Psi'$. On the other hand, for a fixed $\alpha \in \Psi'$ we have that $X(\alpha) = X(-\tau(\alpha))$, but $X(\alpha) \neq X(\beta)$ for all $\beta \in \Psi$ different from α, and $X(\alpha) \neq X(-\beta)$ for any $\beta \in \Psi$ different from $\tau(\alpha)$. Therefore

$$\begin{aligned} \langle \rho_{G,X}(X(\alpha)), \gamma\rangle &= \langle \rho_{G,X}(X(\alpha)), \alpha\rangle + \langle \rho_{G,X}(X(\alpha)), \tau(\alpha)\rangle + \left\langle \rho_{G,X}(X(\alpha)), \sum_{\beta \in \Psi', \beta \neq \alpha, \tau(\alpha)} \beta \right\rangle \\ &= -1 + 1 + 0 = 0. \end{aligned}$$

We obtain that $\operatorname{supp} \mathcal{F}_G(X)$ is contained in the hyperplane $\gamma^\perp$, which is absurd because X is complete. □

Proposition 8.8. *As an A-variety, X is spherical (not necessarily toroidal). The set of its A-stable prime divisors is*

$$\partial_A X = \partial_G X \setminus \{X(\alpha) \mid \alpha \in \Phi\},$$

and these are exactly the G-stable prime divisors D such that $\rho_{G,X}(D) \in \Psi^\perp$. Given the identification $\mathcal{X}(\theta_{G,X}(G)) = \mathcal{X}(B_A)$, we have an inclusion

$$\iota\colon \Lambda_A(X) \to \Lambda_G(X)$$

whose image is the sublattice

$$\{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp \subseteq \Lambda(X). \tag{8.5}$$

The restriction map $\iota^*\colon \mathrm{N}_G(X) \to \mathrm{N}_A(X)$ induces an isomorphism

$$\iota^*|_{\Psi^\perp}\colon \Psi^\perp \xrightarrow{\cong} \mathrm{N}_A(X).$$

For any B_A-stable prime divisor $D \subset X$ we have $\rho_{A,X}(D) = \iota^* \rho_{G,X}(D)$, and the set of A-colors of X is the following:

$$\Delta_A(X) = \{X(\alpha) \mid \alpha \in (-\Psi)\} \setminus \{X(\alpha) \mid \alpha \in \Psi\}.$$

Finally, let $\alpha \in \Psi$ with $X(-\alpha) \in \Delta_A(X)$. For all $\beta \in \Phi_+$ different from α, we have $X(-\alpha) \neq X(\beta)$ and $X(\alpha) \neq X(\beta)$. In particular, if in addition $\beta \in \Psi$, we also have $\rho_{G,X}(X(-\alpha)) \in \beta^\perp$.

Proof. Since $\theta_{G,X}(G) \subseteq B_A$ has already an open orbit on X, the first statement is obvious. The statement about the A-stable prime divisors is also immediate.

Let us prove that the A-colors are the set $\Delta_A(X)$ as above defined. A color must be $X(\alpha)$ for some $\alpha \in \Phi$ otherwise it is A-stable, and at this point not being of the form $X(\alpha)$ for any $\alpha \in \Phi_+$ is equivalent to be stable under B_A. Then, we conclude using (8.1) and (8.2).

The inclusion ι is given by the simple observation that a B_A-eigenvector in $\mathbb{C}(X)$ is a fortiori a G-eigenvector, with same eigenvalue; the identity $\rho_{A,X}(D) = \iota^*\rho_{G,X}(D)$ for any B_A-stable prime divisor is also obvious.

Let us prove that the image of ι is the lattice (8.5). If $\gamma \in \Lambda_A(X)$, then a corresponding B_A-eigenvector $f_\gamma \in \mathbb{C}(X)$ cannot have zeros nor poles on prime divisors $X(\alpha)$ for $\alpha \in \Psi$, since the latter divisors are not B_A-stable. Hence $\iota(\Lambda_A(X)) \subseteq \{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp$. On the other hand, if $\chi \in \{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp$, then a corresponding G-eigenvector $f_\chi \in \mathbb{C}(X)$ has zeros and poles only on A-stable prime divisors or on colors. It follows that f_χ is also a B_A-eigenvector, and the other inclusion is proved.

We prove now that $\iota^*|_{\Psi^\perp}$ is an isomorphism between $\Psi^\perp$ and $\mathrm{N}_A(X)$. From the first part of the proof, this follows if we prove that

$$\Lambda \otimes_\mathbb{Z} \mathbb{Q} = (\Psi \otimes_\mathbb{Z} \mathbb{Q}) \oplus \left(\{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp \otimes_\mathbb{Z} \mathbb{Q}\right),$$

and this equality is an easy consequence of Lemma 8.7.

Let us check the last statement, so let $\alpha \in \Psi$ be such that $X(-\alpha) \in \Delta_A(X)$, and consider $\beta \in \Phi_+$, $\beta \neq \alpha$. We know that $X(-\alpha) \neq X(\beta)$ because of the definition of $\Delta_A(X)$ together with (8.1). This also implies that $X(\alpha) \neq X(\beta)$, because otherwise we would have $\beta - \alpha \in \Phi_+$ with $X(-\alpha) = X(\beta - \alpha)$, thanks to Lemma 8.6. □

Remark 8.9. The two above results imply in particular that the A-colors of X, seen as elements of $\mathrm{N}_A(X)$, are linearly independent.

Example 8.10. An example where X is not toroidal as an A-variety can be given as follows. Let $X = \mathbb{P}^n$ with $n \geq 2$, under the linear action of the group G of $(n+1) \times (n+1)$ invertible diagonal matrices. Then $\partial_G X$ has $n+1$ elements, each given by the vanishing of an homogeneous coordinate. If $\mathcal{D}$ is the set of all of them except for one, then $\mathrm{Aut}^\circ(X, \mathcal{D})$ is a maximal parabolic subgroup of $\mathrm{PGL}(n+1)$. Its Levi subgroup A containing the image of G acts with a fixed point, contained in all elements of $\mathcal{D}$ therefore contained in any L-color of X.

We can now state the main theorem of this section.

Theorem 8.11. *If we identify* $\mathrm{N}_A(X)$ *and* $\Psi^\perp$ *via the map* $\iota^*|_{\Psi^\perp}$ *of Proposition 8.8, the fan of colored convex cones* $\mathcal{F}_A(X)$ *of* X *as a spherical* A*-variety is obtained from the fan* $\mathcal{F}_G(X)$ *as follows:*

$$\mathcal{F}_A(X) = \left\{ (c \cap \Psi^\perp, d(c)) \;\middle|\; c \in \mathcal{F}_G(X) \right\}.$$

Here $d(c)$ *is the set of* A*-colors* D *of* X *such that if* $\beta \in \Phi_+$ *satisfies* $X(-\beta) = D$*, then both* $\rho_{G,X}(X(\beta))$ *and* $\rho_{G,X}(X(-\beta))$ *lie on* 1*-dimensional faces of* c*.*

Proof. First, we consider $c \in \mathcal{F}_G(X)$ and we show that the colored cone $(c \cap \Psi^\perp, d(c))$ belongs to $\mathcal{F}_A(X)$.

The cone c is equal to $c_{X,Y}$ for some G-orbit Y. We claim that the colored cone associated to the A-orbit AY is given by $(c \cap \Psi^\perp, d(c))$, with $d(c)$ defined as in the theorem. To show the claim, it is enough to prove that:

1. the A-stable prime divisors containing AY are the G-stable prime divisors D such that $D \supseteq Y$ and $\rho_{G,X}(D) \in \Psi^\perp$;
2. the set of the A-colors containing AY is $d(c)$;
3. the convex cone c' generated by the image of elements of (1) and (2) under the map $\iota^* \circ \rho$ is $c \cap \Psi^\perp$.

Part (1) is obvious, thanks to the results on $\partial_A X$ contained in Proposition 8.8. For part (2), let us first prove that a color D not belonging to $d(c)$ doesn't contain AY. If D doesn't contain Y there is nothing to prove, therefore we may assume that $\rho_{G,X}(D)$ lies on a 1-dimensional face of c. Suppose at first that $X(-\beta) = D$ for some $\beta \in \Phi_+$, in such a way that $X(\beta)$ doesn't contain Y.

Let X_c be the affine G-stable open subset of X associated to the cone c, i.e.:

$$X_c = \left\{ x \in X \;\middle|\; \overline{Gx} \supseteq Y \right\}.$$

It is isomorphic to an affine space, and in [Oda88, Proof of Proposition 3.14] it is shown that X_c is stable under the action of $U_{-\beta}$.

More precisely, there exist global coordinates $(x_1, \ldots, x_n)$ on X_c such that $X(-\beta) \cap X_c$ is the hyperplane defined by the equation $x_1 = 0$, and in these coordinates $U_{-\beta}$ acts as follows:

$$u_{-\beta}(\xi)(x_1, x_2, \ldots, x_n) = (x_1 + \xi, x_2, \ldots, x_n). \tag{8.6}$$

One may easily check this formula using [Oda88, Proposition 3.14] and the fact that $X(-\beta)$ is the only G-stable prime divisor that contains Y and where β is non-zero. The hyperplane defined in X_c by $x_1 = 0$ contains Y, but from (8.6) we deduce that it doesn't contain $U_\beta Y$. As a consequence, AY is not contained in $X(-\beta)$.

Now we show that a color D in $d(c)$ contains AY. At first, consider $\beta \in \Phi_+$ such that $X(-\beta) = D$. Both $X(-\beta)$ and $X(\beta)$ contain Y, and we consider again the affine space X_c.

Applying [Oda88, Proposition 3.14] once again, there exist coordinates $(x_1, x_2, \dots, x_n)$ such that $X(-\beta) \cap X_c$ is defined by the equation $x_1 = 0$, and $X(\beta) \cap X_c$ by the equation $x_2 = 0$, and such that

$$u_{-\beta}(\xi)(x_1, x_2, \dots, x_n) = (x_1 + \xi x_2, x_2, \dots, x_n) \tag{8.7}$$

and

$$u_{\beta}(\xi)(x_1, x_2, \dots, x_n) = (x_1, x_2 + \xi x_1, \dots, x_n). \tag{8.8}$$

We obtain that Y is both $U_{-\beta}$-stable and U_β-stable, being contained in the subset of X_c defined by $x_1 = x_2 = 0$. Therefore $X(-\beta) = D$ contains $Y = U_\beta U_{-\beta} Y$.

Now observe that the image of the multiplication map

$$\theta_{G,X}(G) \times \prod_{\gamma \in \Phi} U_\gamma \to A$$

(where the product is taken in any fixed order) is dense in A. It follows that D contains AY, if we prove that D is U_γ-stable for all $\gamma \in \Phi$ such that $\gamma \neq \pm\beta$ for all $\beta \in \Phi_+$ satisfying $X(-\beta) = D$. For $\gamma \in \Phi_-$ there is nothing to prove. But also for $\gamma \in \Phi_+$ we know that $D \neq X(\gamma)$: this fact stems from the last statement of Proposition 8.8 together with (8.1). The proof of (2) is complete.

Let us prove (3). Call S the set of A-stable prime divisors containing AY. Then we can describe a minimal set of generators of c (as a convex cone) as the union of the following subsets:

(a) the set $\rho_{G,X}(S)$;
(b) for each color $D \in d(c)$, the set $\{\rho_{G,X}(D)\} \cup \{\rho_{G,X}(X(\beta)) \mid \beta \in \Phi_+,\ X(-\beta) = D\}$;
(c) other generators, different from any of the above.

We show that $c \cap \Psi^\perp$ is contained in c', and recall that the latter is generated by $\rho_{G,X}(S)$ together with $\iota^*(\rho_{G,X}(d(c)))$. An element $x \in c \cap \Psi^\perp$ is a linear combination with non-negative coefficients of the above generators, and we may assume that the elements of (a) do not contribute. This indeed implies the general case, since $\rho_{G,X}(S) \subseteq c'$.

Also, we may suppose that any generator z involved in the linear combination giving x satisfies $\iota^*(z) \neq 0$. Indeed, otherwise we may suppress it using the fact that $x = \iota^*(x)$. Hence, all generators in the linear combination of x are not of the form $\rho_{G,X}(X(\beta))$ for $\beta \in \Psi$.

It remains the generators $\rho_{G,X}(D)$ where $D \in d(c)$, and generators of (c) of the form $\rho_{G,X}(X(-\alpha))$ for some $\alpha \in \Psi$. In the second case $X(-\alpha)$ is a color, because it cannot be equal to $X(\beta)$ for any $\beta \in \Psi$. Being not in $d(c)$, each such $X(-\alpha)$ admits a positive root β satisfying $X(-\beta) = X(-\alpha)$ and D_β not a generator of c. This implies that β is non-positive on c, and the only chance for x to be in $\beta^\perp$ is that such a generator $X(-\beta) = X(-\alpha)$ doesn't occur.

As a consequence, x is a linear combination of the elements $\rho_{G,X}(D)$ with $D \in d(c)$, and we easily conclude that $x \in c'$ using again $\iota^*(x) = x$.

Finally, let $x \in c'$, and let us show that $x \in c \cap \Psi^\perp$. As before, we ignore the generators of c' lying in $\Psi^\perp$, and we assume that x is a linear combination with non-negative coefficients of $\iota^*(d(c))$. In other words:

$$x = \sum_{\alpha \in \Psi, X(-\alpha) \in d(c)} a_\alpha \iota^* \left(\rho_{G,X}\left(X(-\alpha)\right)\right)$$

with $a_\alpha \geq 0$. Consider a summand $a_\alpha \iota^* \left(\rho_{G,X}\left(X(-\alpha)\right)\right)$. For each positive root $\beta \neq \alpha$ such that $X(-\beta) = X(-\alpha)$, Lemma 8.6 implies that $\gamma = \beta - \alpha$ and $-\gamma$ are also roots in Φ, and that $X(-\alpha) = X(-\beta)$, $X(\alpha) = X(-\gamma)$, $X(\gamma) = X(\beta)$ are three distinct prime divisors. Then, we take the sum

$$y = \sum_{\alpha \in \Psi, X(-\alpha) \in d(c)} a_\alpha y_\alpha \tag{8.9}$$

where

$$y_\alpha = \rho_{G,X}\left(X(-\alpha)\right) + \sum_{\substack{\beta \in \Phi_+, \\ X(-\beta) = X(-\alpha)}} \rho_{G,X}\left(X(\beta)\right).$$

We claim that all simple roots in Ψ are zero on this element, hence $\iota^*(y) = y$ and we immediately conclude that $y = x$. On the other hand, y is in c thanks to the definition of the set $d(c)$, therefore $x \in c \cap \Psi^\perp$.

Let us prove the claim. Let $\gamma \in \Psi$, and pick a y_α. If $\gamma = \alpha$, then it is easy to check using the last assertion of Proposition 8.8 that y_α is the sum of $\rho_{G,X}(X(-\alpha))$ and $\rho_{G,X}(X(\alpha))$, plus other terms where α is zero. It follows $\langle y_\alpha, \gamma \rangle = 0$.

If $\gamma \neq \alpha$, then $\langle \rho_{G,X}(X(-\alpha)), \gamma \rangle = 0$ thanks to Proposition 8.8. Moreover, in this case γ does not appear as a β in the sum expressing y_α, because we know that $X(-\alpha) \neq X(-\gamma)$. Also, if $X(\pm\gamma)$ is different from $\rho_{G,X}(X(\beta))$ for all $\beta \in \Phi_+$ such that $X(-\beta) = X(-\alpha)$, then again $\langle y_\alpha, \gamma \rangle = 0$.

Therefore we may suppose that γ is different from all the β appearing in the expression of y_α, but some of them, say $\beta_{i,\gamma}$ for $i = 1, \ldots, k$, satisfy $X(\beta_{i,\gamma}) = X(\epsilon_{i,\gamma}\gamma)$ where $\epsilon_{i,\gamma} = 1$ or -1. In this case Lemma 8.6 implies that $\beta_{i,\gamma} - \epsilon_{i,\gamma}\gamma$ also appears in the sum, with $X(\beta_{i,\gamma} - \epsilon_{i,\gamma}\gamma) =$

$X(-\epsilon_{i,\gamma}\gamma)$. We obtain:

$$\begin{aligned} y_\alpha &= \rho_{G,X}\left(X(-\alpha)\right) + \sum_{i=1}^{k} \left(\rho_{G,X}\left(X(\beta_{i,\gamma})\right) + \rho_{G,X}\left(X(\beta_{i,\gamma} - \epsilon_{i,\gamma}\gamma)\right)\right) \\ &\quad + \sum_{\substack{\beta\in\Phi_+, X(\beta)\neq X(\pm\gamma) \\ X(-\beta)=X(-\alpha)}} \rho_{G,X}\left(X(\beta)\right) \\ &= \rho_{G,X}\left(X(-\alpha)\right) + \sum_{i=1}^{k} \left(\rho_{G,X}\left(X(\gamma)\right) + \rho_{G,X}\left(X(-\gamma)\right)\right) + \sum_{\substack{\beta\in\Phi_+, X(\beta)\neq X(\pm\gamma) \\ X(-\beta)=X(-\alpha)}} \rho_{G,X}\left(X(\beta)\right). \end{aligned}$$

From this expression it is evident that $\langle y_\alpha, \gamma\rangle = 0$, and the proof of (3) is complete.

To finish the proof of the theorem, we must check that all colored cones of $\mathcal{F}_A(X)$ appear as $(c \cap \Psi^\perp, d(c))$ for some $c \in \mathcal{F}_G(X)$. For this, it is enough to notice that for each A-orbit Z there is a G-orbit Y such that $AY = Z$. □

Corollary 8.12. *The A-variety is horospherical, i.e. $\Sigma_A(X) = \varnothing$.*

Proof. There exists a smooth complete toroidal A-variety Y equipped with a surjective birational A-equivariant morphism $Y \to X$ (it is enough to choose an A-equivariant resolution of singularities of the variety given in [Kn91, Lemma 5.2], where X'' in the proof of *loc.cit.* is our X).

Then $\Sigma_A(Y) = \Sigma_A(X)$, and Y is also a complete G-regular embedding. Applying Theorem 8.11 to Y, it follows that $\operatorname{supp} \mathcal{F}_A(Y)$ is a vector space, and it is equal to $\mathrm{V}_A(Y)$ because Y is toroidal and complete. We conclude that $\Sigma_A(Y) = \varnothing$. □

Remark 8.13. With a slightly more involved proof, one can derive the above corollary directly from Proposition 8.8 and avoid using Theorem 8.11.

Remark 8.14. It is easy to check that $d(c) = \varnothing$ if and only if $c \cap \Psi^\perp$ is a face of c.

Example 8.15. Let us compute the colored fan of $X = \mathbb{P}^2$, as in Example 8.10 with $n = 2$. Choose $\mathcal{E} = \{E_3\}$ where $E_i = \{x_i = 0\}$ and x_1, x_2 and x_3 are homogeneous coordinates on $\mathbb{P}^2$. Then $A = \mathrm{Aut}^\circ(\mathbb{P}^2, \mathcal{D})$ is isomorphic to $\mathrm{SL}(2)$, and we choose the Borel subgroup of A stabilizing the point $[1, 0, 0]$. The lattice $\Lambda_G(\mathbb{P}^2)$ is then the root lattice of $\mathrm{PGL}(3)$, and we have $X_1 = X(\alpha_1) = X(\alpha_1 + \alpha_2)$, $X_2 = X(\alpha_2) = X(-\alpha_1)$ and $X_3 = X(-\alpha_1 - \alpha_2) = X(-\alpha_2)$, where α_1 and α_2 are the simple roots of $\mathrm{PGL}(3)$. The lattice $\Lambda_A(\mathbb{P}^2)$ is $\rho_{G,\mathbb{P}^2}(X_1)^\perp = \mathbb{Z}\alpha_2$, which is the weight lattice of $\mathrm{SL}(2)$, and $\mathbb{P}^2$ has only one A-color, namely X_2. The maximal colored cones of $\mathcal{F}_A(\mathbb{P}^2)$ are $(\mathbb{Q}_{\geq 0}\rho_{G,\mathbb{P}^2}(X_3), \varnothing)$ and $(-\mathbb{Q}_{\geq 0}\rho_{G,\mathbb{P}^2}(X_3), \{X_2\})$.

9. Semisimple case

In this section we assume that G is a semisimple group, i.e. $C = \{e\}$. In this setting the functionals associated to the colors of X generate $\mathrm{N}_G(X)$ as a vector space. Indeed, if $\lambda \in \Lambda_G(X)$ is in $\rho_{G,X}(\Delta_G(X))^\perp$, then a rational function $f \in \mathbb{C}(G/H)^{(B)}_\lambda$ is regular on G/H and nowhere zero. It can be then lifted to a nowhere-vanishing function $F \in \mathbb{C}[G]$, which is then constant since G has no non-trivial character (see [KKV89, Proposition 1.2]). We conclude that $\lambda = 0$, and the claim follows.

This essentially implies the following main result of this section.

Theorem 9.1. *If G is semisimple and $\mathcal{D}$ is any subset of $\partial_G X$, then $\mathrm{Aut}^\circ(X, \mathcal{D} \cup (\partial_G X)^\ell)$ is a Levi subgroup of $\mathrm{Aut}^\circ(X, \mathcal{D})$.*

The proof is at the end of this section. The theorem implies that if G is semisimple then §7 is enough to describe a Levi subgroup of $\mathrm{Aut}^\circ(X, \mathcal{D})$ and its action on X, without any restriction on $\mathcal{D}$.

Recall from §5 the restriction map

$$\kappa_{x'} \colon (\ker \psi_*)^\circ \to \mathrm{Aut}^\circ(X_{x'})$$

where x' lies on the open G-orbit of X', and $X_{x'} = \psi^{-1}(x')$.

Lemma 9.2. *For all x' in the open B-orbit of X', the image of $\kappa_{x'}$ in $\mathrm{Aut}^\circ(X_{x'})$ is very solvable (i.e. contained in a Borel subgroup).*

Proof. To simplify notations we assume that $x' = x'_0$. Let $\mathcal{E}' \subseteq \partial_S X_{x'_0}$ be the following subset:

$$\mathcal{E}' = \{E \cap X_{x'_0} \mid E \in \mathcal{E}\},$$

and define $\mathcal{D}' = \partial_S X_{x'_0} \setminus \mathcal{E}'$. Let us also denote by $K_{x'_0}$ the image of $\kappa_{x'_0}$: it is obviously a subgroup of $\mathrm{Aut}^\circ(X_{x'_0}, \mathcal{D}')$. On the other hand $K_{x'_0}$ contains the maximal torus S of $\mathrm{Aut}^\circ(X_{x'_0})$, hence we only have to compute the root subgroup it contains. Thanks to Lemma 6.4 and Corollary 5.3, they are the root spaces $U_\alpha \subset \mathrm{Aut}^\circ(X_{x'_0})$ for α varying in the set

$$R = \left\{ \gamma|_S \;\middle|\; 0 \neq \gamma \in \Lambda_G(G/H),\ X(\gamma) \text{ exists and } X(\gamma) \in \mathcal{E}^\ell \right\}.$$

From Lemma 6.5, we obtain that R doesn't contain the opposite of any of its elements, therefore $K_{x'_0}$ is very solvable. ☐

Proof of Theorem 9.1. First, observe that $(\ker \psi_*)^\circ$ is solvable. This stems from Lemma 9.2, and the obvious observation that

$$\bigcap_{x' \text{ in the open } B\text{-orbit of } X'} \ker(\kappa_{x'}) = \{\mathrm{id}_X\}. \tag{9.1}$$

Consider now the variety X under the action of $A = \mathrm{Aut}^\circ(X, \mathcal{D} \cup (\partial_G X)^\ell)$. Thanks to Theorem 7.8, the group A is semisimple (because here G is semisimple) and under its action X is a G-regular embedding with boundary $\mathcal{D} \cup (\partial_G X)^\ell$. Corollary 7.19 implies $(\partial_A X)^{n\ell} = \mathcal{D}^{n\ell} \subseteq \mathcal{D}$, and we deduce that $\mathrm{Aut}^\circ(X, \mathcal{D}) \subseteq \mathrm{Aut}^\circ(X, (\partial_A X)^{n\ell})$.

Then we may apply Proposition 5.2 with G replaced by the universal cover of A: the theorem follows. □

Remark 9.3. Let X and G be as in Example 7.9. Then the full automorphism group of X is non-reductive. Indeed, it must fix the point $p \in \mathbb{P}^{n+1}$, and one concludes easily that $\mathrm{Aut}^\circ(X)$ is the corresponding maximal proper parabolic subgroup of $\mathrm{PGL}(n+2) \times \mathrm{PGL}(n+1)$. The unipotent radical $\mathrm{Aut}^\circ(X)^u$ can be studied restricting its elements to the generic fiber $X_{x'_0}$; however, the example shows that for any given fiber the restriction may be non-injective, therefore a global analysis of these restrictions is needed. This goes beyond the scope of the present work.

10. G-stable prime divisors on the linear part of the valuation cone

In this section $G = G' \times C$ is neither abelian nor semisimple. For simplicity, and thanks to §8, we may assume that G' acts non-trivially on X. The variety X' is then not a single point. Recall that S acts on X naturally by G-equivariant automorphisms preserving the fibers of ψ, so we can consider S as a subgroup of $\mathrm{Aut}^\circ(X, \partial_G X) \cap (\ker \psi_*)^\circ$.

We study the automorphism group $\mathrm{Aut}^\circ(X, \mathcal{D})$, where $\mathcal{D} \subseteq \partial_G X$ satisfies $\mathcal{D} \supseteq (\partial_G X)^{n\ell}$. Denote as usual $\mathcal{E} = \partial_G X \setminus \mathcal{D} \subseteq (\partial_G X)^\ell$, and recall that all elements $D \in (\partial_G X)^\ell$ intersect $X_{x'_0}$ in an S-stable prime divisor.

Proposition 10.1. *Let x' in the open G-orbit of X', and $L = L(X, \mathcal{D})$ be a Levi subgroup of $(\mathrm{Aut}^\circ(X, \mathcal{D}) \cap \ker(\psi_*))^\circ$ containing S. Then $L_{x'} = \kappa_{x'}(L)$ is isomorphic to L, and the group*

$$(\theta_{G,X}(G), \theta_{G,X}(G)) \times L(X, \mathcal{D})$$

is locally isomorphic to a Levi subgroup of $\mathrm{Aut}^\circ(X, \mathcal{D})$.

Proof. Thanks to formula (9.1) and the fact that x' is generic in X', we know that the map $L \to L_{x'}$ has unipotent kernel, therefore is an isomorphism. The rest follows from Proposition 5.2. □

Definition 10.2. We define the following group:

$$A = A(X, \mathcal{D}) = (\theta_{G,X}(G), \theta_{G,X}(G)) \times L(X, \mathcal{D}),$$

where $L(X, \mathcal{D})$ is defined as in Proposition 10.1.

We describe now the reductive group $L_{x'_0}$ in terms of the root subspaces it contains with respect to its maximal torus S.

Definition 10.3. We define

$$R = R(X,\mathcal{D}) = \{\gamma|_S \mid 0 \neq \gamma \in \Lambda_G(G/H),\ X(\gamma) \text{ exists and } X(\gamma) \in \mathcal{E}\}\,,$$

and we denote by $\Phi = \Phi(X,\mathcal{D})$ the maximal subset of R such that $-\alpha \in R$ for every $\alpha \in R$.

Proposition 10.4. *The set $\Phi(X,\mathcal{D})$ is a subset of $\Phi(X_{x'_0},\mathcal{D}')$, where $\mathcal{D}' = \{D \cap X_{x'_0} \mid D \in \mathcal{D}^\ell\}$. Moreover, $L_{x'_0} \subseteq \mathrm{Aut}^\circ(X_{x'_0})$ is generated by S together with all subgroups U_α such that $\alpha \in \Phi(X,\mathcal{D})$.*

Proof. For the first assertion, it is enough for any $\alpha = \gamma|_S \in \Phi(X,\mathcal{D})$ to restrict the function $f \in H^0(X,\mathcal{O}_X(X(\gamma)))^{(B)}_\gamma$ to $X_{x'_0}$. Since S is a maximal torus of $\mathrm{Aut}^\circ(X_{x'_0})$, the second assertion follows from Lemma 6.4 and Corollary 5.3. □

This provides a complete description of the group A. It remains now to describe the fan associated to X as an A-variety.

Let $0 \neq \gamma \in \Lambda(G/H)$ be such that $\gamma|_S = \alpha \in \Phi$, and choose $f_\gamma \in H^0(X,\mathcal{O}_X(X(\gamma)))^{(B)}_\gamma$ such that $f_\gamma(x_0) = 1$. Then $\rho_{G,X}(X(\gamma)) \in \mathrm{V}^\ell_G(G/H)$ can be considered as an element of $\mathrm{Hom}_\mathbb{Z}(\mathcal{X}(S),\mathbb{Z})$, and therefore it is canonically associated with a 1-PSG $\mu_\gamma \colon \mathbb{C}^* \to S$. The torus S acts on X through the identification with a subtorus of $T_{G,X}$, as we have seen in §5; in this way μ_γ induces a tangent vector field $\delta_\gamma \in H^0(X,\mathcal{T}_X)$ on X.

Lemma 10.5. *The product $\xi_\gamma = f_\gamma\delta_\gamma$ is a well-defined tangent vector field of X, and it is sent to $H^0(X,\mathcal{O}_X(X(\gamma)))$ via the surjective map of (4.1). Its restriction to $X_{x'_0}$ is a tangent vector field and is a generator of the Lie algebra of $U_\alpha \subset \mathrm{Aut}^\circ(X_{x'_0})$. Moreover, the 1-PSG of $\mathrm{Aut}^\circ(\mathcal{Z}_{G,X})$ induced by ξ_γ is expressed in local coordinates by the formulae of* [Oda88, Proposition 3.14].

Proof. The rational function f_γ has its only pole in $X(\gamma)$, which means that we only have to check the first assertion on points of $X(\gamma)$. On $\mathcal{Z}_{G,X} \cap X(\gamma)$ it can be checked easily using the fact that $\mathcal{Z}_{G,X}$ is a toric $T_{G,X}$-variety, and expressing ξ_γ in local coordinates. This also implies that ξ_γ is a well-defined vector field on $E \cap X_0$, thanks to the $P^u_{G,X}$-invariance of both f_γ and δ_γ. Then the locus where ξ_γ might not be a well-defined vector field has codimension at least 2, which implies the first statement.

Since S acts on X stabilizing both $\mathcal{Z}_{G,X}$ and $X_{x'_0}$, we deduce that ξ_γ can be restricted to a vector field on both these varieties. The rest follows easily by expressing ξ_γ on $\mathcal{Z}_{G,X}$ explicitly in local coordinates. □

Definition 10.6. We choose a Borel subgroup B_A of A such that $\theta_{A,X}(B_A)\cap\theta_{G,X}(G) = \theta_{G,X}(B)$ and such that $B_A \cap L$ is a Borel subgroup of L. Let us also denote by $\Psi = \Psi(X,\mathcal{D}) \subset \Phi(X,\mathcal{D})$ the set of simple roots and by $\Phi_+ = \Phi_+(X,\mathcal{D}) \subset \Phi(X,\mathcal{D})$ the set of positive roots associated to the Borel subgroup $B_{L_{x'_0}} = \kappa_{x'_0}(B_A \cap L)$ of $L_{x'_0}$. Finally, let

$$r \colon \Lambda_G(X) \to \mathcal{X}(S) = \Lambda_S(X_{x'_0})$$

be the restriction of characters of $\Lambda_G(X)$ to S (see §5).

We may apply Proposition 8.8 and Theorem 8.11 to the toric S-variety $X_{x'_0}$ and the sets of roots Φ and Ψ. We obtain a description of $X_{x'_0}$ as an $L_{x'_0}$-variety, and in particular the lattice

$$\Lambda_{L_{x'_0}}(X_{x'_0}) \subseteq \Lambda_S(X_{x'_0}),$$

together with the projection

$$\mathrm{N}_S(X_{x'_0}) \to \mathrm{N}_{L_{x'_0}}(X_{x'_0}).$$

Proposition 10.7. *The restriction of weights from $\theta_{A,X}(B_A)$ to $\theta_{G,X}(B)$ induces an isomorphism*

$$\Lambda_A(X) \cong r^{-1}(\Lambda_{L_{x'_0}}(X_{x'_0})) \subseteq \Lambda_G(X).$$

We denote the corresponding surjective map by

$$s\colon \mathrm{N}_G(X) \to \mathrm{N}_A(X).$$

The set of colors of X as a spherical A-variety is the following disjoint union:

$$\Delta_A(X) = \Delta_G(X) \cup \left\{ E \in \mathcal{E} \;\middle|\; E \cap X_{x'_0} \textit{ is a color of the spherical } L_{x'_0}\textit{-variety } X_{x'_0} \right\},$$

and for each $E \in \Delta_A(X)$, we have

$$\rho_{A,X}(E) = s(\rho_{G,X}(E)).$$

Proof. A B_A-eigenvector in $\mathbb{C}(X)$ is a fortiori a B-eigenvector, thanks to the choice of B_A. This induces an inclusion $\Lambda_A(X) \subseteq \Lambda(X)$.

Moreover, a B-eigenvector $f \in \mathbb{C}(X)$ is also a B_A-eigenvector if and only if its restriction $f|_{X_{x'_0}}$ is a $B_{L_{x'_0}}$-eigenvector, thanks to the structure of A as described in Proposition 10.1. This proves the first assertion.

Secondly, a color of X as an A-variety maps either dominantly onto X', or not. In the first case, its intersection with the (generic) fiber $X_{x'_0}$ is $B_{L_{x'_0}}$-stable but not $L_{x'_0}$-stable (otherwise it would have been A-stable).

In the second case, it maps onto a G-color of X', i.e. it is a color of X with respect to the G action. The second assertion follows. □

Let c be a cone of the fan $\mathcal{F}(X)$. Then c is generated as a convex cone by a set of 1-dimensional faces $F(c)$. We denote by c^ℓ the intersection $c \cap \mathrm{V}^\ell_G(X)$, by $F^\ell(c)$ the 1-dimensional faces of $F(c)$ generating c^ℓ, and $F^{n\ell}(c) = F(c) \setminus F^\ell(c)$.

Since c^ℓ is a cone of the toric S-variety $X_{x'_0}$, it corresponds to an S-orbit Y on $X_{x'_0}$. As in the proof of Theorem 8.11, the corresponding $L_{x'_0}$-orbit $L_{x'_0}Y$ on $X_{x'_0}$ has colored cone $(c^\ell \cap \Psi^\perp, d(c^\ell))$, where the orthogonal $\Psi^\perp$ is taken inside $\mathrm{V}^\ell_G(G/H)$, and $d(c^\ell)$ is a set of $L_{x'_0}$-colors of $X_{x'_0}$.

Definition 10.8. For any $c \in \mathcal{F}(X)$, we define a colored cone $(c_A(c), d_A(c))$, where $c_A(c) \subset \mathrm{N}_A(X)$ and $d_A(c) \subseteq \Delta_A(X)$, as follows. The cone $c_A(c)$ is the convex cone in $\mathrm{N}_A(X)$ generated by $s(F^{n\ell}(c))$ and $s(c^\ell \cap \Psi^\perp)$. The set $d_A(c)$ is the set of colors $E \in \Delta_A(X)$ such that $E \notin \Delta_G(X)$, and $E \cap X_{x_0'} \in d(c^\ell)$.

Theorem 10.9. *The colored fan $\mathcal{F}_A(X)$ as an A-variety is*

$$\mathcal{F}_A(X) = \{(c_A(c), d_A(c)) \mid c \in \mathcal{F}_G(X)\}.$$

Proof. Let Y be a G-orbit of X, with associated cone $c = c_{X,Y}$. We claim that the colored cone associated to the A-orbit AY is $(c_A(c), d_A(c))$: arguing as in the proof of Theorem 8.11, this is enough to show the theorem.

To prove the claim, first we show that the set d' of A-colors containing AY is equal to $d_A(c)$. Since X is toroidal, no G-color contains Y, nor AY. Therefore any A-color E in d' is indeed a G-stable prime divisor whose functional lies in $\mathrm{V}_G^\ell(X)$. It intersects $X_{x_0'}$ in an $L_{x_0'}$-color of $X_{x_0'}$, by Proposition 10.7, and we only have to show that $E \cap X_{x_0'}$ is in $d(c^\ell)$.

We check this fact using the definition of $d(c^\ell)$. Take a positive root $\beta \in \Phi_+$ of $X_{x_0'}$, the prime divisors $X_{x_0'}(\beta)$, $X_{x_0'}(-\beta)$ of $X_{x_0'}$ as in Definition 8.1, and suppose that $X_{x_0'}(-\beta) = E \cap X_{x_0'}$, so $\rho_{S,X_{x_0'}}(X_{x_0'}(-\beta))$ lies on a 1-codimensional face of c^ℓ. We have to show that $\rho_{S,X_{x_0'}}(X_{x_0'}(\beta))$ also lies on a 1-codimensional face of c^ℓ, in other words that $X_{x_0'}(\beta)$ contains the S-orbit of $X_{x_0'}$ associated c^ℓ.

Now $E = E_1$ and some other element $E_2 \in \mathcal{E}$ satisfy $E_1 \cap X_{x_0'} = X_{x_0'}(-\beta)$, $E_2 \cap X_{x_0'} = X_{x_0'}(\beta)$, and $-\beta$ and β are the restrictions to S of resp. $\gamma_1, \gamma_2 \in \Lambda_G(X)$, such that $X(\gamma_i) = E_i$ for $i = 1, 2$. Suppose that E_2 doesn't contain Y. Then we consider $\mathcal{Z}_{G,X}$: intersecting it with E_1, E_2 and Y two $T_{G,X}$-stable prime divisors and a $T_{G,X}$-orbit, such that $E_1 \cap \mathcal{Z}_{G,X} \supseteq Y \cap \mathcal{Z}_{G,X}$ and $E_2 \cap \mathcal{Z}_{G,X} \not\supseteq Y \cap \mathcal{Z}_{G,X}$.

At this point we follow the same approach of the proof of Theorem 8.11, statement (2), applied to the toric variety $\mathcal{Z}_{G,X}$ and the automorphisms induced by the tangent vector field ξ_{γ_1} (as defined in Lemma 10.5). This yields the formula (8.6) for ξ_{γ_1}, which shows that $E_1 \cap \mathcal{Z}_{G,X}$ doesn't contain $AY \cap \mathcal{Z}_{G,X}$: a contradiction. As a consequence $E_2 \supseteq Y$, so $X_{x_0'}(\beta)$ contains the S-orbit of $X_{x_0'}$ associated c^ℓ. This concludes the proof of the inclusion $d' \subseteq d_A(c)$.

Let now $D \in d_A(c)$. Then, by Theorem 8.11, the intersection $D \cap X_{x_0'}$ contains the $L_{x_0'}$-orbit of $X_{x_0'}$ corresponding to $(c^\ell \cap \Psi^\perp, d(c^\ell))$. Let y be a point on this orbit: then D contains $\overline{Ay}$.

On the other hand, from the proof of Theorem 8.11, we see that $\overline{L_{x_0'} y}$ contains the S-orbit of $X_{x_0'}$ corresponding to $c^\ell \subset \mathrm{N}_S(X_{x_0'})$. It follows that $\overline{Ay}$ contains the G-orbit of X associated to $c^\ell \subset \mathrm{N}_G(X)$, and thus also the G-orbit Y associated to $c \subset \mathrm{N}_G(X)$. Being A-stable, $\overline{Ay}$ must then contain AY too, and since D is closed, we obtain $D \supseteq AY$. I.e., D is in d'.

We now prove that the convex cone c' associated to AY is $c_A(c)$. First observe that Y and AY are contained in the same elements of $(\partial_G X)^{n\ell}$, since L stabilizes all fibers of ψ. Therefore

c' is generated by $s(F^{n\ell}(c))$ and its intersection with $s(\mathrm{V}_G^\ell(X))$. It remains to prove that $c' \cap s(\mathrm{V}_G^\ell(X)) = s(c^\ell \cap \Psi^\perp)$.

The cone $c' \cap s(\mathrm{V}_G^\ell(X))$ is generated by $\rho_{A,X}(E)$ where $E \in (\partial_G X)^\ell$ is:

(1) an A-color of X containing AY, i.e. $E \in d_A(c)$, or
(2) an A-stable prime divisor containing AY.

On the other hand the generators of $s(c^\ell \cap \Psi^\perp)$ are the elements $\rho_{A,X}(E)$ where $E \in (\partial_G X)^\ell$ is:

(1') an A-color such that $E \cap X_{x_0'}$ is a color containing the $L_{x_0'}$-orbit Z of $X_{x_0'}$ associated to $(c^\ell \cap \Psi^\perp, d(c^\ell))$, or
(2') an A-stable prime divisor such that $E \cap X_{x_0'}$ is a $L_{x_0'}$-stable prime divisor containing Z.

Thanks to the first part of the proof, the prime divisors E of type (1) and of type (1') are the same.

If E is of type (2') then it contains AZ, whose closure in turn contains AY. Therefore E is of type (2). Let now E be of type (2). Then $E \cap \mathcal{Z}_{G,X}$ is an L-stable (and $T_{G,X}$-stable) prime divisor of $\mathcal{Z}_{G,X}$ containing $Y \cap \mathcal{Z}_{G,X}$, which is the $T_{G,X}$-orbit of $\mathcal{Z}_{G,X}$ associated to c, and $\rho_{T_{G,X},\mathcal{Z}_{G,X}}(E \cap \mathcal{Z}_{G,X})$ lies on $\mathrm{V}_G^\ell(G/H)$. Hence $E \cap X_{x_0'}$ is an $L_{x_0'}$-stable prime divisor of $X_{x_0'}$ containing the S-orbit of $X_{x_0'}$ associated to c^ℓ. Thanks to the proof of Theorem 8.11, we deduce that $E \cap X_{x_0'}$ contains Z, i.e. E is of type (2'). □

Corollary 10.10. $\Sigma_A(X) = \Sigma_G(X)$.

Proof. The proof is similar to the proof of Corollary 8.12. □

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