ON A FRACTIONAL QUANTUM POTENTIAL

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1. INTRODUCTION

For fractals we refer to [13, 23] and for differential equations cf. also [18, 22, 34, 38, 39]. The theme of scale relativity as in [1, 10, 11, 26, 27, 28, 30] provides a profound development of differential calculus involving fractals (cf. also the work of Agop et al in the journal Chaos, Solitons, and Fractals) and for interaction with fractional calculus we mention [2, 17, 25, 35, 36]. There are also connections with the Riemann zeta function which we do not discuss here (see e.g. [24]). Now the recent paper [20] of Kobelev describes a Leibnitz type fractional derivative and one can relate fractional calculus with fractal structures as in [2, 21, 32, 35, 36] for example. On the other hand scale relativity with Hausdorff dimension 2 is intimately related to the Schrödinger equation (SE) and quantum mechanics (QM) (cf. [27]). We show now that if one can write a meaningful Schrödinger equation with Kobelev derivatives (α-derivatives) then there will be a corresponding fractional quantum potential (QP) (see e.g. [17, 22, 35, 36] for a related fractional equation and recall that the classical wave function for the SE has the form $\psi = Rexp(iS/\hbar)$).

Going now to [20] we recall the Riemann-Liouville (RL) type fractional operator (assumed to exist here)

$$(1.1)\quad {}_cD_z^\alpha[f(z)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_c^z (z-\zeta)^{-\alpha-1}f(\zeta)d\zeta & c\in\mathbf{R},\ Re(\alpha)<0 \\ \frac{d^m}{dz^m}\,{}_cD_z^{\alpha-m}[f(z)] & m-1\le\Re\alpha<m \end{cases}$$

(the latter for $m \in \mathbf{N} = \{1, 2, 3, \cdots\}$). For $c = 0$ one writes (**1A**) ${}_0D_z^\alpha[f(z)] = D_z^\alpha[f(z)]$ as in the classical RL operator of order α (or $-\alpha$). Moreover when $c \to \infty$ (1.1) may be identified with the familiar Weyl fractional derivative (or integral) of order α (or $-\alpha$). An ordinary derivative corresponds to $\alpha = 1$ with (**1B**) $(d/dz)[f(z)] = D_z^\alpha[f(z)]$. The binomial Leibnitz rule for derivatives is

$$(1.2)\qquad D_z^1[f(z)g(z)] = g(z)D_z^1[f(z)] + f(z)D_z^1[g(z)]$$

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whose extension in terms of RL operators D_z^α has the form

$$D_z^\alpha[f(z)g(z)] = \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha\\ n\end{array}\right) D_z^{\alpha-n}[f(z)]D_z^n[g(z)] \tag{1.3}$$

$$\left(\begin{array}{c}\alpha\\ k\end{array}\right) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)};\ \ \alpha,\, k\in\mathbf{C}$$

The infinite sum in (1.3) complicates things and the binomial Leibnitz rule of [20] will simplify things enormously. Thus consider first a momomial z^β so that

$$D_z^\alpha[z^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}z^{\beta-\alpha};\ \Re(\alpha)<0;\ \Re(\beta)>-1 \tag{1.4}$$

Thus the RL derivative of z^β is the product

$$D_z^\alpha[z^\beta] = C^*(\beta,\alpha)z^{\beta-\alpha};\ C^*(\beta,ga) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+)} \tag{1.5}$$

Now one considers a new definition of a fractional derivative referred to as an α derivative in the form

$$\frac{d_\alpha}{dz}[z^\beta] = d_\alpha[z^\beta] = C(\beta,\alpha)z^{\beta-\alpha} \tag{1.6}$$

This is required to satisfy the Leibnitz rule (1.2) by definition, given suitable conditions on $C(\beta,\alpha)$. Thus first (**1C**) $z^\beta = f(z)g(z)$ with $f(z)=z^{\beta-\epsilon}$ and $g(z)=z^\epsilon$ for arbitrary ϵ the application of (1.3) implies that

$$\frac{d_\alpha}{dz}[z^\beta] = z^\epsilon\frac{d_\alpha}{dz}z^{\beta-\epsilon} + z^{\beta-\epsilon}\frac{d_\alpha}{dz}z^\epsilon = z^\epsilon C(\beta-\epsilon,\alpha)z^{\beta-\epsilon-\alpha}+ \tag{1.7}$$

$$+z^{\beta-\epsilon}C(\epsilon,\alpha)z^{\epsilon-\alpha} = [C(\beta-\epsilon,\alpha)+C(\epsilon,\alpha)]z^{\beta-\alpha}$$

Comparison of (1.6) and (1.7) yields (**1D**) $C(\beta-\epsilon,\alpha)+C(\epsilon,\alpha)=C(\beta,\alpha)$. To guarantee (1.2) this must be satisfied for any β, ϵ, α. Thus (**1D**) is the basic functional equation and its solution is (**1E**) $C(\beta,\alpha)=A(\alpha)\beta$. Thus for the validity of the Leibnitz rule the α-derivative must be of the form

$$d_\alpha[z^\beta] = \frac{d_\alpha}{dz}[z^\beta] = A(\alpha)\beta z^{\beta-\alpha} \tag{1.8}$$

One notes that $C^*(\beta,\alpha)$ in (1.5) is not of the form (**1E**) and the RL operator D_z^α does not in general possess a Leibnitz rule. One can assume now that $A(\alpha)$ is arbitrary and $A(\alpha)=1$ is chosen. Consequently for any β

$$\frac{d_\alpha}{dz}z^\beta = \beta z^{\beta-\alpha};\ \frac{d_\alpha}{dz}z^\alpha = \alpha;\ \frac{d_\alpha}{dz}z^0 = 0 \tag{1.9}$$

Now let K denote an algebraically closed field of characteristic 0 with $K[x]$ the corresponding polynomial ring and $K(x)$ the field of rational functions. Let $F(z)$ have a Laurent series expansion about 0 of the form

$$F(z) = \sum_{-\infty}^{\infty} c_k z^k;\ F_+(z) = \sum_{0}^{\infty} c_k z^k;\ F_-(z) = \sum_{-\infty}^{-1} c_k z^k;\ c_k \in K \tag{1.10}$$

and generally there is a k_0 such that $c_k = 0$ for $k \leq k_0$. The standard ideas of differentiation hold for $F(z)$ and formal power series form a ring $K[[x]]$ with quotient field $K((x))$ (formal Laurent series). One considers now the union (**1F**) $K << x >>= \cup_1^{\infty} K((x^{1/k}))$. This becomes a field if we set

$$x^{1/1} = x,\ x^{m/n} = (x^{1/n})^m \tag{1.11}$$

Then $K << x >>$ is called the field of fractional power series or the field of Puiseux series. If $f \in K << x >>$ has the form (**1G**) $f = \sum_{k_o}^{\infty} c_k x^{m_k/n_k}$ where $c_1 \neq 0$ and $m_k, n_k \in \mathbf{N} = \{1,2,3,\cdots\}$, $(m_i/n_i) < (m_j/n_j)$ for $i < j$ then the order is (**1H**) $O(f) = m/n$ where $m = m_1,\ n = n_1$ and $f(x) = F(x^{1/n})$. Now given n and z complex we look at functions

$$f(z) = \sum_{-\infty}^{\infty} c_k (z-z_0)^{k/n} = f_+(z) + f_-(z);\ f_+(z) = \sum_{0}^{\infty} c_k (z-z_0)^{k/n}, \tag{1.12}$$

$$f_-(z) = \sum_{-\infty}^{-1} c_k (z-z_0)^{k/n};\ c_k = 0\ (k \leq k_0)$$

(cf. [20] for more algebraic information - there are some misprints).

One considers next the α-derivative for a basis (**1I**) $\alpha = m/n;\ 0 < m < n;\ m,n \in \mathbf{N} = \{1,2,3,\cdots\}$. The α-derivative of a Puiseux function of order $O(f) = 1/n$ is again a Puiseux function of order $(1-m)/n$. For $\alpha = 1/n$ we have

$$f_+ = \sum_{0}^{\infty} c_k z^{k/n} = \sum_{0}^{\infty} c_k z^{\beta};\ \beta = \beta(k) = \frac{k}{n} \tag{1.13}$$

leading to

$$\frac{d_\alpha}{dz} f_+(z) = \sum_{1}^{\infty} \alpha\beta c_k z^{(k-1)/n} = \sum_{0}^{\infty} c_{p+1}\alpha\beta z^{p/m}; \tag{1.14}$$

$$\frac{d_\alpha}{dz} f_-(z) = \sum_{-\infty}^{-1} c_k \alpha\beta z^{(k-1)/n} = \sum_{-\infty}^{-2} c_{p+1}\alpha\beta z^{p/n} = \sum_{-\infty}^{-1} \hat{c}_p z^{p/n};\ \hat{c}_{-1} = 0$$

Similar calculations hold for $\alpha = m/n$ (there are numerous typos and errors in indexing in [20] which we don't mention further). The crucial property however is the Leibnitz rule

$$\frac{d_\alpha}{dz}(fg) = g\frac{d_\alpha}{dz}f + f\frac{d_\alpha}{dz}g;\ \ (d_\alpha \sim \frac{d_\alpha}{dz}) \tag{1.15}$$

which is proved via arguments with Puiseux functions. This leads to the important chain rule

$$\frac{d_\alpha}{dz}F(g_i(z)) = \sum \frac{\partial F}{\partial g_k}\frac{d_\alpha}{dz}g_k(z) \tag{1.16}$$

Further calculation yields (again via use of Puiseux functions)

$$\frac{d_\alpha^m}{dz^m}\left[\frac{d_\alpha^\ell}{dz^\ell}f\right] = \frac{d_\alpha^\ell}{dz^\ell}\left[\frac{d_\alpha^m}{dz^m}f\right] \tag{1.17}$$

$$\int f(z)d_\alpha z = \sum_0^\infty \int z^\beta d_\alpha z;\ \ \int z^\beta d_\alpha z = \frac{z^{\beta+\alpha}}{\beta+\alpha} \tag{1.18}$$

$$\frac{d_\alpha}{dz}\int f(z)d_\alpha z = f(z) = \int \frac{d_\alpha}{dz}d_\alpha z \tag{1.19}$$

where $d_\alpha z$ here is an integration symbol here).

The α-exponent is defined as

$$E_\alpha(z) = \sum_0^\infty \frac{(z^\alpha/\alpha)^k}{\Gamma(\alpha+1)} = \tag{1.20}$$

$$= exp\left(\frac{z^\alpha}{\alpha}\right);\ E_1(z) = e^z;\ E_\alpha(0) = 1\ (0 < \alpha, 1)$$

The definition is motivated by the fact that $E_\alpha(z)$ satisfies the α-differential equation (**1J**) $(d_\alpha/dz)E_\alpha(z) = E_\alpha(z)$ with $E_\alpha(0) = 1$. This is proved by term to term differentiation of (1.20). It is worth mentioning that $E_\alpha(z)$ does not possess the semigroup property (**1K**) $E_\alpha(z_1+z_2) \neq E_\alpha(z_1)E_\alpha(z_2)$.

2. FRACTALS AND FRACTIONAL CALCULUS

For relations between fractals and fractional calculus we refer to [2, 12, 21, 31, 32, 35, 36, 37]. In [2] for example one assumes time and space scale isotropically and writes $[x^\mu] = -1$ for $\mu = 0, 1, \cdots, D-1$ and the standard measure is replaced by (**2A**) $d^Dx \to d\rho(x)$ with $[\rho] = -D\alpha \neq -D$ (note [ ] denotes the engineering dimension in momentum units). Here $0 < \alpha < 1$ is a parameter related to the operational definition of Hausdorff dimension which determines the scaling of a Euclidean volume (or mass distribution) of characteristic size R (i.e. $V(R) \propto R^{d_H}$). Taking $\rho \propto d(r^{D\alpha})$ one has (**2B**) $V(R) \propto \int d\rho_{Euclid}(r) = \propto \int_0^R dr r^{D\alpha-1} \propto R^{D\alpha}$, showing that $\alpha = d_H/D$. In general as cited in [2] the Hausdorff dimension of a random process (Brownian motin) described by a fractional differintegral is proportional to the order α of the differintegral. The same relation holds for deterministic fractals and in general the fractional differintegration of a curve changes its Hausdorff dimension as $d_H \to d_H + \alpha$. Moreover integrals on "net fractals" can be approximated by the left sided RL fractional of a function $L(t($ via

(2.1)
$$\int_0^{\bar{t}} d\rho(t)L(t) \propto {}_0I_{\bar{t}}^{\alpha}L(t) = \frac{1}{\Gamma(t)}\int_0^{\bar{t}} dt(\bar{t}-t)^{\alpha-1}L(t);\ \rho(t) = \frac{\bar{t}^{\alpha}-(\bar{t}-t)^{\alpha}}{\Gamma(\alpha+1)}$$

where α is related to the Hausdorff dimension of the set (cf. [31]. Note that a change of variables $t \to \bar{t} - t$ transforms (2.1) to

$$\frac{1}{\Gamma(\alpha)}\int_0^t dtt^{\alpha-1}L(\bar{t}-t) \tag{2.2}$$

The RL integral above can be mapped into a Weyl integral for $\bar{t} \to \infty$. Assuming $lim_{\bar{t}\to\infty}$ the limit is formal if the Lagrangian L is not autonomous and one assumes therefore that $lim_{\bar{t}\to\infty}L(\bar{t}-t) = L[q(t),\dot{q}(t)]$ (leading to a Stieltjes field theory action). After constructing a "fractional phase space" this analogy confirms the interpretation of the order of the fractional integral as the Hausdorff dimension of the underlying fractal (cf. [35]).

Now for the SE we go to [17, 22, 35, 36]. Thus from [22] (1009.5533) one looks at a Hamiltonian operator

$$H_\alpha(p,r) = D_\alpha|p|^\alpha + V(r)\ (1 < \alpha \le 2) \tag{2.3}$$

When $\alpha = 2$ one has $D_2 = 1/2m$ which gives the standard Hamiltonian operator (**2C**) $\hat{H}(\hat{p},\hat{r}) = (1/2m)\hat{p}^2 + \hat{V}(\hat{r}]$. Thus the fractional QM (FQM) based on the Levy path integral generalizes the standard QM based on the Feynman integral for example. This means that the path integral based on Levy trajectories leads to the fractional SE. For Levy index $\alpha = 2$ the Levy motion becomes Brownian motion so that FQM is well founded. Then via (2.2) one obtains a fractional SE (GSE) in the form

$$i\hbar\partial_t\psi = D_\alpha(-\hbar^2\Delta)^{\alpha/2}\psi + V(r)\psi\ \ (1 < \alpha \le 2) \tag{2.4}$$

with 3D generalization of the fractional quantum Riesz derivative $(-\hbar^2\Delta)^{\alpha/2}$ introduced via

$$(-\hbar^2\Delta)^{\alpha/2}\psi(r,t) = \frac{1}{(2\pi\hbar)^3}\int d^3pe^{\frac{ipr}{\hbar}}|p|^\alpha\phi(p,t) \tag{2.5}$$

where ϕ and ψ are Fourier transforms. The 1D FSE has the form

$$i\hbar\partial_t\psi(x,t) = -D_\alpha(\hbar\nabla)^\alpha\psi + V\psi\ \ (1 < \alpha \le 2) \tag{2.6}$$

The quantum Riesz fractional derivative is defined via

$$(\hbar\nabla)^\alpha\psi(x,t) = -\frac{1}{2pi\hbar}\int_{-\infty}^{\infty} dp\, e^{\frac{ipx}{\hbar}}|p|^\alpha\phi(p,t) \tag{2.7}$$

where

$$\phi(p,t) = \int_{-\infty}^{\infty} dx\, e^{\frac{-ixt}{\hbar}}\psi(x,t) \tag{2.8}$$

with the standard inverse. Evidently (2.6) can be written in operator form as (**2D**) $i\hbar\partial_t\psi = H_\alpha\psi;\ H_\alpha = -D_\alpha(\hbar\nabla)^\alpha + V(x)$

In [17] (0510099) a different approach is used involving the Caputo derivatives (where ${}_c^+D(x)k = 0$ for $k = constant$. Here for (**2E**) $f(kx) = \sum_0^\infty a_n(kx)^{n\alpha}$ one writes $(D \to \bar{D})$

$$ {}_c^+f(kx) = k^\alpha\sum_0^\infty a_{n+1}\frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)}(kx)^{n\alpha} \tag{2.9}$$

Next to extend the definition to negative reals one writes

$$x \to \bar{\chi}(x) = sgn(x)|x|^\alpha;\ \bar{D}(x) = sgn(x){}_c^+D(|x|) \tag{2.10}$$

There is a parity tranformation Π satisfying (**2F**) $\Pi\bar{\chi}(x) = -\bar{\chi}(x)$ and $\Pi\bar{D}(x) = -\bar{D}(x)$. Then one defines (**2G**) $f(\bar{\chi}(kx)) = \sum_0^\infty a_n\bar{\chi}^n(kx)$ with a well defined derivative

$$\bar{D}f(\bar{\chi}(kx)) = sgn(k)|k|^\alpha\sum_0^\infty a_{n+1}\frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)}\bar{\chi}^n(kx) \tag{2.11}$$

This leads to a Hamiltonian H^α with

$$H^\alpha = -\frac{1}{2}mc^2\left(\frac{\hbar}{mc}\right)^{2\alpha}\bar{D}^i\bar{D}_i + V(\hat{X}^1,\cdots,\hat{X}^i,\cdots,\hat{X}^{3N}) \tag{2.12}$$

with a time dependent SE
(2.13)

$$H^\alpha\Psi = \left[-\frac{1}{2}mc^2\left(\frac{\hbar}{mc}\right)^{2\alpha}\bar{D}^i\bar{D}_i + V(\hat{X}^1,\cdots,\hat{X}^i,\cdots\hat{X}^{3N})\right]\Psi = i\hbar\partial_t\Psi$$

3. THE SE WITH α-DERIVATIVE

Now we look at a 1-D SE with α-derivatives $d_\alpha \sim d_\alpha/dx$ (without motivational physics). We write $d_\alpha x^\beta = \beta x^{\beta-\alpha}$ as in (1.9) and posit a candidate SE in the form

$$i\hbar\partial_t\psi = D_\alpha\hbar^2 d_\alpha^2\psi + V(x)\psi \tag{3.1}$$

In [26, 27] for example (cf. also [9]) one deals with a Schrödinger type equation

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi - \frac{\mathcal{W}}{2m}\psi = 0 \tag{3.2}$$

where $\mathcal{D} \sim (\hbar/2m)$ in the quantum situation. Further $\mathcal{D}$ is allowed to have macro values with possible application in biology and cosmology (see Remark 3.1 below).

Consider a possible solution corresponding to $\psi = Rexp(iS/\hbar)$ in the form (**3A**) $\psi = RE_\alpha\ (iS/\hbar)$ with E_α as in (1.20). Then one has for $S = S(x,t)$ (**3B**) $\psi_t = R_tE_\alpha + R\partial_tE_\alpha$ and via (1.15)-(1.16)

$$d_\alpha\left[RE_\alpha\left(\frac{iS}{\hbar}\right)\right] = (d_\alpha R)E_\alpha + RE_\alpha\frac{i}{\hbar}(d_\alpha S) \tag{3.3}$$

$$d_\alpha^2\left[RE_\alpha\left(\frac{iS}{\hbar}\right)\right] = (d_\alpha^2 R)E_\alpha + 2(d_\alpha R)E_\alpha\frac{i}{\hbar}d_\alpha S + \tag{3.4}$$

$$+RE_\alpha(\frac{i}{\hbar}d_\alpha S)^2 + RE_\alpha\frac{i}{\hbar}d_\alpha^2 S$$

$$\partial_t E_\alpha(z) = \partial_t\sum_0^\infty\frac{(z^\alpha/\alpha))^k}{\Gamma(k+1} = \frac{z_t}{\alpha}\sum_1^\infty\frac{(z^\alpha/\alpha)}{\Gamma(k)} = \tag{3.5}$$

$$= \frac{z_t}{\alpha}\sum_0^\infty\frac{(z^\alpha/\alpha)^m}{\Gamma(m+1)} = \frac{z_t}{\alpha}E_\alpha$$

Then from (**3B**), (3.4), (3.3), and (3.5) we combine real and imaginary parts in

$$i\hbar\left[R_tE_\alpha + \frac{iS_t}{\alpha\hbar}RE_\alpha\right] = VRE_\alpha + \tag{3.6}$$

$$D_\alpha\hbar^2\left[(d_\alpha^2R)E_\alpha + 2(d_\alpha R)E_\alpha\frac{i}{\hbar}d_\alpha S - \frac{RSE_\alpha}{\hbar^2}(d_\alpha S)^2 + \frac{iRE_\alpha}{\hbar}d_\alpha^2 S\right]$$

leading to

$$R_tE_\alpha = -2D_\alpha d_\alpha RE_\alpha(d_\alpha S) - D_\alpha RE_\alpha d_\alpha^2 S; \tag{3.7}$$

$$-\frac{1}{\alpha}S_tRE_\alpha = VRE_\alpha + D_\alpha\hbar^2 d_\alpha^2 RE_\alpha - RE_\alpha(d_\alpha S)^2$$

Thus E_α cancels and we have

$$R_t = -2D_\alpha(d_\alpha R)(d_\alpha S) - D_\alpha Rd_\alpha^2 S; \tag{3.8}$$

$$-\frac{1}{\alpha}S_tR = VR + D_\alpha\hbar^2 d_\alpha^2 R - R(d_\alpha S)^2$$

Now recall the classical situation here as (cf. [4, 5])

$$S_t + \frac{S_x^2}{2m} + V - \frac{\hbar^2 R''}{2mR} = 0;\ \partial_t(R^2) + \frac{1}{m}(R^2 S')' = 0 \tag{3.9}$$

This gives an obvious comparison:

(1) Compare $2RR_t + (1/m)(2RR'S' + R^2S'') = 0 \sim 2R_t + (1/m)(2R'S' + RS'') = 0$ with $R_t = -2D_\alpha(d_\alpha R)(d_\alpha S) - D_\alpha R d_\alpha^2 S$
(2) Compare $S_t + (S_x^2/2m) + V - \frac{\hbar^2 R''}{2mR} = 0$ with $-\frac{1}{\alpha}S_t = V - \frac{D_\alpha \hbar^2 d_\alpha^2 R}{R} + (d_\alpha S)^2$

which leads to

THEOREM 3.1. The assumption (3.1) for a 1-D α-derivative Schrödinger type equation leads to a fractional quantum potential

$$Q_\alpha = -\frac{D_\alpha \hbar^2 d_\alpha^2 R}{R} \tag{3.10}$$

For the classical case with $d_\alpha R \sim R'$ (i.e. $\alpha = 1$) one has $D_\alpha = 1/2m$ and one imagines more generally that $D_\alpha \hbar^2$ may have macro values. ■

REMARK 3.1. We note that the techniques of scale relativity (cf. [26, 27] lead to quantum mechanics (QM). In the non-relativistic case the fractal Hausdorff dimension $d_H = 2$ arises and one can generate the standard quantum potential (QP) directly (cf. also [9]). The QP turns out to be a critical factor in understanding QM (cf. [4, 5, 6, 14, 15, 16]) while various macro versions of QM have been suggested in biology, cosmology, etc. (cf. [1, 26, 27, 38, 39]). The sign of the QP serves to distinguish diffusion from an equation with a structure forming energy term (namely QM for $D_\alpha = 1/2m$ and fractal paths of Hausdorff dimension 2). The multi-fractal universe of [2, 3] can involve fractional calculus with various degrees α (i.e. fractals of differing Hausdorff dimension). We have shown that, given a physical input for (3.1) with the α-derivative of Kobelev ([20]), the accompanying α-QP could be related to structure formation in the related theory. ■

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