

THE LOCAL Tb THEOREM WITH ROUGH TEST FUNCTIONS

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ABSTRACT. We prove a version of the local Tb theorem under minimal integrability assumptions, answering a question of S. Hofmann (El Escorial, 2008): Every cube is assumed to support two non-degenerate functions $b_Q^1 \in L^p$ and $b_Q^2 \in L^q$ such that $Tb_Q^1 \in L^{q'}$ and $T^*b_Q^2 \in L^{p'}$, with appropriate uniformity and scaling of the norms. This is sufficient for the L^2 -boundedness of the Calderón–Zygmund operator T , for any $p, q \in (1, \infty)$, a result previously unknown for simultaneously small values of p and q . The proof is based on the technique of suppressed operators from the quantitative Vitushkin conjecture due to Nazarov–Treil–Volberg.

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1. INTRODUCTION

The first Tb theorems were proven by David, Journé and Semmes [5], and McIntosh and Meyer [14]. Their idea was to characterize the L^2 -boundedness of a singular integral operator T ,

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y) dy,$$

where K is a standard Calderón–Zygmund kernel, by its (and its adjoint’s) action just on one sufficiently non-degenerate function b . Thus, they generalized the celebrated $T1$ theorem of David and Journé [6], where this function was required to be $b \equiv 1$.

Another significant step in this type of characterizations was taken by Christ [4], who introduced the idea of a *local Tb theorem*. Rather than testing T and T^* on two globally well-behaved (and hence not so easy to find) functions b_1 and b_2 , the operators can be tested against a family of local functions b_Q^1 and b_Q^2 , indexed by the cubes (say) Q on which they are supported, each of which is only required to satisfy a set of conditions on its ‘own’ cube.

Besides necessary non-degeneracy requirements, Christ’s assumptions on his test functions consisted of the uniform boundedness $b_Q^1, b_Q^2, Tb_Q^1, T^*b_Q^2 \in L^\infty$. Weakening these conditions has been a topic of subsequent developments. Nazarov, Treil and Volberg [15] showed (even in a more general non-doubling context) that it suffices to have $b_Q^1, b_Q^2 \in L^\infty$ and $Tb_Q^1, T^*b_Q^2 \in \text{BMO}$, uniformly in Q . On the other hand, for certain dyadic model operators, Auscher, Hofmann, Muscalu, Tao and Thiele [1] were able to relax these conditions to a substantially lower degree of integrability, namely

$$(1.1) \quad b_Q^1 \in L^p, \quad b_Q^2 \in L^q, \quad Tb_Q^1 \in L^{q'}, \quad T^*b_Q^2 \in L^{p'}$$

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for any $p, q \in (1, \infty)$, where the different L^r norms are appropriately scaled relative to $|Q|$. The question then became whether these testing conditions for the model case also suffice for genuine singular integral operators. This was the first of the four open problems on local Tb theorems formulated by Hofmann during his plenary lectures at the International Conference on Harmonic Analysis and P.D.E. in El Escorial, 2008; it was motivated by possible applications to layer potentials and to free boundary theory (see [8, Section 3.3.1]).

Towards the solution of Hofmann's problem, the following developments have taken place. First, Hofmann's [7] positive result concerning the case $b_Q^i \in L^2$, $T_i b_Q^i \in L^{2+\varepsilon}$. Next, Auscher and Yang's [3] elimination of the $\varepsilon > 0$ by a reduction to the dyadic case. In fact, they settled the result for all 'large enough' pairs of exponents $p, q \in (1, \infty)$, namely, subject to the sub-duality condition $1/p + 1/q \leq 1$. Finally, Auscher and Routin's [2] work on general pairs $p, q \in (1, \infty)$: they gave a direct proof of the sub-duality theorem just mentioned, and obtained a positive result for general exponents under additional side conditions of 'weak boundedness' type (but rather more technical than the usual forms of such assumptions). See also [9, 12, 13] for some related work.

In the paper at hand, we solve Hofmann's problem for all exponents $p, q \in (1, \infty)$. In fact, we are going to view (1.1) as sufficient conditions for another natural set of assumptions stated in terms of the maximal truncated singular integral

$$T_{\#}f(x) := \sup_{\varepsilon > 0} |T_{\varepsilon}f(x)|, \quad T_{\varepsilon}f(x) := \int_{|x-y|>\varepsilon} K(x, y)f(y) dy.$$

We make the assumption that for some $u \in (1, \infty)$, we have

$$(1.2) \quad b_Q^1, \quad b_Q^2, \quad T_{\#}b_Q^1, \quad (T^*)_{\#}b_Q^2 \in L^u,$$

with appropriate uniform scaling. As we will prove, (1.2) for $u < \min\{p, q, p', q'\}$ is a consequence of (1.1). But (1.2) seems more natural in the sense that (unlike (1.1) in the super-duality case $1/p + 1/q > 1$) it is obviously necessary for the L^2 -boundedness of T , which implies the L^u -boundedness of $T_{\#}$ by classical theory. And, we show that (1.2), together with certain necessary off-diagonal estimates, is also a sufficient condition for the L^2 -boundedness, as a proper Tb condition should.

Our method of proving this result is new in the context of Hofmann's problem, although borrowed from other developments in the Tb circle of ideas, in particular, the approach to Vitushkin's conjecture by Nazarov, Treil and Volberg [16]. Namely, we show that it is possible to perturb the rough test functions $b_Q \in L^u$ so as to obtain better functions $\tilde{b}_Q \in L^\infty$, which are still well-behaved under a *suppressed* singular integral

$$T_{\Phi}f(x) := \int_{\mathbb{R}^d} K_{\Phi}(x, y)f(y) dy, \quad K_{\Phi}(x, y) := \frac{|x-y|^{2m}K(x, y)}{|x-y|^{2m} + \Phi(x)^m\Phi(y)^m},$$

for a suitably chosen nonnegative Lipschitz function Φ . We can then run a local Tb argument for the suppressed operator T_{Φ} and the bounded test functions \tilde{b}_Q . Once the boundedness of T_{Φ} has been established, this can be used to construct yet another set of bounded test functions, but now for the original operator T . Another local Tb argument with bounded test functions then allows to deduce the boundedness of T itself.

In the following section, we give a detailed statement of the main theorems and a technical outline of the entire argument, where the main auxiliary propositions are stated without proof. The proofs of these intermediate results are then provided in the subsequent sections.

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2. TECHNICAL OUTLINE

Let T be a linear operator given by

$$(2.1) \quad Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y) \, dy,$$

where K is a Calderón–Zygmund standard kernel:

$$(2.2) \quad |x - y|^d |K(x, y)| + |x - y|^{d+\alpha} \frac{|K(x, y) - K(x', y)|}{|x - x'|^\alpha} + |x - y|^{d+\alpha} \frac{|K(x, y) - K(x, y')|}{|y - y'|^\alpha} \lesssim 1$$

for all x, x', y, y' with $|x - x'| + |y - y'| < \frac{1}{2}|x - y|$ and some fixed $\alpha \in (0, 1]$. For convenience, we assume that K is also bounded, *qualitatively*, so that formulae like (2.1) are meaningful, but this will never be used in the quantitative estimates.

2.3. Definition (Accretive system). Let $p, u \in [1, \infty]$. A (p, u) -accretive system for an operator T is a family of functions b_Q , indexed by all dyadic cubes Q , such that

$$(2.4) \quad \text{supp } b_Q \subseteq Q,$$

$$(2.5) \quad \int_Q b_Q \, dx = 1,$$

$$(2.6) \quad \left(\int_Q |b_Q|^p \, dx \right)^{1/p} \lesssim 1,$$

$$(2.7) \quad \left(\int_Q |Tb_Q|^u \, dx \right)^{1/u} \lesssim 1,$$

with the usual reformulation if p or u is ∞ . We call it a *buffered* (p, u) -accretive system for T if (2.7) is replaced by the stronger condition that

$$(2.8) \quad \left(\int_{2Q} |Tb_Q|^u \, dx \right)^{1/u} \lesssim 1.$$

Solving a problem posed by Hofmann [8, Section 3.3.1], we prove the following:

2.9. Theorem (Solution to Hofmann’s problem). *Let $p, q \in (1, \infty)$. Suppose that there is a buffered (p, q') -accretive system for T , and a buffered (q, p') -accretive system for T^* . Then $\|T\|_{L^2 \rightarrow L^2}$ is bounded by a constant depending only on the implied constants in (2.2), (2.6) and (2.8).*

The first theorem of this flavour was proven for so-called perfect dyadic operators [1]. For Calderón–Zygmund operators, prior to our work, it was known in the subduality case: $1/p + 1/q \leq 1$ [2, 3]. For $1/p + 1/q > 1$, it had only been verified under additional technical assumptions [2].

2.10. Remark. In the subduality case, a (p, q') -accretive system is automatically buffered, i.e., (2.8) already follows from the other conditions by the following Hardy inequality:

$$\int_{3Q \setminus Q} \left(\int_Q \frac{|f(y)|}{|x - y|^d} \, dy \right)^u \, dx \lesssim \|1_Q f\|_{L^u}^u, \quad u \in (1, \infty).$$

Whether one can remove the word ‘buffered’ from Theorem 2.9 for general exponents remains open.

We deduce Theorem 2.9 as a corollary to a variant, where (2.8) is replaced by an assumption on the maximal truncated operator

$$T_{\#} f(x) := \sup_{\varepsilon > 0} |T_{\varepsilon} f(x)|, \quad T_{\varepsilon} f(x) := \int_{|x - y| > \varepsilon} K(x, y)f(y) \, dy.$$

Let us first observe that the above conditions on T imply certain conditions on $T_{\#}$, as defined next:

2.11. Definition (Ample collection; off-diagonal estimates). We say that \mathcal{Q} is an *ample collection* of dyadic subcubes of a given cube Q with *exceptional fraction* $\sigma \in (0, 1)$, if the maximal subcubes $\tilde{Q} \subseteq Q$ with $\tilde{Q} \notin \mathcal{Q}$ satisfy the estimate $\sum |\tilde{Q}| \leq \sigma|Q|$.

We say that an accretive system b_Q for an operator T satisfies *off-diagonal estimates* if for all Q and $\sigma > 0$, there exists a bound C_σ (independent of Q) so that

$$(2.12) \quad \int_{Q'} |T(1_{(3Q')^c} b_Q)| dx \leq C_\sigma,$$

for all Q' in an ample collection of dyadic subcubes of Q with exceptional fraction σ .

2.13. Proposition. *Suppose that b_Q^1 is a buffered (p, q') accretive system for T , and b_Q^2 is a buffered (q, p') accretive system for T^* . Then b_Q^1 is also a (buffered) (p, u) accretive system for $T_\#$, and b_Q^2 is a (buffered) (q, v) accretive system for $(T^*)_\#$, for any $u < \min\{p, q'\}$ and any $v < \min\{q, p'\}$.*

Moreover, the system b_Q^1 satisfies off-diagonal estimates for $T_\#$, and the system b_Q^2 for $(T^)_\#$.*

Since $u < p$, the (p, u) accretive system b_Q^1 is automatically buffered by Hardy's inequality; similarly for b_Q^2 . Thanks to Proposition 2.13, and observing that a (p, u) accretive system is also a (t, t) accretive system for $t \leq \min\{p, u\}$, Theorem 2.9 is a consequence of our main result which reads as follows:

2.14. Theorem (Main theorem). *Let T be a Calderón–Zygmund operator, and suppose that for some $p \in (1, \infty)$ there exist (p, p) -accretive systems with off-diagonal estimates, b_Q^1 for $T_\#$ and b_Q^2 for $(T^*)_\#$. Then $\|T\|_{L^2 \rightarrow L^2}$ is bounded by a constant depending only on the implied constants in (2.2), (2.6), (2.7) and (2.12) with $T_\#$ and b_Q^1 , or $(T^*)_\#$ and b_Q^2 , in place of T and b_Q .*

If T is antisymmetric, the assumption “with off-diagonal estimates” can be dropped.

By antisymmetric, we mean that $T^* = -T$, or in terms of the kernel, that $K(y, x) = -K(x, y)$. This is a slightly easier case than the general one as it makes the off-diagonal estimates redundant. This has no consequence for the solution of Hofmann's problem in Theorem 2.9, since the off-diagonal estimates are implied by its assumptions in any case.

We will consider the antisymmetric case on the side of the general one, pointing out simplifications at selected places. Observe that whenever T is antisymmetric, an accretive system for T (or $T_\#$) is automatically an accretive system for T^* (or $(T^*)_\#$) as well. Thus, when discussing the antisymmetric case, it is always understood that the two accretive systems b_Q^1 and b_Q^2 coincide.

2.15. Remark (Necessity of the conditions). As in the usual Tb theorems, the assumptions of Theorem 2.14 are also necessary. Namely, if T is L^2 -bounded, it follows from standard theory that the maximal truncation $T_\#$ is L^p -bounded for all $p \in (1, \infty)$. Thus, for an L^2 -bounded T , any function b_Q with properties (2.4) and (2.6) will also satisfy (2.7) for $T_\#$ in place of T . For the off-diagonal estimates, we can take the ample collection for a cube Q to consist of all $Q' \subseteq Q$ with $\int_{Q'} [M_p b_Q^1 + T_\# b_Q^1] dx \lesssim 1$ (where, adjusting the implicit constant, the exceptional fraction σ can be forced as small as we like), for then

$$\int_{Q'} T_\#(1_{(3Q')^c} b_Q^1) dx \lesssim \int_{Q'} T_\# b_Q^1 dx + \int_{Q'} T_\#(1_{3Q'} b_Q^1) dx \lesssim 1 + \frac{1}{|Q'|^{1/p}} \|1_{3Q'} b_Q^1\|_p \lesssim 1$$

by Hölder's inequality and the L^p boundedness of $T_\#$.

Let us then discuss the proof of Theorem 2.14. It consists of a reduction to the easier case of Hofmann's conjecture with the help of so-called suppressed operators. For any nonnegative function Φ with Lipschitz constant 1, we define

$$T_\Phi f(x) := \int K_\Phi(x, y) f(y) dy, \quad K_\Phi(x, y) := \frac{|x - y|^{2m} K(x, y)}{|x - y|^{2m} + \Phi(x)^m \Phi(y)^m},$$

where $m \geq d/2$ is fixed. Note that $T_\Phi f = Tf$ if $\text{supp } f \subseteq \{\Phi = 0\}$.

Given the assumptions on $T_\#$, our goal is to construct a better behaved accretive system for the suppressed operator T_Φ . To achieve this, we need to relax the notion of an accretive system a little, so as not to demand the supply of test functions for every cube, but only an appropriate subcollection of them.

2.16. Definition (Accretive system on a sparse family; special off-diagonal estimates). Let Q_0 be a cube. A *sparse family* of dyadic subcubes of Q_0 is a collection \mathcal{Q} , containing Q_0 , such that for some $\tau > 0$ and for all $Q \in \mathcal{Q}$ we have

$$\left| \bigcup_{\substack{\tilde{Q} \in \mathcal{Q} \\ \tilde{Q} \subsetneq Q}} \tilde{Q} \right| \leq (1 - \tau)|Q|,$$

i.e., for all $Q \in \mathcal{Q}$, the family $\{Q\} \cup \{Q' \subsetneq Q : Q' \notin \mathcal{Q}\}$ is an ample collection with exceptional fraction $1 - \tau$.

A (p, u) -*accretive system for T on sparse subcubes of Q_0* is a family of functions b_Q , indexed by a sparse family \mathcal{Q} of dyadic subcubes of Q_0 , with the properties (2.4) the following strengthening of (2.5), (2.6) and (2.7):

$$\begin{aligned} \left| \int_{Q'} b_Q \, dx \right| &\gtrsim 1, \\ \left(\int_{Q'} |b_Q|^p \, dx \right)^{1/p} &\lesssim 1, \\ \left(\int_{Q'} |Tb_Q|^u \, dx \right)^{1/u} &\lesssim 1, \end{aligned}$$

whenever $Q' \subseteq Q \in \mathcal{Q}$ is a dyadic subcube, which is not contained in any smaller $\tilde{Q} \subsetneq Q$ with $\tilde{Q} \in \mathcal{Q}$.

If, for all $Q \in \mathcal{Q}$ and all $Q' \subseteq Q$ as before (which form an ample collection of subcubes of Q , with exceptional fraction $1 - \tau$), there also holds

$$\int_{Q'} |T(1_{(3Q')^c} b_Q)| \, dx \lesssim 1,$$

the system of functions b_Q is said to satisfy *special off-diagonal estimates*.

2.17. Proposition. *Suppose that there is a (p, p) accretive system for $T_\#$. Then, for a fixed $\varrho \in (0, 1)$ and any cube Q_0 , there exists a nonnegative function Φ with Lipschitz constant 1 such that*

$$|\{\Phi > 0\}| \leq \varrho|Q_0|,$$

and there exists an (∞, p) accretive system for T_Φ on sparse subcubes of Q_0 . If the accretive system for $T_\#$ has off-diagonal estimates, then the system for T_Φ can be arranged to have special off-diagonal estimates.

Starting from two (p, p) accretive systems with off-diagonal estimates, b_Q^1 for $T_\#$ and b_Q^2 for $(T^*)_\#$, this gives us two Lipschitz functions Φ_1 and Φ_2 , and two (∞, p) accretive systems on sparse subcubes with special off-diagonal estimates — \tilde{b}_Q^1 for T_{Φ_1} and \tilde{b}_Q^2 for $T_{\Phi_2}^*$ —, but we may arrange the construction so that $\Phi_1 = \Phi_2 =: \Phi$.

Given an accretive system for T on all dyadic subcubes of Q_0 , it follows from a standard stopping time argument that we can extract a subsystem, which is an accretive system for T on a sparse family of subcubes of Q_0 . This is typically one of the first steps in the proof of a local Tb theorem; see [15], for instance. For us, it will be important that it is actually enough to only have an accretive system for the sparse subcubes from the beginning:

2.18. Proposition (Baby Tb theorem). *Let T be an operator with Calderón–Zygmund kernel, let Q_0 be a cube, and suppose that there are (∞, t) accretive systems b_Q^1 for T and b_Q^2 for T^* , on sparse subcubes \mathcal{Q}_1 and \mathcal{Q}_2 of Q_0 , respectively. Assume, moreover, the following weak boundedness property:*

$$(2.19) \quad |\langle T(1_Q b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle| \lesssim |Q|,$$

whenever Q is a dyadic subcube of Q_0 and $Q^{a,i}$ is the minimal member of \mathcal{Q}_i which contains Q . Then

$$|\langle Tf, g \rangle| \lesssim \|f\|_{s'} \|g\|_{s'} |Q_0|^{1-2/s'} \quad s' \in (\max\{t', 2\}, \infty],$$

for all $f, g \in L^{s'}(Q_0)$ ($L^{s'}$ functions supported on Q_0).

Note that (∞, t) accretive systems are in particular (r, r') accretive systems for $r = \max\{4, t'\} < \infty$, $r' = \min\{\frac{4}{3}, t\} \leq t$. Hence, in principle, we are in the setting of Hofmann's conjecture in the known case that $1/p + 1/q = 2/r \leq 1/2 < 1$. But contrary to the usual, we only assume an accretive system on sparse subcubes of Q_0 .

For the verification of the technical hypothesis (2.19), we have the following.

2.20. Proposition. *Under the assumptions of Proposition 2.18, the weak boundedness property (2.19) holds under any one of the following additional sets of hypotheses:*

- *T is antisymmetric, and the accretive systems $\{b_Q^1 : Q \in \mathcal{Q}_1\}$ and $\{b_Q^2 : Q \in \mathcal{Q}_2\}$ coincide,*
- *or the two accretive systems on sparse subcubes also satisfy special off-diagonal estimates,*
- *or the two accretive systems on sparse subcubes are restrictions of accretive systems on all dyadic cubes, by a usual stopping time construction.*

Assuming all the auxiliary results formulated above, we can now give:

Proof of Theorem 2.14. By assumption, for some $p \in (1, \infty)$, there are two (p, p) accretive systems of functions, b_Q^1 for $T_\#$ and b_Q^2 for $(T^*)_\#$, with off-diagonal estimates (or, alternatively, just one system b_Q^1 for $T_\#$, where T is antisymmetric). Without loss of generality, we may assume that $p \in (1, 2)$.

Fix a cube Q_0 . Then, by Proposition 2.17, there exists a nonnegative function Φ with Lipschitz constant 1 such that

$$(2.21) \quad |\{\Phi > 0\}| \leq \rho |Q_0|,$$

for some fixed $\rho \in (0, 1)$ (independent of Q_0), and there exist (∞, p) accretive systems \tilde{b}_Q^1 for T_Φ and \tilde{b}_Q^2 for T_Φ^* on sparse subcubes of Q_0 . Moreover, we either have T_Φ antisymmetric (if T is), or the new accretive systems satisfy special off-diagonal estimates.

By Proposition 2.20 (either the antisymmetric case, or the case of special off-diagonal estimates), the operator T_Φ and these new accretive systems satisfy the weak boundedness property (2.19). Thus Proposition 2.18, applied to T_Φ in place of T , implies that

$$(2.22) \quad |\langle T_\Phi f, g \rangle| \lesssim \|f\|_{s'} \|g\|_{s'} |Q_0|^{1-2/s'} \quad s' \in (p', \infty],$$

for all $f, g \in L^s(Q_0)$.

Let

$$b_{Q_0} := \frac{|Q_0|}{|Q_0 \cap \{\Phi = 0\}|} 1_{Q_0 \cap \{\Phi = 0\}}.$$

By (2.21), we have $|Q_0|/|Q_0 \cap \{\Phi = 0\}| \lesssim 1$, and hence

$$\int_{Q_0} b_{Q_0} dx = 1, \quad \|b_{Q_0}\|_\infty \lesssim 1.$$

Since $\text{supp } b_{Q_0} \subseteq \{\Phi = 0\}$, we have $T_\Phi b_{Q_0} = T b_{Q_0}$ and likewise $T_\Phi^* b_{Q_0} = T^* b_{Q_0}$. By an application of (2.22) to $f = b_{Q_0}$ and an arbitrary $g \in L^{s'}(Q_0)$ of norm 1, we deduce that

$$\left(\int_{Q_0} |T b_{Q_0}|^s dx \right)^{1/s} \lesssim |Q_0|^{1/s'} \cdot 1 \cdot |Q_0|^{1-2/s'} = |Q_0|^{1/s}.$$

Similarly, applying (2.22) to $g = b_{Q_0}$ and an arbitrary $f \in L^{s'}(Q_0)$ of norm 1, we obtain

$$\left(\int_{Q_0} |T^* b_{Q_0}|^s dx \right)^{1/s} \lesssim |Q_0|^{1/s}.$$

The above reasoning applies to any cube Q in place of Q_0 . Hence, for every Q , there exists a function b_Q with

$$\text{supp } b_Q \subseteq Q, \quad \int_Q b_Q dx = 1, \quad \|b_Q\|_\infty \lesssim 1, \quad \int_Q |T b_Q|^s dx + \int_Q |T^* b_Q|^s dx \lesssim 1.$$

In other words, there exists an (∞, s) accretive system for the original operator T and its adjoint T^* on all dyadic cubes. By a standard stopping time construction, for any Q_0 , we can extract (∞, s) accretive systems for T and T^* on sparse subcubes of Q_0 . By Proposition 2.20 (the case

of stopping time restrictions of accretive systems on all dyadic cubes), the weak boundedness property (2.19) holds for T and these accretive systems

Another application of Proposition 2.18, to the operator T itself, shows that

$$|\langle Tf, g \rangle| \lesssim \|f\|_{r'} \|g\|_{r'} |Q_0|^{1-2/r'} \quad r' \in (s', \infty],$$

for all $f, g \in L^{r'}(Q_0)$, for any cube Q_0 . We apply this to $f = 1_{Q_0}$ and an arbitrary $g \in L^{r'}(Q_0)$ of norm 1, and to $g = 1_{Q_0}$ and an arbitrary $f \in L^{r'}(Q_0)$ of norm 1, to deduce that

$$\left(\int_{Q_0} |T1_{Q_0}|^r dx \right)^{1/r} \lesssim |Q_0|^{1/r}, \quad \left(\int_{Q_0} |T^*1_{Q_0}|^r dx \right)^{1/r} \lesssim |Q_0|^{1/r}.$$

But this brings us to the setting of the well-known standard local $T1$ theorem, which gives us the desired bound $\|T\|_{L^2 \rightarrow L^2} \lesssim 1$. This completes the proof. \square

3. PREPARATORY ESTIMATES; PROOF OF PROPOSITION 2.13

We show how to obtain the testing conditions for the maximal truncated operator $T_{\#}$ from testing conditions for T , and provide some auxiliary results on the suppressed operators T_{Φ} for the subsequent sections.

3.1. Lemma. *If b_Q^1 is a buffered (p, u) accretive system for T , then (2.8) improves to the global estimate*

$$\|Tb_Q^1\|_{L^u} \lesssim |Q|^{1/u}.$$

Proof. By (2.8), it only remains to estimate $1_{(2Q)^c} Tb_Q^1$. But

$$\begin{aligned} \|1_{(2Q)^c} Tb_Q^1\|_{L^u} &= \left\| x \mapsto \int_Q K(x, y) b_Q^1(y) dy \right\|_{L^u((2Q)^c)} \\ &\leq \int_Q \|x \mapsto K(x, y)\|_{L^u((2Q)^c)} |b_Q^1(y)| dy \lesssim \int_Q |Q|^{-1/u'} |b_Q^1(y)| dy \lesssim |Q|^{1/u}, \end{aligned}$$

by a straightforward estimate of the $L^u((2Q)^c)$ norm of $x \mapsto |K(x, y)| \lesssim |x - y|^{-d}$ for $x \in Q$. \square

3.A. Maximal truncations. We have the following version of Cotlar's lemma:

3.2. Lemma. *Suppose that there is a buffered (q, v) accretive system for T^* . Then*

$$T_{\#} f \lesssim M_{q'}(Tf) + M_{v'} f.$$

Proof. Fix x_0 and $\varepsilon > 0$. Let Q be the unique dyadic cube containing x_0 and of diameter $\varepsilon/8$; thus $B(x_0, c_d \varepsilon) \subset 2Q \subset B(x_0, \varepsilon)$. For all $x \in Q$, we have

$$T_{\varepsilon} f(x_0) = T_{\varepsilon} f(x_0) - T(f1_{(2Q)^c})(x) + Tf(x) - T(f1_{2Q})(x),$$

where

$$\begin{aligned} |T_{\varepsilon} f(x_0) - T(f1_{(2Q)^c})(x)| &= \left| \int_{|y-x_0|>\varepsilon} K(x_0, y) f(y) dy - \int_{(2Q)^c} K(x, y) f(y) dy \right| \\ &\leq \int_{(2Q)^c} |K(x_0, y) - K(x, y)| |f(y)| dy + \int_{\substack{|y-x_0| \leq \varepsilon \\ y \in (2Q)^c}} |K(x_0, y)| |f(y)| dy \\ &\lesssim \int_{|y-x_0|>c_d \varepsilon} \frac{\varepsilon^\alpha}{|y-x_0|^{d+\alpha}} |f(y)| dy + \int_{c_d \varepsilon < |y-x_0| < \varepsilon} \frac{1}{\varepsilon^d} |f(y)| dy \lesssim Mf(x_0). \end{aligned}$$

Thus

$$\begin{aligned}
T_\varepsilon f(x_0) &= \int_Q b_Q^2 dx \cdot T_\varepsilon f(x_0) \\
&= \mathcal{O}(Mf(x_0)) + \int_Q b_Q^2 \cdot Tf dx - \frac{1}{|Q|} \int T^* b_Q^2 \cdot f 1_{2Q} dx \\
&\lesssim Mf(x_0) + \left(\int_Q |b_Q^2|^q \right)^{1/q} \cdot \left(\int_Q |Tf|^{q'} dx \right)^{1/q'} \\
&\quad + \left(\int_{2Q} |T^* b_Q^2|^v dx \right)^{1/v} \left(\int_{2Q} |f|^{v'} dx \right)^{1/v'} \\
&\lesssim Mf(x_0) + M_{q'}(Tf)(x_0) + M_{v'} f(x_0). \quad \square
\end{aligned}$$

3.3. Lemma. *Let b_Q^1 be a buffered (p, q') accretive system for T , and b_Q^2 a buffered (q, p') accretive system for T^* . Then*

$$|Q|^{-1/u} \|1_{2Q} T_\# b_Q^1\|_{L^u} \lesssim |Q|^{-1/s} \|1_{2Q} T_\# b_Q^1\|_{L^{s,\infty}} \lesssim 1, \quad u < s := \min\{p, q'\}.$$

In particular, b_Q^1 is a buffered (p, u) accretive system for $T_\#$ for all $u < \min\{p, q'\}$.

Proof. We apply Lemma 3.2, the quasi-triangle inequality and the monotonicity in the exponent of the weak norms, the boundedness of the maximal operators $M_p : L^p \rightarrow L^{p,\infty}$ and $M_{q'} : L^{q'} \rightarrow L^{q',\infty}$, Lemma 3.1, and finally the assumptions on b_Q^1 to deduce

$$\begin{aligned}
|Q|^{-1/s} \|1_{2Q} T_\# b_Q^1\|_{L^{s,\infty}} &\lesssim |Q|^{-1/s} \|1_{2Q} [M_{q'}(Tb_Q^1) + M_p b_Q^1]\|_{L^{s,\infty}} \\
&\lesssim |Q|^{-1/q'} \|1_{2Q} M_{q'}(Tb_Q^1)\|_{L^{q',\infty}} + |Q|^{-1/p} \|1_{2Q} M_p b_Q^1\|_{L^{p,\infty}} \\
&\lesssim |Q|^{-1/q'} \|Tb_Q^1\|_{L^{q'}} + |Q|^{-1/p} \|b_Q^1\|_{L^p} \lesssim 1. \quad \square
\end{aligned}$$

Note that we have now completed the proof of the first part of Proposition 2.13.

3.B. Off-diagonal estimates. We turn to the off-diagonal estimate as formulated in (2.12). We first study this estimate for T itself, and then for the maximal truncation $T_\#$.

3.4. Lemma. *Let b_Q^1 be a buffered (p, q') accretive system for T , and b_Q^2 a buffered (q, p') accretive system for T^* . Let $Q' \subseteq Q$ be a dyadic subcube such that*

$$(3.5) \quad \int_{Q'} M_p b_Q^1 dx + \int_{Q'} M_{q'}(Tb_Q^1) dx \lesssim 1.$$

Then

$$\sup_{Q'} |T(1_{(3Q')^c} b_Q^1)| \lesssim 1.$$

Proof. Let $x \in Q'$. Then

$$\begin{aligned}
T(1_{(3Q')^c} b_Q^1)(x) &= \int_{Q'} [T(1_{(3Q')^c} b_Q^1)(x) - T(1_{(3Q')^c} b_Q^1)(y)] b_{Q'}^2(y) dy \\
&\quad + \frac{1}{|Q'|} \langle T(1_{(3Q')^c} b_Q^1), b_{Q'}^2 \rangle \\
&= \int_{Q'} \int_{(3Q')^c} [K(x, z) - K(y, z)] b_Q^1(z) dz b_{Q'}^2(y) dy \\
&\quad + \frac{1}{|Q'|} \langle T(b_Q^1), b_{Q'}^2 \rangle - \frac{1}{|Q'|} \langle 1_{3Q'} b_Q^1, T^* b_{Q'}^2 \rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
|T(1_{(3Q')^c} b_Q^1)(x)| &\lesssim \int_{Q'} \int_{(3Q')^c} \frac{\ell(Q)^\alpha}{|z-x|^{d+\alpha}} |b_Q^1(z)| dz |b_{Q'}^2(y)| dy \\
&\quad + \left(\int_{Q'} |Tb_Q^1|^{q'} \right)^{1/q'} \left(\int_{Q'} |b_{Q'}^2|^q \right)^{1/q} \\
&\quad + \left(\int_{3Q'} |b_Q^1|^p \right)^{1/p} \left(\int_{3Q'} |T^* b_{Q'}^2|^{p'} \right)^{1/p'} \\
&\lesssim \int_{Q'} \inf_Q Mb_Q^1 \cdot |b_{Q'}^2(y)| dy + 1 \lesssim \int_{Q'} Mb_Q^1 + 1 \lesssim 1. \quad \square
\end{aligned}$$

3.6. Lemma. *Let b_Q^1 be a buffered (p, q') accretive system for T , and b_Q^2 a buffered (q, p') accretive system for T^* . Let $Q' \subseteq Q$ be a dyadic subcube such that (3.5) holds. Then*

$$(3.7) \quad \int_{Q'} |T_{\#}(1_{(3Q')^c} b_Q^1)| dx \lesssim 1.$$

Proof. Let $x \in Q'$ and $\varepsilon > 0$. Then

$$(3.8) \quad T_\varepsilon(1_{(3Q')^c} b_Q^1)(x) = \begin{cases} T(1_{(3Q')^c} b_Q^1)(x), & \text{if } \varepsilon < \ell(Q'), \\ T(1_{B(x, \varepsilon)^c \cap (3Q')^c} b_Q^1)(x), & \text{if } \ell(Q') \leq \varepsilon \leq c_d \ell(Q'), \\ T_\varepsilon b_Q^1(x), & \text{if } \varepsilon > c_d \ell(Q'). \end{cases}$$

For $\ell(Q') \leq \varepsilon \leq c_d \ell(Q')$, we can argue as in the previous Lemma 3.4 with $B(x, \varepsilon)^c \cap (3Q')^c$ in place of $(3Q')^c$ to find that

$$\begin{aligned}
T(1_{B(x, \varepsilon)^c \cap (3Q')^c} b_Q^1)(x) &= \int_{Q'} \int_{B(x, \varepsilon)^c \cap (3Q')^c} [K(x, z) - K(y, z)] b_Q^1(z) dz b_{Q'}^2(y) dy \\
&\quad + \frac{1}{|Q'|} \langle T(b_Q^1), b_{Q'}^2 \rangle - \frac{1}{|Q'|} \langle 1_{B(x, \varepsilon) \cup 3Q'} b_Q^1, T^* b_{Q'}^2 \rangle,
\end{aligned}$$

where the second term is exactly as in Lemma 3.4, and the first term has the same bound as there, since it was dominated with absolute values inside, and we now integrate over a smaller set. For the last term, since $\varepsilon \leq c_d \ell(Q)$ and hence $B(x, \varepsilon) \subseteq C_d Q$, we have

$$\frac{1}{|Q'|} |\langle 1_{B(x, \varepsilon) \cup 3Q'} b_Q^1, T^* b_{Q'}^2 \rangle| \leq \left(\int_{C_d Q'} |b_Q^1|^p \right)^{1/p} \left(\int_{C_d Q'} |T^* b_{Q'}^2|^{p'} \right)^{1/p'} \lesssim 1.$$

Thus, using Lemma 3.4 directly for $\varepsilon \leq \ell(Q')$ and the above modification for $\varepsilon \approx \ell(Q')$, we find from (3.8) that

$$T_{\#}(1_{(3Q')^c} b_Q^1)(x) \lesssim 1 + T_{\#} b_Q^1(x) \quad \forall x \in Q'.$$

Hence

$$\int_{Q'} T_{\#}(1_{(3Q')^c} b_Q^1) dx \lesssim 1 + \int_{Q'} T_{\#} b_Q^1 dx \lesssim 1 + \int_{Q'} [M_p b_Q^1 + M_{q'}(Tb_Q^1)] dx \lesssim 1. \quad \square$$

Completion of the Proof of Proposition 2.13. It remains only to verify the off-diagonal estimates. To this end, take a $\lambda > 0$, and consider the maximal dyadic cubes $Q' \subseteq Q$ which violate the condition (3.5) with implies constant λ . Then

$$\begin{aligned}
\sum |Q'| &\leq |\{M^d[1_Q(M_p b_Q^1 + M_{q'}(Tb_Q^1))] > \lambda\}| \\
&\leq \frac{1}{\lambda} \int_Q [M_p b_Q^1 + M_{q'}(Tb_Q^1)] dx \\
&\lesssim \frac{1}{\lambda} \left[|Q|^{1/p'} \|M_p b_Q^1\|_{L^{p, \infty}} + |Q|^{1/q} \|M_{q'}(Tb_Q^1)\|_{L^{q', \infty}} \right] \\
&\lesssim \frac{1}{\lambda} \left[|Q|^{1/p'} \|b_Q^1\|_p + |Q|^{1/q} \|Tb_Q^1\|_{q'} \right] \\
&\lesssim \frac{|Q|}{\lambda} \left[\left(\int_Q |b_Q^1|^p dx \right)^{1/p} + \left(\frac{1}{|Q|} \int |Tb_Q^1|^{q'} dx \right)^{1/q'} \right] \lesssim \frac{|Q|}{\lambda}.
\end{aligned}$$

Thus, by Lemma 3.6, the inequality (3.7) holds for all Q' in an ample collection of dyadic subcubes of Q , where the exceptional fraction C/λ can be made as small as desired, in accordance with the definition of off-diagonal estimates. \square

4. MODIFIED TEST FUNCTIONS; PROOF OF PROPOSITION 2.17

Given a family of rough test functions for $T_{\#}$, we want to construct better test functions for T_{Φ} , where Φ is suitably chosen. We start with some preparations.

4.A. Generalities on suppressed operators.

4.1. Lemma. *For any nonnegative function Φ with Lipschitz constant 1, we have*

$$|T_{\Phi}f| \lesssim T_{\#}f + Mf.$$

Proof.

$$\begin{aligned} T_{\Phi}f(x) &= \int K_{\Phi}(x, y)f(y) \, dy = \int_{|x-y| \leq \frac{1}{2}\Phi(x)} K_{\Phi}(x, y)f(y) \, dy \\ &\quad + \int_{|x-y| > \frac{1}{2}\Phi(x)} (K_{\Phi}(x, y) - K(x, y))f(y) \, dy + T_{\frac{1}{2}\Phi(x)}f(x) \end{aligned}$$

For $|x - y| \leq \frac{1}{2}\Phi(x)$, we have $\Phi(y) \geq \Phi(x) - |x - y| \geq \frac{1}{2}\Phi(x)$, hence

$$|K_{\Phi}(x, y)| \leq |K(x, y)| \frac{|x - y|^{2m}}{(\frac{1}{2}\Phi(x)^2)^m} \lesssim \frac{|x - y|^{2m-d}}{\Phi(x)^{2m}},$$

and thus

$$\begin{aligned} \left| \int_{|x-y| \leq \frac{1}{2}\Phi(x)} K_{\Phi}(x, y)f(y) \, dy \right| &\lesssim \sum_{j=1}^{\infty} \int_{2^{-j-1}\Phi(x) < |x-y| \leq 2^{-j}\Phi(x)} \frac{(2^{-j}\Phi(x))^{2m-d}}{\Phi(x)^{2m}} |f(y)| \, dy \\ &\lesssim \sum_{j=1}^{\infty} 2^{-2jm} \int_{|x-y| \leq 2^{-j}\Phi(x)} |f(y)| \, dy \lesssim Mf(x). \end{aligned}$$

And for $|x - y| > \frac{1}{2}\Phi(x)$, we have $\Phi(y) \leq \Phi(x) + |x - y| < 3|x - y|$, so that

$$|K_{\Phi}(x, y) - K(x, y)| = |K(x, y)| \frac{(\Phi(x)\Phi(y))^m}{|x - y|^{2m} + (\Phi(x)\Phi(y))^m} \lesssim \frac{1}{|x - y|^d} \frac{\Phi(x)^m |x - y|^m}{|x - y|^{2m}},$$

and hence

$$\begin{aligned} &\left| \int_{|x-y| > \frac{1}{2}\Phi(x)} (K_{\Phi}(x, y) - K(x, y))f(y) \, dy \right| \\ &\lesssim \sum_{j=0}^{\infty} \int_{2^{j-1}\Phi(x) < |x-y| \leq 2^j\Phi(x)} \frac{\Phi(x)^m}{(2^j\Phi(x))^{d+m}} |f(y)| \, dy \\ &\lesssim \sum_{j=0}^{\infty} 2^{-jm} \int_{|x-y| \leq 2^j\Phi(x)} |f(y)| \, dy \lesssim Mf(x). \end{aligned}$$

Finally, it is clear that $|T_{\frac{1}{2}\Phi(x)}f(x)| \leq T_{\#}f(x)$. \square

4.2. Lemma. *Let b_Q^1 be a (buffered) (p, u) accretive system for $T_{\#}$, where $p > 1$. Then it is also a (buffered) (p, s) accretive system for any T_{Φ} , where $s = \min(p, u)$.*

Proof. Let $\alpha \in \{1, 2\}$ according to whether the system is buffered ($\alpha = 2$) or not ($\alpha = 1$). By Lemma 4.1 and the boundedness of M on L^p , we have

$$\begin{aligned} |Q|^{-1/s} \|1_{\alpha Q} T_{\Phi} b_Q^1\|_{L^s} &\lesssim |Q|^{-1/s} \|1_{\alpha Q} [T_{\#} b_Q^1 + M b_Q^1]\|_{L^s} \\ &\leq |Q|^{-1/u} \|1_{\alpha Q} T_{\#} b_Q^1\|_{L^u} + |Q|^{-1/p} \|1_{\alpha Q} M b_Q^1\|_{L^p} \lesssim 1. \end{aligned} \quad \square$$

4.B. First step of the modification and key estimates. We turn to the actual construction of the modified test functions \tilde{b}_Q^1 .

Consider a fixed cube Q_0 , and abbreviate $b := b_{Q_0}^1$. Let $\mathcal{B}_1 = \mathcal{B}_1(Q_0)$ be the maximal dyadic subcubes $Q \subseteq Q_0$ with $f_Q |b|^p \gg 1$, and $\Omega := \bigcup_{Q \in \mathcal{B}_1} Q$. Let

$$(4.3) \quad \tilde{b} := \tilde{b}_{Q_0}^1 := 1_{\Omega^c} b + \sum_{Q \in \mathcal{B}_1} \langle b \rangle_Q 1_Q = b - \sum_{Q \in \mathcal{B}_1} (b - \langle b \rangle_Q) 1_Q =: b - \sum_{Q \in \mathcal{B}_1} d_Q$$

be the good part of the usual Calderón–Zygmund decomposition of b .

4.4. Lemma. *If Φ is a 1-Lipschitz function with*

$$(4.5) \quad \Phi(x) \geq \sup_{Q \in \mathcal{B}_1} \text{dist}(x, (3Q)^c),$$

then

$$\begin{aligned} |T_\Phi(b - \tilde{b})(x)| &\leq \sum_{Q \in \mathcal{B}_1} |T_\Phi d_Q(x)| \lesssim \sum_{Q \in \mathcal{B}_1} \left(\frac{\ell(Q)}{\ell(Q) + |x - c_Q|} \right)^{d+\alpha} \\ &=: \sum_{Q \in \mathcal{B}_1} \phi_Q(x) =: e_{Q_0}^1(x), \end{aligned}$$

where for all $u \in [1, \infty)$,

$$\|e_{Q_0}^1\|_{L^u} \lesssim |Q_0|^{1/u}.$$

Proof. For $x \in (2Q)^c$,

$$|T_\Phi d_Q(x)| = \left| \int_Q [K_\Phi(x, y) - K_\Phi(x, c_Q)] d_Q(y) dy \right| \lesssim \frac{\ell(Q)^\alpha}{|x - c_Q|^{d+\alpha}} \|d_Q\|_1 \lesssim \left(\frac{\ell(Q)}{|x - c_Q|} \right)^{d+\alpha}.$$

For $x \in 2Q, y \in Q$, we have $\text{dist}(x, (3Q)^c), \text{dist}(y, (3Q)^c) \gtrsim \ell(Q)$, hence $\Phi(x)\Phi(y) \gtrsim \ell(Q)^2$, and therefore

$$(4.6) \quad |K_\Phi(x, y)| \lesssim \frac{|x - y|^{2m-d}}{\ell(Q)^{2m}} \lesssim \frac{1}{\ell(Q)^d}$$

provided that $m \geq d/2$. Thus

$$|T_\Phi d_Q(x)| = \left| \int_Q K_\Phi(x, y) d_Q(y) dy \right| \lesssim \frac{1}{|Q|} \int_Q |d_Q(y)| dy \lesssim 1.$$

Altogether we have

$$|T_\Phi d_Q(x)| \leq \left(\frac{\ell(Q)}{\ell(Q) + |x - c_Q|} \right)^{d+\alpha} =: \phi_Q(x).$$

By duality, for a suitable $g \geq 0$ with $\|g\|_{u'} = 1$,

$$\left\| \sum_{Q \in \mathcal{B}_1} \phi_Q \right\|_u = \int g \sum_{Q \in \mathcal{B}_1} \phi_Q \lesssim \sum_{Q \in \mathcal{B}_1} |Q| \inf_Q Mg \leq \int Mg \sum_{Q \in \mathcal{B}_1} 1_Q \leq \|Mg\|_{u'} \left\| \sum_{Q \in \mathcal{B}_1} 1_Q \right\|_u,$$

where $\|Mg\|_{u'} \lesssim \|g\|_{u'} = 1$ by the maximal inequality, and

$$\left\| \sum_{Q \in \mathcal{B}_1} 1_Q \right\|_u = \left(\sum_{Q \in \mathcal{B}_1} |Q| \right)^{1/u} \leq |Q_0|^{1/u}$$

by the disjointness of the cubes $Q \in \mathcal{B}_1$. □

Concerning off-diagonal estimates, we have the following:

4.7. Lemma. *If Φ satisfies (4.5) and Q' is a dyadic cube, then*

$$\int_{Q'} |T_\Phi(1_{(3Q')^c}(b - \tilde{b}))| dx \lesssim \int_{Q'} e_{Q_0}^1 dx + \int_{Q'} Mb dx.$$

Proof. Clearly, by Lemma 4.4, we have

$$\int_{Q'} |T_{\Phi}(b - \tilde{b})| dx \lesssim \int_{Q'} e_{Q_0}^1 dx,$$

and hence it suffices to estimate the integral average of

$$|T_{\Phi}(1_{3Q'}(b - \tilde{b}))| \leq \sum_{Q \in \mathcal{B}_1} |T_{\Phi}(1_{3Q'}d_Q)|.$$

Here, we only need to consider those $Q \in \mathcal{B}_1$ with $Q \cap 3Q' \neq \emptyset$. Both Q and Q' are dyadic. If $\ell(Q) \leq \ell(Q')$, the intersection condition implies that $Q \subset 3Q'$. Since the cubes $Q \in \mathcal{B}_1$ are pairwise disjoint, there are at most 2^d cubes Q with $\ell(Q) > \ell(Q')$ that intersect with $3Q'$. Now

$$\sum_{\substack{Q \in \mathcal{B}_1 \\ Q \subset 3Q'}} |T_{\Phi}(1_{3Q'}d_Q)| = \sum_{\substack{Q \in \mathcal{B}_1 \\ Q \subset 3Q'}} |T_{\Phi}(d_Q)| \lesssim e_{Q_0}^1$$

by the previous Lemma 4.4, and the bound for the integral average follows.

For the boundedly many remaining cubes $Q \in \mathcal{B}_1$ with $\ell(Q) > \ell(Q')$ and $Q \cap 3Q' \neq \emptyset$, we have for $x \in Q' \subset 2Q$ and $y \in Q$, as in (4.6), that $|K_{\Phi}(x, y)| \lesssim \ell(Q)^{-d} \leq \ell(Q')^{-d}$, and hence

$$\begin{aligned} |T_{\Phi}(1_{3Q'}d_Q)(x)| &\leq \int_{3Q'} |K_{\Phi}(x, y)| |d_Q(y)| dy \lesssim \int_{3Q'} |d_Q(y)| dy \\ &\leq \int_{3Q'} |b(y)| dy + |\langle b \rangle_Q| \lesssim \int_{Q'} Mb(y) dy. \quad \square \end{aligned}$$

We want to interpret the new function $\tilde{b}_{Q_0}^1$, and similarly constructed functions for subcubes of Q_0 , as test functions for the operator T_{Φ} . Note that the choice of Φ will be fixed only after a stopping time construction, by which we construct the remaining functions \tilde{b}_Q^1 . Before we fix this choice, it is important that all the estimates are valid for any Φ with the property (4.5).

For any such Φ , we have by Lemmas 4.1, 4.4 and 4.7 that

$$(4.8) \quad |T_{\Phi}\tilde{b}_{Q_0}^1| \leq |T_{\Phi}b_{Q_0}^1| + |T_{\Phi}(\tilde{b}_{Q_0}^1 - b_{Q_0}^1)| \lesssim T_{\#}b_{Q_0}^1 + Mb_{Q_0}^1 + e_{Q_0}^1,$$

and

$$(4.9) \quad \begin{aligned} \int_{Q'} |T_{\Phi}(1_{(3Q')^c}\tilde{b}_{Q_0}^1)| dx &\leq \int_{Q'} |T_{\Phi}(1_{(3Q')^c}b_{Q_0}^1)| dx + \int_{Q'} |T_{\Phi}(1_{(3Q')^c}(\tilde{b}_{Q_0}^1 - b_{Q_0}^1))| dx \\ &\lesssim \int_{Q'} T_{\#}(1_{(3Q')^c}b_{Q_0}^1) dx + \int_{Q'} Mb_{Q_0}^1 dx + \int_{Q'} e_{Q_0}^1 dx. \end{aligned}$$

4.C. Different stopping conditions. We refer to the maximal dyadic subcubes $Q \subseteq Q_0$ with $\int_Q |b_{Q_0}^1|^p \geq C/\delta$ as its **b -stopping cubes**. They satisfy

$$\sum_{Q \in \mathcal{B}_1(Q_0)} |Q| = |\{M^d |b_{Q_0}^1|^p > C/\delta\}| \leq \frac{1}{C/\delta} \int |b_{Q_0}^1|^p \leq \delta |Q_0|.$$

The function $e_{Q_0}^1$ depends on these cubes, and thus on δ ; however, the bound $\|e_{Q_0}^1\|_u \lesssim |Q|^{1/u}$ depends only on the parameter δ .

The **Tb -stopping cubes** of Q_0 are defined as the maximal dyadic subcubes $Q \subseteq Q_0$ that satisfy any of the following conditions: either

$$(4.10) \quad \int_Q [T_{\#}b_{Q_0}^1 + Mb_{Q_0}^1 + e_{Q_0}^1]^p > \frac{1}{\varepsilon},$$

or (if we assume the off-diagonal estimates for $T_{\#}$, but not in the antisymmetric case)

$$(4.11) \quad \int_Q T_{\#}(1_{(3Q)^c}b_{Q_0}^1) > C_{\sigma},$$

where C_{σ} is as in Definition 2.11 of off-diagonal estimates, or

$$(4.12) \quad \left| \int_Q \tilde{b}_{Q_0}^1 \right| \leq \eta.$$

The measure of the cubes in (4.10) is at most

$$|\{M^d(1_{Q_0}[T_{\#}b_{Q_0}^1 + Mb_{Q_0}^1 + e_{Q_0}^1]^p) > 1/\varepsilon\}| \leq \varepsilon \int_{Q_0} [T_{\#}b_{Q_0}^1 + Mb_{Q_0}^1 + e_{Q_0}^1]^p \lesssim \varepsilon|Q_0|.$$

For the cubes in (4.11), we have

$$\sum_Q |Q| \leq \sigma|Q_0|$$

as a direct consequence of Definition 2.11 of off-diagonal estimates (if we assumed them). Finally, for the cubes in (4.12), we compute

$$\begin{aligned} |Q_0| &= \left| \int b_{Q_0}^1 \right| = \left| \int \tilde{b}_{Q_0}^1 \right| \\ &\leq \sum_Q \left| \int \tilde{b}_{Q_0}^1 \right| + \int_{Q_0 \setminus \bigcup Q} |\tilde{b}_{Q_0}^1| \\ &\leq \sum_Q \eta|Q| + |Q_0 \setminus \bigcup Q|^{1/p'} \|\tilde{b}_{Q_0}^1\|_p \\ &\leq \eta|Q_0| + C \left(|Q_0| - \sum_Q |Q| \right)^{1/p'} |Q_0|^{1/p}, \end{aligned}$$

by using $\|\tilde{b}_{Q_0}^1\|_p \lesssim \|b_{Q_0}^1\|_p \lesssim |Q_0|^{1/p}$ in the last step. From here one can solve

$$\sum |Q| \leq \left\{ 1 - \left(\frac{1-\eta}{C} \right)^{p'} \right\} |Q_0|.$$

Altogether, taking $\eta < 1$ and ε and σ sufficiently small, the measure of the Tb -stopping cubes is at most a fraction $(1-\tau) < 1$ of $|Q_0|$.

4.D. Iteration of the stopping conditions. We iterate the following algorithm, starting from an arbitrary but fixed dyadic cube Q_0 .

- We choose the b -stopping cubes $\mathcal{B}_1 = \mathcal{B}_1(Q_0)$ of Q_0 , and the Tb -stopping cubes $\mathcal{T}_1 = \mathcal{T}_1(Q_0)$ of Q_0 as explained above.
- Assuming that the collections \mathcal{B}_k and \mathcal{T}_k are already constructed, for every $Q \in \mathcal{T}_k$, we choose (using b_Q^1) the b -stopping cubes $\mathcal{B}_1(Q)$ of Q , which determine the functions \tilde{b}_Q^1 and e_Q^1 , and then (using b_Q^1 , \tilde{b}_Q^1 , and e_Q^1), the Tb -stopping cubes $\mathcal{T}_1(Q)$ of Q . We let

$$\mathcal{B}_{k+1} := \bigcup_{Q \in \mathcal{T}_k} \mathcal{B}_1(Q), \quad \mathcal{T}_{k+1} := \bigcup_{Q \in \mathcal{T}_k} \mathcal{T}_1(Q).$$

By iterating the bounds $\sum_{Q' \in \mathcal{B}_1(Q)} |Q'| \leq \delta|Q|$ and $\sum_{Q' \in \mathcal{T}_1(Q)} |Q'| \leq (1-\tau)|Q|$, we arrive at

$$\sum_{Q \in \mathcal{T}_k(Q_0)} |Q| \leq (1-\tau)^k |Q_0|, \quad \sum_{Q \in \mathcal{B}_k(Q_0)} |Q| \leq \delta \sum_{Q \in \mathcal{T}_{k-1}(Q_0)} |Q| \leq \delta(1-\tau)^{k-1} |Q_0|,$$

where we interpret $\mathcal{T}_0(Q_0) := \{Q_0\}$. Hence the measure of all b -stopping cubes satisfies

$$\sum_{k=1}^{\infty} \sum_{Q \in \mathcal{B}_k(Q_0)} |Q| \leq \sum_{k=1}^{\infty} \delta(1-\tau)^{k-1} |Q_0| = \frac{\delta}{\tau} |Q_0|.$$

The parameter δ can be made small independently of τ , and hence we can make the fraction δ/τ as small as we like.

Then we can define

$$\Phi(x) := \sup \{ \text{dist}(x, (3Q)^c) : Q \in \bigcup_{k=1}^{\infty} \mathcal{B}_k \}.$$

It follows that

$$(4.13) \quad |\{\Phi > 0\}| = \left| \bigcup_{k=1}^{\infty} \bigcup_{Q \in \mathcal{B}_k} 3Q \right| \leq 3^d \frac{\delta}{\tau} |Q_0| =: \varrho |Q_0|,$$

where the fraction ϱ can be made arbitrarily small.

4.E. Construction summary; completion of the proof of Proposition 2.17. Given a cube Q_0 , we find the stopping cubes $\bigcup_{k=1}^{\infty} \mathcal{B}_k(Q_0)$ and $\bigcup_{k=1}^{\infty} \mathcal{T}_k(Q_0)$. Let us also call Q_0 itself a stopping cube, and denote $\mathcal{T}_0(Q_0) := \{Q_0\}$. The stopping cubes determine the function Φ . For each $Q \in \mathcal{T}(Q_0) := \bigcup_{k=0}^{\infty} \mathcal{T}_k(Q_0)$, there is a function \tilde{b}_Q^1 , the good part of the Calderón–Zygmund decomposition of b_Q^1 , thus

$$\|\tilde{b}_Q^1\|_{\infty} \lesssim 1.$$

For every $Q \in \mathcal{T}(Q_0)$, we can apply the estimate (4.8) for Q in place of Q_0 . Indeed, it suffices to check that the chosen Φ satisfies the analogue of (4.5) with $Q \in \mathcal{T}(Q_0)$ in place of Q_0 , namely, that

$$\Phi(x) \geq \sup_{Q' \in \mathcal{B}_1(Q)} \text{dist}(x, (3Q')^c).$$

But this is clear from the definition of Φ , since $\mathcal{B}_1(Q) \subseteq \mathcal{B}(Q_0)$ for all $Q \in \mathcal{T}(Q_0)$, and Φ is the supremum over this larger set. Thus, by (4.8) and (4.9) applied to Q in place of Q_0 , we have

$$|T_{\Phi} \tilde{b}_Q^1| \lesssim T_{\#} b_Q^1 + M b_Q^1 + e_Q^1$$

and

$$\int_{Q'} |T_{\Phi}(1_{(3Q')^c} \tilde{b}_Q^1)| \lesssim \int_{Q'} T_{\#}(1_{(3Q')^c} \tilde{b}_Q^1) + \int_{Q'} M b_Q^1 + \int_{Q'} e_Q^1$$

for any dyadic $Q' \subseteq Q \in \mathcal{T}(Q_0)$, which is not contained in any smaller $Q'' \in \mathcal{T}(Q_0)$, equivalently, not in any $Q'' \in \mathcal{T}_1(Q)$. Consider any such $Q' \subseteq Q \in \mathcal{T}(Q_0)$. Then, by the construction of the Tb -stopping cubes $\mathcal{T}_1(Q)$, this means that

$$\int_{Q'} |T_{\Phi} \tilde{b}_Q^1|^p \lesssim \int_{Q'} [T_{\#} b_Q^1 + M b_Q^1 + e_Q^1]^p \lesssim 1,$$

and, if $T_{\#}$ satisfies off-diagonal estimates,

$$\int_{Q'} |T_{\Phi}(1_{(3Q')^c} \tilde{b}_Q^1)| \lesssim 1,$$

as well as

$$\left| \int_{Q'} \tilde{b}_Q^1 \right| \gtrsim 1, \quad Q' \subseteq Q_0, \quad Q' \not\subseteq Q'' \in \mathcal{T}_1(Q).$$

(We are suppressing the dependence on the parameters ε, η , since they are now fixed and no longer relevant to us.) Recall also that

$$\sum_{Q' \in \mathcal{T}_1(Q)} |Q'| \leq (1 - \tau)|Q|.$$

Summa summarum, associated to every Tb -stopping cube $Q \in \mathcal{T}(Q_0)$, there is a non-degenerate test function $\tilde{b}_Q^1 \in L^{\infty}(Q)$ such that $1_Q T_{\Phi} \tilde{b}_Q^1 \in L^t(Q)$ (with the correct normalization), and the special off-diagonal estimates hold for T_{Φ} , if the off-diagonal estimates hold for $T_{\#}$. Moreover, $|\{\Phi > 0\}| \leq \varrho|Q_0|$.

Of course, starting from the original test functions b_Q^2 and T^* in place of T , we can similarly produce new test functions $\tilde{b}_Q^2 \in L^{\infty}(Q)$ with $1_Q T_{\Phi}^* \tilde{b}_Q^2 \in L^t(Q)$ for another set of Tb -stopping cube $Q \in \mathcal{T}^2(Q_0)$. To have the same Φ both for T and T^* , we should define

$$\Phi(x) := \sup\{\text{dist}(x, (3Q)^c) : Q \in \bigcup_{k=1}^{\infty} (\mathcal{B}_k^1 \cup \mathcal{B}_k^2)\},$$

where \mathcal{B}_k^1 and \mathcal{B}_k^2 are the b -stopping cubes related to b_Q^1 and b_Q^2 , respectively. Clearly, this still satisfies the bound (4.13), with at most twice the original constant ϱ , which we can make arbitrarily small in any case.

This completes the proof of Proposition 2.17.

5. THE BABY Tb THEOREM; PROOF OF PROPOSITION 2.18

Let us denote the reference cube in which we operate by Q^0 , as we will need the notation Q_0 for another purpose below. Let us first deal with just one accretive system b_Q defined on a sparse family \mathcal{Q} ; later on, these results will be applied to both b_Q^1 on \mathcal{Q}_1 and b_Q^2 on \mathcal{Q}_2 . We also refer to the members of \mathcal{Q} as *stopping cubes*. For every $Q \subseteq Q^0$, let Q^a be the minimal stopping cube which contains Q . Then

$$\left| \int_Q b_{Q^a} dx \right| \gtrsim 1, \quad \int_Q |b_{Q^a}|^p dx \lesssim 1, \quad \int_Q |Tb_{Q^a}|^p dx \lesssim 1.$$

We start by recalling the adapted martingale difference framework for a local Tb theorem from [15] and [9, 11]. (Also the subsequent analysis borrows from these papers, even when this is not always stated. On the other hand, we take the opportunity to simplify several details, as we are in the simpler case of the Lebesgue measure, rather than a non-doubling one; this allows us to work with the fixed system of standard dyadic cubes, instead of their random translation.)

We have the expectation (averaging) operators

$$\mathbb{E}_Q^b f := 1_Q b_{Q^a} \frac{\langle f \rangle_Q}{\langle b_{Q^a} \rangle_Q}, \quad \mathbb{E}_Q := 1_Q \langle f \rangle_Q,$$

and the difference operators

$$\begin{aligned} \mathbb{D}_Q^b f &:= \sum_{i=1}^{2^d} \mathbb{E}_{Q_i}^b f - \mathbb{E}_Q^b f \\ &= \sum_{i=1}^{2^d} \left(1_{Q_i} b_{Q_i^a} \frac{1}{\langle b_{Q_i^a} \rangle_{Q_i}} - 1_Q b_{Q^a} \frac{|Q_i|}{|Q| \langle b_{Q^a} \rangle_Q} \right) \langle f \rangle_{Q_i} =: \sum_{i=1}^{2^d} \phi_{Q,i} \langle f \rangle_{Q_i}, \end{aligned}$$

where the i -sum goes through the dyadic children Q_i of Q . A direct computation shows that

$$(\mathbb{D}_Q^b)^2 f = \mathbb{D}_Q^b f - \sum_{i=1}^{2^d} 1_{Q_i} \left(\frac{\langle b_{Q_i^a} \rangle_{Q_i} b_{Q_i^a}^a}{\langle b_{Q_i^a} \rangle_{Q_i}} - b_{Q^a} \right) \frac{\langle f \rangle_Q}{\langle b_{Q^a} \rangle_Q} =: \mathbb{D}_Q^b f - \omega_Q \langle f \rangle_Q.$$

Note that $1_{Q_i} \omega_Q$ is nonzero only if $Q_i^a \neq Q^a$, i.e., only if Q_i is a stopping cube. Thus ω_Q is nonzero only if Q has at least one stopping child. Combining the above formulae, we get

$$(5.1) \quad \mathbb{D}_Q^b f = (\mathbb{D}_Q^b)^2 f + \omega_Q \langle f \rangle_Q = \sum_{i=1}^{2^d} \phi_{Q,i} \langle \mathbb{D}_Q^b f \rangle_{Q_i} + \omega_Q \langle f \rangle_Q =: \sum_{i=0}^{2^d} \phi_{Q,i} \langle \mathbb{D}_{Q,i}^b f \rangle_{Q_i},$$

where $Q_0 := Q$, $\phi_{Q,0} := \omega_Q$ and

$$\mathbb{D}_{Q,i}^b f := \begin{cases} \mathbb{D}_Q^b f, & \text{if } i = 1, \dots, 2^d, \\ \mathbb{E}_Q f, & \text{if } i = 0 \text{ and } Q \text{ has a stopping child,} \\ 0, & \text{if } i = 0 \text{ and } Q \text{ does not have any stopping children.} \end{cases}$$

We have the following important estimates. The L^2 case is from [15], and its generalization to L^r from [11]. (Both these papers deal with more general non-doubling situations, the latter even vector-valued—a generality that do not consider here.)

5.2. Proposition. *For all $r \in (1, \infty)$, $i = 0, 1, \dots, 2^d$, and $f \in L^r$, we have*

$$\left\| \left(\sum_Q |\mathbb{D}_{Q,i}^b f|^2 \right)^{1/2} \right\|_r + \left\| \left(\sum_Q |(\mathbb{D}_{Q,i}^b)^* f|^2 \right)^{1/2} \right\|_r \lesssim \|f\|_r.$$

In particular,

$$\left(\sum_Q \|\mathbb{D}_{Q,i}^b f\|_2^2 \right)^{1/2} + \left(\sum_Q \|(\mathbb{D}_{Q,i}^b)^* f\|_2^2 \right)^{1/2} \lesssim \|f\|_2.$$

For every $f \in L^r(Q^0)$, we have the decomposition

$$f = \mathbb{E}_{Q^0}^b f + \sum_Q \mathbb{D}_Q^b f.$$

To simplify writing, we redefine $\mathbb{D}_{Q^0}^b f$ as $\mathbb{E}_{Q^0}^b f + \mathbb{D}_{Q^0}^b f$, so that we can drop the first term from the sum above. Thus

$$\langle Tf, g \rangle = \sum_{Q,R} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle = \sum_{\substack{Q,R \\ \ell(Q) \leq \ell(R)}} + \sum_{\substack{Q,R \\ \ell(Q) > \ell(R)}}.$$

By symmetry, it suffices to estimate the first half. This we reorganize as follows:

$$\begin{aligned} \sum_{\substack{Q,R \\ \ell(Q) \leq \ell(R)}} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \sum_R \sum_{\substack{Q \subseteq R + \ell(R)m \\ \ell(Q) = 2^{-k}\ell(R)}} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle \\ &= \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \sum_R \langle T\mathbb{D}_{R+\ell(R)m}^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle, \end{aligned}$$

where $(S := R + \ell(R)m)$

$$\mathbb{D}_S^{b_1,k} f := \sum_{\substack{Q \subseteq S \\ \ell(Q) = 2^{-k}\ell(S)}} \mathbb{D}_Q^{b_1} f.$$

Below, we will also use the notation $\mathbb{D}_{S,i}^{b_1,k}$ (with additional subscript $i \in \{0, 1, \dots, 2^d\}$) similarly defined with $\mathbb{D}_{Q,i}^{b_1}$ on the right as well.

5.A. Disjoint cubes. We further analyse the part of the m sum with $m \neq 0$, thus $Q \subseteq R + m\ell(R)$ is disjoint from R . By (5.1), we can expand

$$\begin{aligned} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_{i,j} \langle \mathbb{D}_{Q,i}^{b_1} f \rangle_{Q_i} \langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle \langle \mathbb{D}_{R,j} g \rangle_{R_j}, \quad \text{hence, summing in } Q, \\ \langle T\mathbb{D}_S^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_{i,j} \iint_{R \times S} \mathbb{D}_{S,i}^{b_1,k} f(y) K_{R,S}^{i,j;k}(x,y) \mathbb{D}_{R,j}^{b_2} g(x) \, dx \, dy, \quad \text{where} \\ K_{R,S}^{i,j;k}(x,y) &:= \sum_{\substack{Q \subseteq S \\ \ell(Q) = 2^{-k}\ell(S)}} \frac{1_{Q_i}(y)}{|Q_i|} \langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle \frac{1_{R_j}(x)}{|R_j|}. \end{aligned}$$

5.3. Lemma. *For $m \neq 0$, we have*

$$\|K_{R,R+\ell(R)m}^{i,j;k}\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim 2^{-k \min\{\alpha, \frac{1}{2}\}} (1+k)^{\delta_{\alpha,1/2}} |m|^{-d-\alpha},$$

where $\delta_{\alpha,1/2}$ is Kronecker's delta.

Since we can always decrease the Hölder exponent of the Calderón–Zygmund kernel, we will henceforth assume that $\alpha < \frac{1}{2}$, and write the above bound in the simpler form

$$\|K_{R,R+\ell(R)m}^{i,j;k}\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim 2^{-k\alpha} |m|^{-d-\alpha}.$$

Proof. Note that $Q \neq Q^0$ in this sum. Indeed, $\ell(Q) \leq \ell(R) \leq \ell(Q^0)$ so the only way that we could have $Q = Q^0$ is $\ell(Q) = \ell(R) = \ell(Q^0)$, and then (since Q^0 is the only cube of sidelength $\ell(Q^0)$), $Q = R = Q^0$. But then $m = 0$, a contradiction. Thus $\mathbb{D}_Q^{b_1}$ is always given by the original definition, i.e., without the addition of $\mathbb{E}_{Q^0}^{b_1}$, and then all the $\phi_{Q,i}^1$ have mean zero. Hence we can

write

$$\begin{aligned} \langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle &= \iint_{R \times Q} K(x, y) \phi_{Q,i}^1(y) \phi_{R,j}^2(x) \, dx \, dy \\ &= \iint_{R \cap 3Q \times Q} K(x, y) \phi_{Q,i}^1(y) \phi_{R,j}^2(x) \, dx \, dy \\ &\quad + \iint_{R \setminus (3Q) \times Q} [K(x, y) - K(x, c_Q)] \phi_{Q,i}^1(y) \phi_{R,j}^2(x) \, dx \, dy, \end{aligned}$$

and then estimate

$$|\langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle| \lesssim \iint_{R \cap 3Q \times Q} \frac{dx \, dy}{|x - y|^d} + \iint_{R \setminus (3Q) \times Q} \frac{\ell(Q)^\alpha}{|x - c_Q|^{d+\alpha}} \, dx \, dy.$$

If $|m|_\infty > 1$, the first term vanishes, and estimating the second term we get

$$|\langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle| \lesssim \frac{\ell(Q)^\alpha}{\text{dist}(Q, R)^{d+\alpha}} |Q| |R| \lesssim 2^{-k\alpha} |m|^{-d-\alpha} |Q|,$$

and then $\|K_{R, R+\ell(R)m}^{i,j;k}\|_\infty \lesssim 2^{-k\alpha} |m|^{-d-\alpha} / |R|$.

If $|m|_\infty = 1$, then the first term is nonzero (and then bounded by $|Q|$) only if $\text{dist}(Q, R) = 0$, while the second term is estimated by

$$\frac{\ell(Q)^\alpha |Q|}{\max\{\ell(Q), \text{dist}(Q, R)\}^\alpha}.$$

So altogether we have

$$|\langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle| \lesssim \left(1 + \frac{\text{dist}(Q, R)}{\ell(Q)}\right)^{-\alpha} |Q|,$$

and then

$$|K_{R, R+\ell(R)m}^{i,j;k}(x, y)| \lesssim \frac{1_R(x)}{|R|} \sum_{\substack{Q \subseteq R+\ell(R)m \\ \ell(Q)=2^{-k}\ell(R)}} \left(1 + \frac{\text{dist}(Q, R)}{\ell(Q)}\right)^{-\alpha} 1_Q(y).$$

The number of the cubes Q with $\text{dist}(Q, R) = n\ell(Q)$, ($n = 0, 1, \dots, 2^k - 1$), is $\mathcal{O}(2^{k(d-1)})$, while each of them has measure $|Q| = 2^{-kd}|R|$. Hence

$$\begin{aligned} \|K_{R, R+\ell(R)m}^{i,j;k}\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 &\lesssim \frac{1}{|R|} \sum_{n=0}^{2^k-1} 2^{k(d-1)} \cdot (1+n)^{-2\alpha} \cdot 2^{-kd} |R| \\ &= 2^{-k} \sum_{n=1}^{2^k} n^{-2\alpha} \lesssim 2^{-\min\{1, 2\alpha\}k} (1+k)^{\delta_{1, 2\alpha}}. \quad \square \end{aligned}$$

Lemma 5.3 is enough to estimate the part of the series with $m \neq 0$; indeed

$$\begin{aligned} &\sum_{m \neq 0} \sum_{k=0}^{\infty} \sum_R |\langle T\mathbb{D}_{R+\ell(R)m}^{b_1, k} f, \mathbb{D}_R^{b_2} g \rangle| \\ &\leq \sum_{m \neq 0} \sum_{k=0}^{\infty} \sum_R \sum_{i,j} \|K_{R,m}^{i,j;k}\|_2 \|\mathbb{D}_{R+\ell(R)m}^{b_1, k} f\|_2 \|\mathbb{D}_R^{b_2} g\|_2 \\ &\lesssim \sum_{m \neq 0} |m|^{-d-\alpha} \sum_{k=0}^{\infty} 2^{-\alpha k} \sum_R \|\mathbb{D}_{R+\ell(R)m}^{b_1, k} f\|_2 \|\mathbb{D}_R^{b_2} g\|_2, \end{aligned}$$

where

$$\begin{aligned} & \sum_R \|\mathbb{D}_{R+\ell(R)m}^{b_1,k} f\|_2 \|\mathbb{D}_R^{b_2} g\|_2 \\ & \leq \left(\sum_R \|\mathbb{D}_{R+\ell(R)m}^{b_1,k} f\|_2^2 \right)^{1/2} \left(\sum_R \|\mathbb{D}_R^{b_2} g\|_2^2 \right)^{1/2} \\ & = \left(\sum_Q \|\mathbb{D}_Q^{b_1} f\|_2^2 \right)^{1/2} \left(\sum_R \|\mathbb{D}_R^{b_2} g\|_2^2 \right)^{1/2} \lesssim \|f\|_2 \|g\|_2, \end{aligned}$$

and

$$\sum_{m \neq 0} |m|^{-d-\alpha} \sum_{k=0}^{\infty} 2^{-\alpha k} \lesssim 1.$$

5.B. **Nested cubes.** We are left with the part with $m = 0$, that is,

$$\sum_{k=0}^{\infty} \sum_R \langle T\mathbb{D}_R^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle = \sum_R \langle T\mathbb{D}_R^{b_1} f, \mathbb{D}_R^{b_2} g \rangle + \sum_R \sum_{Q \subsetneq R} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle.$$

5.4. **Lemma.** *We have*

$$\begin{aligned} (5.5) \quad \sum_R \sum_{Q \subsetneq R} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_Q \langle T\mathbb{D}_Q^{b_1} f, b_{Q^{a,2}}^2 \rangle \frac{\langle g \rangle_Q}{\langle b_{Q^{a,2}}^2 \rangle_Q} \\ &+ \sum_{k=0}^{\infty} \sum_R \sum_{\substack{S \subsetneq R \\ \ell(S)=\ell(R)/2}} \sum_{j=0}^{2^d} \langle T\mathbb{D}_S^{b_1,k} f, 1_{S^c} \psi_{R,j;S}^2 \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j}, \end{aligned}$$

for some bounded functions $\psi_{R,j;S}^2$.

Proof. For $Q \subsetneq R$, let R_Q be the unique subcube of R which contains Q . Then

$$\mathbb{D}_R^{b_2} g = 1_{R_Q} \mathbb{D}_{R_Q}^{b_2} g + 1_{R_Q^c} \mathbb{D}_{R_Q^c}^{b_2} g,$$

where further (we temporarily drop the superscript 2)

$$1_{R_Q} \mathbb{D}_{R_Q}^b g = (1 - 1_{R_Q^c}) \left(b_{R_Q^a} \frac{\langle f \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle f \rangle_R}{\langle b_{R^a} \rangle_R} \right)$$

and

$$\begin{aligned} (5.6) \quad & b_{R_Q^a} \frac{\langle f \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle f \rangle_R}{\langle b_{R^a} \rangle_R} \\ &= \frac{b_{R_Q^a}}{\langle b_{R_Q^a} \rangle_{R_Q}} \left(\langle f \rangle_{R_Q} - \langle b_{R^a} \rangle_{R_Q} \frac{\langle f \rangle_R}{\langle b_{R^a} \rangle_R} \right) + \left(\frac{\langle b_{R^a} \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} b_{R_Q^a} - b_{R^a} \right) \frac{\langle f \rangle_R}{\langle b_{R^a} \rangle_R} \\ &= \frac{b_{R_Q^a}}{\langle b_{R_Q^a} \rangle_{R_Q}} \langle \mathbb{D}_{R_Q}^b f \rangle_{R_Q} + \left(\frac{\langle b_{R^a} \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} b_{R_Q^a} - b_{R^a} \right) \frac{\langle \mathbb{D}_{R,0}^b f \rangle_R}{\langle b_{R^a} \rangle_R}. \end{aligned}$$

On the last line we observed that the function in the parentheses is zero unless R_Q is a stopping cube (i.e., $R_Q^a = R_Q \neq R^a$), and thus we may replace f by $\mathbb{D}_{R,0}^b f$ inside the average on R .

Substituting back, it follows that

$$\begin{aligned} \mathbb{D}_R^{b_2} g &= b_{R_Q^a} \frac{\langle f \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle f \rangle_R}{\langle b_{R^a} \rangle_R} \\ &+ 1_{R_Q^c} \left\{ \mathbb{D}_{R_Q}^b g - \frac{b_{R_Q^a}}{\langle b_{R_Q^a} \rangle_{R_Q}} \langle \mathbb{D}_{R_Q}^b g \rangle_{R_Q} - \left(\frac{\langle b_{R^a} \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} b_{R_Q^a} - b_{R^a} \right) \frac{\langle \mathbb{D}_{R,0}^b f \rangle_R}{\langle b_{R^a} \rangle_R} \right\} \\ &= \left(b_{R_Q^a} \frac{\langle f \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle f \rangle_R}{\langle b_{R^a} \rangle_R} \right) + \sum_j \psi_{R,j;R_Q}^b \langle \mathbb{D}_{R,j}^b g \rangle_{R_j}, \end{aligned}$$

where (recalling the formula $\mathbb{D}_R^b g = \sum_j \phi_{R,j} \langle \mathbb{D}_{R,j}^b \rangle_{R_j}$)

$$\psi_{R,j;S} := \phi_{R,j} - \begin{cases} \left(\langle b_{R^a} \rangle_S / \langle b_{S^a} \rangle_S \cdot b_{S^a} - b_{R^a} \right) / \langle b_{R^a} \rangle_R, & \text{if } j = 0, \\ b_{S^a} / \langle b_{S^a} \rangle_S, & \text{if } j \geq 1 \text{ and } R_j = S, \\ 0. & \text{else} \end{cases}$$

are bounded functions.

Pairing with $T\mathbb{D}_Q^{b_1} f$, we obtain (changing the summation order, and observing that Q is the smallest R_Q , as well as the telescoping cancellation)

$$\sum_Q \sum_{R \supseteq Q} \left\langle \mathbb{D}_Q^{b_1} f, b_{R_Q^{a,2}}^2 \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q^{a,2}}^2 \rangle_{R_Q}} - b_{R_Q^{a,2}}^2 \frac{\langle g \rangle_R}{\langle b_{R_Q^{a,2}}^2 \rangle_R} \right\rangle = \sum_Q \langle \mathbb{D}_Q^{b_1} f, b_{Q^{a,2}}^2 \frac{\langle g \rangle_Q}{\langle b_{Q^{a,2}}^2 \rangle_Q} \rangle.$$

For the other, we introduce the auxiliary summation variable $S := R_Q$, regroup the summation according to the relative size of Q and S , and recall the notation

$$\mathbb{D}_S^{b_1, k} f := \sum_{\substack{Q \subseteq S \\ \ell(Q) = 2^{-k} \ell(S)}} \mathbb{D}_Q^{b_1} f$$

to arrive at the asserted formula. \square

The last summand in (5.5) can be written as

$$\begin{aligned} \langle T\mathbb{D}_S^{b_1, k} f, 1_{S^c} \psi_{R,j;S}^{b_2} \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j} &= \sum_{\substack{Q \subseteq S; i=0, \dots, 2^d \\ \ell(Q) = 2^{-k} \ell(S)}} \langle \mathbb{D}_{Q,i}^{b_1} f \rangle_{Q_i} \langle T\phi_{Q,i}^1, 1_{S^c} \psi_{R,j;S}^{b_2} \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j} \\ &= \iint \mathbb{D}_{S,i}^{b_1, k}(y) \tilde{K}_{R,S}^{i,j;k}(x,y) \mathbb{D}_{R,j}^{b_2} g(x) \, dx \, dy, \quad \text{where} \\ \tilde{K}_{R,S}^{i,j;k}(x,y) &:= \sum_{\substack{Q \subseteq S; i=0, \dots, 2^d \\ \ell(Q) = 2^{-k} \ell(S)}} \frac{1_{Q_i}(y)}{|Q_i|} \langle T\phi_{Q,i}^1, 1_{S^c} \psi_{R,j;S}^{b_2} \rangle \frac{1_{R_j}(x)}{|R_j|}. \end{aligned}$$

Just as in Lemma 5.3 (case $|m|_\infty = 1$) we check that

$$\|\tilde{K}_{R,S}^{i,j;k}\|_{L^2(\mathbb{R}^{2d})} \lesssim 2^{-k\alpha}, \quad \alpha < \frac{1}{2},$$

and therefore

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_R \sum_{\substack{S \subseteq R \\ \ell(S) = \ell(R)/2}} \sum_j |\langle T\mathbb{D}_S^{b_1, k} f, 1_{S^c} \psi_{R,j;S}^{b_2} \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j}| \\ &\lesssim \sum_{k=0}^{\infty} \sum_R \sum_{\substack{S \subseteq R \\ \ell(S) = \ell(R)/2}} \sum_{j,i} 2^{-k\alpha} \|\mathbb{D}_{S,i}^{b_1, k} f\|_2 \|\mathbb{D}_{R,j}^{b_2} g\|_2 \\ &\lesssim \sum_{k=0}^{\infty} \sum_R \sum_{j,i} 2^{-k\alpha} \|\mathbb{D}_{R,i}^{b_1, k+1} f\|_2 \|\mathbb{D}_{R,j}^{b_2} g\|_2 \lesssim \|f\|_2 \|g\|_2, \end{aligned}$$

which completes this part of the estimate.

5.C. **The paraproduct.** The other part in (5.5), which still requires attention, is

$$\begin{aligned}
& \sum_R |\langle T\mathbb{D}_R^{b_1} f, b_{R^{a,2}}^2 \rangle \frac{\langle g \rangle_R}{\langle b_{R^{a,2}}^2 \rangle_R}| \lesssim \sum_R |\langle (\mathbb{D}_R^{b_1})^2 + \omega_R^1 \mathbb{D}_{R,0}^{b_1} f, T^* b_{R^{a,2}}^2 \rangle \langle g \rangle_R| \\
& \leq \sum_R |\langle \mathbb{D}_R^{b_1} f, (\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2 \rangle \langle g \rangle_R| + \sum_R |\langle \mathbb{D}_{R,0}^{b_1} f, \omega_R^1 T^* b_{R^{a,2}}^2 \rangle \langle g \rangle_R| \\
& \leq \left\| \left(\sum_R |\mathbb{D}_R^{b_1} f|^2 \right)^{1/2} \right\|_{L^{s'}} \left\| \left(\sum_R |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2 \langle g \rangle_R|^2 \right)^{1/2} \right\|_{L^s} \\
& \quad + \left\| \left(\sum_R |\mathbb{D}_{R,0}^{b_1} f|^2 \right)^{1/2} \right\|_{L^{s'}} \left\| \left(\sum_R |\omega_R^1 T^* b_{R^{a,2}}^2 \langle g \rangle_R|^2 \right)^{1/2} \right\|_{L^s} \\
& \lesssim \|f\|_{L^{s'}} \|g\|_{L^s} \sup_S |S|^{-1/s} \left\| \left(\sum_{R \subseteq S} |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \\
& \quad + \|f\|_{L^{s'}} \|g\|_{L^s} \sup_S |S|^{-1/s} \left\| \left(\sum_{R \subseteq S} |\omega_R^1 T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s},
\end{aligned}$$

where in the last step we used Proposition 5.2 and the following L^s version of the Carleson embedding theorem from [10, Theorem 8.2]:

5.7. **Proposition.**

$$\left\| \left(\sum_R |\theta_R \langle g \rangle_R|^2 \right)^{1/2} \right\|_{L^s} \lesssim \|g\|_{L^s} \sup_S |S|^{-1/s} \left\| \left(\sum_{R \subseteq S} |\theta_R|^2 \right)^{1/2} \right\|_{L^s}, \quad s \in (1, 2].$$

It remains to estimate the two Carleson norms above.

5.8. **Lemma.**

$$\left\| \left(\sum_{R \subseteq S} |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \lesssim |S|^{1/s}.$$

Proof.

$$(5.9) \quad \sum_{R \subseteq S} = \sum_{\substack{R \subseteq S \\ R^{a,2} = S^{a,2}}} + \sum_{\substack{P \subseteq S \\ P^{a,2} = P}} \sum_{\substack{R \subseteq P \\ R^{a,2} = P}} = \sum_{k=0}^{\infty} \sum_{P \in \mathcal{P}^k(S)} \sum_{\substack{R \subseteq P \\ R^{a,2} = P^{a,2}}},$$

where $\mathcal{P}^0(S) := \{S\}$, $\mathcal{P}^1(S)$ consists of the maximal $P \subsetneq S$ with $P = P^{a,2}$, and recursively $\mathcal{P}^{k+1}(S) := \bigcup_{G \in \mathcal{P}^k(S)} \mathcal{P}^1(G)$. Since all $P \in \mathcal{P}^k(S)$ are disjoint for a fixed k , we get

$$\begin{aligned}
\left\| \left(\sum_{R \subseteq S} |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} & \leq \sum_{k=0}^{\infty} \left(\sum_{P \in \mathcal{P}^k(S)} \left\| \left(\sum_{\substack{R \subseteq P \\ R^{a,2} = P^{a,2}}} |(\mathbb{D}_R^{b_1})^* T^* b_{P^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s}^s \right)^{1/s} \\
& \lesssim \sum_{k=0}^{\infty} \left(\sum_{P \in \mathcal{P}^k(S)} \|1_P T^* b_{P^{a,2}}^2\|_{L^s}^s \right)^{1/s} \lesssim \sum_{k=0}^{\infty} \left(\sum_{P \in \mathcal{P}^k(S)} |P| \right)^{1/s} \\
& \leq \sum_{k=0}^{\infty} \left((1-\tau)^{k-1} |S| \right)^{1/s} \lesssim |S|^{1/s}. \quad \square
\end{aligned}$$

5.10. **Lemma.**

$$\left\| \left(\sum_{R \subseteq S} |\omega_R^1 T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \lesssim |S|^{1/s}.$$

Proof. This is proven by a similar splitting but, instead of the square function estimate for $(\mathbb{D}_R^{b_1})^*$, using

$$\begin{aligned} \left\| \left(\sum_{R \subseteq P} |\omega_R^1 T^* b_{P^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} &= \left\| \left(\sum_{R \subseteq P} |\omega_R^1|^2 \right)^{1/2} |T^* b_{P^{a,2}}^2| \right\|_{L^s} \\ &\leq \left\| \left(\sum_{R \subseteq P} |\omega_R^1|^2 \right)^{1/2} \right\|_{L^u} \|1_P T^* b_{P^{a,2}}^2\|_{L^t}, \quad \frac{1}{s} = \frac{1}{u} + \frac{1}{t}. \end{aligned}$$

Then, using a decomposition as in (5.9) but in terms of the stopping cubes on the b_1 -side rather than b_2 side,

$$\begin{aligned} \left\| \left(\sum_{R \subseteq P} |\omega_R^1|^2 \right)^{1/2} \right\|_{L^u} &\lesssim \left\| \left(\sum_{\substack{R \subseteq P \\ R^a = R}} 1_R \right)^{1/2} \right\|_{L^u} \leq \sum_{k=0}^{\infty} \left(\sum_{G \in \mathcal{P}^k(P)} |G| \right)^{1/u} \\ &\leq \sum_{k=0}^{\infty} \left((1-\tau)^{k-1} |P| \right)^{1/u} \lesssim |P|^{1/u}. \end{aligned}$$

Since $\|1_P T^* b_{P^{a,2}}^2\|_{L^t} \lesssim |P|^{1/t}$, we get the asserted bound $|P|^{1/u} |P|^{1/t} = |P|^{1/s}$. \square

5.D. The diagonal. It only remains to estimate

$$\begin{aligned} \sum_R |\langle T \mathbb{D}_R^{b_1} f, \mathbb{D}_R^{b_2} g \rangle| &\leq \sum_R \sum_{i,j=1}^{2^d} |\langle T(1_{R_i} \mathbb{D}_R^{b_1} f), 1_{R_j} \mathbb{D}_R^{b_2} g \rangle| \\ &\lesssim \sum_R \sum_{i,j:i \neq j} \|1_{R_i} \mathbb{D}_R^{b_1} f\|_2 \|1_{R_j} \mathbb{D}_R^{b_2} g\|_2 + \sum_R \sum_j |\langle T(1_{R_j} \mathbb{D}_R^{b_1} f), 1_{R_j} \mathbb{D}_R^{b_2} g \rangle|, \end{aligned}$$

where the unequal subcubes were estimated by Hardy's inequality, and this part is readily bounded by $\|f\|_2 \|g\|_2$. For the final part, we write, as in (5.6),

$$1_{R_j} \mathbb{D}_R^{b_1} f = \frac{1_{R_j} b_{R_j^a}}{\langle b_{R_j^a} \rangle_{R_j}} \langle \mathbb{D}_R^{b_1} f \rangle_{R_j} + 1_{R_j} \left(\frac{\langle b_{R^a} \rangle_{R_j}}{\langle b_{R_j^a} \rangle_{R_j}} b_{R_j^a} - b_{R^a} \right) \frac{\langle \mathbb{D}_{R,0}^{b_1} f \rangle_R}{\langle b_{R^a} \rangle_R}.$$

and similarly for $1_{R_j} \mathbb{D}_R^{b_2} g$. Thus

$$\begin{aligned} &|\langle T(1_{R_j} \mathbb{D}_R^{b_1} f), 1_{R_j} \mathbb{D}_R^{b_2} g \rangle| \\ &\lesssim \left(\sum_{i,h \in \{0,j\}} |\langle T(1_{R_j} b_{R_i^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle| \right) \left(\sum_{i,h \in \{0,j\}} |\langle \mathbb{D}_{R,i}^{b_1} f \rangle_{R_i}| |\langle \mathbb{D}_{R,h}^{b_2} g \rangle_{R_h}| \right), \end{aligned}$$

and it remains to check that the first term is dominated by $|R|$, for then e.g.

$$|R|^{1/2} |\langle \mathbb{D}_{R,i}^{b_1} f \rangle_{R_i}| \lesssim \|\mathbb{D}_{R,i}^{b_1} f\|_2,$$

and it is easy to conclude by Proposition 5.2.

Let us first handle the easy cases when $i = j$ and $R_j^{a,1} = R_j$, or $h = j$ and $R_j^{a,2} = R_j$. Consider for example the first case. Then $b_{R_j^{a,1}}^1 = b_{R_j}^1$ is supported on R_j , and hence

$$|\langle T(1_{R_j} b_{R_j^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle| = |\langle T b_{R_j}^1, 1_{R_j} b_{R_h^{a,2}}^2 \rangle| \leq \|1_{R_j} T b_{R_j}^1\|_1 \|b_{R_h^{a,2}}^2\|_\infty \lesssim |R_j| \cdot 1.$$

The case when $R_j^{a,2} = R_j$ and $h = j$ is similar by duality.

Hence it only remains to treat $i = 0$ or $i = j$ and $R_j^{a,1} = R^{a,1}$, where in either case $b_{R_i^{a,1}}^1 = b_{R_j^{a,1}}^1$, and similarly we may take $b_{R_h^{a,2}}^2 = b_{R_j^{a,2}}^2$. So we need to estimate

$$|\langle T(1_{R_j} b_{R_j^{a,1}}^1), 1_{R_j} b_{R_j^{a,2}}^2 \rangle|.$$

But the boundedness of this quantity by $|R_j| \leq |R|$ is precisely the assumed weak boundedness property (2.19). This completes the proof of the ‘‘baby Tb theorem’’, Proposition 2.18.

6. SUFFICIENT CONDITIONS FOR WEAK BOUNDEDNESS; PROOF OF PROPOSITION 2.20

6.A. **The antisymmetric case.** Recall that now $b_Q^1 = b_Q^2 = b_Q$, and also $Q^{a,1} = Q^{a,2} = Q^a$. But then

$$\langle T(1_Q b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle = \langle T(1_Q b_{Q^a}), 1_Q b_{Q^a} \rangle = \langle T\phi, \phi \rangle$$

for $\phi = 1_Q b_{Q^a}$, and clearly

$$\langle T\phi, \phi \rangle = \int K(x, y)\phi(x)\phi(y) dx dy = \int -K(y, x)\phi(x)\phi(y) dx dy = -\langle T\phi, \phi \rangle = 0$$

when T is antisymmetric. Thus the weak boundedness property trivializes in this case.

6.B. **The case of special off-diagonal estimates.** Now we do not have any trivial cancellation, but instead the special off-diagonal estimates. These can be used as follows:

$$(6.1) \quad \begin{aligned} |\langle T(1_Q b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle| &= |\langle T(b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle - \langle T(1_{3Q \setminus Q} b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle \\ &\quad - \langle T(1_{(3Q)^c} b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle| \\ &\leq \|1_Q T b_{Q^{a,1}}^1\|_1 \|b_{Q^{a,2}}^2\|_\infty + \|1_{3Q \setminus Q} b_{Q^{a,1}}^1\|_2 \|1_Q b_{Q^{a,2}}^2\|_2 \\ &\quad + \|1_Q T(1_{(3Q)^c} b_{Q^{a,1}}^1)\|_1 \|b_{Q^{a,2}}^2\|_\infty, \end{aligned}$$

where the second was estimated by Hardy's inequality. Using the bounds

$$\int_Q |T b_{Q^{a,1}}^1| \lesssim 1, \quad \int_Q |T(1_{(3Q)^c} b_{Q^{a,1}}^1)| \lesssim 1,$$

where the latter is an instance of the special off-diagonal estimates, and the boundedness of the functions $b_{Q^{a,i}}^i$, it is easy to conclude.

6.C. **The case of accretive systems on all dyadic cubes.** We will reduce this case to the previous one by showing that the existence of test functions b_Q^i for all dyadic cubes actually implies the special off-diagonal estimates. More precisely, using the existence of a test function b_Q^2 on Q , we write, for $x \in Q$,

$$\begin{aligned} T(1_{(3Q)^c} b_{Q^{a,1}}^1)(x) &= \int_Q [T(1_{(3Q)^c} b_{Q^{a,1}}^1)(x) - T(1_{(3Q)^c} b_{Q^{a,1}}^1)(y)] b_Q^2(y) dy \\ &\quad + \frac{1}{|Q|} \langle T(1_{(3Q)^c} b_{Q^{a,1}}^1), b_Q^2 \rangle \\ &= \int_Q \int_{(3Q)^c} [K(x, z) - K(y, z)] b_{Q^{a,1}}^1(z) b_Q^2(y) dz dy \\ &\quad + \frac{1}{|Q|} \left(\langle T b_{Q^{a,1}}^1, b_Q^2 \rangle - \langle T(1_{3Q \setminus Q} b_{Q^{a,1}}^1), b_Q^2 \rangle - \langle 1_Q b_{Q^{a,1}}^1, T^* b_Q^2 \rangle \right). \end{aligned}$$

The double integral is estimated by

$$\int_Q \int_{(3Q)^c} \frac{\ell(Q)^\alpha}{|x-z|^{d+\alpha}} dz dy \lesssim 1,$$

and we also have the bounds

$$\begin{aligned} |\langle T b_{Q^{a,1}}^1, b_Q^2 \rangle| &\lesssim \|1_Q T b_{Q^{a,1}}^1\|_1 \lesssim |Q|, \\ |\langle 1_{3Q \setminus Q} b_{Q^{a,1}}^1, T^* b_Q^2 \rangle| &\lesssim |Q| \quad \text{by Hardy's inequality, and} \\ |\langle 1_Q b_{Q^{a,1}}^1, T^* b_Q^2 \rangle| &\lesssim \|1_Q T^* b_Q^2\|_1 \lesssim |Q|. \end{aligned}$$

Thus

$$|\langle T(1_{(3Q)^c} b_{Q^{a,1}}^1), 1_Q b_{Q^{a,2}}^2 \rangle| \lesssim \|T(1_{(3Q)^c} b_{Q^{a,1}}^1)\|_\infty |Q| \lesssim |Q|.$$

Combined with the case of the off-diagonal bounds that we already treated, we have completed the proof of Proposition 2.20.

7. CONCLUDING REMARK

We have completed the proofs of Theorems 2.9 and 2.14, dealing with the boundedness of singular integral operators on \mathbb{R}^d with the Lebesgue measure. Using the dyadic cubes of Christ [4] in place of the standard dyadic cubes, these results extend to a metric space with a doubling measure without difficulty. In particular, a Hardy inequality is also valid for Christ's dyadic cubes, as observed by Auscher and Routin [2].

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