ON DYADIC NONLOCAL SCHRÖDINGER EQUATIONS WITH BESOV INITIAL DATA

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ABSTRACT. In this paper we characterize a dyadic type Besov space as an adequate setting to solve the Schrödinger-Dirac type equation $i\frac{\partial u}{\partial t} = D^\beta u$ with $u(x,0) = u^0$ pointwise. Here D^β is the fractional derivative of order β associated to the dyadic distance δ on $(0,1)$.

1. INTRODUCTION

In quantum mechanics time dependent Schrödinger type equations with space derivatives of order less than two, have been considered since the introduction of the Dirac operator which is actually local and of first order [6]. More recently some fractional nonlocal Riemann-Liouville calculus, and some other nonlocal cases, have also been considered in the literature. See for instance [10], [8] and [2].

The differential operator in the space variable that we shall consider is an analogous of the nonlocal fractional derivative of order $\beta > 0$

$$\int \frac{f(x)-f(y)}{|x-y|^{1+\beta}}dy. \tag{1.1}$$

The basic difference is given by the fact that we substitute the Euclidean distance $|x-y|$ by the dyadic distance δ from x to y. To introduce our main result let us start by defining the basic metric δ and the Besov type spaces induced by δ on the interval $(0,1)$.

Let $\mathfrak{D} = \cup_{j\geq 0}\mathfrak{D}^j$ be the family of the standard dyadic intervals in $[0,1)$ intersected with $(0,1)$. In other words $I \in \mathfrak{D}$ if $I = I_k^j = [k2^{-j},(k+1)2^{-j}) \cap (0,1)$, $j \geq 0$, $k = 0,\dots,2^j-1$. Each $\mathfrak{D}^j$ contains the intervals of the j-th level, for $I \in \mathfrak{D}^j$, $|I| = 2^{-j}$. For $I \in \mathfrak{D}^j$ we shall denote by I^+ and I^- the right and left halves of I, which belong to $\mathfrak{D}^{j+1}$. Given two points x and y in $(0,1)$ its dyadic distance $\delta(x,y)$, is defined as the length of the smallest dyadic interval $J \in \mathfrak{D}$ which contains x and y. On the diagonal Δ of $\mathbb{R}^2$, δ vanishes.

Since for x fixed $\delta^{-1-\beta}(x,y)$ is not integrable, the analogous to (1.1) with $\delta(x,y)$ instead of $|x-y|$ in $(0,1)$ is well defined as an absolutely convergent integral, only on a subspace of somehow regular functions. For $0 < \lambda < 1$, with $B^\lambda_{2,dy}$ we denote the class of all L_2 complex valued functions f defined on $(0,1)$ such that

$$\iint_Q \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy < \infty,$$

2010 *Mathematics Subject Classification.* Primary 35Q41 , 46E35.
Key words and phrases. Schrödinger equation, Besov spaces, Haar bases, Nonlocal derivatives.

where Q is the square $(0,1)^2$. For f and g both in $B^{\lambda}_{2,dy}$, the inner product

$$\int_0^1 fg dx + \iint_Q \frac{f(x)-f(y)}{\delta(x,y)^{\lambda}} \frac{g(x)-g(y)}{\delta(x,y)^{\lambda}} \frac{dxdy}{\delta(x,y)},$$

gives a Hilbert structure on $B^{\lambda}_{2,dy}$.

Since, as it is easy to check from the definition of δ, $|x-y| \leq \delta(x,y)$ when $(x,y) \in Q$, we have that the standard Besov space B^{λ}_2 on $(0,1)$ is a subspace of $B^{\lambda}_{2,dy}$. See [11] for the classical theory of Besov spaces.

For $I \in \mathfrak{D}$ we shall write h_I to denote the Haar wavelet adaptated to I. In other words $h_I = |I|^{-\frac{1}{2}} (\mathcal{X}_{I^-} - \mathcal{X}_{I^+})$ where, as usual $\mathcal{X}_E$ is the indicator function of the set E. Sometimes, when the parameters of scale and position j and k, need to be emphasized, we shall write h^j_k to denote h_I for $I = I^j_k$. In the sequel the scale parameter j of I will be denoted by $j(I)$. As it is well known $\{h_I : I \in \mathfrak{D}\}$ is an orthonormal basis for L_2 on $(0,1)$.

As a consequence of Theorem 9 in Section 3, we shall obtain the next result.

Theorem 1. *Let $0 < \beta < 1$ and $u^0 \in L_2$ with $\int_0^1 u^0 dx = 0$, be given. Assume that u^0 is a function in B^{λ}_2 with $\beta < \lambda < 1$, then the function defined by*

$$u(x,t) = \sum_{I \in \mathfrak{D}} e^{it|I|^{-\beta}} \left\langle u^0, h_I \right\rangle h_I(x)$$

solves the problem

$$(P) \qquad \begin{cases} i\dfrac{\partial u}{\partial t}(x,t) = \displaystyle\int_{(0,1)} \frac{u(x,t)-u(y,t)}{\delta(x,y)^{1+\beta}}\, dy & x \in (0,1), t > 0 \\ u(x,0) = u^0(x) & x \in (0,1); \end{cases}$$

where the initial condition is verified pointwise almost everywhere.

The identification of function spaces with low regularity, for which the pointwise convergence to the initial data for solutions of the time dependent free particle Schrödinger equation is a hard problem. Some basic fundamental steps in this direction are contained in [1], [4], [7], [3], [13], [15], [14].

The paper is organized as follows. In Section 2 we introduce the basic operator and the corresponding Besov space and its wavelet characterization in terms of the Haar system. In Section 3 we prove the main result, which contains a detailed formulation of Theorem 1.

As a final remark we would like to mention that the results obtained in [12], suggest that our problem *(P)* could be of help at solving the analogous problem with the operator (1.1) on the right hand side of the differential equation in *(P)*.

2. Nonlocal dyadic differential operators and dyadic Besov spaces

Let $0 < \beta < 1$ be given. We shall deal with the operator D^{β} whose spectral form in the Haar system is given by $D^{\beta} h_I = |I|^{-\beta} h_I$ for $I \in \mathfrak{D}$.

Let $\mathcal{S}(\mathscr{H})$ be the linear span of the Haar system $\mathscr{H} = \{h_I : I \in \mathfrak{D}\}$. The space $\mathcal{S}(\mathscr{H})$ is dense in L_2.

The operator D^β is well defined from $\mathcal{S}(\mathscr{H})$ into itself and is given by

$$D^\beta f = \sum_{I\in\mathfrak{D}} |I|^{-\beta} \langle f, h_I\rangle h_I$$

for $f \in \mathcal{S}(\mathscr{H})$. Observe that D^β is unbounded with the L_2 norm.

In the next result we show that D^β has the structure of a nonlocal differential operator if we change the Euclidean distance by the dyadic distance on $(0,1)$.

Theorem 2. *Let $0 < \beta < 1$ be given, then for $f \in \mathcal{S}(\mathscr{H})$ we have*

$$D^\beta f(x) = \int_0^1 \frac{f(x) - f(y)}{\delta(x,y)^{1+\beta}} dy, \tag{2.1}$$

where the integral on the right hand side is absolutely convergent.

Before proving Theorem 2, we collect some basic properties of δ.

Lemma 3.

(3.a) Q is the disjoint union of the diagonal Δ and the level sets $\Lambda_j = \{(x,y) \in Q : \delta(x,y) = 2^{-j}\}$ of δ, for $j \geq 0$. (See Figure 1)

(3.b) For $\gamma \in \mathbb{R}$, $\delta^\gamma = \sum_{j\geq 0} 2^{-j\gamma}\mathcal{X}_{\Lambda_j}$.

(3.c) Each Λ_j is the disjoint union of the sets $B(I) = (I^+ \times I^-) \cup (I^- \times I^+)$ for $I \in \mathfrak{D}^j$.

(3.d) For $f \in \mathcal{S}(\mathscr{H})$, set $F(x,y) = f(x) - f(y)$, then $\inf\{\delta(x,y) : (x,y) \in \operatorname{supp} F\} > 0$.

(3.e) Let $\alpha > -1$. Then for every $x \in (0,1)$, $\delta(x,y)^\alpha$ is integrable as a function of $y \in (0,1)$. Moreover, $\int_0^1 \delta(x,y)^\alpha dy$ is bounded by $(2^{1+\alpha} - 1)^{-1}$.

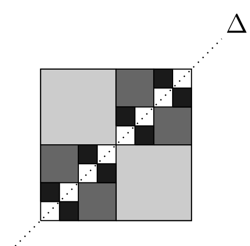

FIGURE 1. The picture depicts schematically the level sets Λ_j of δ for $j = 0$ (lightgray), for $j = 1$ (darkgray) and $j = 2$ (black).

Proof of Lemma 3.

Proof of (3.a). Given a point $(x,y) \in Q$ which does not belong to Δ, since for some $J \in \mathfrak{D}^0$, $(x,y) \in J \times J$ and since $x \neq y$, there exists one and only one subinterval I of J such that x and y belong both to I but not both to the same half of I. In other words $(x,y) \in B(I)$. Since $I \subset J$ then, $j(I) \geq 0$ and $\delta(x,y) = 2^{-j(I)}$, so that $(x,y) \in \Lambda_{j(I)}$.

Proof of (3.b). Follows directly from *(3.a)*.

Proof of (3.c). Notice first that if I and J are two different intervals on $\mathfrak{D}^j$, then $I^+ \cap J^+ = \emptyset$ and $I^- \cap J^- = \emptyset$ hence $B(I) \cap B(J) = \emptyset$. On the other hand, if $(x,y) \in B(I)$ for some $I \in \mathfrak{D}^j$, then $x \in I^+$ and $y \in I^-$ or $x \in I^-$ and $y \in I^+$, so that the smallest dyadic interval containing both x and y is I itself. This means that $\delta(x,y) = 2^{-j}$, in other words $(x,y) \in \Lambda_j$. Assume now that (x,y) is any point in Λ_j, then $\delta(x,y) = 2^{-j}$. This means that there exists $I \in \mathfrak{D}^j$ such that $(x,y) \in I \times I$ but x and y do not belong to the same half of I. In other words $(x,y) \in I \times I$ but $(x,y) \notin (I^- \times I^-) \cup (I^+ \times I^+)$. Hence $(x,y) \in B(I)$.

Proof of (3.d). Since any $f \in \mathcal{S}(\mathscr{H})$ is finite linear combination of some of the h_I's, all we need to prove is that $\inf\{\delta(x,y) : (x,y) \in \operatorname{supp} H_I\} > 0$ for every $I \in \mathfrak{D}$, where $H_I(x,y) = h_I(x) - h_I(y)$. Take $I \in \mathfrak{D}$, then $I \in \mathfrak{D}^j$ for some $j \geq 0$ and H_I vanishes on $(I^- \times I^-) \cup (I^+ \times I^+)$ hence $\delta(x,y) \geq 2^{-j}$ for every $(x,y) \in \operatorname{supp} H_I$.

Proof of (3.e). The desired properties are trivial for $\alpha \geq 0$. Assume then that $-1 < \alpha < 0$ and $x \in (0,1)$. Then

$$\begin{aligned}\int_0^1 \delta(x,y)^\alpha dy &= \sum_{k=0}^{\infty} \int_{\{y \in (0,1): 2^{-k-1} < \delta(x,y) < 2^{-k}\}} \delta(x,y)^\alpha dy \\ &\leq \sum_{k=0}^{\infty} 2^{-\alpha(k+1)} \left|\{y \in (0,1) : \delta(x,y) < 2^k\}\right| \\ &\leq \sum_{k=0}^{\infty} 2^{-(1+\alpha)(k+1)} = (2^{1+\alpha} - 1)^{-1}.\end{aligned}$$

□

Proof of Theorem 2. It is enough to check (2.1) for $f = h_I$ with $I = I^{j(I)}_{k(I)} \in \mathfrak{D}$. From *(3.b)* and *(3.c)* we have

$$\begin{aligned}&\int_0^1 \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\beta}} dy \\ &= \int_0^1 \Big(\sum_{j \geq 0} 2^{j(1+\beta)} \mathcal{X}_{\Lambda_j}(x,y)\Big)(h_I(x) - h_I(y)) dy \\ &= \sum_{j \geq 0} 2^{j(1+\beta)} \int_0^1 \mathcal{X}_{\Lambda_j}(x,y)(h_I(x) - h_I(y)) dy \\ &= 2^{j(I)(1+\beta)} \int_0^1 \mathcal{X}_{B(I)}(x,y)(h_I(x) - h_I(y)) dy \\ &= 2^{j(I)(1+\beta)} 2^{\frac{j(I)}{2}} \int_{\{y \in (0,1): (x,y) \in B(I)\}} [h(2^{j(I)}x - k(I)) - h(2^{j(I)}y - k(I))] dy.\end{aligned}$$

On the other hand

$$\int_{\{y \in (0,1): (x,y) \in B(I)\}} [h(2^{j(I)}x - k(I)) - h(2^{j(I)}y - k(I))] dy = \begin{cases} 2\frac{1}{2} 2^{-j(I)}, & x \in I^-; \\ -2\frac{1}{2} 2^{-j(I)}, & x \in I^+; \\ 0, & x \notin I, \end{cases}$$

which equals $2^{-\frac{3}{2}j(I)} h_I$. Hence

$$\int_0^1 \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\beta}} dy = 2^{j(I)(1+\beta)} 2^{\frac{j(I)}{2}} 2^{-\frac{3}{2}j(I)} h_I(x) = 2^{j(I)\beta} h_I(x) = D^\beta h_I(x).$$

□

A basic identity to obtain a characterization of the Besov type spaces in terms of the Haar system is contained in Theorem 4.

Theorem 4. *Let $0 < \lambda < 1$, be given, then the identity*

$$\iint_Q \frac{|f(x) - f(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy = \sum_{I \in \mathfrak{D}} |\langle f, h_I \rangle|^2 \left[(2 + c_\lambda) |I|^{-2\lambda} - c_\lambda \right] \tag{2.2}$$

holds for every function $f \in \mathcal{S}(\mathscr{H})$ and $c_\lambda = 2(2^{2\lambda} - 1)^{-1}$.

Theorem 4 will be a consequence of some elementary geometric properties of the dyadic system and the distance δ.

Lemma 5.

(5.a) *Set $C(J) = [(J \times (0,1)) \cup ((0,1) \times J)] \setminus (J \times J)$ for $J \in \mathfrak{D}$, then $B(I) \cap C(J) = \emptyset$ for $j(I) \geq j(J)$.*

(5.b) *For every $I \in \mathfrak{D}$ and every $j = 0, 1, \ldots, j(I) - 1$ there exists one and only one $J \in \mathfrak{D}^j$ for which $B(J)$ intersects $C(I)$.*

(5.c) *For each $I \in \mathfrak{D}$ we have*

$$\sum_{j \geq 0} 2^{j(1+2\lambda)} \sum_{J \in \mathfrak{D}^j} m(B(J) \cap C(I)) = c_\lambda |I| (|I|^{-2\lambda} - 1),$$

where m is the area measure in $\mathbb{R}^2$, $\lambda > 0$ and $c_\lambda = 2(2^{2\lambda} - 1)^{-1}$.

(5.d) *For $j \geq 0$, $I \in \mathfrak{D}$ and $J \in \mathfrak{D}$, with $I \neq J$,*

$$\mathcal{I}(j, I, J) := \iint_Q \mathcal{X}_{\Lambda_j}(x,y)[h_I(x) - h_I(y)][h_J(x) - h_J(y)] dxdy = 0.$$

(5.e) *For each $I \in \mathfrak{D}$,*

$$\sum_{j \geq 0} 2^{j(1+2\lambda)} \mathcal{I}(j, I, I) = (2 + c_\lambda) |I|^{-2\lambda} - c_\lambda.$$

Let us start by proving Theorem 4 assuming the results in Lemma 5.

Proof of Theorem 4. Let f be a finite linear combination of some of the Haar functions h_I for $I \in \mathfrak{D}$, i.e. $f = \sum_{I \in \mathfrak{D}} \langle f, h_I \rangle h_I$ with $\langle f, h_I \rangle = 0$ except for a finite number of I in $\mathfrak{D}$. From *(3.a)* and *(3.b)* in Lemma 3, *(5.c)*, *(5.d)* and *(5.e)* in Lemma 5 we get

$$
\begin{aligned}
\iint_Q \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy
&= \iint_Q \Big(\sum_{j\geq 0} 2^{j(1+2\lambda)}\mathcal{X}_{\Lambda_j}(x,y)\Big)\Big(\sum_{I\in\mathfrak{D}}\sum_{J\in\mathfrak{D}} \langle f,h_I\rangle \langle f,h_J\rangle \\
&\qquad \times [h_I(x)-h_I(y)][h_J(x)-h_J(y)]\Big)dxdy \\
&= \sum_{j\geq 0} 2^{j(1+2\lambda)} \sum_{I\in\mathfrak{D}}\sum_{J\in\mathfrak{D}} \langle f,h_I\rangle \langle f,h_J\rangle \iint_Q \mathcal{X}_{\Lambda_j}(x,y) \\
&\qquad \times [h_I(x)-h_I(y)][h_J(x)-h_J(y)]dxdy \\
&= \sum_{j\geq 0} 2^{j(1+2\lambda)} \sum_{I\in\mathfrak{D}} |\langle f,h_I\rangle|^2 \iint_{\Lambda_j} [h_I(x)-h_I(y)]^2 dxdy \\
&= \sum_{I\in\mathfrak{D}} |\langle f,h_I\rangle|^2 \sum_{j\geq 0} 2^{j(1+2\lambda)} \iint_{\Lambda_j} [h_I(x)-h_I(y)]^2 dxdy \\
&= \sum_{I\in\mathfrak{D}} |\langle f,h_I\rangle|^2 \sum_{j\geq 0} 2^{j(1+2\lambda)}\mathcal{I}(j,I,I) \\
&= \sum_{I\in\mathfrak{D}} |\langle f,h_I\rangle|^2 \left[(2+c_\lambda)\,|I|^{-2\lambda}-c_\lambda\right].
\end{aligned}
$$

□

Proof of Lemma 5.

Proof of (5.a). Since $j(I)\geq j(J)$ we have $I\subseteq J$ or $I\cap J=\emptyset$, we divide our analysis in these two cases. When $I\cap J=\emptyset$, then $I^+\cap J=\emptyset$ and $I^-\cap J=\emptyset$ and $B(I)\cap C(J)=$. Assume now that $I\subseteq J$, then $B(I)\subset J\times J$ which is disjoint from $C(J)$.

Proof of (5.b). Let $I\in\mathfrak{D}$ and $j=0,1,\dots,j(I)-1$ be given. Let J be the only dyadic interval in $\mathfrak{D}^j$ such that $J\supsetneqq I$. Then $C(I)\cap B(J)\neq\emptyset$. In fact, since $J\supsetneqq I$, then $I\subset J^+$ or $I\subset J^-$. Assume for example that $I\subset J^+$, then any point (x,y) with $x\in I$ and $y\in J^-$ belongs to both $C(I)$ and $B(J)$. So that, since for $J\in\mathfrak{D}^j$ and $j<j(I)$, arguing as in the proof of *(5.a)*, the condition $J\supset I$ is necessary for $B(J)\cap C(I)\neq\emptyset$, we get the result.

Proof of (5.c). Let $I\in\mathfrak{D}$ be given. For $j=0,1,\dots,j(I)-1$ set $J(j,I)$ to denote the only $J\in\mathfrak{D}^j$ for which $B(J)\cap C(I)\neq\emptyset$, provided by *(5.b)*. Now from *(5.a)* we have

$$
\begin{aligned}
\sum_{j\geq 0} 2^{j(1+2\lambda)} \sum_{J\in\mathfrak{D}^j} m(B(J)\cap C(I)) &= \sum_{j=0}^{j(I)-1} 2^{j(1+2\lambda)} \sum_{J\in\mathfrak{D}^j} m(B(J)\cap C(I)) \\
&= \sum_{j=0}^{j(I)-1} 2^{j(1+2\lambda)} m(B(J(j,I))\cap C(I)).
\end{aligned}
$$

But, as it is easy to see, $m(B(J(j,I)) \cap C(I)) = 2\,|I|\,2^{-j}$. Hence

$$\sum_{j\geq 0} 2^{j(1+2\lambda)} \sum_{J\in\mathfrak{D}^j} m(B(J)\cap C(I)) = 2\,|I| \sum_{j=0}^{j(I)-1} 2^{j(1+2\lambda)} 2^{-j} = c_\lambda\,|I|\,(|I|^{-2\lambda} - 1).$$

Proof of (5.d). From *(3.c)* it is enough to show that $\iint_{B(K)} k_{IJ}(x,y)dxdy = 0$ for every I, J and $K \in \mathfrak{D}$ with $I \neq J$, where $k_{IJ}(x,y) = (h_I(x) - h_I(y))(h_J(x) - h_J(y))$. We shall divide our analysis in two cases according to the relative positions of I and J.

Assume first that $I \cap J = \emptyset$, more precisely, assume that I is to the left of J. Then $k_{IJ}(x,y) = \sqrt{|I|\,|J|}[(\mathcal{X}_{I^-\times J^+}(x,y) - \mathcal{X}_{I^-\times J^-}(x,y) + \mathcal{X}_{I^+\times J^-}(x,y) - \mathcal{X}_{I^+\times J^+}(x,y)) + (\mathcal{X}_{J^-\times I^+}(x,y) - \mathcal{X}_{J^-\times I^-}(x,y) + \mathcal{X}_{J^+\times I^-}(x,y) - \mathcal{X}_{J^+\times I^+}(x,y))]$ whose support is $(I \times J) \cup (J \times I)$. See Figure 2.

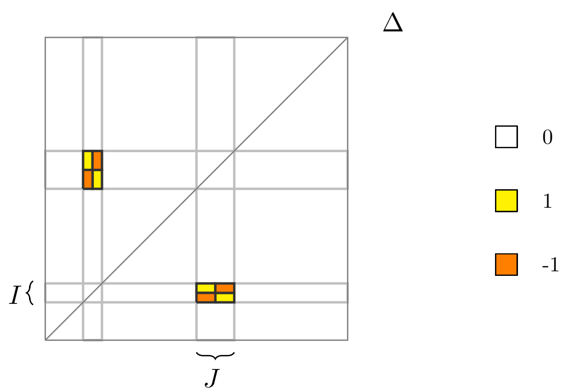

FIGURE 2. Values of $\frac{k_{IJ}}{\sqrt{|I||J|}}$

Notice that while $I \times J$ lies above the diagonal, $J \times I$ is contained in $\{y < x\}$. When $B(K)$ does not intersect $(I \times J) \cup (J \times I)$ then $\iint_{B(K)} k_{IJ}(x,y)dxdy = 0$. Assume now that $B(K) \cap [(I\times J) \cup (J \times I)] \neq \emptyset$. Since $\iint_{(0,1)^2} k_{IJ}(x,y)dxdy = 0$, if we show that $B(K) \cap [(I \times J) \cup (J \times I)] \neq \emptyset$ implies $(I \times J) \cup (J \times I) \subseteq B(K)$ we have $\iint_{B(K)} k_{IJ} dxdy = 0$. Since the set $B(K) \cap [(I \times J) \cup (J \times I)] = [(K^- \times K^+) \cup (K^+ \times K^-)] \cap [(I \times J) \cup (J \times I)]$ is nonempty, we see that $(K^- \times K^+) \cap (I \times J) \neq \emptyset$. Since $K^- \cap I \neq \emptyset$ and $K^+ \cap J \neq \emptyset$ and K, I and J are dyadic intervals with $I \cap J = \emptyset$, we must have that $K^- \supset I$ and $K^+ \supset J$. Therefore $B(K) \supset [(I \times J) \cup (J \times I)]$.

Let us assume now that I and J are nested. For example that $I \subsetneqq J$. Figure 3 depicts in this situation the normalize kernel $\frac{k_{IJ}}{\sqrt{|I||J|}}$.

Since $k_{IJ}(x,y) = k_{IJ}(y,x)$ and $B(K)$ is symmetric, we only need to show that $\iint_{K^+\times K^-} k_{IJ}(x,y)dxdy = 0$. When $j(K) \geq j(J) + 1$, $\operatorname{supp} k_{IJ} \cap B(K) = \emptyset$ and $\iint_{B(K)} k_{IJ}(x,y)dxdy = 0$. Assume on the other hand that $0 \leq j(K) \leq j(J)$. In this case the intersection of the support of k_{IJ} and $B(K)$ can still be empty or, if not, the kernel $k_{IJ}(x,y)$ on $K^+ \times K^-$ takes only two opposite constant non trivial values on subsets of the same area. Hence, again, $\iint_{B(K)} k_{IJ}(x,y)dxdy = 0$. See Figure 4 where two possible positions of K when $I \subsetneqq J$ are illustrated.

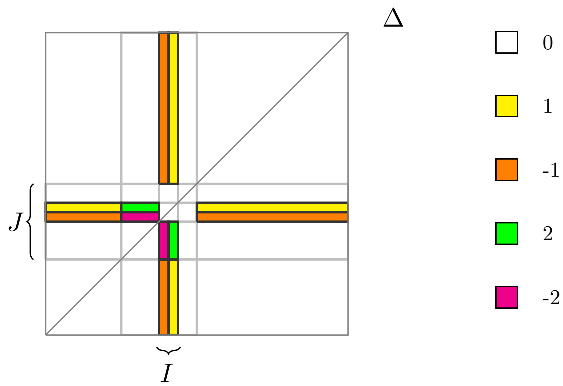

FIGURE 3. Values of $\frac{k_{IJ}}{\sqrt{|I||J|}}$

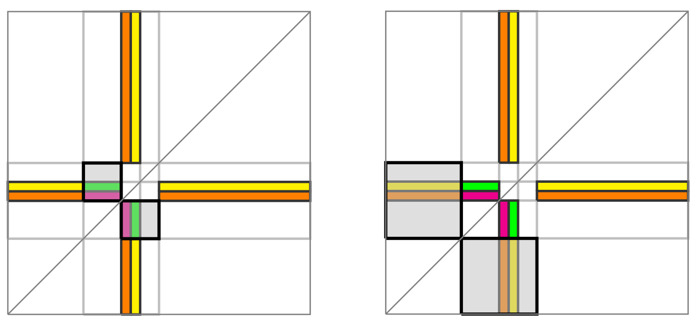

FIGURE 4. On the left, K equals J, and on the right, K is the father of J.

Proof of (5.e). Let us start by computing $\mathcal{I}(j, I, I)$ for $j \geq 0$ and $I \in \mathfrak{D}$. From *(3.c)* we get

$$\begin{aligned}\mathcal{I}(j, I, I) &= \iint_{\Lambda_j} [h_I(x) - h_I(y)]^2 dx dy \\ &= |I|^{-1} \sum_{J \in \mathfrak{D}^j} \iint_{B(J)} [4\mathcal{X}_{B(I)} + \mathcal{X}_{C(I)}] dx dy \\ &= 4\,|I|^{-1} \sum_{J \in \mathfrak{D}^j} m(B(J) \cap B(I)) + |I|^{-1} \sum_{J \in \mathfrak{D}^j} m(B(J) \cap C(I))\end{aligned}$$

for $j \geq 0$ and $I \in \mathfrak{D}$. Hence, since from *(3.a)* and *(3.c)*, $B(J) \cap B(I) = \emptyset$ for $I \neq J$ and then applying *(5.c)*

$$
\begin{aligned}
&\sum_{j\geq 0} 2^{j(1+2\lambda)} \mathcal{I}(j, I, I) \\
&\quad = \sum_{j\geq 0} 2^{j(1+2\lambda)} \left\{ 4\,|I|^{-1} \sum_{J\in\mathfrak{D}^j} m(B(J)\cap B(I)) + |I|^{-1} \sum_{J\in\mathfrak{D}^j} m(B(J)\cap C(I)) \right\} \\
&\quad = 4\,|I|^{-1}\, 2^{j(I)(1+2\lambda)} \frac{|I|^2}{2} + c_\lambda (|I|^{-2\lambda} - 1) \\
&\quad = (2 + c_\lambda)\,|I|^{-2\lambda} - c_\lambda.
\end{aligned}
$$

□

For $0 < \lambda < 1$, a function $f \in L_2$ is said to belong to the **Besov space** $B^\lambda_{2,dy}$ if the function $\frac{f(x)-f(y)}{\delta(x,y)^\lambda}$ belongs to $L_2(Q, \frac{dxdy}{\delta(x,y)})$. In other words, $f \in B^\lambda_{2,dy}$ if and only if

$$\|f\|^2_{B^\lambda_{2,dy}} = \|f\|^2_{L_2} + \iint_Q \frac{|f(x) - f(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy$$

is finite.

For our purposes the main result concerning $B^\lambda_{2,dy}$ is the following Haar wavelet characterization of the Besov space. For the classical nondyadic Euclidean case see for example [9].

Theorem 6. *Let $0 < \lambda < 1$ be given. The space $B^\lambda_{2,dy}$ coincides with the set of all square integrable functions on $(0,1)$ for which*

$$\sum_{I\in\mathfrak{D}} \left| \frac{\langle f, h_I\rangle}{|I|^\lambda} \right|^2 < \infty.$$

Moreover, $\|f\|_{L_2} + \left(\sum_{I\in\mathfrak{D}} \left| \frac{\langle f,h_I\rangle}{|I|^\lambda} \right|^2 \right)^{\frac{1}{2}}$ *is equivalent to* $\|f\|_{B^\lambda_{2,dy}}$.

Proof. Assume first the f is an L_2 function such that $\sum_{I\in\mathfrak{D}} \frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}} < \infty$. Let $\mathcal{F}_n$ be an increasing sequence of finite subfamilies of $\mathfrak{D}$ with $\cup_{n=1}^\infty \mathcal{F}_n = \mathfrak{D}$ and if $f_n = \sum_{I\in\mathcal{F}_n} \langle f, h_I\rangle\, h_I$ we have both the L_2 and a.e. pointwise convergence of f_n to f. Then from Fatou's Lemma we have that

$$
\begin{aligned}
\iint_Q \frac{|f(x) - f(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy &= \iint_Q \lim_{n\to\infty} \frac{|f_n(x) - f_n(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy \\
&\leq \liminf_{n\to\infty} \iint_Q \frac{|f_n(x) - f_n(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy.
\end{aligned}
$$

Now, since each $f_n \in \mathcal{S}(\mathscr{H})$, from Theorem 4 we get

$$
\begin{aligned}
\iint_Q \frac{|f_n(x) - f_n(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy &= \sum_{I\in\mathcal{F}_n} |\langle f, h_I\rangle|^2 \left[(2 + c_\lambda)\,|I|^{-2\lambda} - c_\lambda\right] \\
&\leq 2 \sum_{I\in\mathfrak{D}} \frac{|\langle f, h_I\rangle|^2}{|I|^{2\lambda}},
\end{aligned}
$$

hence

$$\iint_Q \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy \leq 2 \sum_{I\in\mathfrak{D}} \frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}.$$

In order to prove the opposite inequality let us start by noticing that the identity (2.2) in Theorem 4 provides, by polarization, the following formula which holds for every φ and $\psi \in \mathcal{S}(\mathscr{H})$

$$\iint_Q \frac{\varphi(x)-\varphi(y)}{\delta(x,y)^\lambda} \frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda} \frac{dxdy}{\delta(x,y)} = \sum_{I\in\mathfrak{D}} \langle\varphi,h_I\rangle \langle\psi,h_I\rangle [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda]. \quad (2.3)$$

Assume that $f \in B^\lambda_{2,dy}$. Since for any $\psi \in \mathcal{S}(\mathscr{H})$ by *(3.d)*, the function $\frac{\psi(x)-\psi(y)}{\delta(x,y)^{1+2\lambda}}$ has support at a positive δ-distance of the diagonal Δ, we have that it is bounded on Q. Hence $\frac{\psi(x)-\psi(y)}{\delta(x,y)^{1+2\lambda}} \in L_2(Q,dxdy)$. Taking in (2.3) $f_n = \sum_{I\in\mathcal{F}_n} \langle f,h_I\rangle h_I$ instead of φ with $\mathcal{F}_n$ as before, we get

$$\iint_Q \frac{f_n(x)-f_n(y)}{\delta(x,y)^\lambda} \frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda} \frac{dxdy}{\delta(x,y)} = \sum_{\substack{I\in\mathcal{F}_n,\\ \langle\psi,h_I\rangle\neq 0}} \langle f,h_I\rangle \langle\psi,h_I\rangle [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda].$$

Now since $f_n(x)-f_n(y)$ tends $f(x)-f(y)$ in $L_2(Q,dxdy)$ and $\psi \in \mathcal{S}(\mathscr{H})$ we get

$$\iint_Q \frac{f(x)-f(y)}{\delta(x,y)^\lambda} \frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda} \frac{dxdy}{\delta(x,y)} = \sum_{I\in\mathfrak{D}} \langle f,h_I\rangle \langle\psi,h_I\rangle [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda].$$

We have to prove that $\sum \frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}$ is finite. This quantity can be estimated by duality, since

$$\left(\sum_{I\in\mathfrak{D}} \frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}\right)^{\frac{1}{2}} = \sup \sum_{I\in\mathfrak{D}} \frac{\langle f,h_I\rangle}{|I|^\lambda} b_I$$

where the supremum is taken on the family of all sequences (b_I) with $\sum_{I\in\mathfrak{D}} b_I^2 \leq 1$ and $b_I = 0$ except for a finite number of I's in $\mathfrak{D}$. Notice that every such sequence (b_I) can be uniquely determined by the sequence of Haar coefficient of the function $\psi = \sum b_I |I|^{-\lambda} [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda]^{-1} h_I \in \mathcal{S}(\mathscr{H})$. In fact, $b_I = \langle\psi,h_I\rangle |I|^\lambda [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda]$. Hence the condition $\sum_{I\in\mathfrak{D}} b_I^2 \leq 1$ becomes $\sum_{I\in\mathfrak{D}} \langle\psi,h_I\rangle^2 |I|^{2\lambda} [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda]^2 \leq 1$. So

$$\begin{aligned}
\sum_{I\in\mathfrak{D}} \frac{\langle f,h_I\rangle}{|I|^\lambda} b_I &= \sum_{I\in\mathfrak{D}} \langle f,h_I\rangle \langle\psi,h_I\rangle [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda] \\
&= \iint_Q \frac{f(x)-f(y)}{\delta(x,y)^\lambda} \frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda} \frac{dxdy}{\delta(x,y)} \\
&\leq \left[\iint_Q \left(\frac{|f(x)-f(y)|}{\delta(x,y)^\lambda}\right)^2 \frac{dxdy}{\delta(x,y)}\right]^{\frac{1}{2}} \iint_Q \left(\frac{|\psi(x)-\psi(y)|}{\delta(x,y)^\lambda}\right)^2 \frac{dxdy}{\delta(x,y)} \\
&\leq \|f\|_{B^\lambda_{2,dy}},
\end{aligned}$$

since $\iint_Q \left(\frac{|\psi(x)-\psi(y)|}{\delta(x,y)^\lambda}\right)^2 \frac{dxdy}{\delta(x,y)} = \sum_{I\in\mathfrak{D}} \langle\psi,h_I\rangle^2 |I|^{2\lambda} [(2+c_\lambda)|I|^{-2\lambda} - c_\lambda]^2 \leq 1$ for $\psi \in \mathcal{S}(\mathscr{H})$. $\square$

As a corollary of Theorem 6 we easily obtain the following density result.

Corollary 7. *For $f \in B^\lambda_{2,dy}$ and $f_n = \sum_{I\in\mathcal{F}_n} \langle f,h_I\rangle h_I$ with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $\mathcal{F}_n$ finite and $\cup_{n=1}^\infty \mathcal{F}_n = \mathfrak{D}$, we have $f_n \to f$ in $B^\lambda_{2,dy}$ as $n \to \infty$.*

The above result allows to extend Theorem 2 to the general case of dyadic Besov functions.

Theorem 8. *Let $0 < \beta < \lambda < 1$ be given. Then for each $f \in B^\lambda_{2,dy}$ we have*

$$\sum_{I\in\mathfrak{D}} |I|^{-\beta} \langle f,h_I\rangle h_I(x) = \int_0^1 \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy \tag{2.4}$$

as functions in L_2.

Proof. Set $f_n = \sum_{I\in\mathcal{F}_n} \langle f,h_I\rangle h_I$ as in Corollary 7. From Theorem 2,

$$D^\beta f_n(x) = \sum_{I\in\mathcal{F}_n} |I|^{-\beta} \langle f,h_I\rangle h_I(x) = \int_0^1 \frac{f_n(x)-f_n(y)}{\delta(x,y)^\lambda} \frac{dy}{\delta(x,y)^{1-\lambda+\beta}}$$

for every n. Notice that since $\beta < \lambda$ the series $\sum \frac{|\langle f,h_I\rangle|^2}{|I|^{2\beta}}$ converges, hence the series $\sum_{I\in\mathfrak{D}} |I|^{-\beta} \langle f,h_I\rangle h_I$ converges in L_2. And then $D^\beta f_n \to \sum_{I\in\mathfrak{D}} |I|^{-\beta} \langle f,h_I\rangle h_I$ in L_2. On the other hand, from the fact $B^\lambda_{2,dy} \supset B^\beta_{2,dy}$,

$$\begin{aligned} &\int_0^1 \left| \int_0^1 \frac{f_n(x)-f_n(y)}{\delta(x,y)^{1+\beta}} dy - \int_0^1 \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy \right|^2 dx \\ &\quad = \int_0^1 \left| \int_0^1 \frac{(f_n-f)(x)-(f_n-f)(y)}{\delta(x,y)^\lambda} \frac{1}{\delta(x,y)^{\beta-\lambda}} \frac{dy}{\delta(x,y)} \right|^2 dx \\ &\quad \leq \int_0^1 \left(\int_0^1 \frac{|(f_n-f)(x)-(f_n-f)(y)|^2}{\delta(x,y)^{2\lambda}} dy \right) \left(\int_0^1 \frac{dy}{\delta(x,y)^{1-2(\lambda-\beta)}} \right) dx. \end{aligned}$$

Since $1-2(\lambda-\beta) < 1$, from *(3.e)* in Lemma 3 we see that the last integral is uniformly bounded from above, so that applying Corollary 7 and taking the limit for $n \to \infty$ we get (2.4). $\square$

Theorem 8 gives a way of extending the nonlocal differential operator D^β to the spaces $B^\lambda_{2,dy}$ with $0 < \beta < \lambda < 1$ by

$$D^\beta f := \sum_{I\in\mathfrak{D}} |I|^{-\beta} \langle f,h_I\rangle h_I = \int_0^1 \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy. \tag{2.5}$$

3. The main result

In this section we state and prove a detailed formulation of Theorem 1. With the operator D^β and the spaces $B^\lambda_{2,dy}$ introduced in Section 2 the problem can now be formally written in the following way

$$(P)\qquad \begin{cases} i\dfrac{\partial u}{\partial t} = D^\beta u & \text{in } (0,1)\times\mathbb{R}^+ \\ u(0) = u^0 & \text{in } (0,1). \end{cases}$$

Theorem 9. *For $0<\beta<\lambda<1$ and $u^0 \in B^\lambda_2$ with $\int_0^1 u^0 dx = 0$, define*

$$u(t) = -\sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}} \left\langle u^0, h_I\right\rangle h_I, \quad t\geq 0. \tag{3.1}$$

Then,

(9.a) u is continuous as a function of $t\in[0,\infty)$ with values in $B^\lambda_{2,dy}$ and $u(0)=u^0$. In other words, $\|u(t)-u(s)\|_{B^\lambda_{2,dy}} \to 0$ for $s\to t$ and $t\geq 0$;

(9.b) u is differentiable as a function of $t\in(0,\infty)$ with respect to the norm $\|\cdot\|_{B^{\lambda-\beta}_{2,dy}}$, and $\frac{du}{dt} = -iD^\beta u$. Precisely, $\left\|\frac{u(t+h)-u(t)}{h} + iD^\beta u\right\|_{B^{\lambda-\beta}_{2,dy}} \to 0$ when $h\to 0$;

(9.c) there exists $Z\subset(0,1)$ with $|Z|=0$ such that the series (3.1) defining $u(t)$ converges pointwise for every $t\in[0,1)$ outside Z;

(9.d) $u(t)\to u^0$ pointwise almost everywhere on $(0,1)$ when $t\to 0$.

Notice that pointwise convergence is not a consequence of convergence in the $B^\lambda_{2,dy}$ norm. In fact, with the standard notation for the Haar system $h^j_k(x) = 2^{\frac{j}{2}} h(2^j x - k)$, we define a sequence of functions supported in $(0,1)$ in the following way. Let n be a given positive integer. Then there exists one and only one $j = 0,1,2,\dots$ such that $2^j \leq n < 2^{j+1}$. Set $f_n = 2^{-\frac{j}{2}} h^j_{n-2^j}$. Then $\|f_n\|_{L_2} = 2^{-\frac{j}{2}}$ which tends to 0 as $n\to\infty$. Since $D^\lambda f_n = 2^{-\frac{j}{2}} D^\lambda h^j_{n-2^j} = 2^{j\lambda} 2^{-\frac{j}{2}} h^j_{n-2^j} = 2^{-j(\frac{1}{2}-\lambda)} h^j_{n-2^j}$, we see that for $0<\lambda<\frac{1}{2}$, $\left\|D^\lambda f_n\right\|_{L_2} \to 0$ as $n\to\infty$. Hence $f_n \to 0$ in the $B^\lambda_{2,dy}$ sense. Nevertheless f_n does not converge pointwise.

Before proving Theorem 9 we shall obtain some basic maximal estimates involved in the proofs of *(9.c)* and *(9.d)*. With M_{dy} we denote the Hardy-Littlewood dyadic maximal operator given by

$$M_{dy} f(x) = \sup \frac{1}{|I|}\int_I |f(y)|\, dy$$

where the supremum is taken on the family of all dyadic intervals $I\in\mathfrak{D}$ for which $x\in I$. Calderón's sharp maximal operator of order λ is defined by

$$M^\#_\lambda f(x) = \sup_J \frac{1}{|J|^{1+\lambda}} \int_J |f(y)-f(x)|\, dy,$$

where the supremum is taken on the family of all subintervals (dyadic or not) J of $(0,1)$ such that $x\in J$. In [5], see Corollary 11.6, DeVore and Sharpley prove that the L_p norm of $M^{\#}_{\lambda}f$ is bounded by B^{λ}_p norm of f. For our purposes the case $p=2$ is of particular interest,

$$\left\|M^{\#}_{\lambda}f\right\|_{L_2} \leq A\,\|f\|_{B^{\lambda}_2}. \tag{3.2}$$

When dealing with *(9.c)* and *(9.d)* two maximal operators related to the series (3.1) are also relevant. For $t>0$ set

$$S^*_t f(x) = \sup_{N\in\mathbb{N}} \left|S^N_t f(x)\right|, \text{ where } S^N_t f(x) = \sum_{j=0}^{N}\sum_{k=0}^{2^j-1} e^{it2^{j\beta}} \left\langle f, h^j_k\right\rangle h^j_k(x).$$

Set

$$S^* f(x) = \sup_{0<t<1} S^*_t f(x).$$

The next result contains the basic estimates of S^*_t and S^* in terms of M_{dy} and $M^{\#}_{\lambda}$.

Lemma 10. *Let $f\in B^{\lambda}_2$ with $0<\beta<\lambda<1$. Then with $C := 2^{\lambda-\beta+1}(2^{\lambda-\beta}-1)$ we have*

(10.a) $S^*_t f(x) \leq CtM^{\#}_{\lambda}f(x) + 2M_{dy}f(x)$ *for $t\geq 0$ and $x\in(0,1)$;*
(10.b) $S^* f(x) \leq CM^{\#}_{\lambda}f(x) + 2M_{dy}f(x)$ *for $x\in(0,1)$;*
(10.c) $\left\|S^* f\right\|_{L_2} \leq (AC+2)\,\|f\|_{B^{\lambda}_2}$*, where A is the constant in* (3.2).

Proof. For $f\in B^{\lambda}_2$, $t\geq 0$ and $N\in\mathbb{N}$, we have

$$\left|S^N_t f(x)\right| \leq \left|S^N_t f(x) - S^N_0 f(x)\right| + \left|S^N_0 f(x)\right|. \tag{3.3}$$

Since $S^N_0 f(x) = P_N f(x) - \int_0^1 f(y)dy$, where P_N is the projection over the space V_N of functions which are constant on each $I\in\mathfrak{D}^N$, we have $\sup_N \left|S^N_0 f(x)\right| \leq 2M_{dy}f(x)$. Let us now estimate the first term on the right hand side of (3.3). For $x\in(0,1)$ and $j\in\mathbb{N}$, let $k(x,j) = 0,1,\ldots,2^j-1$, be the only index for which $x\in I^j_{k(x,j)}$,

$$\begin{aligned}
\left|S^N_t f(x) - S^N_0 f(x)\right| &\leq \left|\sum_{j=0}^{N}\sum_{k=0}^{2^j-1}(e^{it2^{j\beta}}-1)\left\langle f,h^j_k\right\rangle h^j_k(x)\right| \\
&= \left|\sum_{j=0}^{N}(e^{it2^{j\beta}}-1)\left(\int_{I^j_{k(j,x)}}[f(y)-f(x)]h^j_{k(j,x)}(y)\right)h^j_{k(j,x)}(x)\right| \\
&\leq \sum_{j=0}^{\infty}\left|e^{it2^{j\beta}}-1\right|\frac{1}{\left|I^j_{k(j,x)}\right|}\int_{I^j_{k(j,x)}}|f(y)-f(x)|\,dy \\
&= t\sum_{j=0}^{\infty}\frac{\left|e^{it2^{j\beta}}-1\right|}{t2^{j\lambda}}\frac{1}{\left|I^j_{k(j,x)}\right|^{1+\lambda}}\int_{I^j_{k(j,x)}}|f(y)-f(x)|\,dy \\
&\leq 2t\left(\sum_{j=0}^{\infty}2^{-(\lambda-\beta)j}\right)M^{\#}_{\lambda}f(x),
\end{aligned} \tag{3.4}$$

which proves *(10.a)*. Estimate *(10.b)* follows from *(10.a)* by taking supremum for $t < 1$. To show *(10.c)* we invoque (3.2), and the L_2 boundedness of the Hardy-Littlewood dyadic maximal operator. □

The next lemma gives the pointwise convergence of $S_t^N g(x)$ for every $x \in (0,1)$ in a dense subspace of B_2^λ.

Lemma 11. *Let g be a Lipschitz function defined on $(0,1)$. Then*

$$S_t^N g(x) = \sum_{j=0}^{N} \sum_{k=0}^{2^j-1} e^{it2^{j\beta}} \left\langle g, h_k^j \right\rangle h_k^j(x)$$

converges when $N \to \infty$, for every $x \in (0,1)$ and every $t \geq 0$.

Proof. Fix $t \geq 0$ and $x \in (0,1)$. We shall prove that $(S_t^N g(x) : N = 1, 2, \ldots)$ is a Cauchy sequence of complex numbers. In fact, for $1 \leq M \leq N$,

$$\begin{aligned}
\left|S_t^N g(x) - S_t^M g(x)\right| &= \left| \sum_{j=M+1}^{N} \sum_{k=0}^{2^j-1} e^{it2^{j\beta}} \left\langle g, h_k^j \right\rangle h_k^j(x) \right| \\
&= \left| \sum_{j=M+1}^{N} \sum_{k=0}^{2^j-1} e^{it2^{j\beta}} \left(\int_0^1 [g(y) - g(x)] h_k^j(y) dy \right) h_k^j(x) \right| \\
&\leq \sum_{j=M+1}^{N} \sum_{k=0}^{2^j-1} \|g'\|_\infty \, 2^j \int_{I_k^j} |x-y| \, dy \mathcal{X}_{I_k^j}(x) \\
&= \|g'\|_\infty \sum_{j=M+1}^{N} 2^j \int_{I_{k(j,x)}^j} |x-y| \, dy \\
&\leq \|g'\|_\infty \sum_{j=M+1}^{N} 2^{-j}.
\end{aligned}$$

□

Proof of Theorem 9.

Proof of (9.a). From Theorem 6 we see that for each $t > 0$, $u(t) \in B_{2,dy}^\lambda$, since $u^0 \in B_2^\lambda \subset B_{2,dy}^\lambda$. Moreover, for $t, s \geq 0$,

$$\begin{aligned}
\|u(t) - u(s)\|_{B_{2,dy}^\lambda} &= \left\| \sum_{I \in \mathfrak{D}} \left(e^{it|I|^{-\beta}} - e^{is|I|^{-\beta}} \right) \left\langle u^0, h_I \right\rangle h_I \right\|_{B_{2,dy}^\lambda} \\
&= \sum_{I \in \mathfrak{D}} \left| e^{it|I|^{-\beta}} - e^{is|I|^{-\beta}} \right|^2 \left| \left\langle u^0, h_I \right\rangle \right|^2 + \sum_{I \in \mathfrak{D}} \left| e^{it|I|^{-\beta}} - e^{is|I|^{-\beta}} \right|^2 \frac{\left\langle u^0, h_I \right\rangle^2}{|I|^{2\lambda}}
\end{aligned}$$

which tends to zero if $s \to t$.

Proof of (9.b). Let us prove that the formal derivative of $u(t)$ is actually the derivative in the sense of $B^{\lambda-\beta}_{2,dy}$. In fact, for $t>0$ and h such that $t+h>0$

$$\begin{aligned}
&\left\| \frac{u(t+h)-u(t)}{h} - i\sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}}\,|I|^{-\beta}\left\langle u^0,h_I\right\rangle h_I \right\|^2_{B^{\lambda-\beta}_{2,dy}} \\
&= \left\| \sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}} \left[\frac{e^{ih|I|^{-\beta}}-1}{h} - i\,|I|^{-\beta}\right]\left\langle u^0,h_I\right\rangle h_I \right\|^2_{B^{\lambda-\beta}_{2,dy}} \\
&\le c\left\{ \left\| \sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}} \left[\frac{e^{ih|I|^{-\beta}}-1}{h} - i\,|I|^{-\beta}\right]\left\langle u^0,h_I\right\rangle h_I \right\|^2_{L_2} \right. \\
&\qquad\qquad \left. + \sum_{I\in\mathfrak{D}} \left|\frac{e^{ih|I|^{-\beta}}-1}{h} - i\,|I|^{-\beta}\right|^2 \frac{\left|\left\langle u^0,h_I\right\rangle\right|^2}{|I|^{2(\lambda-\beta)}} \right\} \\
&\le c\sum_{I\in\mathfrak{D}} |I|^{2\beta} \left|\frac{e^{ih|I|^{-\beta}}-1}{h} - i\,|I|^{-\beta}\right|^2 \frac{\left|\left\langle u^0,h_I\right\rangle\right|^2}{|I|^{2\lambda}}.
\end{aligned}$$

Since, from Theorem 6, $\sum_{I\in\mathfrak{D}} \frac{|\langle u^0,h_I\rangle|^2}{|I|^{2\lambda}} < \infty$, and $|I|^{2\beta}\left|\frac{e^{ih|I|^{-\beta}}-1}{h} - i\,|I|^{-\beta}\right|^2 = \left|\frac{e^{ih|I|^{-\beta}}-1}{|I|^{-\beta}h} - i\right|^2 \to 0$ as $h\to 0$ for each $I\in\mathfrak{D}$, we obtain the result.

On the other hand since $u(t)\in B^{\lambda}_{2,dy}$ and since $\lambda>\beta$, $D^\beta u(t)$ is well defined and from (2.5) it is given by

$$D^\beta u(t) = D^\beta\left(\sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}}\left\langle u^0,h_I\right\rangle h_I\right) = \sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}}\,|I|^{-\beta}\left\langle u^0,h_I\right\rangle h_I = -i\frac{du}{dt}.$$

Hence $u(t)$ is a solution of the nonlocal equation and *(9.b)* is proved.

Proof of (9.c). The boundedness properties of S^*_t and S^* and the pointwise convergence on a dense subset of B^λ_2 allows us to use standard arguments for the a.e. pointwise convergence of $S^N_t u^0$ for general $u^0\in B^\lambda_2$. We shall prove that the set Z of all points x in $(0,1)$ such that for some $t\in(0,1)$

$$\overline{L_t}(x) := \inf_N \sup_{n,m\ge N}\left|S^n_t u^0(x) - S^m_t u^0(x)\right| > 0$$

has measure zero. It is enough to show that for each $\varepsilon>0$, the Lebesgue measure of the set $\{x\in(0,1): \overline{L_t}(x)>\varepsilon \text{ for some } t\in(0,1)\}$ vanishes. Since, for any Lipschitz function v defined on $(0,1)$ and every $t\in(0,1)$,

$$\left|S^n_t u^0(x) - S^m_t u^0(x)\right| \le \left|S^n_t(u^0-v)(x)\right| + \left|S^n_t v(x) - S^m_t v(x)\right| + \left|S^m_t(v-u^0)(x)\right|,$$

from Lemma 11, we have $\overline{L_t}(x)\le 2S^*(u^0-v)(x)$. So that, from *(10.c)* we obtain

$$\begin{aligned}
\left|\left\{x\in(0,1): \overline{L_t}(x)>\varepsilon \text{ for some } t\in(0,1)\right\}\right| &\le \left|\left\{x\in(0,1): S^*(u^0-v)(x) > \frac{\varepsilon}{2}\right\}\right| \\
&\le \frac{4}{\varepsilon^2}\left\|S^*(u^0-v)\right\|^2_{L_2} \\
&\le \frac{4(AC+2)^2}{\varepsilon^2}\left\|u^0-v\right\|^2_{B^\lambda_2}.
\end{aligned}$$

Since v is an arbitrary Lipschitz function in $(0,1)$ we get that $|Z| = 0$. Hence for every $t \in [0,1)$ and every $x \notin Z$, $(S_t^n u^0(x) : n = 1, 2, \ldots)$ is a Cauchy sequence which must converge to its L_2 limit, i.e. $u(t)(x)$ for $x \notin Z$ and $t \in [0,1)$.

Proof of (9.d). For $x \notin Z$, taking the limit as $N \to \infty$ in (3.4) we get the maximal estimate

$$\sup_{t\in(0,1)} \frac{\left|u(t)(x) - u^0(x)\right|}{t} \leq 2\frac{2^{\lambda-\beta}}{1-2^{-(\lambda-\beta)}} M_\lambda^\# u^0(x).$$

Since $M_\lambda^\# u^0$ belongs to L_2, the left hand side is finite almost everywhere, hence $u(t)(x) \to u^0(x)$ as $t \to 0$ almost everywhere. □

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