Brown's criterion in Bredon homology

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We translate Brown's criterion for homological finiteness properties to the setting of Bredon homology.

Bredon cohomology has become an important algebraic tool for studying classifying spaces $E_{\mathfrak{F}}\Gamma$ of discrete groups Γ with stabilisers in a given family $\mathfrak{F}$ of subgroups of Γ. It has been defined for finite groups by Bredon [Bre67] and the definition has been extended to arbitrary groups and families of subgroups by Lück [Lüc89]. The basic idea in passing from classical cohomology to Bredon cohomology is to replace Γ, regarded as a small category, by the orbit category $\mathcal{O}_{\mathfrak{F}}\Gamma$.

More precisely, let Γ be a discrete group. By a *family* of subgroups of Γ we mean a non-empty set $\mathfrak{F}$ of subgroups of Γ which is closed under conjugation. The transitive Γ-sets Γ/Λ with $\Lambda \in \mathfrak{F}$ and Γ-maps between them form the *orbit category* $\mathcal{O}_{\mathfrak{F}}\Gamma$. A *right (left) Bredon module over $\mathcal{O}_{\mathfrak{F}}\Gamma$* is a contravariant (covariant) functor from $\mathcal{O}_{\mathfrak{F}}\Gamma$ to the category $\mathfrak{Ab}$ of abelian groups. A morphism of Bredon modules of the same variance is a natural transformation. Right (left) Bredon modules and the morphisms between them form a category which is denoted by Mod-$\mathcal{O}_{\mathfrak{F}}\Gamma$ ($\mathcal{O}_{\mathfrak{F}}\Gamma$-Mod).

Mod-$\mathcal{O}_{\mathfrak{F}}\Gamma$ and $\mathcal{O}_{\mathfrak{F}}\Gamma$-Mod are functor categories and thefore they inherit many properties from the category $\mathfrak{Ab}$. Among others they are abelian categories in which all small limits and colimits exist, they have enough projectives and there exists a notion of being finitely generated. Details are given in Section 1 below.

Let $n \in \mathbb{N} \cup \{\infty\}$. An $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M is said to be of *type $\mathfrak{F}$-FP$_n$* if there exists a resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

of M by projective $\mathcal{O}_{\mathfrak{F}}\Gamma$-modules such that P_k is finitely generated for every $k \leq n$. The trivial $\mathcal{O}_{\mathfrak{F}}\Gamma$-module $\underline{\mathbb{Z}}$ maps every object of $\mathcal{O}_{\mathfrak{F}}\Gamma$ to $\mathbb{Z}$ and every morphism of $\mathcal{O}_{\mathfrak{F}}\Gamma$ to the identity. A group Γ is said to be of *type $\mathfrak{F}$-FP$_n$* if $\underline{\mathbb{Z}}$ is of type FP$_n$ as a right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module.

In the special case that $\mathfrak{F} = \{1\}$ consists only of the trivial group Bredon cohomology reduces to classical cohomology of groups. The finiteness properties FP$_n$ in this case have been extensively studied. The classical proofs sometimes also carry over to the case where $\mathfrak{F} = \mathcal{FIN}$ is the family of finite subgroups. This is true for example of hyperbolic groups,

arithmetic groups, mapping class groups, and outer automorphism groups of finitely generated free groups, see [Lüc05, Sections 4.7,4.8] (and trivially for torsion free groups). On the other hand Leary and Nucinkis [LN03] showed how much finiteness properties with respect to $\mathcal{FIN}$ can differ from the classical ones.

The next family of interest is the family $\mathcal{VC}$ of virtually cyclic subgroups. One result here is that for an elementary amenable group being of type $\mathcal{VC}$-FP_∞ is equivalent being virtually cyclic [KMPN11].

In the classical setting Brown's Criterion [Bro87, Theorem 2.2] has been fruitful in the study of the properties FP_n. Our main result is a translation of this criterion to the Bredon setting. In order to state it, some more definitions are needed.

A Γ-*CW-complex* X is a CW-complex on which Γ acts by cell-permuting homeomorphisms such that the stabilizer of a cell fixes that cell pointwise. We let $\underline{C}_*(X)$ denote the Bredon cellular chain complex of X, cf. [MV03, p. 11]. The *Bredon homology modules* $\underline{H}_*(X)$ of X are defined to be the homology modules of the Bredon chain complex $\underline{C}_*(X)$. Evaluated at $\Gamma/\Lambda \in \mathcal{O}_\mathfrak{F}\Gamma$ these Bredon modules give

$$\underline{H}_*(X)(\Gamma/\Lambda) = H_*(X^\Lambda)$$

where the right hand side is the ordinary homology of the fixed point complex X^Λ. This definition is functorial. Analogously to the classical case we define the *reduced Bredon homology modules* $\underline{\tilde{H}}_*(X)$ to be the kernel of the morphism $\underline{H}_*(X) \to \underline{H}_*(\mathrm{pt.})$ which is induced by the map from X to the singleton space. We say that X is $\mathfrak{F}$-*acyclic* up to dimension n if $\underline{\tilde{H}}_k(X) = 0$ for every $k \leq n$. Note that being $\mathfrak{F}$-acyclic up to dimension -1 is equivalent to the condition that $X^\Lambda \neq \emptyset$ for every $\Lambda \in \mathfrak{F}$.

The following is completely analogous to Brown's original article [Bro87]: Let $n \in \mathbb{N}$. A Γ-CW-complex X is said to be $\mathfrak{F}$-*n-good* if the following two conditions hold:

(i) X is $\mathfrak{F}$-acyclic up to dimension $n-1$, and

(ii) for every p-cell σ of X, $p \leq n$, $\mathfrak{F} \cap \Gamma_\sigma \subset \mathfrak{F}$ and the stabiliser Γ_σ of σ is of type $(\mathfrak{F} \cap \Gamma_\sigma)$-$\mathrm{FP}_{n-p}$.

A filtration $(X_\alpha)_{\alpha \in I}$ of a Γ-CW-complex X by Γ-invariant subcomplexes is said to be of *finite n-type* if the n-skeleta $X_\alpha^{(n)}$ are cocompact for all $\alpha \in I$.

A directed system of Bredon modules $(M_\alpha)_{\alpha \in I}$ is said to be *essentially trivial* if for every $\alpha \in I$ there exists $\beta \geq \alpha$ such that the homomorphism $M_\alpha \to M_\beta$ is trivial.

Main Theorem. *Let Γ be a group and $\mathfrak{F}$ a family of subgroups of Γ. Let X be an $\mathfrak{F}$-n-good Γ-CW-complex and let $(X_\alpha)_{\alpha \in I}$ be a filtration by Γ-invariant subcomplexes of finite n-type.*

Then Γ is of type $\mathfrak{F}$-FP_n if and only if the directed system $(\underline{\tilde{H}}_k(X_\alpha))_{\alpha \in I}$ of reduced Bredon homology modules is essentially trivial for all $k < n$.

The importance of a directed system being essentially trivial stems from the following fact, which is the analogue of [Bro87, Lemma 2.1].

Observation 1. *A directed system* $(M_\alpha)_{\alpha \in I}$ *of* $\mathcal{O}_{\mathfrak{F}}\Gamma$-*modules is essentially trivial if and only if*

$$\varinjlim_\alpha \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} M_\alpha(\Gamma/\Lambda) = 0$$

for every family of cardinals $(J_\Lambda)_{\Lambda \in \mathfrak{F}}$.

1 Basic definitions and results on Bredon modules

This section is to collect basic definitions and facts related to Bredon modules for further reference. Unless stated otherwise the results can be found in [Lüc89, pp. 162–169] or [MV03, p. 7–27]. By a Bredon module we mean either a left or a right Bredon module unless the variance is explicitly mentioned.

Since the category of Bredon modules is a functor category it follows that limits and colimits of Bredon modules are calculated component wise. In particular a sequence of Bredon modules $0 \to M' \to M \to M'' \to 0$ is exact if and only if the corresponding sequence

$$0 \to M'(\Gamma/\Lambda) \to M(\Gamma/\Lambda) \to M''(\Gamma/\Lambda) \to 0$$

of abelian groups is exact for every $\Gamma/\Lambda \in \mathcal{O}_{\mathfrak{F}}\Gamma$.

For subgroups Ξ and Λ of Γ we denote by $[\Gamma/\Xi, \Gamma/\Lambda]_\Gamma$ the set of all Γ-maps $\Gamma/\Xi \to \Gamma/\Lambda$. For a fixed subgroup Λ of Γ we denote by $\mathbb{Z}[-, \Gamma/\Lambda]_\Gamma$ the right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module which sends $\Gamma/\Xi \in \mathcal{O}_{\mathfrak{F}}\Gamma$ to the free abelian group $\mathbb{Z}[\Gamma/\Xi, \Gamma/\Lambda]_\Gamma$ on the basis $[\Gamma/\Xi, \Gamma/\Lambda]_\Gamma$. The left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module $\mathbb{Z}[\Gamma/\Lambda]_\Gamma$ is defined analogously. The *free objects* in Mod-$\mathcal{O}_{\mathfrak{F}}\Gamma$ are now precisely the direct sums of $\mathbb{Z}[-, \Gamma/\Lambda]_\Gamma$ with $\Lambda \in \mathfrak{F}$. Likewise the *free objects* in $\mathcal{O}_{\mathfrak{F}}\Gamma$-Mod are the direct sums of the Bredon modules $\mathbb{Z}[\Gamma/\Lambda, -]_\Gamma$ with $\Lambda \in \mathfrak{F}$. In either case a free Bredon module is *finitely generated* if the direct sum can be taken to be finite. An arbitrary Bredon module is finitely generated if it is the surjective image of a finitely generated free module.

For any two Bredon modules M and N of the same variance the set of morphisms between them is denoted by $\mathrm{mor}_{\mathfrak{F}}(M, N)$. A Bredon module P is *projective* if the functor $\mathrm{mor}_{\mathfrak{F}}(P, -)$ is exact. This is the case if and only if P is a direct sumand of a free Bredon module.

The categorical tensor product [Sch70, p. 45] gives rise to a *tensor product over* $\mathfrak{F}$. It assigns to a right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module N and left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M an abelian group $N \otimes_{\mathfrak{F}} M$. The $\mathcal{O}_{\mathfrak{F}}\Gamma$-module N is said to be *flat* if the functor $N \otimes_{\mathfrak{F}} -$ is exact. Every projective Bredon module is flat.

There exists also the *tensor product over* $\mathbb{Z}$. For two Bredon modules M and N of the same variance it is defined to be the Bredon module $M \otimes N$, which evaluated at any $\Gamma/\Lambda \in \mathcal{O}_{\mathfrak{F}}\Gamma$ is given by $(M \otimes N)(\Gamma/\Lambda) = M(\Gamma/\Lambda) \otimes N(\Gamma/\Lambda)$.

Lemma 1.1. *Let N be a right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module and $\Lambda \in \mathfrak{F}$. Then there exists an isomorphism*

$$N \otimes_{\mathfrak{F}} \mathbb{Z}[\Gamma/\Lambda, -]_{\Gamma} \cong N(\Gamma/\Lambda)$$

which is natural in N.

This statement follows from a Yoneda type argument, see for example [MV03, p. 9].

If Λ is a subgroup of Γ such that $\mathfrak{F} \cap \Lambda := \{\Xi \cap \Lambda \mid \Xi \in \mathfrak{F}\} \subset \mathfrak{F}$, then there exists a functor

$$I_{\Lambda}\colon \mathcal{O}_{\mathfrak{F}\cap\Lambda}\Lambda \to \mathcal{O}_{\mathfrak{F}}\Gamma$$

which sends Λ/Ξ to Γ/Ξ for every $\Xi \in \mathfrak{F} \cap \Lambda$. The induction functor

$$\mathrm{Ind}_{\Lambda}^{\Gamma}\colon \mathcal{O}_{\mathfrak{F}\cap\Lambda}\Lambda \to \mathcal{O}_{\mathfrak{F}}\Gamma$$

and the restriction functor

$$\mathrm{Res}_{\Lambda}^{\Gamma}\colon \mathcal{O}_{\mathfrak{F}}\Gamma \to \mathcal{O}_{\mathfrak{F}\cap\Lambda}\Lambda$$

with respect to I_{Λ} are defined in [Lüc89, p. 166].

Lemma 1.2. *The functor $\mathrm{Ind}_{\Lambda}^{\Gamma}$ preserves the properties of being finitely generated and being projective. Furthermore, it is an exact functor and $\mathrm{Ind}_{\Lambda}^{\Gamma}\underline{\mathbb{Z}} = \mathbb{Z}[-, \Gamma/\Lambda]_{\Gamma}$.*

The first statement is from [Lüc89, p. 169]. The second statement is Lemma 2.9 and Lemma 2.7 in [Sym05, p. 268].

Lemma 1.3. *The functor $\mathrm{Res}_{\Lambda}^{\Gamma}$ is exact and preserves being projective.*

For the first part of the this statement see [Lüc89, p. 169], the remaining part is [MP02, Lemma 3.7]. We also need the following special case of Proposition 3.5 in [MP02]:

Lemma 1.4. *There exists a natural isomorphism*

$$(N \otimes \mathbb{Z}[-, \Gamma/\Lambda]_{\Gamma}) \otimes_{\mathfrak{F}} M \cong \mathrm{Res}_{\Lambda}^{\Gamma} N \otimes_{\mathfrak{F}\cap\Lambda} \mathrm{Res}_{\Lambda}^{\Gamma} M$$

for any right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module N and any left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M.

If Δ is a Γ-set, then we denote by $\mathfrak{F}(\Delta)$ the set of stabilisers of Δ. We denote by $\mathbb{Z}[-, \Delta]_{\Gamma}$ the right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module which sends $\Gamma/\Lambda \in \mathcal{O}_{\mathfrak{F}}\Gamma$ to the free abelian group on the basis $[\Gamma/\Lambda, \Delta]_{\Gamma}$ which is by definition the set of all Γ-maps from $\Gamma/\Lambda \to \Delta$.

Lemma 1.5. *Let Δ be a Γ-set such that $\mathfrak{F} \cap \Lambda \subset \mathfrak{F}$ for every $\Lambda \in \mathfrak{F}(\Delta)$. For every projective right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module Q the $\mathcal{O}_{\mathfrak{F}}\Gamma$-module $Q \otimes \mathbb{Z}[-, \Delta]_{\Gamma}$ is flat.*

Proof. Since tensoring over $\mathbb{Z}$ is an additive functor, it is enough to verify the claim in the case that $\Delta = \Gamma/\Lambda$ for some Λ. Since Q is projective, it follows that the $\mathcal{O}_{\mathfrak{F}\cap\Lambda}\Lambda$-module $\mathrm{Res}_{\Lambda}^{\Gamma} Q$ is projective and hence flat. Furthermore $\mathrm{Res}_{\Lambda}^{\Gamma}$ is an exact functor. Thus the functor, which sends any left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M to $\mathrm{Res}_{\Lambda}^{\Gamma} Q \otimes_{\mathfrak{F}\cap\Lambda} \mathrm{Res}_{\Lambda}^{\Gamma} M$ is exact. Hence, in the light of the natural isomorphism of Lemma 1.4 it follows that tensoring $Q \otimes \mathbb{Z}[-, \Gamma/\Lambda]_{\Gamma}$ over $\mathfrak{F}$ is an exact functor, that is $Q \otimes \mathbb{Z}[-, \Gamma/\Lambda]_{\Gamma}$ is flat. $\square$

For left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M the left derived functors of $-\otimes_{\mathfrak{F}} M$ are denoted by $\mathrm{Tor}_*^{\mathfrak{F}}(-, M)$. The following is a key ingredient to our proof and can be found as Theorem 5.4 in [MPN11]:

Proposition 1.6 (Bieri–Eckmann Criterion for Bredon homology)**.** *Let N be a right $\mathcal{O}_{\mathfrak{F}}\Gamma$-module and let $n \in \mathbb{N}$. The following are equivalent:*

(i) *N is of type $\mathfrak{F}$-FP_n.*

(ii) *Let $(J_\Lambda)_{\Lambda \in \mathfrak{F}}$ be a family of cardinals. The natural map*

$$\mathrm{Tor}_k^{\mathfrak{F}}(N, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \to \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathrm{Tor}_k^{\mathfrak{F}}(N, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$.

Note that $\mathrm{Tor}_k^{\mathfrak{F}}(N, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) = 0$ for every $\Lambda \in \mathfrak{F}$ and $k \geq 1$. Thus the requirement in (ii) that the natural map is an epimorphism is automatically satisfied for $k \geq 1$.

The *Bredon homology* $H_*^{\mathfrak{F}}(\Gamma; M)$ of Γ with coefficients in the left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M are defined to be the groups $\mathrm{Tor}_*^{\mathfrak{F}}(\mathbb{Z}, M)$. Analogous to the classical case [Bro82, p. 172], we define the *equivariant Bredon homology* $H_*^{\mathfrak{F}}(X, M)$ of a Γ-CW-complex with coefficients in the left $\mathcal{O}_{\mathfrak{F}}\Gamma$-module M as follows, cf. [DPT11]. Let Q_* be a projective resolution of the trivial $\mathcal{O}_{\mathfrak{F}}\Gamma$-module $\mathbb{Z}$ by right $\mathcal{O}_{\mathfrak{F}}\Gamma$-modules. Then we have the bigraded complex

$$(\underline{C}_*(X) \otimes Q_{**}) \otimes_{\mathfrak{F}} M$$

of abelian groups. We define $H_*^{\mathfrak{F}}(X, M)$ to be the homology of the total complex of this bicomplex. Note that $H_*^{\mathfrak{F}}(\Gamma, M) = H_*^{\mathfrak{F}}(\mathrm{pt.}, M)$.

2 The case $n = 0$

In the classical case, being of type FP_0 for a group is an empty condition. In the context of Bredon homology this is not true any more. Kochloukova, Martínez-Pérez and Nucinkis [KMPN11, Lemma 2.3] have given a characterisation of when a group is of type $\mathfrak{F}$-FP_0:

Proposition 2.1. *A group Γ is of type $\mathfrak{F}$-FP_0 if and only if there is a finite subset $\mathfrak{F}_0$ of $\mathfrak{F}$ such that every $\Lambda \in \mathfrak{F}$ is subconjugate to some element of $\mathfrak{F}_0$, i.e. there is a $g \in \Gamma$ and a $\Xi \in \mathfrak{F}_0$ such that $\Lambda^g \leq \Xi$.*

Using this result, the case $n = 0$ of the Main Theorem is readily verified:

Proof of the Main Theorem for $n = 0$. First assume that the directed system $(\underline{\tilde{H}}_{-1}(X_\alpha))_{\alpha \in I}$ is essentially trivial. Then there is a $\beta \in I$ such that X_β is $\mathfrak{F}$-acyclic up to dimension -1, i.e. X^Λ is non-empty for every $\Lambda \in \mathfrak{F}$. By assumption $X_\beta^{(0)}$ is finite modulo Γ. The stabilizer Γ_x of every $x \in X_\beta^{(0)}$ is of type $(\mathfrak{F} \cap \Gamma_x)$-$\mathrm{FP}_0$. Hence there is a finite subset $\mathfrak{F}_{x,0}$ of

$\mathfrak{F} \cap \Gamma_x$ such that every $\Lambda \in \mathfrak{F} \cap \Gamma_x$ is subconjugate to some element of $\mathfrak{F}_{x,0}$. Let Σ_0 be a set of representatives for $X_\beta^{(0)}$ modulo Γ an let

$$\mathfrak{F}_0 = \bigcup_{x \in \Sigma_0} \mathfrak{F}_{x,0}$$

which is a finite subset of $\mathfrak{F}$. If $\Lambda \in \mathfrak{F}$ is arbitrary, then X_β^Λ is non-empty. Hence Λ fixes some point x of $X_\beta^{(0)}$ and therefore is subconjugate to some element of $\mathfrak{F}_{x,0}$.

Conversely assume that Γ is of type $\mathfrak{F}$-FP$_0$. Let $\mathfrak{F}_0 \subseteq \mathfrak{F}$ be finite such that every element of $\mathfrak{F}$ is subconjugate to some element of $\mathfrak{F}_0$. For arbitrary $\alpha \in I$ let $\beta \geq \alpha$ be such that X_β contains a fixed point of each element of $\mathfrak{F}_0$. Let $\Lambda \in \mathfrak{F}$ be arbitrary and $\Lambda^g \leq \Xi \in \mathfrak{F}_0$. If $x \in X_\beta$ is a fixed point of Ξ, then $g.x \in X_\beta$ is a fixed point of Λ. $\qquad \square$

3 Proof of the Main Theorem

The following proposition is contained in [DPT11] for the case $n = \infty$ and the proof is essentially the same. We reproduce it for convenience.

Proposition 3.1. *Let X be a Γ-CW-complex which is $\mathfrak{F}$-acyclic up to dimension $n - 1$ and let M be a left $\mathcal{O}_\mathfrak{F}\Gamma$-module. Then the natural isomorphism*

$$H_k^{\widetilde{\mathfrak{F}}}(X, M) \to H_k^{\widetilde{\mathfrak{F}}}(\Gamma, M)$$

(induced by the projection of X to a point) is an isomorphism for $k < n$.

Proof. Let C_* be the chain complex of Bredon modules X and let Q_* be a projective resolution of $\underline{\mathbb{Z}}$. By definition there is a spectral sequence

$$E_{pq}^1 = H_q((C_* \otimes Q_p) \otimes_\mathfrak{F} M) \Rightarrow H_k^{\widetilde{\mathfrak{F}}}(X, M) \ .$$

We claim that

$$C_n \otimes Q_p \to C_{n-1} \otimes Q_p \to \ldots \to C_0 \otimes Q_p \to Q_p \to 0 \tag{3.1}$$

is a partial flat resolution of Bredon modules. By acyclicity of X up to dimension $n - 1$ the sequence

$$C_n \to C_{n-1} \to \ldots \to C_0 \to \underline{\mathbb{Z}} \to 0$$

is exact. To see that (3.1) is exact we have to see that it is exact evaluated at every orbit Γ/Λ. This is true because $Q_p(\Gamma/\Lambda)$ is free abelian. Flatness follows from Lemma 1.5 because every C_p is of the form $\mathbb{Z}[-, \Delta]_\Gamma$.

It follows from (3.1) that $E_{pq}^1 = \mathrm{Tor}_q^{\widetilde{\mathfrak{F}}}(Q_p, M)$ for $q < n$. Since Q_p is projective we get

$$\mathrm{Tor}_q^{\widetilde{\mathfrak{F}}}(Q_p, M) = \begin{cases} Q_p \otimes_\mathfrak{F} M & q = 0 \\ 0 & 0 < q < n \ . \end{cases}$$

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But $Q_* \otimes_{\mathfrak{F}} M$ can be used to compute $H_*^{\mathfrak{F}}(\Gamma, M)$, therefore

$$E_{pq}^2 = \begin{cases} H_p^{\mathfrak{F}}(\Gamma, M) & q = 0 \\ 0 & 0 < q < n \end{cases}.$$

Since the triangle $p + q < n$ remains stable, this closes the proof. $\qquad\square$

Proposition 3.2. *Let X be a Γ-CW-complex with cocompact n-skeleton. Assume that every p-cell σ of X, $p \leq n$, the following two condition hold: $\mathfrak{F} \cap \Gamma_\sigma \subset \mathfrak{F}$, and Γ_σ is of type $(\mathfrak{F} \cap \Gamma)_\sigma$-$\mathrm{FP}_{n-p}$. Then for $k \leq n$ and every family of cardinals $(J_\Lambda)_{\Lambda \in \mathfrak{F}}$ there exists an isomorphism*

$$H_k^{\mathfrak{F}}(X, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \to \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \underline{H}_k(X)(\Gamma/\Lambda)$$

that is natural in X.

Proof. As in the previous proof let $C_* = \underline{C}_*(X)$ and let $Q_* \twoheadrightarrow \underline{\mathbb{Z}}$ be a projective resolution of the trivial $\mathcal{O}_{\mathfrak{F}}\Gamma$-module.

There exists a spectral sequence converging to $H_*^{\mathfrak{F}}(X, M)$ whose E^1-sheet is given by

$$E_{pq}^1 = H_q((C_p \otimes Q_*) \otimes_{\mathfrak{F}} M).$$

Since $C_p(\Gamma/\Lambda)$ is a free abelian group for every $\Lambda \in \mathfrak{F}$ it follows that $C_p \otimes Q_*$ is a resolution of $C_p \otimes \mathbb{Z} = C_p$. Moreover this resolution is flat by Lemma 1.5 and thus there exist isomorphisms

$$H_q((C_p \otimes Q_*) \otimes_{\mathfrak{F}} M) \cong \mathrm{Tor}_q^{\mathfrak{F}}(C_p, M)$$

which are natural in X and M.

Next we show that C_p is of type FP_{n-p} for $p \leq n$. Note that the last statement of Lemma 1.2 implies

$$C_p \cong \coprod_{\sigma \in \Sigma_p} \mathrm{Ind}_{\Gamma_\sigma}^{\Gamma} \mathbb{Z}$$

where Σ_p is a set of representatives for the p-cells of X modulo Γ. Note also, that Σ_p is finite. By assumption $\underline{\mathbb{Z}}$ is of type FP_{n-p} as an $\mathcal{O}_{\mathfrak{F} \cap \Gamma_\sigma}\Gamma_\sigma$-module for every p-cell $\sigma \in \Sigma_p$. The claim now follows from Lemma 1.2.

Now take M to be $\prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma$ and consider the spectral sequence above. Since C_p is of type FP_{n-p} the Bieri–Eckmann Criterion, Proposition 1.6, implies that

$$E_{pq}^1 = \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathrm{Tor}_q^{\mathfrak{F}}(C_p, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma)$$

which is 0 for $q > 0$. The entry E_{p0}^1 is natural isomorphic to $\prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} C_p \otimes_{\mathfrak{F}} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma$ and the differentials are induced by the differentials of the chain complex C_*. Therefore, one can read off the second page of the spectral sequence that

$$H_k^{\mathfrak{F}}(X, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \cong \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} H_k(C_* \otimes_{\mathfrak{F}} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \cong \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} H_k(C_*(\Gamma/\Lambda))$$

for $k < n$ where the last isomorphism is the isomorphism from Lemma 1.1. But $H_k(C_*(\Gamma/\Lambda))$ is just $\underline{H}_k(X)(\Gamma/\Lambda)$ and this concludes the proof. $\qquad\square$

Lemma 3.3. *Let X be a Γ-CW-complex and let $(X_\alpha)_{\alpha \in I}$ be a filtration of X by Γ-invariant subcomplexes. Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism*

$$\varinjlim_\alpha H_*^{\mathfrak{F}}(X_\alpha, M) \to H_*^{\mathfrak{F}}(X, M)$$

for all left $\mathcal{O}_{\mathfrak{F}}\Gamma$-modules M.

Proof. This is due to the fact that $\varinjlim_\alpha$ is a filtered colimit and in particular exact. $\square$

Proof of the Main Theorem. Since we have already covered the case $n = 0$ we may and do assume that $n \geq 1$. By the Bieri–Eckmann Criterion Γ is of type $\mathfrak{F}$-FP_n if and only if for every family $(J_\Lambda)_{\Lambda \in \mathfrak{F}}$ the natural map

$$\varphi \colon H_k^{\mathfrak{F}}\Big(\Gamma, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma\Big) \to \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} H_k^{\mathfrak{F}}(\Gamma, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \tag{$*$}$$

is an isomorphism for $0 \leq k < n$ and an epimorphism for $k = n$. Since the right hand side is 0 for $k > 0$ and since we are assuming that $n \geq 1$, the statement about the epimorphism is trivially satisfied.

Since the codomain of φ is trivial for $k \geq 1$ we first show that also the domain of φ is trivial for $0 < k < n$ (which is a special case of the proof for $k = 0$ below). We have the isomorphisms

$$H_k^{\mathfrak{F}}\Big(\Gamma, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma\Big) \cong H_k^{\mathfrak{F}}\Big(X, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma\Big)$$

from Proposition 3.1

$$H_k^{\mathfrak{F}}\Big(X, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma\Big) \cong \varinjlim_\alpha H_k^{\mathfrak{F}}\Big(X_\alpha, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma\Big)$$

from Lemma 3.3 and

$$\varinjlim_\alpha H_k^{\mathfrak{F}}\Big(X_\alpha, \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma\Big) \cong \varinjlim_\alpha \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \underline{H}_k(X_\alpha)(\Gamma/\Lambda)$$

from Proposition 3.2. By Observation 1 this is trivial if and only if the system $(\underline{H}_k(X_\alpha))_{\alpha \in I}$ of $\mathcal{O}_{\mathfrak{F}}\Gamma$-modules is essentially trivial.

For the remaining case $k = 0$ consider the following commuting diagram (where we dropped the index sets for readability):

$$
\begin{array}{ccc}
H_0^{\tilde{\mathfrak{F}}}(\Gamma, \prod \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) & \xrightarrow{\ \varphi\ } & \prod H_0^{\tilde{\mathfrak{F}}}(\Gamma, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \\
\uparrow & & \uparrow \\
H_0^{\tilde{\mathfrak{F}}}(X, \prod \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) & \longrightarrow & \prod H_0^{\tilde{\mathfrak{F}}}(\mathrm{pt.}, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \\
\uparrow & & \uparrow \\
\varinjlim_\alpha H_0^{\tilde{\mathfrak{F}}}(X_\alpha, \prod \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) & \longrightarrow & \varinjlim_\alpha \prod H_0^{\tilde{\mathfrak{F}}}(\mathrm{pt.}, \mathbb{Z}[\Gamma/\Lambda, -]_\Gamma) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \prod \underline{H}_0(X_\alpha)(\Gamma/\Lambda) & \xrightarrow{\ \psi\ } & \varinjlim_\alpha \prod \underline{H}_0(\mathrm{pt.})(\Gamma/\Lambda)
\end{array}
$$

The vertical arrows of the top square are isomorphisms by Proposition 3.1. The vertical arrows of the middle square are induced by the inclusions $X_\alpha \hookrightarrow X$ and the indentity on the one point space respectively; it follows from Lemma 3.3 that they are isomorphisms. Finally, the vertical arrows of the bottom square are the isomorphisms from Proposition 3.2.

Since all the vertical arrows in the diagram are isomorphisms it follows that φ is an isomorphism if and only if ψ is an isomorphisms. But ψ fits into the short exact sequence

$$
0 \to \varinjlim_\alpha \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \tilde{\underline{H}}_0(X_\alpha)(\Gamma/\Lambda) \to \varinjlim_\alpha \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \underline{H}_0(X_\alpha)(\Gamma/\Lambda) \xrightarrow{\psi} \varinjlim_\alpha \prod_{\Lambda \in \mathfrak{F}} \prod_{J_\Lambda} \mathbb{Z} \to 0
$$

and it follows Observation 1 that ψ (and therefore φ) is an isomorphism if and only if the system $(\tilde{\underline{H}}_0(X_\alpha))_{\alpha \in I}$ of $\mathcal{O}_{\mathfrak{F}}\Gamma$-modules is essentially trivial. $\qquad\square$

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