A NOTE ON THE UNRAMIFIED BRAUER GROUP OF A HOMOGENEOUS SPACE

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ABSTRACT. We give a new proof of the theorem stating that for any connected linear algebraic group G over an algebraically closed field k of characteristic 0 and for any closed connected subgroup H of G, the unramified Brauer group of G/H vanishes.

1. INTRODUCTION

In this note k always denotes an algebraically closed field of characteristic 0. For an irreducible algebraic variety X over k, we denote by $k(X)$ the field of rational functions on X. We denote by $\mathrm{Br_{nr}}\, k(X)$, or just by $\mathrm{Br_{nr}}\, X$, the unramified Brauer groups of $k(X)$ with respect to k, see [CTS, Def. 5.3].

We give a new proof of the following theorem:

Theorem 1 ([BDH, Thm. 5.1])**.** *Let G be a connected linear algebraic group over an algebraically closed field k of characteristic 0, and let $H \subset G$ be a closed connected subgroup. Then* $\mathrm{Br_{nr}}\, k(G/H) = 0$.

The case when G is simply connected is a classical result of Bogomolov [Bog, Thm. 2.4], see Colliot-Thélène and Sansuc [CTS, Thm. 9.13]. Bogomolov considered the case $k = \mathbf{C}$, but the general case of an algebraically closed field k of characteristic 0 reduces to the case $k = \mathbf{C}$, see [CTS], beginning of § 9. Theorem 1 answers affirmatively a question of Colliot-Thélène and Sansuc in [CTS, Rem. 9.14] and a question after Theorem 1.4 in the paper [CTK] by Colliot-Thélène and Kunyavskiĭ. Theorem 1 was recently proved by a number-theoretical method in [BDH, Thm. 5.1], together with certain generalizations to nonzero characteristic. Here we deduce this theorem from Bogomolov's theorem.

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2. NOTATION AND PRELIMINARIES

By k we always denote an algebraically closed field of characteristic 0. Let G be a connected linear algebraic group over k. We use the following notation:

G^{u} is the unipotent radical of G;

$G^{\mathrm{red}} = G/G^{\mathrm{u}}$, it is reductive;

$G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, it is semisimple;

$G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, it is a torus;

$G^{\mathrm{ssu}} = \ker[G \to G^{\mathrm{tor}}]$, it is an extension of a connected semisimple group by a unipotent group.

Note that G^{tor} is the largest quotient torus of G and that G^{ssu} is connected and character-free. Note also that $\mathrm{Pic}\, G = 0$ if and only if G^{ss} is simply connected, cf. [Sa], Lemma 6.9 and Cor. 6.11.

Let X be a smooth integral variety over k. If V is a smooth compactification of X (existing by Hironaka's theorem), then we can identify $\mathrm{Br}_{\mathrm{nr}} X$ with $\mathrm{Br}\, V$. We regard $\mathrm{Br}_{\mathrm{nr}} X = \mathrm{Br}\, V$ as a subgroup of $\mathrm{Br}\, X$, cf. [CTS, Thm. 5.11]. If $f\colon X_1 \to X_2$ is a morphism of smooth integral varieties defined over k, one can extend f to a morphism of suitable smooth compactifications $f'\colon V_1 \to V_2$, where V_i is a smooth compactification of X_i $(i = 1, 2)$, see [BK, § 1.2.2] (again, one uses Hironaka's theorem). It follows that f induces a homomorphism of the unramified Brauer groups $f^{\mathrm{nr}}\colon \mathrm{Br}_{\mathrm{nr}} X_2 \to \mathrm{Br}_{\mathrm{nr}} X_1$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}_{\mathrm{nr}} X_2 & \xrightarrow{f^{\mathrm{nr}}} & \mathrm{Br}_{\mathrm{nr}} X_1 \\ \downarrow & & \downarrow \\ \mathrm{Br}\, X_2 & \xrightarrow{f^{*}} & \mathrm{Br}\, X_1. \end{array} \tag{1}$$

3. PROOF OF THEOREM 1

Let G be a connected linear algebraic group G defined over k, and let $H \subset G$ be a connected closed subgroup. We set $X = G/H$.

Consider the map $G \to G/H$. Since G is a rational variety, we have $\mathrm{Br}_{\mathrm{nr}} G = 0$, and we see from diagram (1) that

$$\mathrm{Br}_{\mathrm{nr}}(G/H) \subset \ker[\mathrm{Br}\,(G/H) \to \mathrm{Br}\, G]. \tag{2}$$

First reduction. It is well known (see e.g. [Bor, Lemma 5.2]) that there exists a connected linear algebraic group G' over k with $\mathrm{Pic}\, G' = 0$ and a connected closed subgroup $H' \subset G'$, such that the varieties G/H and G'/H' are isomorphic. Therefore, we may and shall assume that G in Theorem 1 satisfies $\mathrm{Pic}\, G = 0$.

Second reduction. Set $X' = G^{\mathrm{u}} \backslash X$, then X' is a homogeneous space of the reductive group $G^{\mathrm{red}} := G/G^{\mathrm{u}}$ (satisfying $\mathrm{Pic}\, G^{\mathrm{red}} = 0$) with connected stabilizer $H/(H\cap G^{\mathrm{u}})$. We have $\mathrm{Br}_{\mathrm{nr}}X\simeq \mathrm{Br}_{\mathrm{nr}}X'$, see [BDH], proof of Theorem 5.1, Step 2. Therefore, we may and shall also assume in Theorem 1 that G is reductive.

Third reduction. Consider the homomorphism $H\to H^{\mathrm{red}}$ (where $H^{\mathrm{red}}:=H/H^{\mathrm{u}}$). By Mostow's theorem (see [Mo, Thm. 7.1]) this homomorphism admits a splitting (homomorphic section) $s\colon H^{\mathrm{red}}\to H$. Set $H^{\mathrm{r}}=s(H^{\mathrm{red}})\subset H$. We have $\mathrm{Br}_{\mathrm{nr}}(G/H^{\mathrm{r}})\simeq \mathrm{Br}_{\mathrm{nr}}(G/H)$, see [BDH], proof of Theorem 5.1, Step 2. Therefore, we may and shall also assume in Theorem 1 that H is reductive.

Fourth reduction. Consider the subgroup $H^{\mathrm{ssu}}=H^{\mathrm{ss}}$ of the reductive group H. The map $G/H^{\mathrm{ss}}\to G/H$ is a torsor under the torus H^{tor}. By Hilbert's Theorem 90 this torsor admits a local section, hence G/H^{ss} is birationally equivalent to $H^{\mathrm{tor}}\times_k G/H$, and by [CTS, Prop. 5.7] we have $\mathrm{Br}_{\mathrm{nr}}(G/H^{\mathrm{ss}})\simeq \mathrm{Br}_{\mathrm{nr}}(G/H)$. Therefore, we may and shall also assume in Theorem 1 that H is semisimple.

Reduction to Bogomolov's theorem. Using the previous reductions, we now assume that G is reductive with $\mathrm{Pic}\, G=0$, and that $H\subset G$ is connected and semisimple. Set $G_1=G^{\mathrm{ss}}$, then G_1 is simply connected because $\mathrm{Pic}\, G=0$. Since H is semisimple, we have $H\subset G_1$.

Let $i\colon G_1\hookrightarrow G$ denote the inclusion homomorphism. Consider the following commutative diagram of morphisms of varieties:

(3)
$$\begin{array}{ccc} G_1 & \xrightarrow{\ i\ } & G \\ \downarrow & & \downarrow \\ G_1/H & \xrightarrow{\ i_*\ } & G/H. \end{array}$$

By functoriality (see § 2) this diagram defines a homomorphism $i^{\mathrm{nr}}\colon \mathrm{Br}_{\mathrm{nr}}(G/H)\to \mathrm{Br}_{\mathrm{nr}}(G_1/H)$ fitting into a commutative diagram

(4)
$$\begin{array}{ccc} \mathrm{Br}_{\mathrm{nr}}(G/H) & \xrightarrow{\ i^{\mathrm{nr}}\ } & \mathrm{Br}_{\mathrm{nr}}(G_1/H) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Br}\,(G/H) & \xrightarrow{\ i^*\ } & \mathrm{Br}\,(G_1/H). \end{array}$$

Note that the map $i\colon G_1\to G$ in diagram (3) is an H-equivariant map from the H-torsor G_1 over G_1/H to the H-torsor G over G/H. Sansuc's exact sequence [Sa, (6.10.1)], applied to this diagram, gives a commutative diagram with exact rows

(5)
$$\begin{array}{ccccccc} 0=\mathrm{Pic}\, G & \longrightarrow & \mathrm{Pic}\, H & \longrightarrow & \mathrm{Br}\,(G/H) & \longrightarrow & \mathrm{Br}\, G \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i^*} & & \downarrow \\ 0=\mathrm{Pic}\, G_1 & \longrightarrow & \mathrm{Pic}\, H & \longrightarrow & \mathrm{Br}\,(G_1/H) & \longrightarrow & \mathrm{Br}\, G_1. \end{array}$$

We see from (5) that the homomorphism i^* restricted to $\ker[\mathrm{Br}\,(G/H)\to \mathrm{Br}\, G]$ is injective, and we see from (2) that i^* restricted to $\mathrm{Br}_{\mathrm{nr}}(G/H)$ is injective. Now it follows from diagram (4) that the homomorphism

$$i^{\mathrm{nr}}\colon \mathrm{Br}_{\mathrm{nr}}(G/H) \to \mathrm{Br}_{\mathrm{nr}}(G_1/H)$$

is injective. Since G_1 is semisimple and simply connected and H is connected and semisimple, we have $\mathrm{Br}_{\mathrm{nr}}(G_1/H) = 0$ by Bogomolov's theorem [CTS, Thm. 9.13]. We conclude that $\mathrm{Br}_{\mathrm{nr}}(G/H) = 0$, which proves Theorem 1. □

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