Local Hölder continuity property of the Densities of Solutions of SDEs with Singular Coefficients *

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Abstract

We prove that the weak solution of a uniformly elliptic stochastic differential equation with locally smooth diffusion coefficient and Hölder continuous drift has a Hölder continuous density function. This result complements recent results of Fournier-Printems [3], where the density is shown to exist if both coefficients are Hölder continuous and exemplifies the role of the drift coefficient in the regularity of the density of a diffusion.

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1 Introduction

Malliavin calculus is well known as a method to prove the regularity of a solution of a SDE (stochastic differential equation). Especially, if we assume that the coefficients of a hypoelliptic SDE are bounded functions with bounded derivatives of any order, then the solution has a smooth density (see, for example, Nualart[11]). In recent years, one of the directions in this area is to develop tools to deal with the case of non-smooth coefficients.

In this article, we consider the one dimensional SDE of the form $dX_t = \sigma(X_t)dB_t + b(X_t)dt$ on a probability space $(\Omega, \mathcal{F}, Q)$, where $\{B_t\}_{0\le t}$ is a one dimensional standard Brownian motion. The main purpose of this paper is to prove the local smoothness of the density of the SDE under some weak assumptions on the drift coefficient b.

Our assumptions, roughly speaking, are local boundedness of the coefficients, Hölder continuity of b, uniformly ellipticity and local smoothness of σ. More details about the assumptions will be given later.

Under these assumptions, we will see that the density of the solution of the above SDE exists on the set in which σ is smooth. Furthermore, we also show that the density is γ-Hölder continuous, with $\gamma \in (0, \alpha)$ and α is the exponent of the Hölder continuity of b. This shows that the drift coefficient may be a determining factor in the regularity of the density.

Some related results have already been obtained for this problem, for example, Fournier and Printems [3] proved in the case that σ is α-Hölder continuous with $\alpha > \frac{1}{2}$ and b is at most linear growth then the density of X_t exists. Their approach is very simple. The key idea is to consider the following random variable which approximates X_t; $Z_\varepsilon := X_{t-\varepsilon} + \sigma(X_{t-\varepsilon})(B_t - B_{t-\varepsilon})$ for $\varepsilon \in (0, 1)$ and using some classical lemmas about the existence of the density and conditions of the coefficients. In that case, they showed the existence of the density on the set $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$. A careful analysis of their method shows that the argument for the proof can not be used to obtain any further properties of the density (such as the Hölder continuity of the density).

For a multi-dimensional SDE whose coefficients depends on time, Kusuoka [7] introduced a space denoted by V_h which is larger than the usual Sobolev space and showed the relation between the space V_h and absolute continuity of random variables. According to [7], one can show the existence of the density of X_t on the set $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$ when the coefficients are bounded, σ is twice continuously differentiable on $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$ and b is Lipschitz continuous on $\mathbb{R}$.

Our result uses a probabilistic approach to the regularity problem of fundamental solution to parabolic equations. In the theory of parabolic equations, there are some regularity results which we briefly compare here. In [2], one can find some classical results on the existence and regularity of fundamental solutions of parabolic equations under global Hölder continuity assumptions on the coefficients of the parabolic equation. In particular, the Hölder continuity of coefficients yields higher order smoothness of the solution to parabolic equations.

In the modern theory of parabolic equations, these equations are solved in Sobolev spaces and by using embedding theorems, one can find a modification of a solution such that this solution might have Holder continuous derivatives (see [8] or [6]). Thie arguments in this approach are somewhat global.

On the other hand, in this paper, we focus our attention on the local regularity problem: Does the local regularity of coefficients yield the same property of solution to parabolic equations? In particular, except for the existence and uniqueness (in law) of weak solution to stochastic differential equation, our assumptions are restricted only on a neighborhood of some point. In [1], the reader can find some sufficient conditions so that the stochastic differential equation under consideration admits a unique weak solution.

The main tool of our approach is Malliavin calculus, but in general, due to our local hypotheses, the stochastic process X will not be differentiable in the Malliavin sense. To solve this problem, we use Girsanov's theorem in order to reduce our study to the solution of the equation $dX_t = \sigma(X_t)dW_t$ where W is a new Brownian motion under a new probability measure P. In order to deal with the local smoothness of the diffusion coefficient, we use stopping times in order to introduce a localization argument. This localization will allow us to change the process X by a regularized version $\bar{X}$ for which Malliavin Calculus is applicable.

The remaining problem is how to deal with the change of measure which contains the non-smooth function b which implies that this random variable is non differentiable. For this reason, we introduce an approximation of the change of measure which is differentiable. Finally, to end the argument we only need to measure the distance between the change of measure and its approximation by using the Hölder property of b.

As in [3], we believe that the method introduced here can be generalized to other situations such as SDE's with random coefficients or Lévy driven SDE with Brownian component. For examples of applications of the results obtained here, we refer the reader to [3] and [9].

2 Preliminaries and Notation

In this chapter, we introduce some notations and give a brief introduction to Malliavin calculus.

2.1 Some Basic Notations

For $n \in \mathbb{N}$, we denote by C_b^n the class of bounded and n-times continuously differentiable functions with bounded derivatives defined in $\mathbb{R}$ taking values in $\mathbb{R}$. Similarly, we define C_b^∞ as the class of bounded smooth functions defined in $\mathbb{R}$ and taking values in $\mathbb{R}$ with bounded derivatives of any order and C_p^∞ as the class of all infintely continuously differentiable functions defined in $\mathbb{R}$ and taking values in $\mathbb{R}$ such that the function and its derivatives have at most polynomial growth. For a bounded function $f : \mathbb{R} \to \mathbb{R}$, we denote by $\|f\|_\infty$ the supremum norm of f.

Let X be a random variable on the probability space $(\Omega, \mathcal{F}, P)$. For $1 \le p < \infty$, we denote

$$\|X\|_{L^p(P)} := E_P[|X|^p]^{\frac{1}{p}},$$

where $E_P[X]$ means the expectation of X with respect to P.

2.2 Brief Introduction to Malliavin Calculus

Now we turn to introduce Malliavin calculus. For the proofs of the following results and more details about Malliavin calculus, see [11]. In this chapter, we abbreviate $\|\cdot\|_{L^p(P)}$ by $\|\cdot\|_{L^p}$.

Fix $T > 0$. For any measurable function $h \in \mathcal{H} := L^2([0,T];\mathbb{R})$, we denote its stochastic integral by

$$W(h) := \int_0^T h(s) dW_s,$$

where $\{W_t\}_{t \ge 0}$ is a one dimensional Brownian motion.

Define

$$\mathcal{S} := \{F : F = f(W(h_1), \cdots, W(h_n)); h_1, \cdots, h_n \in \mathcal{H}, f \in C_p^\infty(\mathbb{R})\}.$$

For $F \in \mathcal{S}$ and $t \in [0,T]$, we define the H-derivative (or the Malliavin derivative) as

$$D_t F := \sum_{i=1}^n \partial_i f(W(h_1), \cdots, W(h_n)) h_i(t)$$

and for $k \in \mathbb{Z}_+$ and $p \ge 1$, define the norm $\|\cdot\|_{k,p}$ by

$$\|F\|_{k,p} := \left\{ E[|F|^p] + \sum_{j=1}^k E[\|D^j F\|_{\mathcal{H}^{\otimes j}}^p] \right\}^{\frac{1}{p}},$$

where

$$\|D^j F\|_{\mathcal{H}^{\otimes j}} := \int_0^T \cdots \int_0^T |D_{s_1} \cdots D_{s_j} F|^2 ds_1 \cdots ds_j.$$

As usual, $\|F\|_{0,p} := \|F\|_{L^p}$.

We will denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$ and by $\mathbb{D}^\infty := \cap_{k,p} \mathbb{D}^{k,p}$. Similarly, for a Hilbert space V and V-valued random variables, one can define $\mathbb{D}^{k,p}(V)$ and $\mathbb{D}^\infty(V) := \cap_{k,p} \mathbb{D}^{k,p}(V)$. In particular, for a $\mathbb{R}$-valued stochastic process $\{u_s\}_{0 \le s \le T}$, we define the norm

$$\|u\|_{k,p} := \left\{ E[\|u\|_{\mathcal{H}}^p] + \sum_{j=1}^k E[\|D^j u\|_{\mathcal{H}^{\otimes j}}^p] \right\}^{\frac{1}{p}}.$$

We define the Skorokhod integral, as the dual operator of D and denote it by δ.

Let $\{\mathcal{F}_s\}_{0\le s\le T}$ be the filtration generated by our Brownian motion $\{W_s\}_{0\le s\le T}$. It is a well known fact that for $\{\mathcal{F}_s\}_{0\le s\le T}$-adapted L^2 stochastic process $\{u_s\}_{0\le s\le T}$, its Skorokhod integral coincides with its Itô integral. That is,

$$\delta(u) = \int_0^T u_s dW_s.$$

Moreover, if $\{u_s\}_{0\le s\le T}$ belongs to the domain of δ (for example, $u \in \mathbb{D}^{1,2}(\mathcal{H})$), $F \in \mathbb{D}^{1,2}$ and they satisfy $E[F^2 \int_0^T u_s^2 ds]$ is finite, then

$$\delta(Fu_t) = F\delta(u) - \int_0^t (D_sF)u_s ds$$

is hold provided the right hand side of the above equation is square integrable.

For $F = (F^1, \cdots, F^d) \in (\mathbb{D}^{1,2})^d$, define the $d \times d$-matrix M_F by

$$M_F^{ij} := \langle DF^i, DF^j\rangle_{\mathcal{H}}.$$

This M_F is called Malliavin covariance matrix. The random vector F is nondegenerate if for any $p \ge 1$,

$$E[(\det M_F)^{-p}] < +\infty.$$

The following proposition, so called integration by parts formula (in Malliavin's sense), plays an important role in this paper.

Proposition 1. *(Integration by parts formula)*

Let $F, G \in \mathbb{D}^\infty$ be nondegenerate and $\varphi \in C_p^\infty$. Then for any $n \in \mathbb{N}$, there exists random variable $H_n \in \mathbb{D}^\infty$ such that

$$E\left[\varphi^{(n)}(F)G\right] = E\left[\varphi(F)H_n(F,G)\right].$$

Moreover H_n is recursively given by

$$H_1(F,G) := \delta(G{M_F}^{-p}DF)$$
$$H_k(F,G) := H_1(F, H_{k-1}(F,G)) \text{ for } 2 \le k \le n$$

and for $1 \le p < q < +\infty$, we have

$$\|H_n(F,G)\|_{L^p} \le c_{p,q}\|{M_F}^{-1}DF\|^n_{n,r2^{n-1}}\|G\|_{n,q}$$

where r satisfies that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $c_{p,q}$ is a constant depends only on p and q.

3 Preparatory Lemmas

The basic argument to study the density of a random variable follows from the study of its characteristic function. The first basic result is the following.

Theorem 1. (*Lévy's inversion theorem*) *Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a $\mathbb{R}$-valued random variable defined on that space. If $\varphi(\theta) := E[e^{i\theta X}]$, the characteristic function of the X, belongs to $L^1(\mathbb{R})$, then f_X, the density function of the law of X, exists and is continuous. Moreover,*

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \varphi(\theta) d\theta$$

for any x in $\mathbb{R}$.

This result is very well-known result which is called "Lévy's inversion theorem" for the proof of this, see e.g. [12]. The following corollary gives us a more precise criterion for the Hölder continuity of the density.

Corollary 1. *Let X be a random variable under the same setting as in Theorem 1 and φ be its characteristic function. Assume that the following inequality holds for some positive constant C and $0 < \gamma < 1$.*

$$|\varphi(\theta)| \leq 1 \wedge (C|\theta|^{-(1+\gamma)}).$$

Then the density function of the law of X exists and is α-Hölder continuous for any $0 < \alpha < \gamma$.

Proof. Let $\alpha \in (0, \gamma)$. The existence and continuity of the density immediately follows by Theorem 1. We only show that the density is α-Hölder continuous. Let f_X be the density of the law of X. Then by Theorem 1, we have

$$\begin{aligned} |f_X(x) - f_X(y)| &\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |e^{-i\theta x} - e^{-i\theta y}||\varphi(\theta)| d\theta \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |e^{-i\theta y}||e^{-i\theta(x-y)} - 1||\varphi(\theta)| d\theta \\ &\leq \frac{C_\alpha}{2\pi} \int_{-\infty}^{+\infty} |\theta x - \theta y|^\alpha |\varphi(\theta)| d\theta \\ &= |x-y|^\alpha \frac{C_\alpha}{2\pi} \int_{-\infty}^{+\infty} |\theta|^\alpha |\varphi(\theta)| d\theta. \end{aligned}$$

By the hypothesis, the last integral is finite. Hence, f_X is α-Hölder continuous. □

Now we define the notion of local density function.

Definition 1. *Let ε be a positive number and $y_0 \in \mathbb{R}$. The random variable X has a (local) density function p on the set $B_\varepsilon(y_0)$ if*

$$E[f(X)] = \int_{\mathbb{R}} f(x) p(x) dx$$

holds for any bounded continuous function f whose support in $B_\varepsilon(y_0)$.

Remark 1. *The above function p corresponds to the density function of X on the set $B_\varepsilon(y_0)$ provided X has a density function, but p may exist when X does not have a density function. For example, if $X = 0$ almost surely, then X clearly does not have a density function. However, for any $y_0 \in \mathbb{R} \setminus \{0\}$ and $0 < \varepsilon < |y_0|$, the constant function $p = 0$ satisfies the above definition.*

Although Corollary 1 gives us a useful criterion about the global existence and continuity of the density function, we need another lemma which is used to show the local existence of the density function.

Lemma 1. *Assume that X is a random variable under the same setting as in Theorem 1. Let $\varepsilon > 0$ and ϕ_ε be an element of C_b^∞ which satisfies that*

$$1_{B_\varepsilon(0)} \leq \phi_\varepsilon \leq 1_{B_{2\varepsilon}(0)}.$$

Fix $y_0 \in \mathbb{R}$ and set $m_0 := E[\phi_\varepsilon(X - y_0)]$. If $m_0 > 0$, we define $\mathcal{L}_{y_0}$ as the probability measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f(y)\mathcal{L}_{y_0}(dy) = \frac{1}{m_0} E[f(X)\phi_\varepsilon(X - y_0)],$$

for all continuous and bounded function f.

If $\mathcal{L}_{y_0}$ possesses a density $\tilde{p}_{y_0}$ then $p_{y_0} := m_0 \tilde{p}_{y_0}$ is the density function of X on $B_\varepsilon(y_0)$.

If $m_0 = 0$, then the constant function $\tilde{p}_{y_0} = 0$ is a density function of X on $B_\varepsilon(y_0)$ even if $\mathcal{L}_{y_0}$ does not have a density.

Proof. Let $m_0 > 0$ and f be a continuous and bounded function whose support is a subset of $B_\varepsilon(y_0)$. By the definition of p_{y_0}, we have

$$\begin{aligned}\int_{\mathbb{R}} f(y)p_{y_0}(y)dy &= m_0 \int_{\mathbb{R}} f(y)\tilde{p}_{y_0}(y)dy \\ &= m_0 \int_{\mathbb{R}} f(y)\mathcal{L}_{y_0}(dy) \\ &= E[f(X)].\end{aligned}$$

This implies that p_{y_0} is a density function of X on $B_\varepsilon(y_0)$.

On the other hand, if $m_0 = 0$ then it is clear that

$$E[f(X)] = 0.$$

Therefore $p_{y_0} = 0$ is a density function of X on $B_\varepsilon(y_0)$. ☐

Remark 2. *The function ϕ_ε in Lemma 1 can be constructed as follows. Let $a \in (\varepsilon, 2\varepsilon)$. Define the function*

$$f_{a,2\varepsilon}(x) := \begin{cases} \exp(\frac{1}{x-2\varepsilon} - \frac{1}{x-a}); & \text{for } x \in (a, 2\varepsilon) \\ 0; & \text{for } x \notin (a, 2\varepsilon). \end{cases}$$

and

$$g_{a,2\varepsilon}(x) := \frac{\int_a^x f_{a,2\varepsilon}(y)dy}{\int_a^{2\varepsilon} f_{a,2\varepsilon}(y)dy}.$$

Then $g_{a,2\varepsilon} \in C_b^\infty$ *and*

$$g_{a,2\varepsilon} = \begin{cases} 0 & (x \le a) \\ 1 & (x \ge 2\varepsilon). \end{cases}$$

Hence ϕ_ε *may be defined as* $\phi_\varepsilon := g_{-2\varepsilon,-a}(1 - g_{a,2\varepsilon})$.

Before stating and proving our main result, we remind the reader that according to Corollary 1 and Lemma 1, if

$$|E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)]| \le 1 \wedge (C|\theta|^{-(1+\gamma)}) \ (\forall|\theta| \ge 1) \tag{1}$$

holds for some positive constants C and γ, then for any $\gamma' \in (0,\gamma)$ the density function of the X exists and is γ'-Hölder continuous on $B_\varepsilon(y_0)$ at time t. Here, ϕ_ε is an element of $C_b^\infty(\mathbb{R})$ which satisfies the conditions of Lemma 1.

4 Main result

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, Q)$ be a probability space, where $\{\mathcal{F}_t\}_{t\ge0}$ is the filtration generated by the one dimensional standard Wiener process $B := \{B_t\}_{t\ge0}$ on $(\Omega, \mathcal{F}, Q)$. Consider the following SDE;

$$X_t = x_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \quad t \in [0,T] \tag{2}$$

for a finite $T > 0$ and $x_0 \in \mathbb{R}$, where σ and b are Borel measurable functions.

4.1 Assumptions

$(H1)$: There exists some $y_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that σ and b are bounded on the open ball $B_{6\varepsilon}(y_0) := \{y \in \mathbb{R}; |y - y_0| < 6\varepsilon\}$. Moreover, $\inf_{x \in B_{6\varepsilon}(y_0)} |\sigma(x)| > \sigma_0 > 0$ for some constant σ_0.
$(H2)$ $\sigma \in C_b^\infty(B_{6\varepsilon}(y_0))$.
$(H3)$: $\sigma^{-1}b := \frac{b}{\sigma}$ is α-Hölder continuous on $B_{6\varepsilon}(y_0)$, where $\alpha \in (0,1)$.

Remark 3. *The assumption* $(H3)$ *implies that the function* b *is* α*-Hölder continuous if* σ *belongs to* C_b^1.

If our assumptions $(H1)$, $(H2)$ and $(H3)$ are satified on $\mathbb{R}$, coefficients σ and b also satisfy the assumptions in Fournier and Primtems [3]. However their method does not apply if one wants to study the smoothness of the density.

We assume throughout the article the weak existence of solutions for (2). Sufficient conditions are stated in e.g. [1]. Our main result is the following theorem.

Theorem 2. *Assume* $(H1), (H2)$ *and* $(H3)$. *Then for any initial value* x_0, *any* $0 < t \leq T$ *and any* $0 < \gamma < \alpha$, *the distribution of* X_t *has a* γ*-Hölder continuous density on* $B_\varepsilon(y_0)$.

Remark 4. *We define*

$$I := \{y \in \mathbb{R}; P(T_y < \infty) > 0\},$$

where

$$T_y := \inf\{t > 0; X_t = y\}.$$

Then I *forms an interval when* I *is not a point (see Section* 3.5 *(page* 92*) in Itô-McKean[4]). The process* X *does not go out from* I, *hence the support of the distribution of* X_t *is contained in the closure of* I. *Thus we may concentrate our attention on the interval* I, *although in assumption* $(H1)$ *we may pick* $y_0 \in \mathbb{R}$ *which belongs to the complement of* I *and obtain the existence of a density (which is zero).*

5 Estimate of the characteristic function

We assume without loss of generality that the α-Hölder continuity constant of $\sigma^{-1}b$ is equal to one. Now we start the study of the characteristic function of X_t.

5.1 Change of the measure and localization

Fix $0 < t < T$.

We define the coefficients $\bar{\sigma}(y) := \sigma(\lambda(y))$ and $\bar{b}(y) := b(\lambda(y))$ where $\lambda \in C_b^\infty$ (a truncation function) is defined by

$$\lambda(y) = \begin{cases} y; & \text{if } |y - y_0| \leq 4\varepsilon \\ y_0 + 5\varepsilon \frac{y-y_0}{|y-y_0|}; & \text{if } |y - y_0| \geq 5\varepsilon \end{cases}$$

and $\lambda(y) \in \overline{B_{5\varepsilon}(y_0)}$ for all $y \in \mathbb{R}$. As a consequence of $(H1)$ and $(H2)$, $\bar{\sigma}$ is an C_b^∞ extension of $\sigma|_{B_{4\varepsilon}(y_0)}$ and $\bar{\sigma}^{-1}\bar{b}$ is α-Hölder continuous on $\mathbb{R}$.

Let $0 < \delta < (t \wedge 1)$. Define

$$\bar{X}_s(v, y) := y + \int_v^s \bar{\sigma}(\bar{X}_u(v, y))dB_u + \int_v^s \bar{b}(\bar{X}_u(v, y))du, \tag{3}$$

$$\nu := \inf\{s \geq t - \delta; X_s \in \overline{B_{3\varepsilon}(y_0)}\}$$

and

$$\tau := \inf\{s \geq \nu; X_s \notin \overline{B_{4\varepsilon}(y_0)}\}.$$

Define the sets

$$A := \{\phi_\varepsilon(X_t - y_0) > 0; \nu = t - \delta, t < \tau\}$$

and

$$C := \{\phi_\varepsilon(X_t - y_0) > 0; \sup_{0\leq s\leq \delta} |\bar{X}_{\nu+s}(\nu, X_\nu) - X_\nu| \geq \varepsilon\} \setminus A.$$

Then we have $\{\phi_\varepsilon(X_t - y_0) > 0\} = A \cup C$. Hence, as $A \cap C = \emptyset$, then

$$E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)] = E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)1_C] + E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)1_A]. \quad (4)$$

The next step in the proof is to remove the coefficient $\bar{b}$ from (3) in the case of $(v, y) = (t - \delta, X_{t-\delta})$ by changing the measure. Define the stochastic processes for $t - \delta \leq s \leq T$

$$W_s := B_s + \int_{t-\delta}^{s} (\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t - \delta, X_{t-\delta}))du,$$

$$Z_s := \exp\left(\int_{t-\delta}^{s} (\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t - \delta, X_{t-\delta}))dB_u + \frac{1}{2}\int_{t-\delta}^{s} |(\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t - \delta, X_{t-\delta}))|^2 du\right)$$

and introduce the probability measure P as

$$\frac{dP}{dQ}\Big|_{\mathcal{F}_s} = Z_s^{-1} \ (t - \delta \leq s \leq T). \quad (5)$$

Then $\bar{X}(t - \delta, X_{t-\delta})$ satisfies the following SDE;

$$\bar{X}_s(t - \delta, X_{t-\delta}) = X_{t-\delta} + \int_{t-\delta}^{s} \bar{\sigma}(\bar{X}_u(t - \delta, X_{t-\delta}))dW_u.$$

Remark 5. *Due to $(H1)$ and the boundedness of $\bar{\sigma}^{-1}\bar{b}$, Z^{-1} satisfies the Novikov condition. Hence, under the measure P, W is a one dimensional Wiener process. In order to apply Malliavin Calculus in the setting given in Section 2.2 we may change probability spaces without any further mention.*

Let us remark some general properties of stochastic processes of exponential type.

Lemma 2. *Z satisfies the following SDE:*

$$Z_t = 1 + \int_{t-\delta}^{t} Z_s(\bar{\sigma}^{-1}\bar{b})(\bar{X}_s(t - \delta, X_{t-\delta}))dW_s. \quad (6)$$

In general, for predictable bounded processes ψ (the lowest upper bound is denoted by $\|\psi\|_\infty$), we have that processes of the type

$$Z_t = 1 + \int_{t-\delta}^{t} Z_s\psi(s)dW_s = \exp\left(\int_{t-\delta}^{t} \psi(s)dW_s - \frac{1}{2}\int_{t-\delta}^{t} |\psi(s)|^2 ds\right)$$

satisfy that

$$E[Z_t^p] \leq \exp\left(\frac{p(p-1)}{2}\delta\|\psi\|_\infty^2\right).$$

Proof. For the first property, it is enough to note that $dB_s = dW_s + \bar{\sigma}^{-1}\bar{b}(\bar{X}_s(t-\delta, X_{t-\delta}))ds$.

Since W is a Wiener process under P, Z is a $\mathcal{F}$-martingale under P and hence for any $p > 1$,

$$
\begin{aligned}
E[Z_t^p] &\leq E\left[\exp\left(\int_{t-\delta}^t p\psi(s)dW_s - \frac{1}{2}\int_{t-\delta}^t |p\psi(s)|^2 ds + \frac{p(p-1)}{2}\int_{t-\delta}^t |\psi(s)|^2 ds\right)\right] \\
&\leq \exp\left(\frac{p(p-1)}{2}\delta\|\psi\|_\infty^2\right).
\end{aligned}
$$

□

5.2 Proof of the main theorem

Proof. Now we turn to the proof of Theorem 1. By Lemmas 3 and 4 in the Appendix applied to (4), we obtain that for some positive constants K_n, M_n, C_{ε,n_2}, C_α and $\tilde{C}_{\varepsilon,n_2}$ the following inequality is satisfied

$$
\begin{aligned}
&|E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)]| \\
&\leq 2\varepsilon^{-2n}K_n\left(M_n\|\bar{\sigma}\|_\infty^{2n}\delta^n + \delta^{2n}\|\bar{b}\|_\infty^{2n}\right) \\
&+ C_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2} + C_\alpha\delta^{\frac{1+\alpha}{2}} + \|\bar{\sigma}^{-1}\bar{b}\|_\infty\tilde{C}_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2},
\end{aligned} \tag{7}
$$

Since $\delta \in (0, t\wedge 1)$ is an arbitrary number, we can take

$$\delta := |\theta|^{-\beta},$$

for $|\theta| > (t\wedge 1)^{-\frac{1}{\beta}}$ and any $\beta > 0$. If we denote by $\bar{C}_{\varepsilon,n,n_2}$ the maximum of all the coefficients of θ appearing in (7), we rewrite that inequality as

$$|E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t - y_0)]| \leq \bar{C}_{\varepsilon,n,n_2}(|\theta|^{-n\beta} + |\theta|^{-2n\beta} + |\theta|^{-\frac{(2-\beta)n_2}{2}} + |\theta|^{-\frac{(1+\alpha)\beta}{2}})$$

for $(t\wedge 1)^{-\frac{1}{\beta}} < |\theta|$ and $\frac{2}{1+\alpha} < \beta < 2$. Since n and n_2 are arbitrary, if we choose $\gamma \in (0,\alpha)$, β as

$$\frac{2(1+\gamma)}{1+\alpha} < \beta < 2$$

and sufficiently large n and n_2, then by (1), X_t has a γ-Hölder continuous density on $B_\varepsilon(y_0)$. □

Remark 6. *1. Note that as β is chosen closer to 2, n_2 has to be chosen bigger. Therefore in comparison with the classical proofs of the regularity of the density, we need higher regularity of σ in order to obtain γ-Hölder properties of the density for γ closer to α.*

2. Note that the term that decided the rate of decrease for the localized characteristic function was $\delta^{\frac{1+\alpha}{2}}$ which is the approximation term for the Girsanov change of measure and which strongly uses the Hölder continuity of b (see the proof of Lemma 4. Therefore even if the other terms may have a faster rate of decrease this will not improve the final result.

6 Conclusions

We have proved that the regularity of the diffusion coefficient can help transfer the irregularity of the drift to the density function in contrast to the role played by the drift in [3] and [9]. In both of these results the drift seems does not seem to play any important role. In this article, we intended to point out that this is not the case and that the regularity of the drift may play an important role in determining the regularity of the density. This is the point where the integration by parts formula of Malliavin Calculus plays an important role in comparison with the previously mentioned results.

In fact, in a related research, we intend to show, using a more complicated technique (this involves a more complex version of the technique introduced in [5]) in the case that the diffusion coefficient is constant, that there are situations where the drift is the determining factor in the regularity of the density of X_t.

7 Appendix

7.1 Estimate of (4) on the event C

Lemma 3. *Under* $(H1)$ *and* $(H2)$, *we have the following estimate:*

$$|E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t-y_0)1_C]| \le \varepsilon^{-2n}K_n\left(M_n\|\bar\sigma\|_\infty^{2n}\delta^n+\delta^{2n}\|\bar b\|_\infty^{2n}\right), \qquad (8)$$

where K_n *and* M_n *are constants depend only on* n.

Proof. Using Markov's inequality, we have

$$\begin{aligned} Q(C) &\le Q\left(\sup_{0\le s\le\delta}|\bar X_{\nu+s}(\nu,X_\nu)-X_\nu|\ge\varepsilon\right)\\ &\le \varepsilon^{-2n}E_Q\left[\sup_{0\le s\le\delta}|\bar X_{\nu+s}(\nu,X_\nu)-X_\nu|^{2n}\right]\\ &\le \varepsilon^{-2n}K_n\left(E_Q\left[\sup_{0\le s\le\delta}|\int_\nu^{s+\nu}\bar\sigma(\bar X_u(\nu,X_\nu))dB_u|^{2n}\right]+E_Q\left[\sup_{0\le s\le\delta}|\int_\nu^{s+\nu}\bar b(\bar X_u(\nu,X_\nu))du|^{2n}\right]\right), \end{aligned}$$

where K_n is a constant which depends only on n.

Since $\bar\sigma$ and $\bar b$ are bounded, by Doob's inequality and Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} &\varepsilon^{-2n}K_n\left(E_Q\left[\sup_{0\le s\le\delta}|\int_\nu^{s+\nu}\bar\sigma(\bar X_u(\nu,X_\nu))dB_u|^{2n}\right]+E_Q\left[\sup_{0\le s\le\delta}|\int_\nu^{s+\nu}\bar b(\bar X_u(\nu,X_\nu))du|^{2n}\right]\right)\\ &\le \varepsilon^{-2n}K_n\left(M_nE_Q\left[\left\{\int_\nu^{\nu+\delta}(\bar\sigma(\bar X_u(\nu,X_\nu))^2du\right\}^n\right]+(\delta\|\bar b\|_\infty)^{2n}\right)\\ &\le \varepsilon^{-2n}K_n\left(M_n\|\bar\sigma\|_\infty^{2n}\delta^n+(\delta\|\bar b\|_\infty)^{2n}\right) \end{aligned}$$

for any $n\in\mathbb{N}$, where M_n is a constant depends only on n. Therefore (8) follows. □

7.2 Estimate of (4) on the event A

Now we turn to estimate the second term of (4).

Lemma 4. *Under* $(H1), (H2)$ *and* $(H3)$*, we have the following estimate:*

$$|E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t-y_0)1_A]| \le \varepsilon^{-2n}K_n\left(M_n\|\bar\sigma\|_\infty^{2n}\delta^n+\delta^{2n}\|\bar b\|_\infty^{2n}\right) + C_{\varepsilon,n_2}|\theta\delta^{\frac12}|^{-n_2}+C_\alpha\delta^{\frac{1+\alpha}{2}}+\|\bar\sigma^{-1}\bar b\|_\infty\tilde C_{\varepsilon,n_2}|\theta\delta^{\frac12}|^{-n_2}.$$

Proof. By the definition of $\bar X$, on the event A,

$$X_t=\bar X_t(t-\delta,X_{t-\delta}).$$

Hence, we obtain that

$$|E_Q[e^{i\theta X_t}\phi_\varepsilon(X_t-y_0)1_A]|=|E_Q[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)1_A]|.$$

Since

$$\begin{aligned}1_{\{\nu=t-\delta;t<\tau\}}&=1_{\{\nu=t-\delta;t<\tau\}}1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}\\&=(1-1_{\{\nu=t-\delta;\tau\le t\}})1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}\\&=1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}-1_{\{\nu=t-\delta;\tau\le t\}},\end{aligned}$$

we have

$$\begin{aligned}&|E_Q[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)1_A]| \\ &\le |E_Q[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]| \\ &+|E_Q[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)1_{\{\nu=t-\delta;\tau\le t\}}]|.\end{aligned}\tag{9}$$

By the definitions of ν and τ, we have

$$\{\nu=t-\delta;\tau\le t\}\subseteq\{\sup_{0\le s\le\delta}|\bar X_{t-\delta+s}(t-\delta,X_{t-\delta})-X_{t-\delta}|\ge\varepsilon\}.$$

So, as in Lemma 3 we obtain that

$$Q(\nu=t-\delta;\tau\le t)\le\varepsilon^{-2n}K_n\left(M_n\|\bar\sigma\|_\infty^{2n}\delta^n+\delta^{2n}\|\bar b\|_\infty^{2n}\right).$$

Therefore, we have the following upper bound for the second term in (9)

$$\begin{aligned}&|E_Q[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)1_{\{\nu=t-\delta;\tau\le t\}}]| \\ &\le\varepsilon^{-2n}K_n\left(M_n\|\bar\sigma\|_\infty^{2n}\delta^n+\delta^{2n}\|\bar b\|_\infty^{2n}\right).\end{aligned}\tag{10}$$

For the first term in (9), we change the probability measure from Q to P defined by (5). That is,

$$\begin{aligned}&E_Q[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}] \\ &=E_P[e^{i\theta \bar X_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar X_t(t-\delta,X_{t-\delta})-y_0)Z_t1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}].\end{aligned}$$

Then we have

$$
\begin{aligned}
&|E_P[e^{i\theta \bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)Z_t 1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]| \qquad (11)\\
&\le |E_P[e^{i\theta \bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)(Z_t-1)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]|\\
&+|E_P[e^{i\theta \bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]|.
\end{aligned}
$$

Since $1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}$ is $\mathcal{F}_{t-\delta}$-measurable, using conditional expectation and the Markov property for $\bar{X}$, we have

$$
\begin{aligned}
&|E_P[e^{i\theta \bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]|\\
&=|E_P[E_P[e^{i\theta \bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)|\mathcal{F}_{t-\delta}]1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]|\\
&\le \sup_{y\in\overline{B_{3\varepsilon}(y_0)}}|E_P[e^{i\theta \bar{X}_t(t-\delta,y)}\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)]|.
\end{aligned}
$$

As in Proposition 1, the integration by parts formula of Malliavin calculus in the interval $[t-\delta,t]$, implies that for any $n_2\in\mathbb{N}$ and $y\in\overline{B_{3\varepsilon}(y_0)}$, there exists a random variable $H_{n_2}(\bar{X}_t(t-\delta,y),\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0))\in\mathbb{D}^\infty$ such that

$$
\begin{aligned}
&E_P\left[\frac{d^{n_2}}{dx^{n_2}}\left(e^{i\theta x}\right)\Big|_{x=\bar{X}_t(t-\delta,y)}\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)\right]\\
&=E_P\left[e^{i\theta X_t(t-\delta,y)}H_{n_2}(\bar{X}_t(t-\delta,y),\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0))\right].
\end{aligned}
$$

Furthermore, by Theorem 2.3. and Corollary 1 of [10] (which are consequences of the application of Proposition 1 to our situation), there exists a constant C_{ε,n_2} which depends on ε, n_2 and derivatives of $\bar{\sigma}$ up to the order n_2 such that for any $y\in\overline{B_{3\varepsilon}(y_0)}$,

$$
\|H_{n_2}(\bar{X}_t(t-\delta,y),\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0))\|_{L^2(P)}\le C_{\varepsilon,n_2}\delta^{-\frac{n_2}{2}}. \qquad (12)
$$

In fact, Theorem 2.3. of [10] tells us that there exists some constant $C^\star_{\varepsilon,n}$ such that

$$
\|H_{n_2}(\bar{X}_t(t-\delta,y),\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0))\|_{L^2(P)}\le C^\star_{\varepsilon,n_2}\|\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)\|_{n_2,2^{n_2+1}}\delta^{-\frac{n_2}{2}}.
$$

On the other hand, thanks to $(H1)$, Corollary 1 of [10] implies that there exists some constant $C^\dagger_{\varepsilon,n_2}$ such that

$$
\|\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)\|_{n_2,2^{n_2+1}}\le C^\dagger_{\varepsilon,n_2}.
$$

The above constant C_{ε,n_2} is the product of these constants $C^\star_{\varepsilon,n_2}$ and $C^\dagger_{\varepsilon,n_2}$.

By (12) and recalling that Z is a non-negative martingale with mean one, for any $n_2\in\mathbb{N}$, we obtain the following inequality.

$$
\begin{aligned}
&|E_P[e^{i\theta \bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]| \qquad (13)\\
&\le C_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2}.
\end{aligned}
$$

However, since $Z_t - 1$ is not $\mathcal{F}_{t-\delta}$-measurable and we do not assume the smoothness of the coefficient b, we can not apply the integration by parts formula for the first term in (11). Instead, we rewrite

$$\begin{aligned} Z_t - 1 = &\int_{t-\delta}^{t} (\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t-\delta, X_{t-\delta}))Z_u - (\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})dW_u \\ &+ (\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})(W_t - W_{t-\delta}). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\left| E_P\left[e^{i\theta \bar{X}_t(t-\delta, X_{t-\delta})} \phi_\varepsilon(\bar{X}_t(t-\delta, X_{t-\delta}) - y_0)(Z_t - 1) 1_{\{X_{t-\delta} \in \overline{B_{3\varepsilon}(y_0)}\}} \right] \right| \qquad (14) \\ &\le E_P\left[\int_{t-\delta}^{t} |(\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t-\delta, X_{t-\delta}))Z_u - (\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})|^2 du \right]^{\frac{1}{2}} \\ &+ |E_P[e^{i\theta \bar{X}_t(t-\delta, X_{t-\delta})} \phi_\varepsilon(\bar{X}_t(t-\delta, X_{t-\delta}) - y_0)(\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})(W_t - W_{t-\delta}) 1_{\{X_{t-\delta} \in \overline{B_{3\varepsilon}(y_0)}\}}]|. \end{aligned}$$

For the first term, by the Hölder continuity of $\bar{\sigma}^{-1}\bar{b}$, (6) and Hölder's inequality, we have

$$\begin{aligned} &E_P\left[\int_{t-\delta}^{t} |(\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t-\delta, X_{t-\delta}))Z_u - (\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})|^2 du \right]^{\frac{1}{2}} \qquad (15) \\ &\le \sqrt{2}\Bigg[\int_{t-\delta}^{t} E_P\left[|(\bar{\sigma}^{-1}\bar{b})(\bar{X}_u(t-\delta, X_{t-\delta})) - (\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})|^2 Z_u^2 \right] du \\ &+ \int_{t-\delta}^{t} E_P[|(\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})|^2 (Z_u - 1)^2] du \Bigg]^{\frac{1}{2}} \\ &\le \sqrt{2}\Bigg[\int_{t-\delta}^{t} E_P[|\bar{X}_u(t-\delta, X_{t-\delta}) - X_{t-\delta}|^{2\alpha} Z_u^2] du \\ &+ \|\bar{\sigma}^{-1}\bar{b}\|_\infty^2 \int_{t-\delta}^{t} \int_{t-\delta}^{u} E_P[|(\bar{\sigma}^{-1}\bar{b})(\bar{X}_v(t-\delta, X_{t-\delta}))Z_v|^2] dv du \Bigg]^{\frac{1}{2}} \\ &\le \sqrt{2}\left[\int_{t-\delta}^{t} E_P[|\bar{X}_u(t-\delta, X_{t-\delta}) - X_{t-\delta}|^2]^\alpha E_P[Z_u^{\frac{2}{1-\alpha}}]^{1-\alpha} du + \frac{\|\bar{\sigma}^{-1}\bar{b}\|_\infty^4}{2} \|Z_t\|_{L^2(P)}^2 \delta^2 \right]^{\frac{1}{2}} \\ &\le \sqrt{2}\left[\frac{2}{1+\alpha} \|Z_t^2\|_{L^{\frac{1}{1-\alpha}}(P)} \|\bar{\sigma}\|_\infty^{2\alpha} \delta^{1+\alpha} + \frac{\|\bar{\sigma}^{-1}\bar{b}\|_\infty^4}{2} \|Z_t\|_{L^2(P)}^2 \delta^2 \right]^{\frac{1}{2}} \\ &\le C_\alpha \delta^{\frac{1+\alpha}{2}}, \end{aligned}$$

where

$$C_\alpha := \left(\frac{2}{\sqrt{1+\alpha}} \|Z_t\|_{L^{\frac{2}{(1-\alpha)}}(P)} \|\bar{\sigma}\|_\infty^\alpha \right) \vee \left(\|\bar{\sigma}^{-1}\bar{b}\|_\infty^2 \|Z_t\|_{L^2(P)} \right).$$

For the second term of (14), we proceed as in (13). Since $(\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})$ is bounded and $\mathcal{F}_{t-\delta}$-measurable, we have

$$
\begin{aligned}
&|E_P[e^{i\theta\bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)(\bar{\sigma}^{-1}\bar{b})(X_{t-\delta})(W_t-W_{t-\delta})1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]| \\
&\leq \|\bar{\sigma}^{-1}\bar{b}\|_\infty \sup_{y\in\overline{B_{3\varepsilon}(y_0)}} |E_P[e^{i\theta\bar{X}_t(t-\delta,y)}\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)(W_t-W_{t-\delta})]|.
\end{aligned}
$$

Now we can apply the integration by parts formula which implies that for any $n_2\in\mathbb{N}$ and $y\in\overline{B_{3\varepsilon}(y_0)}$ there exists a random variable $H_{n_2}(\bar{X}_t(t-\delta,y),\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)(W_t-W_{t-\delta}))\in\mathbb{D}^\infty$ such that

$$
\begin{aligned}
&E_P\left[\frac{d^{n_2}}{dx^{n_2}}(e^{i\theta x})\bigg|_{x=\bar{X}_t(t-\delta,y)}\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)(W_t-W_{t-\delta})\right] \\
&= E_P\left[e^{i\theta\bar{X}_t(t-\delta,y)}H_{n_2}(\bar{X}_t(t-\delta,y),\phi_\varepsilon(\bar{X}_t(t-\delta,y)-y_0)(W_t-W_{t-\delta}))\right]
\end{aligned}
$$

and by the Hölder inequality for the stochastic Sobolev norms (see Proposition 1.5.6 of [11]), its $L^2(P)$-norm is bounded by $C_{\varepsilon,n_2}\delta^{-\frac{n_2}{2}}c_{n_2}\|(W_t-W_{t-\delta})\|_{n_2,2^{n_2+1}}$, where c_{n_2} is a constant depends only on n_2.

However, the k-th order H-derivatives of $W_t-W_{t-\delta}$ vanish when $k\geq 2$. Therefore, there exists a positive constant C (independent of n_2) such that

$$
\|(W_t-W_{t-\delta})\|_{n_2,2^{n_2+1}} = \|(W_t-W_{t-\delta})\|_{1,2^{n_2+1}} \leq C \tag{16}
$$

and hence, we have

$$
\begin{aligned}
&|E_P[e^{i\theta\bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)(Z_t-1)1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]| \\
&\leq C_\alpha\delta^{\frac{1+\alpha}{2}} + \|\bar{\sigma}^{-1}\bar{b}\|_\infty\tilde{C}_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2},
\end{aligned} \tag{17}
$$

where $\tilde{C}_{\varepsilon,n_2} := 2C_{\varepsilon,n_2}c_{n_2}$.

Substituting (17) and (13) into (11), we have

$$
\begin{aligned}
&|E_P[e^{i\theta\bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)Z_t1_{\{X_{t-\delta}\in\overline{B_{3\varepsilon}(y_0)}\}}]| \\
&\leq C_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2} + C_\alpha\delta^{\frac{1+\alpha}{2}} + \|\bar{\sigma}^{-1}\bar{b}\|_\infty\tilde{C}_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2}.
\end{aligned} \tag{18}
$$

As a result, we have

$$
\begin{aligned}
&|E_Q[e^{i\theta\bar{X}_t(t-\delta,X_{t-\delta})}\phi_\varepsilon(\bar{X}_t(t-\delta,X_{t-\delta})-y_0)1_A]| \\
&\leq \varepsilon^{-2n}K_n\left(M_n\|\bar{\sigma}\|_\infty^{2n}\delta^n + \delta^{2n}\|\bar{b}\|_\infty^{2n}\right) \\
&+ C_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2} + C_\alpha\delta^{\frac{1+\alpha}{2}} + \|\bar{\sigma}^{-1}\bar{b}\|_\infty\tilde{C}_{\varepsilon,n_2}|\theta\delta^{\frac{1}{2}}|^{-n_2}
\end{aligned}
$$

by substituting (18) and (10) into (9). □

Remark 7. *The above estimate* (16) *for the Sobolev norm of the Wiener process is clearly non-optimal. However, as the term appearing in* (15) *decreases slowly, improving the estimate in* (16) *will not change the final result. The same comment applies to other terms such as* (10).

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