A new Kontorovich-Lebedev-like transformation

Semyon YAKUBOVICH

Abstract

A different application of the familiar integral representation for the modified Bessel function drives to a new Kontorovich-Lebedev-like integral transformation of a general complex index. Mapping and operational properties, a convolution operator and inversion formula are established. Solvability conditions and explicit solutions of the corresponding class of convolution integral equations are exhibited.

Keywords: *Kontorovich-Lebedev transform, modified Bessel functions, Mellin transform, Laplace transform, convolution, integral equations of the convolution type*

AMS subject classification: 44A15, 33C05, 33C10, 33C15

1 Introduction

As it is known [2], Vol. II, the modified Bessel function $K_z(2\sqrt{x})$ can be represented by the following integral

$$K_z(2\sqrt{x}) = \frac{x^{-z/2}}{2}\int_0^\infty e^{-t-\frac{x}{t}} t^{z-1} dt,\ x > 0, \tag{1.1}$$

where $z = \nu + i\tau$ is a complex number. As it is easily seen, integral (1.1) converges absolutely for any $x \in \mathbb{R}_+, z \in \mathbb{C}$ and represents an entire function by z. Formula (1.1) can be written with the use of the Parceval relation for the Mellin transform [6], which leads to the integral representation

$$2x^{z/2}K_z(2\sqrt{x}) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+z)\Gamma(s)x^{-s}ds,\ x > 0, \tag{1.2}$$

where $\Gamma(w)$ is Euler's gamma function [2], Vol. 1 and $\gamma > \max(0, -\text{Re}z)$. Reciprocally, we have the direct Mellin transform of the modified Bessel function, namely

$$\Gamma(s+z)\Gamma(s) = 2\int_0^\infty K_z(2\sqrt{x})x^{s+z/2-1}dx. \tag{1.3}$$

The left-hand side of (1.2) has the following asymptotic behavior near the origin $x \to 0+$

$$x^{z/2}K_z(2\sqrt{x}) = \begin{cases} O(1), & \text{if Re}z > 0,\\ O(x^{\text{Re}z}), & \text{if Re}z < 0,\\ O\left(\log\left(\frac{1}{x}\right)\right), & \text{if } z = 0 \end{cases}$$

and $x^{z/2}K_z(2\sqrt{x}) = O(e^{-2\sqrt{x}}x^{(\text{Re}z-1/2)/2}),\ x \to +\infty$.

Let us consider the following integral transformation with respect to an index $z \in \mathbb{C}$ of the modified Bessel function

$$(Ff)(z) = 2\int_0^\infty x^{z/2}K_z(2\sqrt{x})f(x)dx. \tag{1.4}$$

This transformation looks like the Kontorovich-Lebedev transform [5], [8], [9]. However, it is a completely different operator and cannot be reduced to the Kontorovich-Lebedev integral by any change of variables and functions. As far as the author is aware, the transform (1.4) was not studied yet, taking into account his mapping properties and inversion formula in an appropriate class of functions.

Our goal is to do this involving a special class of functions related to the Mellin transform and its inversion, which was introduced in [7]. Indeed, we have

Definition 1. Denote by $\mathcal{M}^{-1}(L_c)$ the space of functions $f(x), x \in \mathbb{R}_+$, representable by inverse Mellin transform of integrable functions $f^*(s) \in L_1(c)$ on the vertical line $c = \{s \in \mathbb{C} : \text{Re}s = c_0\}$:

$$f(x) = \frac{1}{2\pi i}\int_c f^*(s)x^{-s}ds. \tag{1.5}$$

The space $\mathcal{M}^{-1}(L_c)$ with the usual operations of addition and multiplication by scalar is a linear vector space. If the norm in $\mathcal{M}^{-1}(L_c)$ is introduced by the formula

$$||f||_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |f^*(c_0 + it)|dt, \tag{1.6}$$

then it becomes a Banach space.

Definition 2 ([7], [8]). Let $c_1, c_2 \in \mathbb{R}$ be such that $2\text{sign } c_1 + \text{sign } c_2 \geq 0$. By $\mathcal{M}^{-1}_{c_1,c_2}(L_c)$ we denote the space of functions $f(x), x \in \mathbb{R}_+$, representable in the form (1.5), where $s^{c_2}e^{\pi c_1|s|}f^*(s) \in L_1(c)$.

It is a Banach space with the norm

$$||f||_{\mathcal{M}^{-1}_{c_1,c_2}(L_c)} = \frac{1}{2\pi}\int_c e^{\pi c_1|s|}|s^{c_2}f^*(s)ds|.$$

In particular, letting $c_1 = c_2 = 0$ we get the space $\mathcal{M}^{-1}(L_c)$. Moreover, it is easily seen the inclusion

$$\mathcal{M}^{-1}_{d_1,d_2}(L_c) \subseteq \mathcal{M}^{-1}_{c_1,c_2}(L_c)$$

when $2\text{sign}(d_1 - c_1) + \text{sign}(d_2 - c_2) \geq 0$.

2 Mapping properties and an inversion formula

We begin with the following result.

Theorem 1. *Let $f \in \mathcal{M}^{-1}(L_c)$ and $c_0 < 1$. Then transformation* (1.4) *is well-defined and* $(Ff)(z)$ *is analytic in the half-plane* $\text{Re}z > c_0 - 1$. *Further,*

$$(Ff)(z) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \Gamma(1-s+z)\Gamma(1-s)f^*(s)ds, \tag{2.1}$$

and the operator $F : \mathcal{M}^{-1}(L_c) \to L_1(\text{Re}z - i\infty, \text{Re}z + i\infty)$, $\text{Re}z > c_0 - 1$ *is bounded with the norm satisfying the estimate*

$$||F|| \leq \Gamma(1-c_0)\int_{-\infty}^{\infty} |\Gamma(1-c_0+\text{Re}z+i\tau)|d\tau.$$

Proof. In fact, substituting (1.5) into (1.4) and changing the order of integration by Fubini's theorem, we call (1.3) to prove (2.1). The inversion of the order of integration is guaranteed by the estimate (see (1.3))

$$2\int_0^{\infty} \left|x^{z/2}K_z(2\sqrt{x})\right| \int_c |f^*(s)x^{-s}ds|dx$$

$$\leq 2\int_0^\infty x^{(\text{Re}z-2c_0)/2}K_{\text{Re}z}(2\sqrt{x})dx\int_c |f^*(s)ds|$$

$$= \Gamma\left(1-c_0+\text{Re}z\right)\Gamma(1-c_0)\int_c |f^*(s)ds| < +\infty,\ \text{Re}z > c_0-1,\ c_0<1$$

and the asymptotic behavior of the modified Bessel function at infinity and near the origin (see above). Furthermore, integral (1.4) converges absolutely in the half-plane $\text{Re}z > c_0-1$ and uniformly in $\text{Re}z \geq a_0 > c_0 1$. Since for each $x>0$ the function $x^{z/2}K_z(2\sqrt{x})$ is analytic by z, we have that $F(z)$ is well-defined and represents an analytic function in the half-plane $\text{Re}z > c_0-1$. Finally, the straightforward estimate takes place

$$||Ff||_1 = \int_{-\infty}^{\infty} |(Ff)(\text{Re}z+i\tau)|d\tau$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Gamma(1-c_0+\text{Re}z+i(\tau-t))\Gamma(1-c_0-it)f^*(c_0+it)|dtd\tau$$

$$\leq \Gamma(1-c_0)\ ||f||_{\mathcal{M}^{-1}(L_c)}\int_{-\infty}^{\infty} |\Gamma(1-c_0+\text{Re}z+i\tau)|d\tau,$$

which completes the proof of the theorem. □

For the subspace $\mathcal{M}^{-1}_{0,n}(L_c) \subseteq \mathcal{M}^{-1}(L_c),\ n\in\mathbb{N}_0$ we have

Theorem 2. *Let $n\in\mathbb{N}_0,\ f\in\mathcal{M}^{-1}_{0,n}(L_c)$ and $c_0<1-n$. Then $f(x),\ x\in\mathbb{R}_+$ is n times continuously differentiable, $\left(Ff^{(n)}\right)(z)$ is analytic in the half-plane $\text{Re}z>c_0+n-1$ and $\left(Ff^{(n)}\right)(z)=(Ff)(z-n)$. Finally, for any arbitrary $y\in\mathbb{R}_+$ the following representation holds*

$$(Ff)_y(z) = 2\int_y^\infty x^{z/2}K_z(2\sqrt{x})f(x)dx$$

$$= 2\sum_{m=0}^{n-1}(-1)^m y^{(z+m+1)/2}K_{z+m+1}(2\sqrt{y})f^{(m)}(y) + (-1)^n\left(Ff^{(n)}\right)_y(z+n),\ y>0, \tag{2.2}$$

where the empty sum ($n=0$) is equal to zero.

Proof. Clearly, from representation (1.1) after differentiation and integration n times with respect to x under the integral sign we come out, accordingly, with the identities

$$2\frac{d^n}{dx^n}\left[x^{z/2}K_z(2\sqrt{x})\right] = (-1)^n\int_0^\infty e^{-t-\frac{x}{t}}t^{z-n-1}dt = 2(-1)^n x^{(z-n)/2}K_{z-n}(2\sqrt{x}), \tag{2.3}$$

$$\frac{2}{(n-1)!}\int_y^\infty (x-y)^{n-1}x^{z/2}K_z(2\sqrt{x})dx = \int_0^\infty e^{-t-\frac{y}{t}}t^{z+n-1}dt$$

$$= 2y^{(z+n)/2}K_{z+n}(2\sqrt{y}),\ y>0. \tag{2.4}$$

Further, from Definition 2 it follows that f is n times continuously differentiable and via (1.5) it has

$$f^{(n)}(x) = \frac{(-1)^n}{2\pi}\int_c (s)_n f^*(s)x^{-s-n}ds, \tag{2.5}$$

where $(a)_n$ is Pochhammer's symbol. Hence, considering $(Ff^n)(z)$, we integrate by parts in the corresponding integral (1.4), taking into account that the integrated terms are vanished owing to the asymptotic behavior of the modified Bessel function, the estimate $f^{(n)} = O(x^{-c_0-n}),\ x > 0$ (see (2.5)) and limit relations

$$\lim_{x\to 0+} x^{1-c_0-j(\mathrm{Re}z-i)/2} K_{\mathrm{Re}z-i}(2\sqrt{x}) = 0,\ i,j \in \mathbb{N}_0, i+j=n,$$

which take place by virtue of the conditions $c_0 < 1-n,\ \mathrm{Re}z > c_0+n-1$. Thus calling (2.3) we prove the equality $\left(Ff^{(n)}\right)(z) = (Ff)(z-n)$ and similar to the proof of Theorem 1 we easily justify the analyticity of $G(z) = \left(Ff^{(n)}\right)(z)$ in the half-plane $\mathrm{Re}z > c_0+n-1$. Finally, the proof of (2.2) follows immediately, appealing to (2.4) and integrating n times by parts in its left-hand side. □

In order to establish an inversion formula for the transformation (1.4) we employ an operational technique, which was used formally by Sneddon [5], Ch. 6 to deduce the inversion formula for the Kontorovich-Lebedev transform. We start multiplying both sides of the equality (2.1) by $x^z,\ x > 0$ and integrating with respect to z over the line $(\gamma - i\infty, \gamma + i\infty),\ \gamma > c_0 - 1$. Changing the order of integration in the right-hand side of the obtained equality, which is possible via Theorem 1 and calculating the corresponding inverse Mellin transform of the gamma-function, we derive

$$\int_{\gamma-i\infty}^{\gamma+i\infty} (Ff)(z)x^z dz = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(1-s+z)\Gamma(1-s)f^*(s)x^z dzds$$

$$= e^{-1/x}\int_{c_0-i\infty}^{c_0+i\infty} \Gamma(1-s)f^*(s)x^{s-1}ds.$$

Hence, taking into account that $f \in \mathcal{M}^{-1}(L_c)$, we apply the Mellin -Parseval identity to the right-hand side of the latter equality. Thus

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} (Ff)(z)e^{1/x}x^z dz = (Lf)(x) = \int_0^\infty e^{-xt}f(t)dt,\ x > 0 \tag{2.6}$$

and the right-hand side of the latter equality represents the Laplace transform denoted by $(Lf)(x)$. In the meantime, relation (2.15.5.4) in [4], Vol. 2 gives the key integral involving the modified Bessel function of the third kind $I_\nu(w)$ [2], Vol. II

$$e^{1/x}x^z = \int_0^\infty e^{-xt} I_{-(1+z)}\left(2\sqrt{t}\right)\ t^{-(1+z)/2}dt,\ x > 0,\ \mathrm{Re}z < 0.$$

Substituting this integral into the left-hand side of (2.6) and assuming an additional condition

$$(Ff)(\gamma + i\tau) \in L_1\left(|\tau| > 1; |\tau|^{\gamma+1/2}\ e^{\pi|\tau|/2}d\tau\right),\ \gamma \in (c_0-1, 0), \tag{2.7}$$

we change of integration by Fubini's theorem and arrive at the equality

$$\int_0^\infty e^{-xt}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} I_{-(1+z)}\left(2\sqrt{t}\right)\ t^{-(1+z)/2}\ (Ff)(z)dzdt = \int_0^\infty e^{-xt}f(t)dt,\ x > 0. \tag{2.8}$$

Indeed, the motivation of the inversion of the order of integration in (2.8) is given due to the representation of the modified Bessel function $I_{-(1+z)}\left(2\sqrt{t}\right)$ in terms of the series

$$I_{-(1+z)}\left(2\sqrt{t}\right) = \sum_{n=0}^\infty \frac{t^{n-(1+z)/2}}{n!\ \Gamma(n-z)} \tag{2.9}$$

and an absolute integrability by $\tau \in \mathbb{R}$ of the product $(Ff)(\gamma + i\tau)I_{-(1+\gamma+i\tau)}\left(2\sqrt{t}\right)$ under condition (2.7), since $\Gamma(n-\gamma-i\tau) = O(|\tau|^{n-\gamma-1/2}e^{-\pi|\tau|/2}),\ |\tau| \to \infty,\ n \in \mathbb{N}_0$ via Stirling's formula [2], Vol. I. Finally, we observe that equality (2.8) is true for all $x > 0$, where functions under the convergent Laplace integrals in its both sides are continuous on $\mathbb{R}_+$ owing to condition $f \in \mathcal{M}^{-1}(L_c)$ and assumption (2.7). Therefore one can cancel the Laplace transform in (2.8) by virtue of the uniqueness theorem (see in [3]) to get the inversion formula for the Kontorovich-Lebedev transformation (1.4). Thus we have proved

Theorem 3. *Let $f(t) \in \mathcal{M}^{-1}(L_c),\ c_0 < 1$ and condition (2.7) holds. Then for all $t > 0$ the following inversion formula for the transformation (1.4) takes place*

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} I_{-(1+z)}\left(2\sqrt{t}\right) t^{-(1+z)/2}\ (Ff)(z)dz,\ \gamma \in (c_0-1, 0), \tag{2.10}$$

where the integral is absolutely convergent.

3 Expansion of an arbitrary function in terms of the Kontorovich-Lebedev-like integral

In this section we will prove that any function from the space $\mathcal{M}^{-1}_{0,(|\varepsilon|+\varepsilon)/2}(L_c),\ c_0 < 1,\ 2c_0 - 1 < \varepsilon < c_0$ can be expanded in terms of the following integral

$$f(x) = \frac{1}{\pi i}\frac{d}{dx}\int_{\gamma-i\infty}^{\gamma+i\infty} I_{-z}\left(2\sqrt{x}\right) x^{-z/2} \int_0^\infty t^{z/2}K_z(2\sqrt{t})f(t)\ dtdz,\ x > 0, \tag{3.1}$$

where γ is taken from the interval $(c_0 - 1, (\varepsilon-1)/2)$.

Precisely, we have

Theorem 4. *Let $c_0 < 1,\ 2c_0 - 1 < \varepsilon < c_0$ and $f \in \mathcal{M}^{-1}_{0,(|\varepsilon|+\varepsilon)/2}(L_c)$. Then for any $x > 0$ formula (3.1) is true, where the interior integral with respect to t converges absolutely and the exterior integral by z is understood in the improper sense of Riemann.*

Proof. In fact, since $\mathcal{M}^{-1}_{0,(|\varepsilon|+\varepsilon)/2}(L_c) \subseteq \mathcal{M}^{-1}(L_c)$, the absolute convergence of the interior integral in (3.1) follows from Theorem 1. Moreover, equality (2.1) holds. Hence writing the modified Bessel function $I_{-z}\left(2\sqrt{x}\right)$ similar to (2.9) and substituting the right-hand side of (2.1) into (3.1), we come out with the following iterated integral

$$I(x) = -\frac{1}{4\pi^2}\int_{\gamma-i\infty}^{\gamma+i\infty}\sum_{n=0}^{\infty}\frac{x^{n-z}}{n!\ \Gamma(1+n-z)}\int_{c_0-i\infty}^{c_0+i\infty}\Gamma(1-s+z)\Gamma(1-s)f^*(s)dsdz. \tag{3.2}$$

Meanwhile, appealing to the Stirling formula for gamma-functions [2], Vol. I, we find for any $n \in \mathbb{N}$

$$\left|\frac{\Gamma(1-s+z)}{\Gamma(1+n-z)}\right| = \left|B(1-s+z, s-\varepsilon)B(n, 1-z)\frac{\Gamma(1-\varepsilon+z)}{\Gamma(s-\varepsilon)\Gamma(1-z)(n-1)!}\right|$$

$$\leq \frac{B(1-c_0+\gamma, c_0-\varepsilon)\Gamma(1-\gamma)}{\Gamma(1+n-\gamma)}\left|\frac{\Gamma(1-\varepsilon+z)}{\Gamma(s-\varepsilon)\Gamma(1-z)}\right| = O\left(\frac{|z|^{2\gamma-\varepsilon}}{|\Gamma(s-\varepsilon)|}\right),\quad |\mathrm{Im}z| \to \infty,$$

where $B(a,b)$ is Euler's beta-function, $c_0 - 1 < \gamma < (\varepsilon-1)/2$. Hence from (3.2) for each fixed $x > 0$ we obtain the estimate

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\sum_{n=0}^{\infty}\left|\frac{x^{n-z}}{n!\ \Gamma(1+n-z)}\right|\int_{c_0-i\infty}^{c_0+i\infty}|\Gamma(1-s+z)\Gamma(1-s)f^*(s)dsdz|$$

$$\leq \ B(1-c_0+\gamma, c_0-\varepsilon)\Gamma(1-\gamma)x^{-\gamma/2}I_{-\gamma}(2\sqrt{x})$$
$$\times \int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma(1-\varepsilon+z)}{\Gamma(1-z)}\right| \int_{c_0-i\infty}^{c_0+i\infty}\left|\frac{\Gamma(1-s)}{\Gamma(s-\varepsilon)}f^*(s)dsdz\right|$$
$$= O\left(\int_{\gamma-i\infty}^{\gamma+i\infty}|z|^{2\gamma-\varepsilon}|dz|\int_{c_0-i\infty}^{c_0+i\infty}|s|^{\varepsilon}|f^*(s)ds|\right) < +\infty.$$

Consequently, the change of the order of integration and summation is possible in (3.2). After calculation of the integral with respect to z using relation (8.4.19.1) in [4], Vol. 3 it becomes

$$I(x) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\sum_{n=0}^{\infty}\frac{x^n}{n!}J_{n+1-s}(2\sqrt{x})\ \Gamma(1-s)f^*(s)x^{(1-s)/2}ds, \tag{3.3}$$

where $J_\mu(w)$ is the Bessel function of the first kind [2], Vol. II. But the series inside (3.3) is calculated in [4], Vol. 2, relation (5.7.6.7), namely

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}J_{n+1-s}(2\sqrt{x}) = \frac{x^{(1-s)/2}}{\Gamma(2-s)}.$$

Thus substituting this value into (3.3) and applying the reduction formula for gamma-function, we arrive at the equality

$$I(x) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}f^*(s)\frac{x^{1-s}}{1-s}ds. \tag{3.4}$$

Hence the differentiation with respect to $x > 0$ under integral sign in (3.4) is permitted via the absolute and uniform convergence since $f^*(s) \in L_1(c)$ (see Definition 2). Thus we establish equality (3.1) and complete the proof. □

As we see, expansion (3.1) generates the following reciprocal inversion formula of the index transform (1.4)

$$f(x) = \frac{1}{2\pi i}\frac{d}{dx}\int_{\gamma-i\infty}^{\gamma+i\infty}I_{-z}\left(2\sqrt{x}\right)x^{-z/2}(Ff)(z)dz,\ x > 0. \tag{3.5}$$

Corollary 1. *Let, in addition, condition* (2.7) *hold. Then formula* (3.5) *can be written in the form* (2.10).

Proof. Indeed, in this case the differentiation under integral sign in (3.5) is allowed via the absolute and uniform convergence. Hence using the identity for derivatives of Bessel functions [2], Vol. II

$$\frac{d}{dx}\left[I_{-z}\left(2\sqrt{x}\right)x^{-z/2}\right] = I_{-(z+1)}\left(2\sqrt{x}\right)x^{-(z+1)/2},$$

we arrive at the result. □

Corollary 2. *Let* $c_0 < 1,\ 2c_0 - 1 < \varepsilon < c_0$ *and* $f \in \mathcal{M}^{-1}_{0,(|\varepsilon|+\varepsilon)/2}(L_c)$*. Then the homogeneous integral equation*

$$\int_0^\infty t^{z/2}K_z(2\sqrt{t})f(t)dt = 0$$

has only the trivial solution.

Expansion (3.1) gives a new source of index integrals involving the modified Bessel function $I_\nu(w)$. It can be obtained employing the corresponding integrals (1.4) for concrete functions f from [4], Vol. 2. In fact, making a simple substitution in (1.4) and then using relation (2.16.6.4) in [4], Vol. 2 we calculate the value of the index integral

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} I_{-z}\left(2\sqrt{x}\right)\frac{\Gamma(z)}{2z+1}x^{-z/2}\ dz = e^{-2\sqrt{x}},\ x>0;\ \nu<1/2.$$

Meanwhile, relation (2.16.33.2) in [4], Vol. 2 leads us to the value of the reciprocal index integral

$$\frac{1}{4\pi i}\int_{\nu-i\infty}^{\nu+i\infty} I_{-z}\left(2\sqrt{x}\right)\frac{\Gamma\left(z+\frac{\mu}{2}\right)\Gamma\left(z-\frac{\mu}{2}\right)}{\Gamma(z+1)}x^{-z/2}\ dz$$

$$= K_\mu(2\sqrt{x})\left[\Gamma\left(1+\frac{\mu}{2}\right)\Gamma\left(1-\frac{\mu}{2}\right)\right]^{-1},\ x>0;\ |\mathrm{Re}\mu|/2<\nu<1/2.$$

More curious example can be calculated, for instance, via relation (2.16.8.4) in [4], Vol. 2. Indeed, we have

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} I_{-z}\left(2\sqrt{x}\right)W_{-z/2,\ (z-1)/2}\left(\frac{1}{4p}\right)\Gamma\left(z\right)(4px)^{-z/2}\ dz$$

$$= e^{-4px-\frac{1}{8p}},\ x,p>0;\ 1/2<\nu<\varepsilon+1/2,\ \varepsilon\in(0,1/4).$$

where $W_{\mu,\nu}(w)$ is the Whittaker function [2], Vol. I.

4 A convolution operator and integral equations of the convolution type

In this section we will construct a convolution operator, which is related to the transformation (1.4) and the Mellin transform [6]

$$\left(\mathcal{M}f\right)(z) = \int_0^\infty f(x)x^{z-1}dx. \tag{4.1}$$

Our construction will be based on the convolution properties of the Mellin transform in L_1 (see [5], [6], Th. 44) and representation (1.1). Indeed, considering (1.1) of the same parameter z and different positive arguments x and y, we deduce the following representation of the product of these integrals, namely

$$4(xy)^{z/2}K_z(2\sqrt{x})K_z(2\sqrt{y}) = \int_0^\infty e^{-t-\frac{x}{t}}t^{z-1}dt\int_0^\infty e^{-u-\frac{y}{u}}u^{z-1}du$$

$$= \int_0^\infty v^{z-1}\left(\int_0^\infty e^{-\ \frac{t(y+v)}{v}-\ \frac{x+v}{t}}\frac{dt}{t}\right)dv = 2\int_0^\infty K_0\left(2\sqrt{\frac{(x+v)(y+v)}{v}}\right)v^{z-1}dv,$$

where the change of the order of integration is allowed by the Fubini theorem via the absolute convergence. So we find the product integral formula for the kernel of transformation (1.4)

$$2(xy)^{z/2}K_z(2\sqrt{x})K_z(2\sqrt{y}) = \int_0^\infty K_0\left(2\sqrt{\frac{(x+v)(y+v)}{v}}\right)v^{z-1}dv,\ (x,y)\in\mathbb{R}_+^2,\ z\in\mathbb{C}. \tag{4.2}$$

Definition 3. We will call the following bilinear form $(f*g)(x),\ x\in\mathbb{R}_+$

$$(f*g)(x) = 2\int_{\mathbb{R}_+^2} K_0\left(2\sqrt{\frac{(x+u)(x+v)}{x}}\right)f(u)g(v)dudv \tag{4.3}$$

a convolution operator for the transformation (1.4) whenever it exists.

Let us consider the weighted L_1-space $L_1(\mathbb{R}_+; 2x^{\alpha/2}K_\alpha(2\sqrt{x})dx),\ \alpha \in \mathbb{R}$ with the norm

$$||f||_{L_1(\mathbb{R}_+;2x^{\alpha/2}K_\alpha(2\sqrt{x})dx)} = 2\int_0^\infty |f(x)|x^{\alpha/2}K_\alpha(2\sqrt{x})dx.$$

Similar to (2.6), we prove first the composition representation of the transformation (1.4) in terms of the Mellin and Laplace integrals.

Theorem 5. *Let* $f \in L_1(\mathbb{R}_+; x^{(\alpha-|\alpha|)/2}dx),\ \alpha \neq 0$. *Then* $(Ff)(z)$ *is analytic in the right half-plane*

$$\text{Re } z \geq \begin{cases} 0, & \textit{if } \alpha > 0, \\ \alpha, & \textit{if } \alpha < 0 \end{cases}$$

and can be represented there by the composition of the Mellin and Laplace transforms as follows

$$(Ff)(z) = \mathcal{M} \circ \left(e^{-t}(Lf)(1/t)\right)(z). \tag{4.4}$$

Proof. The proof is straightforward by Fubini's theorem with the use of integral representation (1.1), asymptotic behavior of the modified Bessel function and the estimates

$$|x^{z/2}K_z(2\sqrt{x})| \leq x^{\text{Re}z/2}K_{\text{Re}z}(2\sqrt{x}) \leq Cx^{\beta/2}K_\beta(2\sqrt{x}),\ x > 0, \tag{4.5}$$

where $C > 0$ is an absolute constant when

$$\text{Re } z \geq \begin{cases} 0, & \text{if } \beta \geq 0, \\ \beta, & \text{if } \beta < 0, \end{cases}$$

$$2x^{\alpha/2}K_\alpha(2\sqrt{x}) \leq x^{(\alpha-|\alpha|)/2}\Gamma(|\alpha|),\ \alpha \neq 0, \tag{4.6}$$

$$\int_0^\infty |x^{z/2}K_z(2\sqrt{x})||f(x)|dx \leq C\int_0^\infty |f(x)|\int_0^\infty e^{-t-\frac{x}{t}}t^{\alpha-1}dtdx$$

$$\leq C\ \Gamma(|\alpha|)\int_0^\infty x^{(\alpha-|\alpha|)/2}|f(x)|dx < \infty.$$

□

Theorem 6. *Let* $f, g \in L_1(\mathbb{R}_+; 2x^{\alpha/2}K_\alpha(2\sqrt{x})dx),\ \alpha \in \mathbb{R}$. *Then convolution* (4.3) *exists and belongs to the space* $L_1(\mathbb{R}_+; x^{\alpha-1}dx)$, *satisfying the Young type inequality*

$$||f * g||_{L_1(\mathbb{R}_+;x^{\alpha-1}dx)} \leq ||f||_{L_1(\mathbb{R}_+;2x^{\alpha/2}K_\alpha(2\sqrt{x})dx)}||g||_{L_1(\mathbb{R}_+;2x^{\alpha/2}K_\alpha(2\sqrt{x})dx)}. \tag{4.7}$$

Moreover, this form is commutative and the following factorization equality holds in terms of transformations (1.4), (4.1)

$$\left(\mathcal{M}(f * g)\right)(z) = (Ff)(z)(Fg)(z), \tag{4.8}$$

where z *belongs to the half-plane*

$$\text{Re } z \geq \begin{cases} 0, & \text{if } \alpha \geq 0, \\ \alpha, & \text{if } \alpha < 0. \end{cases}$$

Proof. In fact, the existence of the convolution (4.3) for almost all $x > 0$ follows from Fubini's theorem and the estimate

$$\int_0^\infty |(f*g)(x)|x^{\alpha-1}dx \le 2\int_0^\infty x^{\alpha-1}\int_0^\infty\int_0^\infty K_0\left(2\sqrt{\frac{(x+u)(x+v)}{x}}\right)|f(u)g(v)|dudvdx$$

$$= 4\int_0^\infty u^{\alpha/2}K_\alpha(2\sqrt{u})|f(u)|du\int_0^\infty v^{\alpha/2}K_\alpha(2\sqrt{v})|g(v)|dv.$$

This also drives us to the Young type inequality (4.6). Hence the factorization equality (4.7) is an immediate consequence of (4.2), (4.5) with $\beta = \alpha$ and straightforward calculations. □

Letting $\alpha = 1$ and using inequality (4.6) we obtain as a corollary the L_1-property of the convolution (4.3).

Corollary 3. *Let $f, g \in L_1(\mathbb{R}_+; dx)$. Then convolution* (4.3) *exists and belongs to $L_1(\mathbb{R}_+; dx)$, yielding the corresponding Young inequality*

$$||f*g||_{L_1} \le ||f||_{L_1}||g||_{L_1}. \tag{4.9}$$

Moreover, the convolution is commutative and associative, satisfying the factorization equality (4.8) *in the half-plane* $\mathrm{Re}z \ge 0$.

Further, appealing to Corollary 2 we prove an analog of Titchmarsh's theorem about the absence of divisors of zero for convolution (4.3).

Theorem 7. *Let $f, g \in L_1(\mathbb{R}_+; dx)$. Then the equality $(f*g)(x) = 0$ yields that at least one of the functions $f(x)$ and $g(x)$ is equal to zero for all $x > 0$.*

Proof. In fact, both functions $(Ff)(z), (Fg)(z)$ are analytic in the half plane $\mathrm{Re}z > 0$ and via equality (4.8) at least on of them is identically equal to zero. Then the result follows from Theorem 5 due to the uniqueness theorems in L_1 for the Mellin and Laplace transforms. □

The Parseval type equality for convolution (4.3) is an immediate consequence of the Plancherel L_2-theory of the Mellin transform [6]. We have

Theorem 8. *Let $f, g \in L_1(\mathbb{R}_+; dx)$. Then $(f*g)(x) \in L_2(\mathbb{R}_+; x^{2\alpha-1}dx)$, $\alpha > 0$ and the Parseval type equality holds*

$$\int_0^\infty |(f*g)(x)|^2 x^{2\alpha-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty \left|(Ff)(\alpha+it)(Fg)(\alpha+it)\right|^2 dt. \tag{4.10}$$

Proof. Indeed by virtue of the generalized Minkowskii inequality and relation (8.4.23.27) in [4], Vol. 3 we derive

$$||f*g||_{L_2(\mathbb{R}_+;x^{2\alpha-1}dx)} = 2\left(\int_0^\infty \left|\int_{\mathbb{R}_+^2} K_0\left(2\sqrt{\frac{(x+u)(x+v)}{x}}\right)f(u)g(v)dudv\right|^2 x^{2\alpha-1}dx\right)^{1/2}$$

$$\le 2\int_{\mathbb{R}_+^2}|f(u)g(v)|\left(\int_0^\infty K_0^2\left(2\sqrt{\frac{(x+u)(x+v)}{x}}\right)x^{2\alpha-1}dx\right)^{1/2}dudv$$

$$\le \left(\int_0^\infty K_0^2\left(2\sqrt{x}\right)x^{2\alpha-1}dx\right)^{1/2}||f||_{L_1}||g||_{L_1} = 2^{-2\alpha-1/2}\pi^{1/4}\frac{\Gamma^{3/2}(2\alpha)}{\Gamma^{1/2}(2\alpha+1/2)}||f||_{L_1}||g||_{L_1} < \infty.$$

Hence factorization equality (4.8) and Theorem 71 in [6] give the result. □

Finally, let us consider a class of convolution integral equations of the first kind generated by (4.3)

$$\int_0^\infty k_h(x,y)f(y)dy = g(x),\ x>0, \tag{4.11}$$

where

$$k_h(x,y) = 2\int_0^\infty K_0\left(2\sqrt{\frac{(x+y)(x+u)}{x}}\right) h(u)du, \tag{4.12}$$

h, g are given functions and f is to be determined.

Theorem 9. *Let $f \in \mathcal{M}^{-1}(L_c)$, $c_0 < 1$, $h \in L_1(\mathbb{R}_+; 2x^{\alpha/2}K_\alpha(2\sqrt{x})dx)$, $0 > \alpha > c_0 - 1$ and $g \in L_1(\mathbb{R}_+; x^{\alpha-1}dx)$. Let also transformation (1.4) of h $(Fh)(z)$ has no zeros in the strip $\mathrm{Re} z \in (\alpha, 0)$ and the quotient $(\mathcal{M}g)(z)/(Fh)(z)$, where $(\mathcal{M}g)(z)$ is the Mellin transform (4.1) of g, satisfies condition (2.7) in this strip. Then a solution of integral equation (4.12) has the form*

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} I_{-(1+z)}\left(2\sqrt{x}\right) x^{-(1+z)/2}\ \frac{(\mathcal{M}g)(z)}{(Fh)(z)}dz,\ x>0, \gamma\in(\alpha,0). \tag{4.13}$$

Proof. Clearly, by straightforward estimate of the norm we verify that if $f \in \mathcal{M}^{-1}(L_c)$, $c_0 < 1$ and $\alpha \in (c_0 - 1,\ 0)$, then $f \in L_1(\mathbb{R}_+; 2x^{\alpha/2}K_\alpha(2\sqrt{x})dx)$. Therefore Theorem 6 and formula (4.8) are valid for convolution $(f * h)(x)$. Hence since $(Fh)(z) \neq 0$ in the strip $\mathrm{Re}z \in (\alpha, 0)$ it has the equality

$$(Ff)(z) = \frac{(\mathcal{M}g)(z)}{(Fh)(z)}.$$

Consequently, appealing to Theorem 3 and formula (2.10), we complete the proof of the theorem. □

An interesting example of the equation (4.12) and its solution can be found, taking, for instance, $h(x) = x^{-1/2}$. In this case one can calculate the kernel (4.12) via relation (2.16.3.10) in [4], Vol. 2 and we obtain

$$k_h(x,y) = \frac{\pi\sqrt{x}}{\sqrt{x+y}}e^{-2\sqrt{x+y}}.$$

Moreover, it has $(Fh)(z) = \sqrt{\pi}\Gamma(z+1/2)$ by formula (1.3). Hence Theorem 9 says that a solution of the integral equation

$$\pi\sqrt{x}\int_0^\infty \frac{e^{-2\sqrt{x+y}}}{\sqrt{x+y}} f(y)dy = g(x),\ x>0,$$

is given by the integral

$$f(x) = \frac{1}{2\pi\sqrt{\pi}i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{I_{-(1+z)}\left(2\sqrt{x}\right)}{\Gamma(z+1/2)} x^{-(1+z)/2}\ (\mathcal{M}g)(z)dz,\ x>0,$$

where $\gamma \in (\alpha, 0)$, α is chosen from the interval $\alpha \in (\max(c_0 - 1, -1/2),\ 0)$ and

$$(\mathcal{M}g)(\gamma + i\tau) \in L_1\left(|\tau| > 1;\ |\tau|^{1/2}\ e^{\pi|\tau|}d\tau\right).$$

This example can be generalized, considering $h(x) = x^{\beta-1}$, $\beta > 0$. Hence using relation (2.16.3.8) in [4], Vol. 2, we find

$$k_h(x,y) = 2\Gamma(\beta)\left(\frac{x}{\sqrt{x+y}}\right)^\beta K_\beta\left(2\sqrt{x+y}\right).$$

Moreover, $(Fh)(z) = \Gamma(\beta)\Gamma(\beta + z)$ and a solution of the equation

$$2\Gamma(\beta)\int_0^{\infty}\left(\frac{x}{\sqrt{x+y}}\right)^{\beta}K_{\beta}\left(2\sqrt{x+y}\right)f(y)dy = g(x),\ x>0,$$

is

$$f(x) = \frac{1}{2\pi\Gamma(\beta)i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_{-(1+z)}\left(2\sqrt{x}\right)}{\Gamma(z+\beta)}x^{-(1+z)/2}\ \left(\mathcal{M}g\right)(z)dz,\ x>0,$$

where $\gamma \in (\alpha, 0)$, α is chosen from the interval $\alpha \in (\max(c_0 - 1, -\beta),\ 0)$ and

$$\left(\mathcal{M}g\right)(\gamma + i\tau) \in L_1\left(|\tau| > 1;\ |\tau|^{1-\beta}\ e^{\pi|\tau|}d\tau\right).$$

Finally we write this solution in terms of the Neumann type series. In fact, substituting the value of the modified Bessel function $I_{-(1+z)}\left(2\sqrt{x}\right)$ by series (2.9), we change the order of summation and integration via the absolute convergence to obtain

$$f(x) = \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{x^{n-1}}{n!\ \Gamma(n+\beta)}\left\{x^{\beta}(1+x)^{-\beta-n}\right\}^{-1}g,$$

where by the symbol

$$\left\{x^{\beta}(1+x)^{-\beta-n}\right\}^{-1}g = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\left(\mathcal{M}g\right)(z)}{\Gamma(n-z)\Gamma(z+\beta)}x^{-z}dz$$

the generalized inverse Stieltjes transform is denoted (see details in [1]).

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S.Yakubovich
Department of Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto
Portugal
E-Mail: syakubov@fc.up.pt