Old and new reductions of dispersionless Toda hierarchy

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Abstract

Two types of finite-variable reductions of the dispersionless Toda hierarchy are considered in the geometric perspectives. The reductions are formulated in terms of "Landau-Ginzburg potentials" that play the role of reduced Lax functions. One of them is a generalization of Dubrovin and Zhang's trigonometric polynomial. The other is intended to be a Toda version of the waterbag model of the dispersionless KP hierarchy. The two types of Landau-Ginzburg potentials are shown to satisfy (a radial version of) the Lönwer equations with respect to the critical values of the Landau-Ginzburg potentials. Integrability conditions of these Löwner equations are (a radial version of) the Gibbons-Tsarev equations. These equations are used to formulate hodograph solutions of the reduced hierarchy. Geometric aspects of the Gibbons-Tsarev equations are explained in the language of classical differential geometry (Darboux equations, Egorov metrics and Combescure transformations). Frobenius structures on the parameter space of the Landau-Ginzburg potentials are introduced, and flat coordinates are constructed explicitly.

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1 Introduction

The problem of finite-variable reductions of dispersionless integrable hierarchies [1] has rich geometric contents that range from the classical differential geometry of orthogonal curvilinear coordinates [2] to the modern theory of Frobenius manifolds [3]. Moreover, the notion of Löwner equations [4], first introduced to solve the Bieberbach conjecture in the univalent function theory, also plays a role therein. In this paper, we study two types of finite-variable reductions of the dispersionless Toda hierarchy in these geometric perspectives.

As in the cases of the dispersionless KP and universal Whitham hierarchies [5], a finite-variable reduction of the dispersionless Toda hierarchy can be formulated as a system of dispersionless Lax equations for a single Lax function $\lambda(p)$. This reduced Lax function can be interpreted as a "Laudau-Ginzburg potential" in the "B-model" side of topological field theories [6]. In this sense, the two types of reductions addressed in this paper have Landau-Ginzburg potentials of the following form:

Case (I)

$$\lambda(p) = p^{-N} \prod_{i=1}^{M} (p - b_i)^{\kappa_i}, \quad \text{where} \sum_{i=1}^{M} \kappa_i - N > 0,\ \kappa_i \neq 0,\ N \neq 0. \tag{1}$$

Case (II)

$$\lambda(p) = \prod_{i=1}^{M} (p - b_j)^{\kappa_i} \exp\left(\sum_{k=1}^{N} c_k p^{-k} \right), \quad \text{where} \sum_{i=1}^{M} \kappa_j > 0,\ \kappa_i \neq 0,\ N > 0. \tag{2}$$

(1) is a generalization of the Laurent polynomial studied by Dubrovin and Zhang [7] in the context of Frobenius manifolds. Dubrovin and Zhang's Laurent polynomial corresponds to the case where $N > 0$ and $\kappa_1 = \cdots = \kappa_M = 1$. This case contains, as particular examples, the dispersionless limit of the 1D Toda and bigraded Toda hierarchies [8]. By allowing κ_i's and N to take negative values, we can include therein, for example, the dispersionless limit of the Ablowitz-Ladik hierarchy [9] and its possible generalizations. Thus (1) provides a stage for comparing a number of old and new dispersionless integrable systems on an equal footing.

In contrast, (2) is presumably a new example that has never been studied in the literature. We propose it as a "waterbag model" in the dispersionless Toda hierarchy. The waterbag model was first presented in the celebrated work of Gibbons and Tsarev [10, 11] on reductions of the Benney hierarchy, and has been further studied in the dispersionless KP hierarchy from various points of view [12, 13, 14, 15, 16]. The reduced Lax function of the waterbag model is a sum of polynomial and logarithmic terms. In a sense, the logarithm

$$\log \lambda(p) = \sum_{i=1}^{M} \kappa_i \log(p - b_i) + \sum_{k=1}^{N} c_k p^{-k} \tag{3}$$

of (2) amounts to the reduced Lax function of the waterbag model. (in particular, the most general Lax function studied by Ferguson and Strachan [16])) in the Benney and dispersionless KP hierarchies.

Let us mention that Chang [17] proposed three types of "waterbag models" for the dispersionless Toda hierarchy. (1) is a slight generalization of Chang's first model (which amounts to the case where $N = 1$, and for which Chang considered a Frobenius structure). The logarithmic expression (3) of our second case resembles the other two of Chang's models. More careful inspection, however, reveals that they are actually different: p^{-k}'s in (3) are replaced by p^k's in one of them, and $\log(p - b_i)$'s in (3) are replaced by $\log(p^{-1} - b_i)$ in the other model. This is by no means a negligible difference. What is more important, however, is that the exponentiated form (2) is more natural than the logarithmic expression, at least in the case of the dispersionless Toda hierarchy. For these reasons, we adopt (1) and (2) as the subjects of our case studies.

This paper is organized as follows. Section 2 is devoted to the Lax formalism. The Landau-Ginzburg potentials (1) and (2) are shown to define consistent reductions of the Lax equations of the dispersionless Toda hierarchy. The two-variable cases cover some well known dispersionless integrable hierarchies. Possible applications of those integrable systems to mathematical physics are also reviewed. Section 3 is focussed on the Löwner equations and the associated Gibbons-Tsarev equations. These equations are a "radial" version of the "chordal" Löwner and Gibbons-Tsarev equations that were first introduced in the aforementioned work of Gibbons and Tsarev [10, 11]. These equations are used here to formulate the generalized hodograph method [18, 19] for finite-variable reductions of the dispersionless Toda hierarchy. The two Landau-Ginzburg potentials in question indeed turn out to satisfy the Löwner equations. Therefore the reduced hierarchy can be solved by the generalized hodograph method. Section 4 presents some geometric implications of the Gibbons-Tsarev equations in the language of Darboux equations, Egorov metrics and Combescure transformations . Three particular Egorov metrics are shown to underlie the radial Gibbons-Tsarev equations. These metrics are re-considered in Section 5. Section 5 deals with

Frobenius (or flat) structures in the parameter spaces of the two Landau-Ginzburg potentials. The two "dual" Frobenius structures of Dubrovin and Zhang [7] are generalized to the parameter space of (1), and flat coordinates for these Frobenius structure are constructed. As regards the case of (2), a single Frobenius structure without a dual is introduced, and flat coordinates are constructed. The method and the result in the case of (2) exhibits remarkable similarities with the work of Ferguson and Strachan [16] on the waterbag model.

2 Lax equations

2.1 Lax formalism of dispersionless Toda hierarchy

The Lax equations of the dispersionless Toda hierarchy [1] are formulated by two Lax functions $z(p), \bar{z}(p)$ [1] of a spatial variable s, a momentum variable p, and two sets of time variables $\boldsymbol{t} = (t_1, t_2, \ldots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$ [2]. p is a "classical limit" of the shift operator $e^{\partial/\partial s}$ that satisfies the twisted canonical commutation relation

$$[e^{\partial/\partial s}, s] = e^{\partial/\partial s}. \tag{4}$$

In the classical (or "long-wave") limit, this commutation relation turns into the Poisson commutation relation

$$\{p, s\} = p. \tag{5}$$

This Poisson bracket can be extended to arbitrary functions of s and p as

$$\{f, g\} = p\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial s} - \frac{\partial f}{\partial s}\frac{\partial g}{\partial p}\right).$$

In the most general formulation, the Lax functions $z(p)$ and $\bar{z}(p)$ are understood to be mutually independent formal Laurent series of p of the form

$$z(p) = p + u_1 + u_2 p^{-1} + \cdots,$$
$$\bar{z}(p) = \bar{u}_0 p^{-1} + \bar{u}_1 + \bar{u}_2 p + \cdots.$$

The coefficients $u_n = u_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ and $\bar{u}_n = \bar{u}_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ are dynamical variables. The leading coefficient $\bar{u}_0$ is assumed to take the exponential form

$$\bar{u}_0 = e^{\phi}, \quad \phi = \phi(s, \boldsymbol{t}, \bar{\boldsymbol{t}}).$$

[1] z and $\bar{z}$ amount to $\mathcal{L}$ and $\bar{\mathcal{L}}^{-1}$ in our previous notations [1].

[2] Throughout this paper, the overline " $^-$ " does not mean complex conjugation. For example, t_n and $\bar{t}_n$ are independent variables.

Let us define the polynomials $B_n(p)$ and $\bar{B}_n(p)$ in p, p^{-1} as

$$B_n(p) = (z(p)^n)_{\geq 0}, \quad \bar{B}_n(p) = (\bar{z}(p)^n)_{<0},$$

where $(\quad)_{\geq 0}$ and $(\quad)_{<0}$ are projection operators acting on the linear space of Laurent series as

$$\left(\sum_{n=-\infty}^{\infty} a_n p^n\right)_{\geq 0} = \sum_{n\geq 0} a_n p^n, \quad \left(\sum_{n=-\infty}^{\infty} a_n p^n\right)_{<0} = \sum_{n<0} a_n p^n.$$

Time evolutions are generated by the Lax equations

$$\begin{aligned} \frac{\partial z(p)}{\partial t_n} &= \{B_n(p), z(p)\}, \quad \frac{\partial z(p)}{\partial \bar{t}_n} = \{\bar{B}_n(p), z(p)\}, \\ \frac{\partial \bar{z}(p)}{\partial t_n} &= \{B_n(p), \bar{z}(p)\}, \quad \frac{\partial \bar{z}(p)}{\partial \bar{t}_n} = \{\bar{B}_n(p), \bar{z}(p)\}. \end{aligned} \tag{6}$$

It is convenient to introduce the complementary generators

$$B_n^c(p) = (z(p)^n)_{<0}, \quad \bar{B}_n^c(p) = (\bar{z}(p)^n)_{\geq 0}$$

as well. The Lax equations can be thereby rewritten as

$$\begin{aligned} \frac{\partial z(p)}{\partial t_n} &= \{z(p), B_n^c(p)\}, \quad \frac{\partial z(p)}{\partial \bar{t}_n} = \{z(p), \bar{B}_n^c(p)\}, \\ \frac{\partial \bar{z}(p)}{\partial t_n} &= \{\bar{z}(p), B_n^c(p)\}, \quad \frac{\partial \bar{z}(p)}{\partial \bar{t}_n} = \{\bar{z}(p), \bar{B}_n^c(p)\}. \end{aligned} \tag{7}$$

2.2 Landau-Ginzburg potential as reduced Lax function

We now specialize the Lax equations to the case where the formal (or local) Lax functions $z(p)$ and $\bar{z}(p)$ are linked with the globally defined Landau-Ginzburg potential $\lambda(p)$ as follows:

(I) For the Landau-Ginzburg potential (1),

$$\begin{aligned} z(p) &= \lambda(p)^{1/\tilde{M}} \quad \text{as } p \to \infty, \\ \bar{z}(p) &= \lambda(p)^{1/N} \quad \text{as } p \to 0, \end{aligned} \tag{8}$$

where

$$\tilde{M} = \sum_{i=1}^{M} \kappa_i - N.$$

Recall that $\tilde{M}$ is assumed to be positive. N can be both positive and negative.

(II) For the Landau-Ginzburg potential (2),

$$
\begin{aligned}
z(p) &= \lambda(p)^{1/\tilde{M}} \quad \text{as } p \to \infty, \\
\bar{z}(p) &= (\log \lambda(p))^{1/N} \quad \text{as } p \to 0,
\end{aligned} \tag{9}
$$

where

$$
\tilde{M} = \sum_{i=1}^{M} \kappa_i .
$$

Recall that $\tilde{M}$ and N is assumed to be positive.

The Lax equations (6) and its complementary form (7) thus reduce to the Lax equations

$$
\begin{aligned}
\frac{\partial \lambda(p)}{\partial t_n} &= \{B_n(p), \lambda(p)\} = \{\lambda(p), B_n^c(p)\}, \\
\frac{\partial \lambda(p)}{\partial \bar{t}_n} &= \{\bar{B}_n(p), \lambda(p)\} = \{\lambda(p), \bar{B}_n^c(p)\}
\end{aligned} \tag{10}
$$

for $\lambda(p)$. We can confirm that this is a consistent reduction procedure in the following sense.

Theorem 1. *The Lax equations (10) are equivalent to a system of first order evolutionary equations of the form*

$$
\frac{\partial b_i}{\partial t_n} = F_{in}, \quad \frac{\partial b_i}{\partial \bar{t}_n} = \bar{F}_{in} \tag{11}
$$

for the Landau-Ginzburg potential (1) and

$$
\frac{\partial b_i}{\partial t_n} = F_{in}, \quad \frac{\partial c_k}{\partial t_n} = G_{kn}, \quad \frac{\partial b_i}{\partial \bar{t}_n} = \bar{F}_{in}, \quad \frac{\partial c_k}{\partial \bar{t}_n} = \bar{G}_{kn} \tag{12}
$$

for the Landau-Ginzburg potential (2), where F_{in}, $\bar{F}_{in}$, *etc. are functions of* $b_1, \ldots, b_M$, $c_1, \ldots, c_N$ *and their first order derivatives with respect to* s.

Proof. Since the two cases can be treated in the same way, let us consider the case (II) only. We can rewrite the Lax equations (10) in terms of $\log \lambda(p)$ as

$$
\begin{aligned}
\frac{\partial \log \lambda(p)}{\partial t_n} &= \{B_n(p), \log \lambda(p)\} = \{\log \lambda(p), B_n^c(p)\}, \\
\frac{\partial \log \lambda(p)}{\partial \bar{t}_n} &= \{\bar{B}_n(p), \log \lambda(p)\} = \{\log \lambda(p), \bar{B}_n^c(p)\}.
\end{aligned}
$$

The left hand side are linear combinations of $1/(p-b_i)$'s and p^{-k}'s:

$$\frac{\partial \log \lambda(p)}{\partial t_n} = -\sum_{i=1}^{M} \frac{\partial b_i}{\partial t_n} \frac{\kappa_i}{p-b_i} + \sum_{k=1}^{N} \frac{\partial c_k}{\partial t_n} p^{-k},$$
$$\frac{\partial \log \lambda(p)}{\partial \bar{t}_n} = -\sum_{i=1}^{M} \frac{\partial b_i}{\partial \bar{t}_n} \frac{\kappa_i}{p-b_i} + \sum_{k=1}^{N} \frac{\partial c_k}{\partial \bar{t}_n} p^{-k}.$$

Since $B_n(p)$ and $\bar{B}_n(p)$ are polynomials in p and p^{-1}, the right hand sides are rational functions of p. Actually, because of the two complementary expressions, they turn out to be linear combinations of $1/(p-b_i)$'s and p^{-k}'s. For example, $\{B_n(p), \log \lambda(p)\}$ can be expanded as

$$\begin{aligned}\{B_n(p), \log \lambda(p)\} &= \sum_{i=1}^{M} \{B_n(p), \kappa_i \log(p-b_i)\} + \sum_{k=1}^{N} \{B_n(p), c_k p^{-k}\} \\ &= \sum_{i=1}^{M} p \left(-\frac{\partial B_n(p)}{\partial p} \frac{\partial b_j}{\partial s} \frac{\kappa_i}{p-b_i} - \frac{\partial B_n(p)}{\partial s} \frac{\kappa_i}{p-b_i} \right) \\ &\quad + \sum_{k=1}^{N} p \left(\frac{\partial B_n(p)}{\partial p} \frac{\partial c_k}{\partial s} p^{-k} + \frac{\partial B_n(p)}{\partial s} k c_k p^{-k-1} \right),\end{aligned}$$

which has first order poles at $p = b_i$, an N-th order pole at $p = 0$ and no poles in the finite part of the Riemann sphere. On the other hand, since $B_n^c(p) = O(p^{-1})$ as $p \to \infty$, the complementary expression $\{\log \lambda(p), B_n^c(p)\}$ is also $O(p^{-1})$. Consequently, the right hand side of the Lax equation is a rational function of the form

$$\{B_n(p), \log \lambda(p)\} = \{\log \lambda(p), B_n^c(p)\} = \sum_{i=1}^{M} \frac{\kappa_i F_{in}}{p-b_i} + \sum_{k=1}^{N} G_{kn} p^{-k},$$

where F_{in}'s and G_{in}'s are functions of b_is, c_k's and their first order derivatives with respect to s. Thus the Lax equations reduce to evolution equations as stated in the theorem. □

2.3 Examples: Two-variable reductions

The simplest nontrivial cases are two-variable reductions. They contain the following old and new examples of integrable hierarchies.

(i) Landau-Ginzburg potential (1) with $N = 1$, $M = 2$, $\kappa_1 = \kappa_2 = 1$:

$$\lambda(p) = p^{-1}(p-b_1)(p-b_2) = p + b + cp^{-1}. \tag{13}$$

This is the reduced Lax function of the dispersionless 1D Toda hierarchy. It is well known that this hierarchy plays a fundamental role in a wide range of issues of mathematical physics. In a literal sense, this Laurent polynomial is used for the "Landau-Ginzburg" (or "mirror") description of the topological sigma model of $\mathbf{CP}^1$ [20, 21]. The associated Frobenius structure is also a significant example of Dubrovin's duality [35, 36].

(ii) Landau-Ginzburg potential (1) with $N = 1$, $M = 2$, $\kappa_1 = 1$, $\kappa_2 = -1$:

$$\lambda(p) = p\frac{p-b}{p-c}. \tag{14}$$

This is the reduced Lax function of the dispersionless Ablowitz-Ladik hierarchy. This hierarchy and the dispersive version have found new applications in universality classes of nonlinear waves [22] and local Gromov-Witten invariants of the resolved conifold [23, 24].

(iii) Landau-Ginzburg potential (2) with $N = M = 1$, $\kappa_1 = 1$:

$$\lambda(p) = (p-b)e^{cp^{-1}}. \tag{15}$$

This seems to give a new dispersionless integrable hierarchy.

As regards the third example, one can further impose the condition $b = 0$ and obtain a hierarchy with the reduced Lax function

$$\lambda(p) = pe^{cp^{-1}}. \tag{16}$$

This is an interesting case in itself, because the inverse function of $\lambda(p)$ is Lambert's W-function that plays a role in the theory of Hurwitz numbers (see Takasaki paper [25] and references cited therein). This "one-variable reduction" can be generalized to Landau-Ginzburg potentials of the form

$$\lambda(p) = p^M \exp\left(\sum_{k=1}^{N} c_k p^{-k}\right), \tag{17}$$

where M is an arbitrary positive integer. They should be classified as "case (III)", though we shall not pursue this case in this paper.

3 Löwner equations

3.1 General scheme of finite-variable reductions

According to the general scheme [26, 27, 28, 29], finite-variable reductions of the dispersionless Toda hierarchy are characterized by the equations

$$\frac{\partial z(p)}{\partial \lambda_n} = \frac{\alpha_n p}{p-\gamma_n}\frac{\partial z(p)}{\partial p}, \quad \frac{\partial \bar{z}(p)}{\partial \lambda_n} = \frac{\alpha_n p}{p-\gamma_n}\frac{\partial \bar{z}(p)}{\partial p} \tag{18}$$

(referred to as "Löwner equations" in the following) for the reduced Lax functions

$$z(p) = z(p;\lambda_1,\ldots,\lambda_K), \quad \bar{z}(p) = \bar{z}(p;\lambda_1,\ldots,\lambda_K). \tag{19}$$

These equations are a variant of the radial Löwner equations that were first introduced by Löwner [4].

The reduced Lax functions depend on the space-time variables through the reduced dynamical variables $\lambda_n = \lambda_n(s,\boldsymbol{t},\bar{\boldsymbol{t}})$. The reduced dynamical variables, in turn, are required to satisfy the "hydrodynamic system"

$$\frac{\partial \lambda_n}{\partial t_k} = V_{kn}\frac{\partial \lambda_n}{\partial s}, \quad \frac{\partial \lambda_n}{\partial \bar{t}_k} = \bar{V}_{kn}\frac{\partial \lambda_n}{\partial s}. \tag{20}$$

The "characteristic speeds" $V_{kn} = V_{kn}(\lambda_1,\ldots,\lambda_K)$ and $\bar{V}_{kn} = \bar{V}_{kn}(\lambda_1,\ldots,\lambda_K)$ are defined as

$$V_{kn} = \gamma_n B_k'(\gamma_n), \quad \bar{V}_{kn} = \gamma_n \bar{B}_k'(\gamma_n),$$

where the prime denotes the derivative, i.e., $B_k'(p) = \partial B_k(p)/\partial p$ and $\bar{B}_k'(p) = \partial \bar{B}_k(p)/\partial p$. Since $B_1(p) = p + u_1$, the characteristic speeds for the t_1-flow coincides with γ_n:

$$V_{1n} = \gamma_n. \tag{21}$$

Thus, once the reduced Lax functions are given as a solution of the Löwner equations, the Lax equations are transformed to the hydrodynamic system (20) for the "Riemann invariants" $\lambda_1,\ldots,\lambda_K$. A precise statement of this fact reads as follows [26, 27, 28, 29]:

Theorem 2. *If $z(p)$ and $\bar{z}(p)$ satisfy the Löwner equations (25) with respect to λ_n's, and λ_n's satisfy the hydrodynamic system (20) with respect to the space-time variable, then $z(p)$ and $\bar{z}(p)$ satisfy the Lax equations (6) with respect to the space-time variables.*

α_n's and γ_n's in (18) are functions of $(\lambda_1, \cdots, \lambda_K)$ to be determined in the reduction procedure. Note that they are not arbitrary functions. Integrability conditions of (18) yield the differential equations

$$\frac{\partial \gamma_n}{\partial \lambda_m} = \frac{\alpha_m \gamma_n}{\gamma_m - \gamma_n}, \quad \frac{\partial \alpha_n}{\partial \lambda_m} = \frac{\alpha_m \alpha_n (\gamma_m + \gamma_n)}{(\gamma_m - \gamma_n)^2}, \quad m \neq n, \tag{22}$$

which are referred as "Gibbons-Tsarev equations" or, more precisely, "radial Gibbons-Tsarev equations". These equations are a radial version of the celebrated Gibbons-Tsarev equations [10, 11] (see the remarks below).

It is interesting that the product and quotient of α_n, γ_n satisfy the following equations:

$$\begin{aligned} \frac{\partial}{\partial \lambda_m}(\alpha_n \gamma_n) &= \frac{2(\alpha_m \gamma_m)(\alpha_n \gamma_n)}{(\gamma_m - \gamma_n)^2}, \\ \frac{\partial}{\partial \lambda_m}\left(\frac{\alpha_n}{\gamma_n}\right) &= \frac{2\gamma_m \gamma_n}{(\gamma_m - \gamma_n)^2} \frac{\alpha_m}{\gamma_m} \frac{\alpha_n}{\gamma_n}. \end{aligned} \tag{23}$$

The right hand side of these equations are symmetric with respect to m and n. This implies the existence of potentials.

It is easy to identify these potentials. Expands both hand sides of (18) into Laurent series at $p = \infty$ and pick out the p^1 and p^2 terms from the first equation and the p^0 terms from the second equation. One can thus find the relations

$$\alpha_n = \frac{\partial u_1}{\partial \lambda_n}, \quad \alpha_n \gamma_n = \frac{\partial u_2}{\partial \lambda_n}, \quad \frac{\alpha_n}{\gamma_n} = \frac{\partial \phi}{\partial \lambda_n}, \tag{24}$$

which show that u_1, u_2 and $\phi = \log \bar{u}_0$ play the role of potentials. By the first expression of α_n in (24), one can rewrite (18) as

$$\frac{\partial z(p)}{\partial \lambda_n} = \frac{p z'(p)}{p - \gamma_n} \frac{\partial u_1}{\partial \lambda_n}, \quad \frac{\partial \bar{z}(p)}{\partial \lambda_n} = \frac{p \bar{z}'(p)}{p - \gamma_n} \frac{\partial u_1}{\partial \lambda_n}. \tag{25}$$

It is these equations that can be obtained directly from the Lax equations under the finite-variable ansatz (19).

The problem is now converted to solving (20). This problem can be treated by the generalized hodograph method [18, 19].

Remark 1. The original form of the Gibbons-Tsarev equations [10, 11] read

$$\frac{\partial \gamma_n}{\partial \lambda_m} = \frac{\alpha_n}{\gamma_m - \gamma_n}, \quad \frac{\partial \alpha_n}{\partial \lambda_m} = \frac{2\alpha_m \alpha_n}{(\gamma_m - \gamma_n)^2}. \tag{26}$$

They are integrable conditions of the chordal Löwner equations

$$\frac{\partial z(p)}{\partial \lambda_n} = \frac{\alpha_n}{p - \gamma_n} \frac{\partial z(p)}{\partial p} \tag{27}$$

for the Lax function

$$z(p) = p + u_2 p^{-1} + \cdots$$

of the Benney hierarchy or, more generally, of the dispersionless KP hierarchy [32]. Note that the u_1-term is absent here, and u_2 plays the role of a potential for the coefficients α_n:

$$\alpha_n = \frac{\partial u_2}{\partial \lambda_n}. \tag{28}$$

Remark 2. Ferapontov et al. [30] presented the radial Gibbons-Tsarev equations (22) (written in a trigonometric form) in their work on finite-variable reductions of the Boyer-Finley equation. The Boyer-Finley equation is the lowest 2D part of the dispersionless Toda hierarchy.

3.2 Hodograph solutions

The generalized hodograph method [18, 19] is based on the fact that the characteristic speeds satisfy the equations

$$\frac{1}{V_{km} - V_{kn}} \frac{\partial V_{kn}}{\partial \lambda_m} = \frac{1}{\bar{V}_{km} - \bar{V}_{kn}} \frac{\partial \bar{V}_{kn}}{\partial \lambda_m} = \frac{\alpha_m \gamma_n}{(\gamma_m - \gamma_n)^2} \tag{29}$$

for $k = 1, 2, \ldots$. Note that these equations include the special case

$$\frac{1}{\gamma_m - \gamma_n} \frac{\partial \gamma_n}{\partial \lambda_m} = \frac{\alpha_m \gamma_n}{(\gamma_m - \gamma_n)^2} \tag{30}$$

associated with $V_{1n} = \gamma_n$. One can derive these equations directly from the definition of $B_k(p)$ and $\bar{B}_k(p)$ [26] (see the remarks below) or by generating functions of these polynomials [28, 29]. Having these equations, one can readily apply the generalized hodograph method to the hydrodynamic system (20) as follows [26, 27, 28]:

Theorem 3. *If a set of functions $F_n = F_n(\lambda_1, \ldots, \lambda_K)$ satisfy the equations*

$$\frac{1}{F_m - F_n} \frac{\partial F_n}{\partial \lambda_m} = \frac{\alpha_m \gamma_n}{(\gamma_m - \gamma_n)^2} \tag{31}$$

and the non-degeneracy condition

$$\det\left(\frac{\partial F_n}{\partial \lambda_m}\right)_{m,n=1\cdots K} \neq 0, \tag{32}$$

then a solution of the hydrodynamic system (20) can be obtained from the hodograph relations

$$s + \sum_{k\geq 1} t_k V_{kn} + \sum_{k\geq 1} \bar{t}_k \bar{V}_{kn} = F_n, \quad n = 1, \ldots, K, \tag{33}$$

as an K-tuple of implicit functions $\lambda_n = \lambda_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$, $n = 1, \ldots, K$, in a neighborhood of $(\boldsymbol{t}, \bar{\boldsymbol{t}}) = (\mathbf{0}, \mathbf{0})$.

Remark 3. The characteristic speeds V_{kn} and $\bar{V}_{kn}$ have the contour integral representation

$$V_{kn} = -\gamma_n \oint \frac{z(p)^n}{(p-\gamma_n)^2}\frac{dp}{2\pi i}, \quad \bar{V}_{kn} = -\gamma_n \oint \frac{\bar{z}(p)^n}{(p-\gamma_n)^2}\frac{dp}{2\pi i}, \tag{34}$$

where the contours encircle $p = \infty$ and $p = 0$, respectively, leaving γ_n outside. One can use the Löwner equations and the Gibbons-Tsarev equations to differentiate these contour integrals. Thus, after some algebra, one can derive (29).

Remark 4. Solutions of (31), too, can be obtained as contour integrals. For example, arbitrary linear combinations of V_{kn} and $\bar{V}_{kn}$, which are obvious solutions of (31), can be cast into a contour integral of the form

$$F_n = \gamma_n \oint \frac{F_1(z(p))}{(p-\gamma_n)^2}\frac{dp}{2\pi i} + \gamma_n \oint \frac{F_2(\bar{z}(p))}{(p-\gamma_n)^2}\frac{dp}{2\pi i}, \tag{35}$$

where $F_1(p)$ and $F_2(p)$ are arbitrary (analytic) functions, If $z(p)$ and $\bar{z}(p)$ are obtained from a globally defined Landau-Ginzburg potential $\lambda(p)$, one can unify the two integrals to a single integral

$$F_n = \gamma_n \oint_C \frac{F(\lambda(p))}{(p-\gamma_n)^2}\frac{dp}{2\pi i} \tag{36}$$

along a general cycle C in the domain of definition of $F(\lambda(p))$. This gives a more general solution of (31) as Ferapontov et al. [30] pointed out in their formulation.

3.3 Löwner equations for Landau-Ginzburg potentials

If the Lax functions $z(p)$ and $\bar{z}(p)$ are reduced to a single Landau-Ginzburg potential $\lambda(p)$, the Löwner equations (18) for the Lax functions, too, are reduced to the equations

$$\frac{\partial\lambda(p)}{\partial\lambda_n} = \frac{\alpha_n p}{p-\gamma_n}\frac{\partial\lambda(p)}{\partial p} \tag{37}$$

for $\lambda(p)$.

We show below that the Landau-Ginzburg potentials (1) and (2) do satisfy these equations. The relevant variables λ_n are the critical values of $\lambda(p)$, i.e., the values of $\lambda(p)$ at the critical points γ_n's,

$$\lambda_n = \lambda(\gamma_n), \quad \lambda'(\gamma_n) = 0, \quad n = 1, \ldots, K,$$

where $K = M$ in the case of (1) and $K = M + N$ in the case of (2). We choose these λ_n's as new coordinates on the parameter space of $\lambda(p)$, and treat $\lambda(p)$ as a function $\lambda(p; \lambda_1, \ldots, \lambda_K)$ of p and λ_n's.

Let us show a few technical remarks.

Lemma 1.

$$\left.\frac{\partial\lambda(p)}{\partial\lambda_m}\right|_{p=\gamma_n} = \delta_{mn}. \tag{38}$$

Proof. By the definition of λ_n's and the chain rule of differentiation,

$$\delta_{mn} = \frac{\partial\lambda_n}{\partial\lambda_m} = \left.\frac{\partial\lambda(p)}{\partial\lambda_m}\right|_{p=\gamma_n} + \lambda'(\gamma_m) = \left.\frac{\partial\lambda(p)}{\partial\lambda_m}\right|_{p=\gamma_n}.$$

□

Lemma 2. *If the Löwner equations (37) are satisfied, the coefficients α_n are uniquely determined by the equations themselves as*

$$\alpha_n = \frac{1}{\gamma_n\lambda''(\gamma_n)}. \tag{39}$$

Proof. Let $p \to \gamma_n$ in (37). By (38), the left hand side tends to 1. As regards the right hand side,

$$\lim_{p\to\gamma_n}\frac{\alpha_n p}{p-\gamma_n}\frac{\partial\lambda(p)}{\partial p} = \alpha_n\gamma_n \lim_{p\to\gamma_n}\frac{\lambda'(p)}{p-\gamma_n} = \alpha_n\gamma_n\lambda''(\gamma_n).$$

□

Bearing these technical remarks in mind, let us examine the two cases separately.

Case (I) It is convenient to consider the logarithmic derivative, rather than the derivative, of the Landau-Ginzburg potential (1):

$$\frac{\partial \log \lambda(p)}{\partial p} = -\frac{N}{p} + \sum_{i=1}^{M} \frac{\kappa_i}{p - b_i} = \frac{Q(p)}{p \prod_{i=1}^{M} (p - b_i)}. \tag{40}$$

The numerator $Q(p)$ is a polynomial of the form

$$Q(p) = \tilde{M} p^M + \cdots .$$

We assume that $Q(p)$ has M distinct zeroes γ_n, $n = 1, \ldots, M$,

$$Q(p) = \tilde{M} \prod_{n=1}^{M} (p - \gamma_n), \quad \gamma_m \neq \gamma_n \text{ for } m \neq n. \tag{41}$$

From now on, $\lambda(p)$ is understood to be a function of p and λ_n's. The parameters b_i's, too, become functions of λ_n's. The derivative of $\log \lambda(p)$ with respect to λ_m can be expressed as

$$\frac{\partial \log \lambda(p)}{\partial \lambda_m} = -\sum_{i=1}^{M} \frac{\kappa_i}{p - b_i} \frac{\partial b_i}{\partial \lambda_m} = \frac{Q_m(p)}{\prod_{i=1}^{M} (p - b_i)}, \tag{42}$$

where $Q_m(p)$ is a polynomial of degree less than M. (40) and (42) imply the equality

$$\frac{\partial \log \lambda(p)}{\partial \lambda} = \frac{Q_m(p) p}{Q(p)} \frac{\partial \log \lambda(p)}{\partial p}. \tag{43}$$

The problem is to find an explicit form of the prefactor $Q_m(p)p/Q(p)$.

Since (38) implies that

$$\left. \frac{\partial \log \lambda(p)}{\partial \lambda_m} \right|_{p=\gamma_n} = \frac{1}{\lambda(p)} \left. \frac{\partial \lambda(p)}{\partial \lambda_m} \right|_{p=\gamma_n} = \frac{\delta_{mn}}{\lambda_n},$$

letting $p \to \gamma_n$ in (42) yields that

$$Q_m(\gamma_n) = 0 \quad \text{for } n \neq m.$$

Therefore, by the Lagrange interpolation formula, $Q_m(p)/Q(p)$ can be expressed as

$$\frac{Q_m(p)}{Q(p)} = \sum_{n=1}^{M} \frac{Q_m(\gamma_n)}{Q'(\gamma_n)(p - \gamma_n)} = \frac{Q_m(\gamma_m)}{Q'(\gamma_n)(p - \gamma_m)}. \tag{44}$$

Thus, defining α_m as

$$\alpha_m = Q_m(\gamma_m)/Q'(\gamma_m),$$

we obtain the Löwner equations

$$\frac{\partial \log \lambda(p)}{\partial \lambda_m} = \frac{\alpha_m p}{p - \gamma_m} \frac{\partial \log \lambda(p)}{\partial p} \tag{45}$$

for $\log \lambda(p)$. Of course, they are equivalent to the Löwner equations (37) for $\lambda(p)$. By the second lemma above, α_m turns out to have another expression (39).

Case (II) The logarithmic derivatives of the Landau-Ginzburg potential (2) can be expresses as

$$\frac{\partial \log \lambda(p)}{\partial p} = \sum_{i=1}^{M} \frac{\kappa_i}{p - b_i} - \sum_{k=1}^{N} k c_k p^{-k-1} = \frac{Q(p)}{p^{N+1} \prod_{i=1}^{M} (p - b_i)} \tag{46}$$

and

$$\frac{\partial \log \lambda(p)}{\partial \lambda_m} = -\sum_{i=1}^{M} \frac{\kappa_i}{p - b_i} \frac{\partial b_i}{\partial \lambda_m} + \sum_{k=1}^{N} \frac{\partial c_k}{\partial \lambda_m} p^{-k} = \frac{Q_m(p)}{p^N \prod_{i=1}^{M} (p - b_i)}, \tag{47}$$

where $Q(p)$ is a polynomial of the form

$$Q(p) = \tilde{M} \prod_{n=1}^{M+N} (p - \gamma_n), \quad \gamma_m \neq \gamma_n \text{ for } m \neq n. \tag{48}$$

$Q_m(p)$ is a polynomial of degree less than $M + N$, and the roots γ_n of $Q(p)$ are assumed to be distinct. Starting from these data, one can derive the Löwner equations (37) in much the same way as in the case (I).

We thus obtain the following result:

Theorem 4. $\lambda(p) = \lambda(p; \lambda_1, \dots, \lambda_K)$ *satisfies the Löwner equations (37) with the coefficients* α_n *defined by (39).*

On the basis of this result, we can apply the foregoing scheme of finite-variable reductions to the Landau-Ginzburg potentials (1) and (2).

4 Darboux equations

4.1 Basic notions in classical differential geometry

Given a diagonal metric $ds^2 = \sum_{n=1}^{K}(h_n d\lambda_n)^2$ [3], one can define the rotation coefficients β_{mn}, $m \neq n$, as

$$\beta_{mn} = \frac{1}{h_m}\frac{\partial h_n}{\partial \lambda_m}.$$

h_n's are called "Lamé coefficients" in the theory of orthogonal curvilinear coordinate systems [2, 19]. The Riemann curvature of this metric vanishes if and only if the following equations are satisfied:

$$\frac{\partial \beta_{mn}}{\partial \lambda_k} = \beta_{mk}\beta_{kn} \quad \text{for } k \neq m, n, \tag{49}$$

$$\frac{\partial \beta_{mn}}{\partial \lambda_m} + \frac{\partial \beta_{mn}}{\partial \lambda_n} + \sum_{k=1}^{K}\beta_{km}\beta_{kn} = 0. \tag{50}$$

The first part (49) of these equations are called "Darboux equations" in the literature. Thus the Darboux equations are partial-flatness conditions, and have to be supplemented by the second equations (50) to ensure flatness.

If the rotation coefficients are symmetric, i.e., $\beta_{mn} = \beta_{nm}$, the metric components satisfy the conditions (Egorov conditions)

$$\frac{\partial}{\partial \lambda_m}({h_n}^2) = \frac{\partial}{\partial \lambda_n}({h_m}^2) \tag{51}$$

that ensure the existence of a potential ϕ (Egorov potential) such that

$${h_n}^2 = \frac{\partial \phi}{\partial \lambda_n}. \tag{52}$$

If the Darboux equations and the Egorov condition are satisfied, the flatness condition (50) reduces to the equations

$$\sum_{k=1}^{K}\frac{\partial \beta_{mn}}{\partial \lambda_k} = 0. \tag{53}$$

A diagonal metric that satisfies the Darboux equations and the Egorov conditions is called an "Egorov metric". Thus an Egorov metric is associated with a symmetric ($\beta_{mn} = \beta_{nm}$) solution of the coupled system of the Darboux

[3] We do not use Einstein's convention in this paper.

equations and (53). This system is closely related to the K-wave system [31, 3].

If one starts from a solution of the Darboux equations (49), the Lamé coefficients are recovered as a solution of the equations

$$\frac{\partial h_n}{\partial \lambda_m} = h_m \beta_{mn}. \tag{54}$$

The Darboux equations are integrability conditions of these linear equations. These equations leaves some arbitrariness in the Lamé coefficients. Two sets $h_n, \tilde{h}_n$ of Lamé coefficients have the same rotation coefficients if and only if their ratios $w_n = \tilde{h}_n / h_n$ satisfy the equations

$$\frac{1}{w_m - w_n} \frac{\partial w_n}{\partial \lambda_m} = \frac{\partial \log h_n}{\partial \lambda_m}. \tag{55}$$

Any solution of these equations thus gives a transformation on the set of Lamé coefficients with the same rotational coefficients. This transformation is called "Combescure transformation" in the theory of orthogonal curvilinear coordinate systems [2, 19].

4.2 Implications of Gibbons-Tsarev equations

Let us consider the Gibbons-Tsarev equations (22) in the language of Darboux equations and Egorov metrics. There are three metrics that are of particular interest:

$$\sum_{n=1}^{K} (h_n d\lambda_n)^2, \quad h_n = \sqrt{\alpha_n / \gamma_n}, \tag{56}$$

$$\sum_{n=1}^{K} (\tilde{h}_n d\lambda_n)^2, \quad \tilde{h}_n = \sqrt{\alpha_n \gamma_n}, \tag{57}$$

$$\sum_{n=1}^{K} (\hat{h}_n d\log\lambda_n)^2, \quad \hat{h}_n = \sqrt{\alpha_n \lambda_n / \gamma_n}. \tag{58}$$

(56) and (57) underlie the hodograph solutions of the hydrodynamic equations (20). (56) and (58) are the metrics that we shall consider in the next section in the context of Frobenius structures.

We show below that the rotation coefficients of these metrics are symmetric and satisfy the Darboux equations (with respect to λ_n's in the first and second cases and $\log\lambda_n$'s in the third case). Darboux potentials themselves can be readily identified as one can see from (24):

$$ {h_n}^2 = \frac{\partial \phi}{\partial \lambda_n}, \quad {\tilde{h}_n}^2 = \frac{\partial u_2}{\partial \lambda_n}, \quad {\hat{h}_n}^2 = \frac{\partial \phi}{\partial \log \lambda_n}. \tag{59} $$

The first two cases (56) and (57) are closely related.

Theorem 5. *The Lamé coefficients of (56) and (57) have the same rotation coefficients*

$$ \beta_{mn} = \beta_{nm} = \frac{\sqrt{\alpha_m \alpha_n \gamma_m \gamma_n}}{(\gamma_m - \gamma_n)^2}. \tag{60} $$

These rotation coefficients satisfy the Darboux equations (49).

Proof. Do straightforward calculations using the Gibbons-Tsarev equations (22). The rotation coefficients of h_n's can be calculated as

$$ \begin{aligned} \frac{1}{h_m}\frac{\partial h_n}{\partial \lambda_m} &= \frac{h_n}{h_m}\frac{\partial \log h_n}{\partial \lambda_m} \\ &= \sqrt{\frac{\alpha_n \gamma_m}{\alpha_m \gamma_n}} \left(\frac{1}{2\alpha_n}\frac{\partial \alpha_n}{\partial \lambda_m} - \frac{1}{2\gamma_n}\frac{\partial \gamma_n}{\partial \lambda_m} \right) \\ &= \sqrt{\frac{\alpha_n \gamma_m}{\alpha_m \gamma_n}} \left(\frac{1}{2\alpha_n}\frac{\alpha_m \alpha_n (\gamma_m + \gamma_n)}{(\gamma_m - \gamma_n)^2} - \frac{1}{2\gamma_n}\frac{\alpha_m \gamma_n}{\gamma_m - \gamma_n} \right) \\ &= \frac{\sqrt{\alpha_m \alpha_n \gamma_m \gamma_n}}{(\gamma_m - \gamma_n)^2}. \end{aligned} $$

In much the same way, the rotation coefficients of $\tilde{h}_n$'s can be calculated as

$$ \begin{aligned} \frac{1}{\tilde{h}_n}\frac{\partial \tilde{h}_n}{\partial \lambda_m} &= \frac{\tilde{h}_n}{\tilde{h}_m}\frac{\partial \log \tilde{h}_n}{\partial \lambda_m} \\ &= \sqrt{\frac{\alpha_n \gamma_n}{\alpha_m \gamma_m}} \left(\frac{1}{2\alpha_n}\frac{\partial \alpha_n}{\partial \lambda_m} + \frac{1}{2\gamma_n}\frac{\partial \gamma_n}{\partial \lambda_m} \right) \\ &= \sqrt{\frac{\alpha_n \gamma_n}{\alpha_m \gamma_m}} \left(\frac{1}{2\alpha_n}\frac{\alpha_m \alpha_n (\gamma_m + \gamma_n)}{(\gamma_m - \gamma_n)^2} + \frac{1}{2\gamma_n}\frac{\alpha_m \gamma_n}{\gamma_m - \gamma_n} \right) \\ &= \frac{\sqrt{\alpha_m \alpha_n \gamma_m \gamma_n}}{(\gamma_m - \gamma_n)^2}. \end{aligned} $$

Thus the rotation coefficients of h_n's and $\tilde{h}_n$'s turn out to coincide. Differentiating them with respect to λ_k and doing some algebra, one can derive the Darboux equations. $\square$

Corollary 6. *$\tilde{h}_n$'s are a Combescure transformation of h_n's, and the ratios $\gamma_n = \tilde{h}_n/h_n$ satisfy (55).*

The right hand side of (55) can be calculated explicitly as

$$\frac{\partial \log h_n}{\partial \lambda_m} = \frac{\alpha_m \gamma_n}{(\gamma_m - \gamma_n)^2}, \tag{61}$$

thus (55) coincides with (30). Since (30) is a special case of (29), the characteristic speeds V_{kn} and $\bar{V}_{kn}$, too, generate Combescure transformations of h_n. An analogous fact is known for for the dispersionless KP hierarchy [32] and universal Whitham hierarchy [33] as well (see the remark below). Thus, as stressed by Tsarev [19], the notion of Combescure transformations lies in the heart of integrability of hydrodynamic systems such as (20).

The third case (58) is of somewhat different nature. It is $\log \lambda_n$'s rather than λ_n's that are used to formulate the Darboux equations.

Theorem 7. *The rotation coefficients*

$$\hat{\beta}_{mn} = \frac{1}{\hat{h}_m} \frac{\partial \hat{h}_n}{\partial \log \lambda_m}, \quad m \neq n,$$

of the Lamé coefficients of (58) are related to (60) as

$$\hat{\beta}_{mn} = \sqrt{\lambda_m \lambda_n} \beta_{mn}, \tag{62}$$

and satisfy the Darboux equations

$$\frac{\partial \hat{\beta}_{mn}}{\partial \log \lambda_k} + \hat{\beta}_{mk} \hat{\beta}_{kn} = 0 \quad \textit{for } k \neq m, n. \tag{63}$$

Proof. Do straightforward calculations. Note, in particular, that the left hand side of (63) and (49) are related as

$$\frac{\partial \hat{\beta}_{mn}}{\partial \log \lambda_k} + \hat{\beta}_{mk} \hat{\beta}_{kn} = \sqrt{\lambda_m \lambda_n} \lambda_k \left(\frac{\partial \beta_{mn}}{\partial \lambda_k} + \beta_{mk} \beta_{kn} \right).$$

□

Remark 5. In the chordal case (26), the two sets

$$h_n = \sqrt{\alpha_n}, \quad \tilde{h}_n = \sqrt{\alpha_n} \gamma_n \tag{64}$$

of Lamé coefficients amount to (56) and (57). They have the same rotation coefficients

$$\beta_{mn} = \beta_{nm} = \frac{\sqrt{\alpha_m \alpha_n}}{(\gamma_m - \gamma_n)^2} \tag{65}$$

that satisfy the Darboux equations (49). Consequently, $\tilde{h}_n$'s are a Combescure transformation of h_n's, and the ratios $\tilde{h}_n/h_n = \gamma_n$ satisfy the equations

$$\frac{1}{\gamma_m - \gamma_n}\frac{\partial\gamma_n}{\partial\lambda_m} = \frac{\partial \log h_n}{\partial\lambda_m} = \frac{\alpha_m}{(\gamma_m - \gamma_n)^2}. \tag{66}$$

The last equations are fundamental equations in the hodograph solutions of the Benney equations [10, 11], the dispersionless KP hierarchy [32] and the universal Whitham hierarchy [33, 34].

5 Frobenius structures

5.1 Two flat structures in case (I)

If $\kappa_1 = \cdots = \kappa_M = 1$ and $N > 0$, the Landau-Ginzburg potential (1) reduce to the Laurent polynomial studied by Dubrovin and Zhang [7] [4]. They introduce two types of inner products (or metrics) $\langle\ ,\ \rangle$, $(\ ,\ )$ and cubic forms $\langle\ ,\ ,\ \rangle$, $(\ ,\ ,\ )$ for vector fields on the parameter space of the Laurent polynomial:

$$\begin{aligned}
\langle\partial, \partial'\rangle &= \sum_{n=1}^{M} \operatorname*{res}_{p=\gamma_n}\left[\frac{\partial\lambda(p)\cdot\partial'\lambda(p)}{d\lambda(p)}(d\log p)^2\right],\\
\langle\partial, \partial', \partial''\rangle &= \sum_{n=1}^{M} \operatorname*{res}_{p=\gamma_n}\left[\frac{\partial\lambda(p)\cdot\partial'\lambda(p)\cdot\partial''\lambda(p)}{d\lambda(p)}(d\log p)^2\right],\\
(\partial, \partial') &= \sum_{n=1}^{M} \operatorname*{res}_{p=\gamma_n}\left[\frac{\partial\log\lambda(p)\cdot\partial'\log\lambda(p)}{d\log\lambda(p)}(d\log p)^2\right],\\
(\partial, \partial', \partial'') &= \sum_{n=1}^{M} \operatorname*{res}_{p=\gamma_n}\left[\frac{\partial\log\lambda(p)\cdot\partial'\log\lambda(p)\cdot\partial''\log\lambda(p)}{d\log\lambda(p)}(d\log p)^2\right].
\end{aligned}$$

The cubic forms are used to define two commutative and associative product structures $\circ, \star$ of vector fields:

$$\begin{aligned}
\langle\partial\circ\partial', \partial''\rangle &= \langle\partial, \partial'\circ\partial''\rangle = \langle\partial, \partial', \partial''\rangle,\\
(\partial\star\partial', \partial'') &= (\partial, \partial'\star\partial'') = (\partial, \partial', \partial'').
\end{aligned} \tag{67}$$

They give a pair of dual Frobenius structures in the sense of Dubrovin [35, 36].

The same construction works for the more general Landau-Ginzburg potential (1) as well. Let us mention that the second Frobenius structure was

[4] Actually, Dubrovin and Zhang considered a trigonometric polynomial.

considered by Chang [17] in the case where $N = 1$ and κ_i's are arbitrary. When $\partial, \partial', \partial''$ are derivatives in λ_n's, one can use the Löwner equations (37) to evaluate these inner products and cubic forms as follows:

$$\left\langle \frac{\partial}{\partial\lambda_m}, \frac{\partial}{\partial\lambda_n} \right\rangle = \delta_{mn}\frac{\alpha_n}{\gamma_n}, \quad \left\langle \frac{\partial}{\partial\lambda_k}, \frac{\partial}{\partial\lambda_m}, \frac{\partial}{\partial\lambda_n} \right\rangle = \delta_{kmn}\frac{\alpha_n}{\gamma_n}, \tag{68}$$

$$\left(\frac{\partial}{\partial\lambda_m}, \frac{\partial}{\partial\lambda_n} \right) = \delta_{mn}\frac{\alpha_n}{\gamma_n\lambda_n}, \quad \left(\frac{\partial}{\partial\lambda_k}, \frac{\partial}{\partial\lambda_m}, \frac{\partial}{\partial\lambda_n} \right) = \delta_{kmn}\frac{\alpha_n}{\gamma_n{\lambda_n}^2}, \tag{69}$$

where $\delta_{kmn} = \delta_{km}\delta_{mn}$ (i.e., δ_{kmn} is equal to 1 if $k = m = n$ and 0 otherwise). Thus $\langle\ ,\ \rangle$ and $(\ ,\ )$ correspond to the metrics (56) and (58) considered in the last section.

Speaking more precisely, the definition of a Frobenius manifold requires some more data, in particular, an Euler vector field E and associated scaling properties [3]. In the present setting, the Landau-Ginzburg potential $\lambda(p)$ has natural homogeneity, and we can use the vector field

$$E = \sum_{n=1}^{M} \lambda_m \frac{\partial}{\partial\lambda_n} = \frac{1}{\tilde{M}} \sum_{i=1}^{M} b_i \frac{\partial}{\partial b_i} \tag{70}$$

as an Euler vector field. In the following, we refer to the notion of Frobenius manifolds in a loose way, and focus our consideration on the flatness of metrics.

One can confirm the flatness of the two metrics by constructing flat coordinates explicitly. Flat coordinates for the first metric $\langle\ ,\ \rangle$ can be constructed in exactly the same way as done by Dubrovin and Zhang [7]. Define $z(p)$ and $\bar{z}(p)$ as (8) shows, and construct the new coordinates $q_1, \ldots, q_{\tilde{M}-1}$ and $\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_N$ on the parameter space as

$$q_n = \operatorname*{res}_{p=\infty} \left[\frac{z(p)^n}{n} d\log p \right], \quad \bar{q}_0 = \phi, \quad \bar{q}_n = \operatorname*{res}_{p=0} \left[\frac{\bar{z}(p)^n}{n} d\log p \right]. \tag{71}$$

Theorem 8. *The inner products of $\partial/\partial q_n$'s and $\partial/\partial\bar{q}_n$'s with respect to the first metric can be expressed as*

$$\left\langle \frac{\partial}{\partial q_m}, \frac{\partial}{\partial q_n} \right\rangle = \tilde{M}\delta_{m+n,\tilde{M}}, \quad \left\langle \frac{\partial}{\partial\bar{q}_m}, \frac{\partial}{\partial\bar{q}_n} \right\rangle = N\delta_{m+n,N},$$
$$\left\langle \frac{\partial}{\partial q_m}, \frac{\partial}{\partial\bar{q}_n} \right\rangle = 0. \tag{72}$$

In particular, q_n's and $\bar{q}_n$'s are flat coordinates of the first metric.

Flat coordinates for the second metric (,) are also presented in the work of Dubrovin and Zhang, though somewhat implicitly. A clearer statement, along with an explicit result on the cubic form, can be found in the work of Chang [17] (in the case where $N = 1$).

Theorem 9. *The inner products of $\partial/\partial b_i$'s with respect to the second metric can be expressed as*

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_j}\right) = (1-\delta_{ij})\frac{\kappa_i \kappa_j}{N b_i b_j} + \delta_{ij}\frac{(\kappa_i - N)\kappa_i}{N {b_i}^2}. \tag{73}$$

In particular, $\log b_i$'s are flat coordinates of the second metric.

We refer details of the proof of these facts to the papers of Dubrovin, Zhang and Chang.

5.2 Flat structure in case (II)

Let us turn to the Landau-Ginzburg potential (2). The goal is to present a set of flat coordinates for the inner product

$$(\partial, \partial') = \sum_{n=1}^{M+N} \operatorname*{res}_{p=\gamma_n} \left[\frac{\partial \log\lambda(p) \cdot \partial' \log\lambda(p)}{d\log\lambda(p)} (d\log p)^2\right]$$

and the associated cubic form

$$(\partial, \partial', \partial'') = \sum_{n=1}^{M+N} \operatorname*{res}_{p=\gamma_n} \left[\frac{\partial \log\lambda(p) \cdot \partial' \log\lambda(p) \cdot \partial'' \log\lambda(p)}{d\log\lambda(p)} (d\log p)^2\right].$$

The technical details and the final result are remarkably similar to the case of Ferguson and Strachan [16], though their case is a reduction of the dispersionless KP hierarchy.

Let us start from the natural coordinates $b_1, \ldots, b_M, c_1, \ldots, c_N$ of the parameter space. One can calculate part of the inner product explicitly as follows.

Lemma 3.

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_j}\right) = -\delta_{ij}\frac{\kappa_i}{{b_i}^2}, \quad \left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial c_k}\right) = \delta_{kN}\frac{\kappa_j}{N b_j c_N}. \tag{74}$$

Proof. The derivative of $\log\lambda(p)$ with respect to p is a function as shown in (46). The derivatives with respect to b_i and c_k take the simple form

$$\frac{\partial \log\lambda(p)}{\partial b_i} = -\frac{\kappa_i}{p-b_i}, \qquad \frac{\partial \log\lambda(p)}{\partial c_k} = p^{-k}.$$

Consequently, the inner products in question can be expressed as

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_j}\right) = \sum_{n=1}^{M+N} \operatorname*{res}_{p=\gamma_n} \left[\frac{{\kappa_i}^2}{(p-b_i)^2}\left(\frac{\partial \log\lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right],$$
$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial c_k}\right) = \sum_{n=1}^{M+N} \operatorname*{res}_{p=\gamma_n} \left[-\frac{\kappa_i p^{-k}}{p-b_i}\left(\frac{\partial \log\lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right].$$

Since

$$\left(\frac{\partial \log\lambda(p)}{\partial p}\right)^{-1} = \frac{p^{N+1}\prod_{k=1}^{M}(p-b_k)}{Q(p)}, \quad Q(p) = \tilde{M}\prod_{n=1}^{M+N}(p-\gamma_n),$$

the 1-forms in the residues are rational and have poles of the first order at $p = p_1,\ldots,p_{M+N}$. Other possible poles are located at $p = b_i, b_j, 0$. The latter poles, however, can disappear because of zeros of the numerator in this expression of $(\partial\log\lambda(p)/\partial p)^{-1}$. For example, if $i \neq j$, the first 1-form is non-singular at $p = b_i, b_j$ as well as at $p = 0$. Since the residue theorem says that the sum of all residues is equal to 0, one can conclude that

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_j}\right) = 0 \quad \text{for } i \neq j.$$

By the same reasoning, one can confirm that

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial c_k}\right) = 0 \quad \text{for } k < N.$$

As regards the remaining cases, one can use the residue theorem to rewrite the inner products as

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_i}\right) = -\operatorname*{res}_{p=b_i}\left[\frac{{\kappa_i}^2}{(p-b_i)^2}\left(\frac{\partial \log\lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right],$$
$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial c_N}\right) = -\operatorname*{res}_{p=0}\left[-\frac{\kappa_i p^{-N}}{p-b_i}\left(\frac{\partial \log\lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right].$$

In view of the local expression

$$\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1} = \frac{p-b_i}{\kappa_i} + O((p-b_i)^2 \quad \text{as } p \to b_i$$

and

$$\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1} = -\frac{p^{N+1}}{Nc_N} + O(p^{N+2}) \quad \text{as } p \to 0,$$

one can readily calculate the residues and confirm the statement for the remaining cases. $\square$

This lemma shows that the metric is already flat with respect to $\log b_i$'s. If one can further find a partial change of coordinates $(c_1, \cdots, c_N) \to (\bar{q}_0, \ldots, \bar{q}_{N-1})$ that is independent of b_i's and flat in themselves, the new coordinate system $\log b_1, \ldots, \log b_M, \bar{q}_0, \ldots, \bar{q}_{N-1}$ are totally flat. We shall show that

$$\bar{q}_0 = \phi, \quad \bar{q}_n = \operatorname*{res}_{p=0}\left[\frac{\bar{z}(p)^n}{n} d\log p\right], \quad n = 1, \ldots, N-1, \tag{75}$$

give such coordinates.

To this end, we need to know some properties of $\bar{q}_n$'s (which are defined for $n \geq N$ as well by the same residue formula). Let $\bar{p}(\zeta)$ denote the inverse function of $\zeta = \bar{z}(p)$. It has a Laurent expansion of the form

$$\bar{p}(\zeta) = e^{\phi}\zeta^{-1}(1 + \bar{u}_1\zeta^{-1} + \cdots). \tag{76}$$

Therefore $\log \bar{p}(\zeta)$ is also well-defined as a series of the form

$$\log \bar{p}(\zeta) = -\log \zeta + \phi + \bar{u}_1\zeta^{-1} + \cdots.$$

Lemma 4. *$\bar{q}_n$'s coincide with the coefficients of the expansion of $\log \bar{p}(\zeta)$:*

$$\log \bar{p}(\zeta) = -\log \zeta + \bar{q}_0 + \bar{q}_1\zeta^{-1} + \cdots + \bar{q}_n\zeta^{-n} + \cdots. \tag{77}$$

Proof. One can rewrite the definition of $\bar{q}_n$'s as

$$\begin{aligned}\bar{q}_n &= \operatorname*{res}_{p=0}\left[\frac{\bar{z}(p)^n}{n} d\log p\right] \\ &= \operatorname*{res}_{\zeta=\infty}\left[\frac{\zeta^n}{n} d\log \bar{p}(\zeta)\right] \\ &= -\operatorname*{res}_{\zeta=\infty}\left[\log \bar{p}(\zeta) d\left(\frac{\zeta^n}{n}\right)\right] \\ &= -\operatorname*{res}_{\zeta=\infty}\left[\log \bar{p}(\zeta)\zeta^{n-1} d\zeta\right].\end{aligned}$$

This implies that $\log \bar{p}(\zeta)$ has a Laurent expansion as (77) shows. $\square$

Lemma 5. *For $n = 1, \ldots, N-1$, $\bar{q}_n$ is a polynomial of $c_{N-n}, \ldots, c_{N-1}$ and $e^{-\phi}$ of the form*

$$\bar{q}_n = \frac{1}{N} e^{(n-N)\phi} c_{N-n} + \text{higher orders in } c_{N-n+1}, \ldots, c_{N-1} \tag{78}$$

and q_N is a function of b_i's only:

$$\bar{q}_N = \frac{1}{N} \log \prod_{i=1}^{M} (-b_i)^{\kappa_i}. \tag{79}$$

Proof. Since $\bar{z}(p) = (\log \lambda(p))^{1/N}$ and $c_N = e^{N\phi}$, one can calculate its n-th power as a Laurent series of the form

$$\bar{z}(p)^n = \left(e^{N\phi} p^{-N} + c_{N-1} p^{1-N} + \cdots + c_1 p^{-1} + \log \prod_{i=1}^{M} (-b_i) + O(p) \right)^{n/N}.$$

Extracting the p^0 term yields (78) and (79). □

(78) implies that the map $(c_1, \ldots, c_N) \mapsto (\bar{q}_0, \ldots, \bar{q}_{N-1})$ is invertible. We now choose b_i's and $\bar{q}_0, \ldots, \bar{q}_{N-1}$ as a new coordinate system on the parameter space of $\lambda(p)$, and consider $\lambda(p)$ to be a function of p and these coordinates.

Lemma 6. *The derivatives of $\log \lambda(p)$ with respect to $\bar{q}_n$'s can be expressed as*

$$\frac{\partial \log \lambda(p)}{\partial \bar{q}_n} = (-\zeta^{-n} + O(\zeta^{-N-1}))|_{\zeta = \bar{z}(p)} \frac{d \log \lambda(p)}{d \log p} \tag{80}$$

in a neighborhood of $p = 0$.

Proof. We can use the so called "thermodynamic identity" [3]

$$\frac{\partial \log \lambda(p)}{\partial \bar{q}_n} d \log p = - \left. \frac{\partial \log \bar{p}(\zeta)}{\partial \bar{q}_n} \right|_{\zeta = \bar{z}(p)} d \log \lambda(p) \tag{81}$$

to rewrite the derivatives of $\log \lambda(p)$ with respect to $\bar{q}_n$'s as

$$\frac{\partial \log \lambda(p)}{\partial \bar{q}_n} = - \left. \frac{\partial \log \bar{p}(\zeta)}{\partial \bar{q}_n} \right|_{\zeta = \bar{z}(p)} \frac{d\lambda(p)}{d \log p}.$$

Let us recall the expansion (77) of $\log \bar{p}(\zeta)$. In this expansion, the coefficients of $\zeta^0, \ldots, \zeta^{N-1}$ are $\bar{q}_n$'s themselves and, as (79) shows, the next leading coefficient q_N is a function of b_i's only. Consequently,

$$\frac{\partial \log \bar{p}(\zeta)}{\partial \bar{q}_n} = \zeta^{-n} + O(\zeta^{-N-1}).$$

□

Remark 6. This is a place where the technical details slightly deviate from the case of Ferguson and Strachan [16]. As in their case, the flat coordinates under construction are a mixture of the two types of coordinates, b_i's and $\bar{q}_n$'s, presented in the case (I). There is, however, a delicate difference in the derivation of the vital equalities (80). In their case, they could derive these equalities in a rather straightforward manner. In our case, we need a small piece of extra consideration on the special structure of q_N as explained above.

Theorem 10. *The inner product of the derivatives in b_i's and $\bar{q}_n$'s can be expressed as*

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_j}\right) = -\delta_{ij}\frac{\kappa_i}{{b_i}^2}, \quad \left(\frac{\partial}{\partial \bar{q}_m}, \frac{\partial}{\partial \bar{q}_n}\right) = N\delta_{m+n,N},$$
$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial \bar{q}_n}\right) = \delta_{n0}\frac{\kappa_i}{b_i}. \tag{82}$$

In particular, $\log b_i$*'s and* $\bar{q}_n$*'s are flat coordinates.*

Proof. Since the expression

$$\frac{\partial \log \lambda(p)}{\partial b_i} = -\frac{\kappa_i}{p - b_i}$$

of derivatives with respect to b_i's persists to be true, the foregoing calculations of the inner products of $\partial/\partial b_i$'s are also valid. To consider the inner products containing $\partial/\partial \bar{q}_n$'s, we note the equality

$$\frac{\partial \log \lambda(p)}{\partial \bar{q}_n} = \sum_{k=1}^{N} \frac{\partial c_k}{\partial \bar{q}_n} p^{-k}$$

as well. Thus the 1-forms in the expression

$$\left(\frac{\partial}{\partial \bar{q}_m}, \frac{\partial}{\partial \bar{q}_n}\right) = \sum_{n=1}^{M+N} \operatorname*{res}_{p=\gamma_n} \left[\frac{\partial \log \lambda(p)}{\partial \bar{q}_m}\frac{\partial \log \lambda(p)}{\partial \bar{q}_n}\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right],$$
$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial \bar{q}_n}\right) = \sum_{n=1}^{M+N} \operatorname*{res}_{p=\gamma_n} \left[\frac{\partial \log \lambda(p)}{\partial b_i}\frac{\partial \log \lambda(p)}{\partial \bar{q}_n}\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right]$$

of the inner products are rational, and can have extra poles at $p = 0$ in addition to the first order poles at $p = q_n$'s. By the residue theorem, the sum over the residues at q_n's can be converted to the residues at $p = 0$:

$$\left(\frac{\partial}{\partial \bar{q}_m}, \frac{\partial}{\partial \bar{q}_n}\right) = -\operatorname*{res}_{p=0}\left[\frac{\partial \log \lambda(p)}{\partial \bar{q}_m}\frac{\partial \log \lambda(p)}{\partial \bar{q}_n}\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right],$$

$$\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial \bar{q}_n}\right) = -\operatorname*{res}_{p=0}\left[-\frac{\kappa_i}{p-b_i}\frac{\partial \log \lambda(p)}{\partial \bar{q}_n}\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right].$$

To evaluate the residues of $p = 0$, let us recall (80). The residues in the last equalities can be thereby evaluated as

$$\begin{aligned}
&\operatorname*{res}_{p=0}\left[\frac{\partial \log \lambda(p)}{\partial \bar{q}_m}\frac{\partial \log \lambda(p)}{\partial \bar{q}_n}\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right] \\
&= \operatorname*{res}_{p=0}\left[(-\zeta^{-m} + O(\zeta^{-N-1}))(-\zeta^{-n} + O(\zeta^{-N-1}))|_{\zeta=\bar{z}(p)} d\log \lambda(p)\right] \\
&= \operatorname*{res}_{\zeta=\infty}\left[(-\zeta^{-m} + O(\zeta^{-N-1}))(-\zeta^{-n} + O(\zeta^{-N-1}))N\zeta^{N-1}d\zeta\right] \\
&= -N\delta_{m+n,N}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname*{res}_{p=0}\left[\frac{\partial \log \lambda(p)}{\partial b_i}\frac{\partial \log \lambda(p)}{\partial \bar{q}_n}\left(\frac{\partial \log \lambda(p)}{\partial p}\right)^{-1}\frac{dp}{p^2}\right] \\
&= \operatorname*{res}_{p=0}\left[\left(-\frac{\kappa_i}{p-b_i}\right)(-\zeta^{-n} + O(\zeta^{-N-1}))|_{\zeta=\bar{z}(p)} d\log p\right] \\
&= -\delta_{n0}\frac{\kappa_i}{b_i}.
\end{aligned}$$

This completes the proof. □

Let us conclude the present consideration with a few comments.

(a) As in the case of Ferguson and Strachan [16], we have been unable to find a dual Frobenius structure. The naive prescription replacing $\log \lambda(p) \to \lambda(p)$ does not work, because the 1-forms in the definition of the inner product and the cubic form have essential singularities at $p = 0$, and one cannot use the residue theorem. Another possible idea is to replace $\log \lambda(p) \to \log \log \lambda(p)$, but this will lead to some other difficulties.

(b) Unlike the Landau-Ginzburg potential of Ferguson and Strachan [16], our Landau-Ginzburg potential (2) has natural quasi-homogeneity. We can use the vector field

$$E = \sum_{n=1}^{M+N} \lambda_n \frac{\partial}{\partial \lambda_n} = \frac{1}{\tilde{M}} \sum_{i=1}^{M} b_i \frac{\partial}{\partial b_i} + \frac{1}{\tilde{M}} \sum_{k=1}^{N} k c_k \frac{\partial}{\partial c_k} \tag{83}$$

as an Euler vector field.

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