GEOMETRICAL STRUCTURE OF LAPLACIAN EIGENFUNCTIONS

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Abstract. We review the properties of eigenvalues and eigenfunctions of the Laplace operator in bounded Euclidean domains with Dirichlet, Neumann or Robin boundary condition. We keep the presentation at a level accessible to scientists from various disciplines ranging from mathematics to physics and computer sciences. The main focus is put onto multiple intricate relations between the shape of a domain and the geometrical structure of eigenfunctions.

Key words. Laplace operator, eigenfunctions, eigenvalues, localization

AMS subject classifications. 35J05, 35Pxx, 49Rxx, 51Pxx

Dedicated to Prof. Bernard Sapoval for his forthcoming 75th birthday

Foreword. *Since the theory of the Laplace operator is a vast and actively developing field, any review would be necessarily incomplete. Although we have tried to collect and overview the major achievements on geometrical properties of Laplacian eigenvalues and eigenfunctions in Euclidean bounded domains, some significant works might be overlooked. We would greatly appreciate any comments, critics, corrections and bibliography updates (please send them to the corresponding author). These improvements will be included into the revised version and properly acknowledged.*

1. Introduction.

This review focuses on the classical eigenvalue problem for the Laplace operator $\Delta = \partial^2/\partial x_1^2 + ... + \partial^2/\partial x_d^2$ in an open bounded connected domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3, ...$ being the space dimension),

$$-\Delta u_m(x) = \lambda_m u_m(x) \quad (x \in \Omega), \tag{1.1}$$

with Dirichlet, Neumann or Robin boundary condition on a piecewise smooth boundary $\partial\Omega$:

$$\begin{aligned} u_m(x) &= 0 \quad (x \in \partial\Omega) \quad \text{(Dirichlet)}, \\ \frac{\partial}{\partial n} u_m(x) &= 0 \quad (x \in \partial\Omega) \quad \text{(Neumann)}, \\ \frac{\partial}{\partial n} u_m(x) + h u_m(x) &= 0 \quad (x \in \partial\Omega) \quad \text{(Robin)}, \end{aligned} \tag{1.2}$$

where $\partial/\partial n$ is the normal derivative pointed outwards the domain, and h is a positive constant. The spectrum of the Laplace operator is known to be discrete, the eigenvalues λ_m are positive and ordered in an ascending order by the index $m = 1, 2, 3, ...$,

$$0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le ... \tag{1.3}$$

(with possible multiplicities), while the eigenfunctions $\{u_m(x)\}$ form a complete basis in the functional space $L_2(\Omega)$ of measurable and square-integrable functions on Ω

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[107, 333]. By definition, the function 0 satisfying Eqs. (1.1, 1.2) is excluded from the set of eigenfunctions. Since the eigenfunctions are defined up to a multiplicative factor, it is sometimes convenient to normalize them to get the unit L_2-norm:

$$\|u_m\|_2 \equiv \|u_m\|_{L_2(\Omega)} \equiv \left(\int\limits_{\Omega} dx \; |u_m(x)|^2 \right)^{1/2} = 1 \tag{1.4}$$

(note that there is still ambiguity up to the multiplication by $e^{i\alpha}$, with $\alpha \in \mathbb{R}$).

Laplacian eigenfunctions appear as vibration modes in acoustics, as electron waves functions in quantum waveguides, as natural basis for constructing heat kernels in diffusion, etc. For instance, vibration modes of a thin membrane (a drum) with a fixed boundary are given by Dirichlet Laplacian eigenfunctions u_m, with the drum frequencies $\sqrt{\lambda_m}$. In an experiment, a particular eigenmode can be excited at the corresponding frequency [348–350]. In diffusion theory, an interpretation of eigenfunctions is less explicit. The first eigenfunction represents the long-time asymptotic spatial distribution of particles diffusing in a bounded domain (see below). A conjectural probabilistic representation of higher-order eigenfunctions through a Fleming-Viot type model was developed by Burdzy *et al.* [78, 80].

The eigenvalue problem (1.1, 1.2) is archetypical in the theory of elliptic operators, while the properties of the underlying eigenfunctions have been thoroughly investigated in various mathematical and physical disciplines, including spectral theory, probability and stochastic processes, dynamical systems and quantum billiards, condensed matter physics and quantum mechanics, theory of acoustical, optical and quantum waveguides, computer sciences, etc. Many books and reviews were dedicated to different aspects of Laplacian eigenvalues, eigenfunction and their applications (see, e.g., [7, 20, 21, 27, 47, 96, 190, 199, 233, 304, 315]). The diversity of notions and methods developed by mathematicians, physicists and computer scientists often makes the progress in one discipline almost unknown or hardly accessible to scientists from the other disciplines. One of the goals of the review is to bring together various facts about Laplacian eigenvalues and eigenfunctions and to present them at a level accessible to scientists from various disciplines. For this purpose, many technical details and generalities are omitted in favor to simple illustrations. While the presentation is focused on the Laplace operator in bounded Euclidean domains with piecewise smooth boundaries, a number of extensions are relatively straightforward. For instance, the Laplace operator can be extended to a second order elliptic operator with appropriate coefficients, the piecewise smoothness of a boundary can often be relaxed [170, 259], while Euclidean domains can be replaced by Riemannian manifolds [199]. The main emphasis is put onto the geometrical structure of Laplacian eigenfunctions and on their relation to the shape of a domain. Although the bibliography counts four hundred citations, it is far from being complete, and readers are invited to refer to other reviews and books for further details and references.

The review is organized as follows. We start by recalling in Sec. 2 general properties of the Laplace operator. Explicit representations of eigenvalues and eigenfunctions in simple domains are summarized in Sec. 3. In Sec. 4 we review the properties of eigenvalues and their relation to the shape of a domain: Weyl's asymptotic law, isoperimetric inequalities and the related shape optimization problems, and Kac's inverse spectral problem. Although eigenfunctions are not involved at this step, valuable information can be learned about the domain from the eigenvalues alone. The next step consists in the analysis of nodal lines/surfaces or nodal domains in Sec. 5.

The nodal lines tell us how the zeros of eigenfunctions are spatially distributed, while their amplitudes are still ignored. In Sec. 6, several estimates for the amplitudes of eigenfunctions are summarized. Most of these results were obtained during the last twenty years.

Section 7 is devoted to the property of eigenfunctions known as localization. We start by recalling the notion of localization in quantum mechanics: the strong localization by a potential (Sec. 7.1), Anderson localization (Sec. 7.2) and trapped modes in infinite waveguides (Sec. 7.3). In all three cases, the eigenvalue problem is different from Eqs. (1.1, 1.2), due to either the presence of a potential, or the unboundness of a domain. Nevertheless, these cases are instructive, as similar effects may be observed for the eigenvalue problem (1.1, 1.2). In particular, we discuss in Sec. 7.4 an exponentially decaying estimate for the norm of eigenfunctions in domains with branches of variable cross-sectional profiles. Section 7.5 reviews the properties of low-frequency eigenfunctions in "dumbbell" domains, in which two (or many) subdomains are connected by narrow channels. This situation is convenient for a rigorous analysis as the width of channels plays the role of a small parameter [346]. A number of asymptotic results for eigenvalues and eigenfunctions were derived, for Dirichlet, Neumann and Robin boundary conditions. A harder case of irregular or fractal domains is discussed in Sec. 7.6. Here, it is difficult to identify a relevant small parameter to get rigorous estimates. In spite of numerous numerical examples of localized eigenfunctions (both for Dirichlet and Neumann boundary conditions), a comprehensive theory of localization is still missing. Section 7.7 is devoted to high-frequency localization and the related scarring problems in quantum billiards. We start by illustrating the classical whispering gallery, bouncing ball and focusing modes in circular and elliptical domains. We also provide examples for the case without localization. A brief overview of quantum billiards is presented. In the last Sec. 8, we mention some issues which could not be included into the review, e.g., numerical methods for computation of eigenfunctions or their numerous applications.

2. Basic properties. We start by recalling several basic properties of the Laplacian eigenvalues and eigenfunctions (see [57, 107, 333] or other standard textbooks).

(i) The eigenfunctions are infinitely differentiable inside the domain Ω. For any open subset $V \subset \Omega$, the restriction of u_m on V cannot be strictly 0 [233].

(ii) Multiplying Eq. (1.1) by u_m, integrating over Ω and using the Green's formula yield

$$\lambda_m = \frac{\int\limits_{\Omega} dx\; |\nabla u_m|^2 - \int\limits_{\partial\Omega} dx\; u_m \frac{\partial u_m}{\partial n}}{\int\limits_{\Omega} dx\; u_m^2} = \frac{\|\nabla u_m\|^2_{L_2(\Omega)} + h\|u_m\|^2_{L_2(\partial\Omega)}}{\|u_m\|^2_{L_2(\Omega)}}, \tag{2.1}$$

where ∇ stands for the gradient operator, and we used Robin boundary condition (1.2) in the last equality; for Dirichlet or Neumann boundary conditions, the boundary integral (second term) vanishes. This formula ensures that all eigenvalues are positive.

(iii) Similar expression appears in the variational formulation of the eigenvalue problem, known as the minimax principle [107]

$$\lambda_m = \min\max \frac{\|\nabla v\|^2_{L_2(\Omega)} + h\|v\|^2_{L_2(\partial\Omega)}}{\|v\|^2_{L_2(\Omega)}}, \tag{2.2}$$

where the maximum is over all linear combinations of the form

$$v = a_1\phi_1 + ... + a_m\phi_m,$$

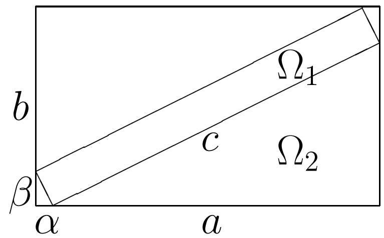

FIG. 2.1. *A counter-example for the property of domain monotonicity for Neumann boundary condition. Although a smaller rectangle Ω_1 is inscribed into a larger rectangle Ω_2 (i.e., $\Omega_1 \subset \Omega_2$), the second eigenvalue $\lambda_2(\Omega_1) = \pi^2/c^2$ is smaller than the second eigenvalue $\lambda_2(\Omega_2) = \pi^2/a^2$ (if $a > b$) when $c = \sqrt{(a-\alpha)^2 + (b-\beta)^2} > a$ (courtesy by N. Saito).*

and the minimum is over all choices of m linearly independent continuous and piecewise-differentiable functions ϕ_1, ..., ϕ_m (said to be in the functional space $H^1(\Omega)$) [107]. Note that the minimum is reached exactly on the eigenfunction u_m. For Dirichlet eigenvalue problem, there is a supplementary condition $v = 0$ on the boundary $\partial\Omega$ so that the second term in Eq. (2.2) is canceled. For Neumann eigenvalue problem, $h = 0$ and the second term vanishes again.

(iv) The minimax principle implies the monotonous increase of the eigenvalues λ_m with h, namely if $h < h'$, then $\lambda_m(h) \leq \lambda_m(h')$. In particular, any eigenvalue $\lambda_m(h)$ of the Robin problem lies between the corresponding Neumann and Dirichlet eigenvalues.

(v) For Dirichlet boundary condition, the minimax principle implies the property of domain monotonicity: eigenvalues monotonously decrease when the domain enlarges, i.e., $\lambda_m(\Omega_1) \geq \lambda_m(\Omega_2)$ if $\Omega_1 \subset \Omega_2$. This property does not hold for Neumann or Robin boundary conditions, as illustrated by a simple counter-example on Fig. 2.1.

(vi) The eigenvalues are invariant under translations and rotations of the domain. This is a key property for an efficient image recognition and analysis [335, 343, 344]. When a domain is expanded by factor α, all the eigenvalues are rescaled by $1/\alpha^2$.

(vii) The first eigenfunction u_1 does not change the sign and can be chosen positive. Because of the orthogonality of eigenfunctions, u_1 is in fact the only eigenfunction not changing its sign.

(viii) The first eigenvalue λ_1 is simple and strictly positive for Dirichlet and Robin boundary conditions; for Neumann boundary condition, $\lambda_1 = 0$ and u_1 is a constant.

(ix) The completeness of eigenfunctions in $L_2(\Omega)$ can be expressed as

$$\sum_m u_m^*(x) u_m(y) = \delta(x-y) \qquad (x, y \in \Omega), \tag{2.3}$$

where asterisk denotes the complex conjugate, and $\delta(x)$ being the Dirac distribution. Multiplying this relation by a function $f \in L_2(\Omega)$ and integrating over Ω yields the decomposition of $f(x)$ over $u_m(x)$:

$$f(x) = \sum_m u_m^*(x) \int_\Omega dy \; f(y) \; u_m(y).$$

(x) The Green function $G(x, y)$ for the Laplace operator which satisfies

$$-\Delta G(x, y) = \delta(x-y) \qquad (x, y \in \Omega) \tag{2.4}$$

(with an appropriate boundary condition), admits the spectral decomposition over the eigenfunctions

$$G(x,y) = \sum_m \lambda_m^{-1} u_m^*(x) u_m(y). \tag{2.5}$$

(for Neumann boundary condition, $\lambda_1 = 0$ has to be excluded; in that case, the Green function is defined up to an additive constant).

Similarly, the heat kernel (or diffusion propagator) $G_t(x,y)$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t} G_t(x,y) - \Delta G_t(x,y) = 0 \qquad (x,y \in \Omega), \\ G_{t=0}(x,y) = \delta(x-y) \end{aligned} \tag{2.6}$$

(with an appropriate boundary condition), admits the spectral decomposition

$$G_t(x,y) = \sum_m e^{-\lambda_m t} u_m^*(x) u_m(y). \tag{2.7}$$

The Green function and heat kernel allow one to solve the standard boundary value and Cauchy problems for the Laplace and heat equations, respectively [90, 108]. The decompositions (2.5, 2.7) are the major tool for getting explicit solutions in simple domains for which the eigenfunctions are known explicitly (see Sec. 3). This representation is also important for the theory of diffusion due to the probabilistic interpretation of $G_t(x,y)dx$ as the conditional probability for Brownian motion started at y to arrive in the dx vicinity of x after a time t [41, 42, 140, 196, 320, 332, 395]. Setting Dirichlet, Neumann or Robin boundary conditions, one can respectively describe perfect absorptions, perfect reflections and partial absorption/reflection on the boundary [163].

For Dirichlet boundary condition, if $\Omega \subset \Omega'$, then $0 \le G_t^{(\Omega)}(x,y) \le G_t^{(\Omega')}(x,y)$ [386]. In particular, taking $\Omega' = \mathbb{R}^d$, one gets

$$0 \le G_t(x,y) \le (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right), \tag{2.8}$$

where the Gaussian heat kernel for free space is written on the right-hand side. The above domain monotonicity for heat kernels may not hold for Neumann boundary condition [43].

3. Eigenbasis for simple domains. We list the examples of "simple" domains, in which symmetries allow for variable separations and thus explicit representations of eigenfunctions in terms of elementary or special functions.

3.1. Intervals, rectangles, parallelepipeds. For rectangle-like domains $\Omega = [0,\ell_1] \times ... \times [0,\ell_d] \subset \mathbb{R}^d$ (with the sizes $\ell_i > 0$), the natural variable separation yields

$$u_{n_1,...,n_d}(x_1,...,x_d) = u_{n_1}^{(1)}(x_1) \dots u_{n_d}^{(d)}(x_d), \qquad \lambda_{n_1,...,n_d} = \lambda_{n_1}^{(1)} + ... + \lambda_{n_d}^{(d)}, \tag{3.1}$$

where the multiple index $n_1...n_d$ is used instead of m, and $u_{n_i}^{(i)}(x_i)$ and $\lambda_{n_i}^{(i)}$ $(i = 1,...,d)$ correspond to the one-dimensional problem on the interval $[0,\ell_i]$. Depending on the boundary condition, $u_n^{(i)}(x)$ are sines (Dirichlet), cosines (Neumann) or their combinations (Robin):

$$
\begin{aligned}
u_n^{(i)}(x) &= \sin(\pi(n+1)x/\ell_i), && \lambda_n^{(i)} = \pi^2(n+1)^2/\ell_i^2, && \text{(Dirichlet)},\\
u_n^{(i)}(x) &= \cos(\pi n x/\ell_i), && \lambda_n^{(i)} = \pi^2 n^2/\ell_i^2, && \text{(Neumann)},\\
u_n^{(i)}(x) &= \sin(\alpha_n x/\ell_i) + \frac{\alpha_n}{h\ell_i}\cos(\alpha_n x/\ell_i), && \lambda_n^{(i)} = \alpha_n^2/\ell_i^2, && \text{(Robin)},
\end{aligned}
\tag{3.2}
$$

where $n = 0, 1, 2, ...$ and the coefficients α_n depend on the parameter h and satisfy the equation obtained by imposing the Robin boundary condition in Eq. (1.2) at $x = \ell_i$:

$$
\frac{2\alpha_n}{h\ell_i}\cos\alpha_n + \left(1 - \frac{\alpha_n^2}{h^2\ell_i^2}\right)\sin\alpha_n = 0. \tag{3.3}
$$

According to the property (iv) of Sec. 2, this equation has the unique solution α_n on each interval $[n\pi, (n+1)\pi]$ $(n = 0, 1, 2, ...)$, that makes its numerical computation by bisection (or other) method easy and fast. All the eigenvalues $\lambda_n^{(i)}$ are simple (not degenerate). The L_2-norm of this function is

$$
\|u_n^{(i)}(x)\|_{L_2((0,\ell_i))} = \left(\frac{\alpha_n^2 + 2h\ell_i + h^2\ell_i^2}{2h^2}\right)^{1/2}. \tag{3.4}
$$

In turn, the eigenvalues $\lambda_{n_1,...,n_d}$ can be degenerate if there exists a rational ratio $(\ell_i/\ell_j)^2$ (with $i \neq j$). For instance, the Dirichlet eigenvalues of the unit square are $2\pi^2$, $5\pi^2$, $5\pi^2$, $8\pi^2$, ... , with the twice degenerate second eigenvalue. An eigenfunction associated to a degenerate eigenvalue is a linear combination of the corresponding functions. For the above example $u(x_1, x_2) = c_1 \sin(\pi x_1)\sin(2\pi x_2) + c_2 \sin(2\pi x_1)\sin(\pi x_2)$ with any c_1 and c_2 such that $c_1^2 + c_2^2 \neq 0$.

3.2. Disk, sector and circular annulus. The rotation symmetry of a circular annulus, $\Omega = \{x \in \mathbb{R}^2 \ : \ R_0 < |x| < R\}$, allows one to write the Laplace operator in polar coordinates,

$$
\begin{cases} x_1 = r\cos\varphi, \\ x_2 = r\sin\varphi, \end{cases} \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}, \tag{3.5}
$$

that leads to variable separation and an explicit representation of eigenfunctions

$$
u_{nkl}(r,\varphi) = \big[J_n(\alpha_{nk} r/R) + c_{nk} Y_n(\alpha_{nk} r/R)\big] \times \begin{cases} \cos(n\varphi), & l = 1, \\ \sin(n\varphi), & l = 2 \ (n \neq 0), \end{cases} \tag{3.6}
$$

where $J_n(z)$ and $Y_n(z)$ are the Bessel functions of the first and second kind [1, 66, 393], and the coefficients α_{nk} and c_{nk} are set by the boundary conditions at $r = R$ and $r = R_0$:

$$
\begin{aligned}
0 &= \frac{\alpha_{nk}}{R}\left[J_n'(\alpha_{nk}) + c_{nk} Y_n'(\alpha_{nk})\right] + h\left[J_n(\alpha_{nk}) + c_{nk} Y_n(\alpha_{nk})\right],\\
0 &= -\frac{\alpha_{nk}}{R}\left[J_n'\big(\alpha_{nk}\frac{R_0}{R}\big) + c_{nk} Y_n'\big(\alpha_{nk}\frac{R_0}{R}\big)\right] + h\left[J_n\big(\alpha_{nk}\frac{R_0}{R}\big) + c_{nk} Y_n\big(\alpha_{nk}\frac{R_0}{R}\big)\right].
\end{aligned}
\tag{3.7}
$$

For each $n = 0, 1, 2, ...$, the system of these equations has infinitely many solutions α_{nk} which are enumerated by the index $k = 1, 2, 3, ...$ [393]. The eigenfunctions are enumerated by the triple index nkl, with $n = 0, 1, 2, ...$ counting the order of Bessel functions, $k = 1, 2, 3, ...$ counting solutions of Eqs. (3.7), and $l = 1, 2$. Since $u_{0k2}(r, \varphi)$ are trivially zero (as $\sin(n\varphi) = 0$ for $n = 0$), they are excluded. The eigenvalues $\lambda_{nk} = \alpha_{nk}^2/R^2$, which are independent of the last index l, are simple for $n = 0$ and twice degenerate for $n > 0$. In the latter case, an eigenfunction is any nontrivial linear combination of u_{nk1} and u_{nk2}. The squared L_2-norm of the eigenfunction is

$$\|u_{nkl}(r,\varphi)\|_2^2 = \frac{\pi(2-\delta_{n,0})R^2}{2\alpha_{nk}^2}\biggl[\Bigl(\alpha_{nk}^2 + h^2R^2 - n^2\Bigr)v_{nk}^2(R) - \Bigl((\alpha_{nk}^2 + h^2R^2)\frac{R_c^2}{R^2} - n^2\Bigr)v_{nk}^2(R_c)\biggr], \tag{3.8}$$

where $v_{nk}(r) = J_n(\alpha_{nk}r/R) + c_{nk}Y_n(\alpha_{nk}r/R)$.

For the special case of a disk ($R_0 = 0$), all the coefficients c_{nk} in front of the Bessel functions $Y_n(z)$ (divergent at 0) are set to 0:

$$u_{nkl}(r,\varphi) = J_n(\alpha_{nk}r/R) \times \begin{cases} \cos(n\varphi), & l = 1, \\ \sin(n\varphi), & l = 2 \ (n \neq 0), \end{cases} \tag{3.9}$$

where α_{nk} are either the positive roots j_{nk} of the Bessel function $J_n(z)$ (Dirichlet), or the positive roots $\tilde{j}_{nk}$ of its derivative $J_n'(z)$ (Neumann), or the positive roots of their linear combination $J_n'(z) + hJ_n(z)$ (Robin). The asymptotic behavior of zeros of Bessel functions was thoroughly investigated. For fixed k and large n, the Olver's expansion holds $j_{nk} \simeq n + \delta_k n^{1/3} + O(n^{-1/3})$ (with known coefficients δ_k) [130, 293, 294], while for fixed n and large k, the McMahon's expansion holds: $j_{nk} \simeq \pi(k + n/2 - 1/4) + O(k^{-1})$ [393]. Similar asymptotic relations are applicable for Neumann and Robin boundary conditions.

For a circular sector of radius R and of angle $\pi\beta$, the eigenfunctions are

$$u_{nk}(r,\varphi) = J_{n/\beta}(\alpha_{nk}r/R) \times \begin{cases} \sin(n\varphi/\beta) & \text{(Dirichlet)} \\ \cos(n\varphi/\beta) & \text{(Neumann)} \end{cases} \qquad (r < R,\ 0 < \varphi < \pi\beta) \tag{3.10}$$

i.e., they are expressed in terms of Bessel functions of fractional order, and α_{nk} are the positive roots of $J_{n/\beta}(z)$ (Dirichlet) or $J_{n/\beta}'(z)$ (Neumann). The Robin boundary condition and a sector of a circular annulus can be treated similarly.

3.3. Sphere and spherical shell. The rotation symmetry of a spherical shell, $\Omega = \{x \in \mathbb{R}^3 \ : \ R_0 < |x| < R\}$, allows one to write the Laplace operator in spherical coordinates,

$$\begin{cases} x_1 = r\sin\theta\cos\varphi, \\ x_2 = r\sin\theta\sin\varphi, \\ x_3 = r\cos\theta, \end{cases} \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\varphi^2}\right), \tag{3.11}$$

that leads to the variable separation and an explicit representation of eigenfunctions

$$u_{nkl}(r,\theta,\varphi) = \bigl[j_n(\alpha_{nk}r/R) + c_{nk}y_n(\alpha_{nk}r/R)\bigr]P_n(\cos\theta)e^{il\varphi}, \tag{3.12}$$

where $j_n(z)$ and $y_n(z)$ are the spherical Bessel functions of the first and second kind,

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z), \qquad y_n(z) = \sqrt{\frac{\pi}{2z}}Y_{n+1/2}(z), \tag{3.13}$$

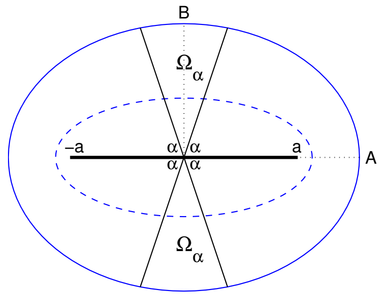

Fig. 3.1. *Two ellipses of radii $R = 0.5$ (dashed line) and $R = 1$ (solid line), with the focal distance $a = 1$. The major and minor semi-axes, $A = a\cosh R$ and $B = a\sinh R$, are shown by black dotted lines. The horizontal thick segment connects the foci.*

$P_n(z)$ are Legendre polynomials, and the coefficients α_{nk} and c_{nk} are set by the boundary conditions at $r = R$ and $r = R_0$ similar to Eq. (3.7). The eigenfunctions are enumerated by the triple index nkl, with $n = 0, 1, 2, ...$ counting the order of spherical Bessel functions, $k = 1, 2, 3, ...$ counting zeros, and $l = -n, -n+1, ..., n$. The eigenvalues $\lambda_{nk} = \alpha_{nk}^2/R^2$, which are independent of the last index l, have the degeneracy $2n + 1$. The squared L_2-norm of the eigenfunction is

$$\begin{aligned}\|u_{nkl}(r,\theta,\varphi)\|_2^2 = \frac{2\pi R^3}{(2n+1)\alpha_{nk}^2}\biggl[\Bigl(\alpha_{nk}^2 + h^2R^2 - hR - n(n+1)\Bigr)v_{nk}^2(R) \\ - \Bigl(\alpha_{nk}^2(R_c/R)^3 + h^2R_c^2 - hR_c - n(n+1)R_c/R\Bigr)v_{nk}^2(R_c)\biggr],\end{aligned} \tag{3.14}$$

where $v_{nk}(r) = j_n(\alpha_{nk}r/R) + c_{nk}y_n(\alpha_{nk}r/R)$.

In the special case of a sphere ($R_0 = 0$), one has $c_{nk} = 0$ and the equations are simplified.

3.4. Ellipse and elliptical annulus. In elliptic coordinates, the Laplace operator reads as

$$\begin{cases} x_1 = a\cosh r\cos\theta, \\ x_2 = a\sinh r\sin\theta, \end{cases} \qquad \Delta = \frac{1}{a^2(\sinh^2 r + \sin^2\theta)}\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial\theta^2}\right), \tag{3.15}$$

where $a > 0$ is the prescribed distance between the origin and the foci, $r \ge 0$ is the radial coordinate that fixes the major and minor semi-axes: $A = a\cosh r$ and $B = a\sinh r$, and $0 \le \theta < 2\pi$ is the angular coordinate (Fig. 3.1). An ellipse is a curve of constant r so that its points (x_1, x_2) satisfy $x_1^2/A^2 + x_2^2/B^2 = 1$. Note that the eccentricity $e = a/A = 1/\cosh r$ is strictly positive. A filled ellipse (i.e., the interior of an given ellipse) can be characterized in elliptic coordinates as $0 \le r < R$ and $0 \le \theta < 2\pi$. Similarly, an elliptical annulus (i.e., the interior between two ellipses with the same foci) is characterized by $R_1 < r < R_2$ and $0 \le \theta < 2\pi$.

In the elliptic coordinates, the variables can be separated, $u(r,\theta) = f(r)g(\theta)$, from which Eq. (1.1) reads as

$$\left(\frac{1}{f(r)}\frac{d^2f}{dr^2} + \frac{\lambda a^2}{2}\cosh(2r)\right) = -\left(\frac{1}{g(\theta)}\frac{d^2g}{d\theta^2} - \frac{\lambda a^2}{2}\cos(2\theta)\right)$$

so that both sides are equal to a constant (denoted c). As a consequence, the angular and radial parts, $g(\theta)$ and $f(r)$, are solutions of the Mathieu equation and the modified Mathieu equation, respectively [97, 271, 405]

$$g''(\theta) + (c - 2q\cos 2\theta)\, g(\theta) = 0, \qquad f''(r) - (c - 2q\cosh 2r)\, f(r) = 0,$$

where $q = \lambda a^2/4$ and the parameter c is called the characteristic value of Mathieu functions. Periodic solutions of the Mathieu equation are possible for specific values of c. They are denoted as $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ (with $n = 0, 1, 2, ...$) and called the angular Mathieu functions of the first and second kind. Each function $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ corresponds to its own characteristic value c (the relation being implicit, see [271]).

For the radial part, there are two linearly independent solutions for each characteristic value c: two modified Mathieu functions $\mathrm{Mc}_n^{(1)}(r, q)$ and $\mathrm{Mc}_n^{(2)}(r, q)$ correspond to the same c as $\mathrm{ce}_n(\theta, q)$, and two modified Mathieu functions $\mathrm{Ms}_{n+1}^{(1)}(r, q)$ and $\mathrm{Ms}_{n+1}^{(2)}(r, q)$ correspond to the same c as $\mathrm{se}_{n+1}(\theta, q)$. As a consequence, there are four families of eigenfunctions (distinguished by the index $l = 1, 2, 3, 4$) in an elliptical domain

$$\begin{aligned}
u_{nk1}(r, \theta) &= \mathrm{ce}_n(\theta, q_{nk1})\mathrm{Mc}_n^{(1)}(r, q_{nk1}),\\
u_{nk2}(r, \theta) &= \mathrm{ce}_n(\theta, q_{nk2})\mathrm{Mc}_n^{(2)}(r, q_{nk2}),\\
u_{nk3}(r, \theta) &= \mathrm{se}_{n+1}(\theta, q_{nk3})\mathrm{Ms}_{n+1}^{(1)}(r, q_{nk3}),\\
u_{nk4}(r, \theta) &= \mathrm{se}_{n+1}(\theta, q_{nk4})\mathrm{Ms}_{n+1}^{(2)}(r, q_{nk4}),
\end{aligned}$$

where the parameters q_{nkl} are determined by the boundary condition. For instance, for a filled ellipse of radius R with Dirichlet boundary condition, there are four individual equations for the parameter q for each $n = 0, 1, 2, ...$

$$\mathrm{Mc}_n^{(1)}(R, q_{nk1}) = 0, \;\; \mathrm{Mc}_n^{(2)}(R, q_{nk2}) = 0, \;\; \mathrm{Ms}_{n+1}^{(1)}(R, q_{nk3}) = 0, \;\; \mathrm{Ms}_{n+1}^{(2)}(R, q_{nk4}) = 0,$$

each of them having infinitely many positive solutions q_{nkl} enumerated by $k = 1, 2, \ldots$ [1, 271]. Finally, the associated eigenvalues are $\lambda_{nkl} = 4q_{nkl}/a^2$.

3.5. Equilateral triangle. Lamé discovered the Dirichlet eigenvalues and eigenfunctions of the equilateral triangle $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \;\; : 0 < x_1 < 1,\; 0 < x_2 < x_1\sqrt{3},\; x_2 < \sqrt{3}(1 - x_1)\}$ by using reflections and the related symmetries [234]:

$$\lambda_{mn} = \frac{16\pi^2}{27}\big(m^2 + n^2 - mn\big) \qquad (m, n \in \mathbb{Z}), \tag{3.16}$$

where 3 divides $m + n$, $m \neq 2n$, and $n \neq 2m$, and the associate eigenfunction is

$$u_{mn}(x_1, x_2) = \sum_{(m', n')} \pm \exp\left[\frac{2\pi i}{3}\left(m' x_1 + (2n' - m')\frac{x_2}{\sqrt{3}}\right)\right], \tag{3.17}$$

where (m', n') runs over $(-n, m-n)$, $(-n, -m)$, $(n-m, -m)$, $(n-m, n)$, (m, n) and $(m, m-n)$ with the $\pm$ sign alternating. Pinsky showed that this set of eigenfunctions is complete in $L_2(\Omega)$ [309, 310]. Note that the conditions $m \neq 2n$ and $n \neq 2m$ should be satisfied for all 6 pairs in the sum that yields one additional condition: $m \neq -n$. The following relations hold: $u_{-m,-n} = u_{mn}^*$, $u_{n,m} = -u_{mn}^*$ and $u_{m,0} = u_{mm}$. Moreover, all symmetric eigenfunctions are enumerated by the index $(m, 0)$. The eigenvalue λ_{mn} corresponds to a symmetric eigenfunction if and only if m is a multiple of 3 [309].

The eigenfunctions for Neumann boundary condition are

$$u_{mn}(x_1, x_2) = \sum_{(m',n')} \exp\left[\frac{2\pi i}{3}\left(m'x_1 + (2n' - m')\frac{x_2}{\sqrt{3}}\right)\right], \tag{3.18}$$

where the only condition is that $m + n$ are multiples of 3 (and no sign change).

4. Eigenvalues.

4.1. Weyl's law. The Weyl's law is one of the first connections between the spectral properties of the Laplace operator and the geometrical structure of a bounded domain Ω. In 1911, Hermann Weyl derived the asymptotic behavior of the Laplacian eigenvalues [396, 397]:

$$\lambda_m \propto \frac{4\pi^2}{(\omega_d \mu_d(\Omega))^{2/d}}\, m^{2/d} \qquad (m \to \infty), \tag{4.1}$$

where $\mu_d(\Omega)$ is the Lebesgue measure of Ω (its surface area in 2D and volume in 3D), and

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \tag{4.2}$$

is the volume of the unit ball in d dimensions ($\Gamma(z)$ being the Gamma function). As a consequence, plotting eigenvalues versus $m^{2/d}$ allows one to extract the area in 2D or the volume in 3D. This result can equivalently be written for the counting function $N(\lambda) = \#\{m \;:\; \lambda_m < \lambda\}$ (i.e., the number of eigenvalues smaller than λ):

$$N(\lambda) \propto \frac{\omega_d \mu_d(\Omega)}{(2\pi)^d}\, \lambda^{d/2} \qquad (\lambda \to \infty). \tag{4.3}$$

Weyl also conjectured the second asymptotic term which in two and three dimensions reads as

$$N(\lambda) \propto \begin{cases} \dfrac{\mu_2(\Omega)}{4\pi}\, \lambda \mp \dfrac{\mu_1(\partial\Omega)}{4\pi}\sqrt{\lambda} & (d = 2) \\ \dfrac{\mu_3(\Omega)}{6\pi^2}\, \lambda^{3/2} \mp \dfrac{\mu_2(\partial\Omega)}{16\pi}\lambda & (d = 3) \end{cases} \qquad (\lambda \to \infty), \tag{4.4}$$

where $\mu_2(\Omega)$ and $\mu_1(\partial\Omega)$ are the surface area and perimeter of Ω in 2D, $\mu_3(\Omega)$ and $\mu_2(\partial\Omega)$ are the volume and surface area of Ω in 3D, and sign "–" (resp. "+") refers to the Dirichlet (resp. Neumann) boundary condition. The correction terms which yield information about the boundary of the domain, were justified, under certain conditions on Ω (e.g., convexity) only in 1980 [208, 273] (see [7] for a historical review and further details).

Alternatively, one can study the heat trace (or partition function)

$$Z(t) \equiv \int\limits_{\Omega} dx\; G_t(x, x) = \sum_{m=1}^{\infty} e^{-\lambda_m t} = \int\limits_0^{\infty} e^{-\lambda t} dN(\lambda) \tag{4.5}$$

(here $G_t(x,y)$ is the heat kernel, cf. Eq. (2.6)), for which the following asymptotic expansion holds [67, 115, 155, 270, 275, 325]

$$Z(t) = (4\pi t)^{-d/2}\left(\sum_{k=0}^{K} c_k t^{k/2} + o(t^{(K+1)/2})\right) \qquad (t \to 0), \tag{4.6}$$

where the coefficients c_k are again related to the geometrical characteristics of the domain

$$c_0 = \mu_d(\Omega), \quad c_1 = -\frac{\sqrt{\pi}}{2}\mu_{d-1}(\partial\Omega), \quad ... \tag{4.7}$$

(see [335] for further discussion). Some estimates for the trace of Dirichlet Laplacian were given by Davies [116].

A number of extensions have been proposed. Berry conjectured that, for irregular boundaries, for which the Lebesgue measure in the correction term is infinite, the correction term should be $\lambda^{H/2}$ instead of $\lambda^{(d-1)/2}$, where H is the Hausdorff dimension of the boundary [51, 52]. However, Brossard and Carmona constructed a counter-example to this conjecture and suggested a modified version, in which the Hausdorff dimension was replaced by Minkowski dimension [68]. The modified Weyl-Berry conjecture discussed at length by Lapidus in [235] who proved it for $d = 1$ [236] (see these references for further discussion). For dimensions d higher than 1, this conjecture was disproved by Lapidus and Pomerance [238]. The correction term to the Weyl's formula for domains with rough boundary (in particular, from Lipschitz class) was studied by Netrusov and Safarov [285]. Levitin and Vassiliev also considered the asymptotic formulas for iterated sets with fractal boundary [250]. Extensions to various manifolds and higher order Laplacians were discussed [122, 123].

The high-frequency Weyl's law and the related short-time asymptotics of the heat kernel have been thoroughly investigated [7]. The dependence of these asymptotic laws on the volume and surface of the domain has found applications in physics. For instance, diffusion-weighted nuclear magnetic resonance experiments were proposed and conducted to estimate the surface-to-volume ratio of mineral samples and biological tissues [164, 184, 197, 240, 241, 277, 278, 358].

The multiplicity of eigenvalues is yet a more difficult problem [281]. From basic properties (see Sec. 2), the first eigenvalue λ_1 is simple. Cheng proved that the multiplicity $m(\lambda_2)$ of the second Dirichlet eigenvalue λ_2 is not greater than 3 [99]. This inequality is sharp since an example of domain with $m(\lambda_2) = 3$ was constructed. For $k \geq 3$, Hoffmann-Ostenhof *et al.* proved the inequality $m(\lambda_k) \leq 2k - 3$ [193, 194].

4.2. Isoperimetric inequalities for eigenvalues. In the low-frequency limit, the relation between the shape of a domain and the associated eigenvalues manifests in the form of isoperimetric inequalities. Since there are many excellent reviews on this topic, we only provide a list of most known inequalities, while further discussion and references can be found in [20, 21, 27, 47, 176, 186, 190, 233, 304, 315, 335].

(i) *The Rayleigh-Faber-Krahn inequality* states that the disk minimizes the first Dirichlet eigenvalue λ_1 among all domains of the same area $\mu_2(\Omega)$, i.e.

$$\lambda_1^D \geq \frac{\pi}{\mu_2(\Omega)}(j_{0,1})^2, \tag{4.8}$$

where $j_{\nu,1}$ is the first positive zero of $J_\nu(z)$ (e.g., $j_{0,1} \approx 2.4048...$). This inequality was conjectured by Lord Rayleigh and proven independently by Faber and Krahn

[138, 225]. The corresponding isoperimetric inequality in d dimensions,

$$\lambda_1^D \geq \left(\frac{\omega_d}{\mu_d(\Omega)}\right)^{2/d} (j_{\frac{d}{2}-1,1})^2, \tag{4.9}$$

was proven by Krahn [226].

Another lower bound for the first Dirichlet eigenvalue for a simply connected planar domain was obtained by Makai [263] and later rediscovered by Hayman [179]

$$\lambda_1^D \geq \frac{\alpha}{\rho^2} \tag{4.10}$$

where α is a constant, and

$$\rho = \max_{x\in\Omega} \min_{y\in\partial\Omega} \{|x-y|\} \tag{4.11}$$

is the inradius of Ω (i.e., the radius of the largest ball inscribed in Ω). The above inequality means that the lowest frequency (bass note) can be made arbitrarily small only if the domain includes an arbitrarily large circular drum (i.e., ρ goes to infinity). The constant α, which was equal to $1/900$ in the Hayman's original proof, was gradually improved, ranging from the value $\alpha = 1/4$ obtained by Osserman [296] to the best value (up to date) $\alpha = 0.6197...$ by Banuelos and Carroll [30]. For convex domains, the lower bound (4.10) with $\alpha = \pi^2/4 \approx 2.4674$ was derived much earlier by Hersch [188], with the equality if and only if Ω is an infinite strip.

An obvious upper bound for the first Dirichlet eigenvalue can be obtained from the domain monotonicity (property (v) in Sec. 2):

$$\lambda_1^D \leq \lambda_1^D(B_\rho) = \rho^{-2}\; j^2_{\frac{d}{2}-1,1}, \tag{4.12}$$

with the first Dirichlet eigenvalue $\lambda_1^D(B_\rho)$ for the largest ball B_ρ inscribed in Ω (here, ρ is the inradius). However, this upper bound is not accurate in general. Pólya and Szegö gave another upper bound for planar star-shaped domains [315]. Freitas and Krejčiřík extended their result to higher dimensions [147]: for a bounded strictly star-shaped domain $\Omega \subset \mathbb{R}^d$ with locally Lipschitz boundary, they proved

$$\lambda_1^D \leq \lambda_1^D(B_1) \frac{F(\Omega)}{d\ \mu_d(\Omega)}, \tag{4.13}$$

where the function $F(\Omega)$ is defined in [147]. From this inequality, they also deduced a weaker but more explicit upper bound which is applicable to any bounded convex domain in $\mathbb{R}^d$:

$$\lambda_1^D \leq \lambda_1^D(B_1)\ \frac{\mu_{d-1}(\partial\Omega)}{d\ \rho\ \mu_d(\Omega)}. \tag{4.14}$$

The second Dirichlet eigenvalue λ_2^D is minimized by the union of two identical balls (see [318]). Note that finding the minimizer among convex planar sets is still an open problem [187]. Bucur and Henrot proved the existence of a minimizer for the third eigenvalue in the family of domains in $\mathbb{R}^d$ of given volume, although its shape remains unknown [71]. The range of the first two eigenvalues was also investigated [399, 400].

The first nontrivial Neumann eigenvalue λ_2^N (as $\lambda_1^N = 0$) also satisfies the isoperimetric inequality

$$\lambda_2^N \le \left(\frac{\omega_d}{\mu_d(\Omega)}\right)^{2/d} (\tilde{j}_{\frac{d}{2},1})^2, \tag{4.15}$$

which states that λ_2^N is maximized by a d-dimensional ball (here $\tilde{j}_{\nu,1}$ is the first positive zero of $J'_\nu(z)$). This inequality was proven for simply-connected planar domains by Szegö [376] and in higher dimensions by Weinberger [394]. Pólya conjectured the following upper bound for all Neumann eigenvalues [317] in planar bounded regular domains (see also [354])

$$\lambda_n^N \le \frac{4(n-1)\pi}{\mu_2(\Omega)} \qquad (n = 2, 3, 4, ...). \tag{4.16}$$

This inequality is true for all domains that tile the plane, e.g., for any triangle and any quadrilateral [319]. For $n = 2$, the inequality (4.16) follows from (4.15). For $n \ge 3$, Pólya's conjecture is still open, although Kröger proved a weaker estimate $\lambda_n^N \le 8\pi(n-1)$ [228]. Recently, Girouard *et al.* obtained a sharp upper bound for the second nontrivial Neumann eigenvalue λ_3^N for a regular simply-connected planar domain [156]:

$$\lambda_3^N \le \frac{2\pi(\tilde{j}_{0,1})^2}{\mu_2(\Omega)}, \tag{4.17}$$

with the equality attained in the limit by a family of domains degenerating to a disjoint union of two identical disks (the domain is called regular if its Neumann eigenspectrum is discrete, see [156] for details).

Payne and Weinberger obtained the lower bound for the second Neumann eigenvalue in d dimensions [303]

$$\lambda_2^N \ge \frac{\pi^2}{\delta^2}, \tag{4.18}$$

where δ is the diameter of Ω:

$$\delta = \max_{x,y\in\partial\Omega} \{|x-y|\}. \tag{4.19}$$

This is the best bound that can be given in terms of the diameter alone in the sense that $\lambda_2^N \delta^2$ tends to π^2 for a parallelepiped all but one of whose dimensions shrink to zero.

Szegö and Weinberger noticed that Szegö's proof of the inequality (4.15) for planar simply connected domains extends to prove the bound

$$\frac{1}{\lambda_2^N} + \frac{1}{\lambda_3^N} \ge \frac{2\mu_2(\Omega)}{\pi(\tilde{j}_{1,1})^2}, \tag{4.20}$$

with equality if and only if Ω is a disk [376, 394]. Ashbaugh and Benguria derived another bound for arbitrary bounded domains in $\mathbb{R}^d$ [18]

$$\frac{1}{\lambda_2^N} + ... + \frac{1}{\lambda_{d+1}^N} \ge \frac{d}{d+2}\left(\frac{\mu_d(\Omega)}{\omega_d}\right)^{2/d} \tag{4.21}$$

In particular, one gets $1/\lambda_2^N + 1/\lambda_3^N \ge \frac{\mu_2(\Omega)}{2\pi}$ for $d = 2$ (see also extensions in [191, 401]).

(ii) *The Payne-Pólya-Weinberger inequality* concerns the ratio between first two Dirichlet eigenvalues and states that

$$\frac{\lambda_2^D}{\lambda_1^D} \le \left(\frac{j_{\frac{d}{2},1}}{j_{\frac{d}{2}-1,1}} \right)^2, \tag{4.22}$$

with equality if and only if Ω is the d-dimensional ball. This inequality (in 2D form) was conjectured by Payne, Pólya and Weinberger [302] and proved by Ashbaugh and Benguria in 1990 [16–19]. A weaker estimate $\lambda_2^D/\lambda_1^D \le 1 + 4/d$ was proved for $d = 2$ in the original paper by Payne, Pólya and Weinberger [302].

(iii) Singer *et al.* derived the upper and lower estimates for the spectral gap between the first two Dirichlet eigenvalues for a smooth convex bounded domain Ω in $\mathbb{R}^d$ (in fact, they considered a more general problem in the presence of a potential):

$$\frac{d\pi^2}{\rho^2} \ge \lambda_2^D - \lambda_1^D \ge \frac{\pi^2}{4\delta^2}, \tag{4.23}$$

where δ is the diameter of Ω and ρ is the inradius [365]. For a convex planar domain, Donnelly proved a sharper lower estimate [128]

$$\lambda_2^D - \lambda_1^D \ge \frac{3\pi^2}{\delta^2}. \tag{4.24}$$

(iv) The isoperimetric inequalities for Robin eigenvalues are less known. Daners proved that among all bounded domains $\Omega \subset \mathbb{R}^d$ of the same volume, the ball B minimizes the first Robin eigenvalue [72, 113]

$$\lambda_1^R(\Omega) \ge \lambda_1^R(B). \tag{4.25}$$

Kennedy showed that among all bounded domains in $\mathbb{R}^d$, a domain B_2 composed of two disjoint balls minimizes the second Robin eigenvalue [218]

$$\lambda_2^R(\Omega) \ge \lambda_2^R(B_2). \tag{4.26}$$

(v) The minimax principle ensures that the Neumann eigenvalues are always smaller than the corresponding Dirichlet eigenvalues: $\lambda_n^N \le \lambda_n^D$. Pólya proved $\lambda_2^N < \lambda_1^D$ [316] while Szegö got a sharper inequality $\lambda_2^N \le c\lambda_1^D$ for a planar domain bounded by an analytic curve, where $c = (\tilde{j}_{1,1}/j_{0,1})^2 \approx 0.5862...$ [376] (note that this result also follows from inequalities (4.8, 4.15)). Payne derived a stronger inequality for a planar domain with a C^2 boundary: $\lambda_{n+2}^N < \lambda_n^D$ for all n [301]. Levine and Weinberger generalized its result for higher dimensions d and proved that $\lambda_{n+d}^N < \lambda_n^D$ for all n when Ω is smooth and convex, and that $\lambda_{n+d}^N \le \lambda_n^D$ if Ω is merely convex [249]. Friedlander proved the inequality $\lambda_{n+1}^N \le \lambda_n^D$ for a general bounded domain with a C^1 boundary [149]. Filonov found a simpler proof of this inequality in a more general situation (see [144] for details).

Many other inequalities can be found in several reviews [20, 21, 47]. It is worth noting that isoperimetric inequalities are related to shape optimization problems [5, 65, 84–86, 311, 367].

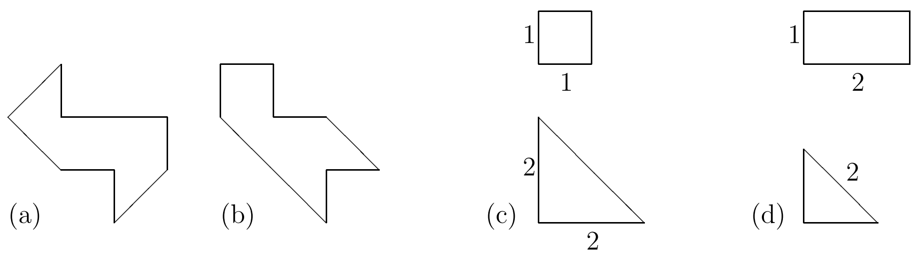

FIG. 4.1. *Two examples of nonisometric domains with the identical Laplace operator eigenspectra (with Dirichlet or Neumann boundary conditions): the original example (shapes 'a' and 'b') constructed by Gordon et al. [159], and a simpler example with disconnected domains (shapes 'c' and 'd') by Chapman [95]. In the latter case, the eigenspectrum is simply obtained as the union of the eigenspectra of two subdomains known explicitly. For instance, the Dirichlet eigenspectrum is* $\{\pi^2(m^2+n^2) \ : \ m,n \in \mathbb{N}\} \cup \{\pi^2((i/2)^2+(j/2)^2) \ : \ i,j \in \mathbb{N}, \ i > j\}$.

4.3. Kac's inverse spectral problem. The problem of finding relations between the Laplacian eigenspectrum and the shape of a domain was formulated in the famous Kac's question "Can one hear the shape of a drum?" [209]. In fact, the drum's frequencies are uniquely determined by the eigenvalues of the Laplace operator in the domain of drum's shape. By definition, the shape of the domain fully determines the Laplacian eigenspectrum. Is the opposite true, i.e., does the set of eigenvalues which appear as "fingerprints" of the shape, uniquely identify the domain? The negative answer to this question for general planar domains was given by Gordon and co-authors [159] who constructed two different (nonisometric) planar polygons (Fig. 4.1a,b) with the identical Laplacian eigenspectra, both for Dirichlet and Neumann boundary conditions (see also [48]). Their construction was based on Sunada's paper on isospectral manifolds [375]. An elementary proof, as well as many other examples of isospectral domains, were provided by Buser and co-workers [83] and by Chapman [95] (see Fig. 4.1c,d). An experimental evidence for this not "hearing the shape" of drums was brought by Sridhar and Kudrolli [370]. In all these examples, isospectral domains are either non-convex, or disjoint. Gordon and Webb addressed the question of existence of isospectral convex connected domains and answered this question positively for domains in Euclidean spaces of dimension $d \geq 4$ [160]. To our knowledge, this question remains open for convex domains in two and three dimensions, as well as for domains with smooth boundaries.

A somewhat similar problem was recently formulated for domains in which one part of the boundary admits Dirichlet boundary condition and the other Neumann boundary condition. Does the spectrum of the Laplace operator determine uniquely which condition is imposed on which part? Jakobsen and co-workers gave the negative answer to this question by assigning Dirichlet and Neumann conditions onto different parts of the boundary of the half-disk (and some other domains), in a way to produce the same eigenspectra [200].

The Kac's inverse spectral problem can also be seen from a different point of view. For a given sequence $0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$, whether does exist a domain Ω in $\mathbb{R}^d$ for which the Laplace operator with Dirichlet or Neumann boundary condition has the spectrum given by this sequence. A similar problem can be formulated for a compact Riemannian manifold with arbitrary Riemannian metrics. Colin de Verdière studied these problems for finite sequences $\{\lambda_n\}_{n=1}^N$ and proved the existence of such domains or manifolds under certain restrictions [106].

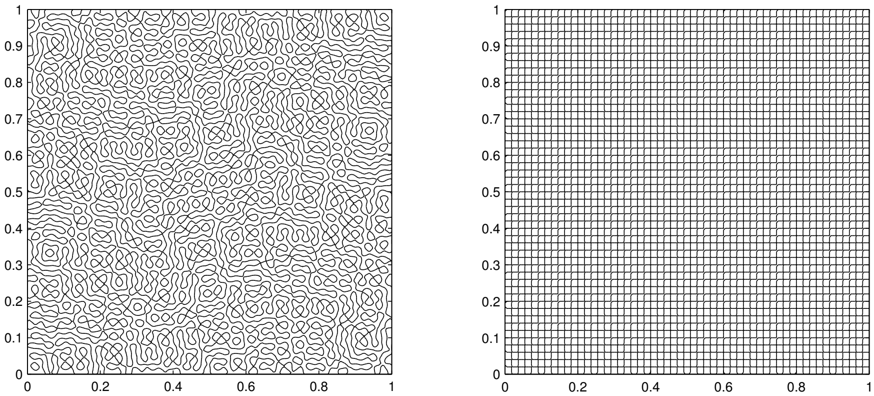

FIG. 5.1. *The nodal lines of a Dirichlet eigenfunction* $u(x_1, x_2)$ *on the unit square, with the associated eigenvalue* $\lambda = 5525\pi^2$ *of multiplicity 12. The eigenfunction was obtained as a linear combination of terms* $\sin(\pi n_1 x_1)\sin(\pi n_2 x_2)$*, with* $n_1^2 + n_2^2 = 5525$ *and randomly chosen coefficients. For comparison, another eigenfunction with the same eigenvalue,* $\sin(50\pi x_1)\sin(55\pi x_2)$*, is shown.*

5. Nodal lines. The first insight onto the geometrical structure of eigenfunctions can be gained from their nodal lines. Kuttler and Sigillito gave a brief overview of the basic properties of nodal lines for Dirichlet eigenfunctions in two dimensions [233] that we partly reproduce here:

"The set of points in Ω where $u_m = 0$ is the nodal set of u_m. By the unique continuation property, it consists of curves that are C^∞ in the interior of Ω. Where nodal lines cross, they form equal angles [107]. Also, when nodal lines intersect a C^∞ portion of the boundary, they form equal angles. Thus, a single nodal line intersects the C^∞ boundary at right angles, two intersect it at 60° angles, and so forth. Courant's nodal line theorem [107] states that the nodal lines of the m-th eigenfunction divide Ω into no more than m subregions (called nodal domains): $\nu_m \leq m$, ν_m being the number of nodal domains. In particular, u_1 has no interior nodes and so λ_1 is a simple eigenvalue (has multiplicity one)."

It is worth noting that any eigenvalue λ_m of the Dirichlet-Laplace operator in Ω is the first eigenvalue for each of its nodal domains. This simple observation allows one to construct specific domains with a prescribed eigenvalue (see [233] for examples). Eigenfunctions with few nodal domains were constructed in [107, 251].

Even for such a simple domain as a square, the nodal lines and domains may have complicated structure, especially for high-frequency eigenfunctions (Fig. 5.1). This is particularly true for degenerate eigenfunctions for which one can "tune" the coefficients of the corresponding linear combination to modify continuously the nodal lines.

Pleijel sharpened the Courant's theorem by showing that the upper bound m for the number ν_m of nodal domains is attained only for a finite number of eigenfunctions [313]. Moreover, he obtained the upper limit: $\overline{\lim\limits_{m\to\infty}}\, \nu_m/m = 4/j_{0,1}^2 \approx 0.691...$. Note that Lewy constructed spherical harmonics of any degree n whose nodal sets have one component for odd n and two components for even n implying that no non-trivial lower bound for ν_m is possible [251].

Blum *et al.* considered the distribution of the (properly normalized) number of nodal domains of the Dirichlet-Laplacian eigenfunctions in 2D quantum billiards and showed the existence of the limiting distribution in the high-frequency limit (i.e., when $\lambda_m \to \infty$) [61]. These distributions were argued to be universal for systems with integrable or chaotic underlying classical dynamics that allows one to distinguish them and thus provides a new criterion for quantum chaos (see Sec. 7.7.4). It was also conjectured that the distribution for chaotic systems coincides with the distribution of nodal domains for Gaussian random functions.

Bogomolny and Schmit proposed a percolation-like model to describe the nodal domains which permitted to perform analytical calculations and agreed well with numerical simulations [63]. This model allows one to apply ideas and methods developed within the percolation theory [371] to the field of quantum chaos. Using the analogy with Gaussian random functions, Bogomolny and Schmit obtained that the mean and variance of the number ν_m of nodal domains grow as m, with explicit formulas for the prefactors. From the percolation theory, the distribution of the area s of the connected nodal domains was conjectured to follow a power law, $n(s) \propto s^{-187/91}$, as confirmed by simulations [63]. In the particular case of random Gaussian spherical harmonics, Nazarov and Sodin rigorously derived the asymptotic behavior for the number ν_n of nodal domains of the harmonic of degree n [284]. They proved that as n grows to infinity, the mean of ν_n/n^2 tends to a positive constant, and that ν_n/n^2 exponentially concentrates around this constant (we recall that the associate eigenvalue is $n(n+1)$).

The geometrical structure of nodal lines and domains has been intensively studied (see [283, 314] for further discussion of the asymptotic nodal geometry). For instance, the length of the nodal line of an eigenfunction of the Laplace operator in two-dimensional Riemannian manifolds was separately investigated by Brüning, Yao and Nadirashvili who obtained its lower and upper bounds [70, 280, 402]. In addition, a number of conjectures about the properties of particular eigenfunctions were discussed in the literature. We mention three of them:

(i) In 1967, Payne conjectured that the second Dirichlet eigenfunction u_2 cannot have a closed nodal line in a bounded planar domain [304, 308]. This conjecture was proved for convex domains [4, 272] and disproved by non-convex domains [192], see also [167, 201].

(ii) The hot spots conjecture formulated by J. Rauch in 1974 says that the maximum of the second Neumann eigenfunction is attained at a boundary point. This conjecture was proved by Banuelos and Burdzy for a class of planar domains [32] but in general the statement is wrong, as shown by several counter-examples [44, 79, 81, 202].

(iii) Liboff formulated several conjectures; one of them states that the nodal surface of the first-excited state of a 3D convex domain intersects its boundary in a single simple closed curve [253].

The analysis of nodal lines that describe zeros of eigenfunctions, can be extended to other level sets. For instance, a level set of the first Dirichlet eigenfunction u_1 on a bounded convex domain $\Omega \in \mathbb{R}^d$ is itself convex [214]. Grieser and Jerison estimated the size of the first eigenfunction uniformly for all convex domains [168]. In particular, they located the place where u_1 achieves its maximum to within a distance comparable to the inradius, uniformly for arbitrarily large diameter. Other geometrical characteristics (e.g., the volume of a set on which an eigenfunction is positive) can also be analyzed [282].

6. Estimates for Laplacian eigenfunctions. The "amplitudes" of eigenfunctions can be characterized either globally by their L_p norms

$$\|u\|_p \equiv \left(\int\limits_\Omega dx\ |u(x)|^p\right)^{1/p} \qquad (p \ge 1), \tag{6.1}$$

or locally by pointwise estimates. Since eigenfunctions are defined up to a multiplicative constant, one often uses $L_2(\Omega)$ normalization: $\|u\|_2 = 1$. Note also the limiting case of L_∞-norm

$$\|u\|_\infty = \max_{x\in\Omega} |u(x)|. \tag{6.2}$$

It is worth recalling the Hölder's inequality for any two measurable functions u and v and for any positive p, q such that $1/p + 1/q = 1$:

$$\|uv\|_1 \le \|u\|_p \|v\|_q. \tag{6.3}$$

In addition, for a bounded domain $\Omega \subset \mathbb{R}^d$ (with a finite Lebesgue measure $\mu_d(\Omega)$), the Jensen's inequality for convex functions yields

$$\|u\|_p \le [\mu_d(\Omega)]^{\frac{1}{p}-\frac{1}{p'}} \|u\|_{p'} \qquad (1 \le p \le p'). \tag{6.4}$$

6.1. First (ground) Dirichlet eigenfunction. The Dirichlet eigenfunction u_1 associated with the first eigenvalue $\lambda_1 > 0$ does not change the sign in Ω and may be taken to be positive. It satisfies the following inequalities.

(i) Payne and Rayner showed in two dimensions that

$$\|u_1\|_2 \le \frac{\sqrt{\lambda_1}}{\sqrt{4\pi}}\ \|u_1\|_1, \tag{6.5}$$

with equality if and only if Ω is a disk [305, 306]. Kohler-Jobin gave an extension of this inequality to higher dimensions [222] (see [103, 223, 306] for other extensions):

$$\|u_1\|_2 \le \frac{\lambda_1^{d/4}}{\sqrt{2d\omega_d [j_{\frac{d}{2}-1,1}]^{d-2}}}\ \|u_1\|_1. \tag{6.6}$$

(ii) Payne and Stakgold derived two inequalities for a convex domain in 2D

$$\frac{\pi}{2\mu_2(\Omega)} \|u_1\|_1 \le \|u_1\|_\infty \tag{6.7}$$

and

$$u_1(x) \le |x - \partial\Omega| \frac{\sqrt{\lambda_1}}{\mu_2(\Omega)} \|u_1\|_1 \qquad (x \in \Omega), \tag{6.8}$$

where $|x - \partial\Omega|$ is the distance from a point x in Ω to the boundary $\partial\Omega$ [307].

(iii) Van Den Berg proved the following inequality for L_2-normalized eigenfunction u_1 when Ω is an open, bounded and connected set in $\mathbb{R}^d$ ($d = 2, 3, \dots$):

$$\|u_1\|_\infty \le \frac{2^{\frac{2-d}{2}}}{\pi^{d/4}\sqrt{\Gamma(d/2)}}\ \frac{\left(j_{\frac{d-2}{2},1}\right)^{\frac{d-2}{2}}}{|J_{\frac{d}{2}}\left(j_{\frac{d-2}{2},1}\right)|}\ \rho^{-d/2}, \tag{6.9}$$

with equality if and only if Ω is a ball, where ρ is the inradius (Eq. (4.11)) [389]. Van Den Berg also conjectured the stronger inequality for an open bounded convex domain $\Omega \subset \mathbb{R}^d$:

$$\|u_1\|_\infty \le C_d \rho^{-d/2} (\rho/\delta)^{1/6}, \tag{6.10}$$

where δ is the diameter of Ω, and C_d is a universal constant independent of Ω.

(iv) Pang investigated how the first Dirichlet eigenvalue and eigenfunction would change when the domain slightly shrinks [297, 298]. For a bounded simply connected open set $\Omega \subset \mathbb{R}^2$, let

$$\Omega_\epsilon \supseteq \{x \in \Omega : |x - \partial\Omega| \ge \epsilon\}$$

be its interior, i.e., Ω without an ϵ boundary layer. Then the Dirichlet eigenvalues λ_m^ϵ and L_2-normalized eigenfunctions u_m^ϵ in Ω_ϵ (with $\lambda_m^0 = \lambda_m$ and $u_m^0 = u_m$ referring to the original domain Ω) satisfy, for all $\epsilon \in (0, \rho/2)$,

$$\begin{aligned} |\lambda_1^\epsilon - \lambda_1| &\le C_1 \epsilon^{1/2}, \\ \|u_1 - T_\epsilon u_1^\epsilon\|_{L_\infty(\Omega)} &\le \Big[C_2 + C_3(\lambda_2 - \lambda_1)^{-1/2} + C_4(\lambda_2 - \lambda_1)^{-1}\Big]\, \epsilon^{1/2}, \end{aligned} \tag{6.11}$$

where ρ is the inradius of Ω (Eq. (4.11)), T_ϵ is the extension operator from Ω_ϵ to Ω, and

$$\begin{aligned} C_1 &= \rho^{-3/2} \beta^{9/4}\, \frac{2^9 \gamma_1^4}{3\pi^{9/4}}, \qquad C_2 = \rho^{-3/2}\beta^{13/4}\, \frac{2^{12}\gamma_1^5}{\pi^{15/4}}, \\ C_3 &= \rho^{-5/2}\beta^4 \left(\frac{2^{15}\gamma_1^6 \gamma_2}{3\sqrt{2\alpha}\pi^{9/2}}\right)\left[1 + \frac{9\gamma_1}{\pi^{3/4}}\beta^{3/4}\right], \\ C_4 &= \rho^{-7/2}\beta^7 \left(\frac{2^{26}\gamma_1^{10}\gamma_2^2}{81\sqrt{2}\ \alpha\ \pi^{15/2}}\right)\left[1 + 18\gamma_1\beta^{3/4} + \frac{81\gamma_1^2}{\pi^{3/2}}\beta^{3/2}\right], \end{aligned}$$

where $\beta = \mu_2(\Omega)/\rho^2$, α is the constant from Eq. (4.10) (for which one can use the best known estimate $\alpha = 0.6197...$ from [30]), and γ_1 and γ_2 are the first and second Dirichlet eigenvalues for the unit disk: $\gamma_1 = j_{0,1}^2 \approx 5.7832$ and $\gamma_2 = j_{1,1}^2 \approx 14.6820$. Moreover, when Ω is the cardioid in $\mathbb{R}^2$, the term $\epsilon^{1/2}$ cannot be improved.[1]

In addition, Davies proved that for a bounded simply connected open set $\Omega \in \mathbb{R}^2$ and for any $\beta \in (0, 1/2)$, there exists $c = c(\beta) \ge 1$ such that [117]

$$|\lambda_1^\epsilon - \lambda_1| \le c\epsilon^\beta \tag{6.12}$$

for all sufficiently small $\epsilon > 0$. Moreover, the estimate also holds for higher Dirichlet eigenvalues.

6.2. Estimates applicable for all eigenfunctions.

6.2.1. Estimates through the Green function. Using the spectral decomposition (2.5) of the Green function $G(x, y)$, one can rewrite Eq. (1.1) as

$$u_m(x) = \lambda_m \int_\Omega G(x, y) u_m(y) dy,$$

[1] In the original paper [298], the coefficient C_4 in Eq. (1.5) should be multiplied by the omitted prefactor $\sqrt{2}|\Omega|$ that follows from the derivation.

from which the Hölder inequality (6.3) yields a family of simple pointwise estimates

$$|u_m(x)| \le \lambda_m \|u_m\|_{\frac{p}{p-1}} \left(\int\limits_\Omega |G(x,y)|^p dy \right)^{1/p}, \tag{6.13}$$

with any $p \ge 1$. Here, a single function of x in the right-hand side bounds all the eigenfunctions. In particular, for $p = 1$, one gets

$$|u_m(x)| \le \lambda_m \|u_m\|_\infty \int\limits_\Omega |G(x,y)| dy. \tag{6.14}$$

For Dirichlet boundary condition, $G(x,y)$ is positive everywhere in Ω so that

$$|u_m(x)| \le \lambda_m \|u_m\|_\infty \, U(x), \qquad U(x) = \int\limits_\Omega G(x,y) dy, \tag{6.15}$$

where $U(x)$ solves the boundary value problem

$$-\Delta U(x) = 1 \quad (x \in \Omega), \qquad U(x) = 0 \quad (x \in \partial\Omega). \tag{6.16}$$

The solution of this equation is known to be the mean first passage time to the boundary $\partial\Omega$ from an interior point x [332]. The inequalities (6.14, 6.15) (or their extensions) were reported by Moler and Payne [279] (Sect. 6.2.2) and were used by Filoche and Mayboroda for determining the geometrical structure of eigenfunctions [143] (Sect. 6.2.6).

6.2.2. Bounds for eigenvalues and eigenfunctions of symmetric operators. Moler and Payne derived simple bounds for eigenvalues and eigenfunctions of symmetric operators by considering their extensions [279]. As a typical example, one can think of the Dirichlet-Laplace operator in a bounded domain Ω (symmetric operator A) and of the Laplace operator without boundary conditions (extension A_*). Let λ_* and u_* be an approximate eigenvalue and eigenfunction of A that are obtained by solving a simpler eigenvalue problem $A_* u_* = \lambda_* u_*$ without boundary condition. If there exists a function w such that $A_* w = 0$ and $w = u_*$ at the boundary of Ω and if $\varepsilon = \dfrac{\|w\|_{L_2(\Omega)}}{\|u_*\|_{L_2(\Omega)}} < 1$, then there exists an eigenvalue λ_k of A satisfying

$$\frac{\lambda_*}{1+\varepsilon} \le |\lambda_k| \le \frac{|\lambda_*|}{1-\varepsilon}. \tag{6.17}$$

Moreover, if $\|u_*\|_{L_2(\Omega)} = 1$ and u_k is the L_2-normalized projection of u_* onto the eigenspace of λ_k, then

$$\|u_* - u_k\|_{L_2(\Omega)} \le \frac{\varepsilon}{\alpha} \left(1 + \frac{\varepsilon^2}{\alpha^2} \right)^{1/2}, \tag{6.18}$$

where $\alpha = \min\limits_{\lambda_n \ne \lambda_k} \frac{|\lambda_n - \lambda_*|}{|\lambda_n|}$.

If u_* is a good approximation to an eigenfunction of the Dirichlet-Laplace operator, then it must be close to zero on the boundary of Ω, yielding small ε and thus accurate lower and upper bounds in (6.17). The accuracy of the bound (6.18) also depends on the separation α between eigenvalues.

In the same work, Moler and Payne also provided pointwise bounds for eigenfunctions that rely on Green's functions (an extension of Sec. 6.2.1).

6.2.3. Estimates for L_p-norms. Chiti extended the Payne-Rayner's inequality (6.5) to the eigenfunctions of linear elliptic second order operators in divergent form, with Dirichlet boundary condition [103]. For the Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^d$, Chiti's inequality for any real numbers $q \geq p > 0$ states:

$$\|u\|_q \leq \|u\|_p \, (d\omega_d)^{\frac{1}{q}-\frac{1}{p}} \lambda^{\frac{q-p}{2pq}d} \frac{\left(\int\limits_0^{j_{\frac{d}{2}-1,1}} dr \; r^{d-1+q(1-d/2)} [J_{\frac{d}{2}-1}(r)]^q \right)^{1/q}}{\left(\int\limits_0^{j_{\frac{d}{2}-1,1}} dr \; r^{d-1+p(1-d/2)} [J_{\frac{d}{2}-1}(r)]^p \right)^{1/p}}. \tag{6.19}$$

6.2.4. Pointwise bounds for Dirichlet eigenfunctions. Banuelos derived a pointwise upper bound for L_2-normalized Dirichlet eigenfunctions [31]

$$|u_m(x)| \leq \lambda_m^{d/4} \qquad (x \in \Omega). \tag{6.20}$$

Van Den Berg and Bolthausen proved the following estimates for L_2-normalized Dirichlet eigenfunctions [387]. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3, \dots)$ be an open bounded domain with boundary $\partial\Omega$ which satisfies an α-uniform capacitary density condition with some $\alpha \in (0, 1]$, i.e.

$$\mathrm{Cap}\{\partial\Omega \cap B(x; r)\} \geq \alpha \mathrm{Cap}\{B(x; r)\}, \quad x \in \partial\Omega, \; 0 < r < \delta, \tag{6.21}$$

where $B(x, r)$ is the ball of radius r centered at x, δ is the diameter of Ω (Eq. (4.19)), and Cap is the logarithmic capacity for $d = 2$ and the Newtonian capacity for $d > 2$. This condition guarantees that all points of $\partial\Omega$ are regular. The following estimates hold

(i) in two dimensions ($d = 2$), for all $m = 1, 2, \dots$ and all $x \in \Omega$ such that $|x - \partial\Omega|\sqrt{\lambda_m} < 1$, one has

$$|u_m(x)| \leq \left\{ \frac{6\lambda_m \ln(\alpha^{2\pi}/2)}{\ln\left(|x - \partial\Omega|\sqrt{\lambda_m}\right)} \right\}^{1/2}. \tag{6.22}$$

(ii) in higher dimensions ($d > 2$), for all $m = 1, 2, \dots$ and all $x \in \Omega$ such that

$$|x - \partial\Omega|\sqrt{\lambda_m} \leq \left(\frac{\alpha^6}{2^{13}} \right)^{1+\gamma(d-1)/(d-2)}, \tag{6.23}$$

with $\gamma = \dfrac{3^{-d-1}\alpha}{\ln(2(2/\alpha)^{1/(d-2)})}$, one has

$$|u_m(x)| \leq 2\lambda_m^{d/4} \left(|x - \partial\Omega|\sqrt{\lambda_m} \right)^{\frac{1}{2}((1/\gamma)+(d-1)/(d-2))^{-1}}. \tag{6.24}$$

(iii) for a planar simply connected domain and all $m = 1, 2, \dots,$

$$|u_m(x)| \leq m \; 2^{9/2}\pi^{1/4} \frac{(\mu_2(\Omega))^{1/4}}{\rho^2} \; |x - \partial\Omega|^{1/2} \qquad (x \in \Omega), \tag{6.25}$$

where ρ is the inradius of Ω (see Eq. (4.11)), and the inequality is sharp.

6.2.5. Upper and lower bounds for normal derivatives of Dirichlet eigenfunctions. Suppose that M is a compact Riemannian manifold with boundary and u is an L_2-normalized Dirichlet eigenfunction with eigenvalue λ. Let ψ be its normal derivative at the boundary. A scaling argument suggests that the L_2-norm of ψ will grow as $\sqrt{\lambda}$ as $\lambda \to \infty$. Hassell and Tao proved that

$$c\sqrt{\lambda} \le \|\psi\|_{L_2(\partial M)} \le C\sqrt{\lambda}, \tag{6.26}$$

where the upper bound holds for any Riemannian manifold, while the lower bound is valid provided that M has no trapped geodesics [177]. The positive constants c and C depend on M, but not on λ.

6.2.6. Estimates for restriction onto a subdomain. For a bounded domain $\Omega \subset \mathbb{R}^d$, Filoche and Mayboroda have obtained the upper bound for the L_2-norm of a Dirichlet-Laplacian eigenfunction u associated to λ, in any open subset $D \subset \Omega$ [143]:

$$\|u\|_{L_2(D)} \le \left(1 + \frac{\lambda}{d_D(\lambda)}\right) \|v\|_{L_2(D)}, \tag{6.27}$$

where the function v solves the boundary value problem in D:

$$\Delta v = 0 \quad (x \in D), \qquad v = u \quad (x \in \partial D).$$

and $d_D(\lambda)$ is the distance from λ to the spectrum of the Dirichlet-Laplace operator in D. Note also that the above bound was proved for general self-adjoint elliptic operators [143]. When combined with Eq. (6.15), this inequality helps to investigate the spatial distribution of eigenfunctions because it is in general much easier to compute or estimate the harmonic function v than the eigenfunction u.

The above estimate can be completed by a lower bound (see Appendix A):

$$\|u\|_{L_2(D)} \ge \frac{\lambda_1(D)}{\lambda + \lambda_1(D)} \|v\|_{L_2(D)}, \tag{6.28}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of the subdomain D.

7. Localization of eigenfunctions. “Localization” is defined in the Webster’s dictionary as “act of localizing, or state of being localized”. The notion of localization appears in various fields of science and often has different meanings. Throughout this review, a function u defined on a domain $\Omega \subset \mathbb{R}^d$, is called L_p-*localized* (for $p \ge 1$) if there exists a bounded subset $\Omega_0 \subset \Omega$ which supports almost all L_p-norm of u, i.e.

$$\frac{\|u\|_{L_p(\Omega\backslash\Omega_0)}}{\|u\|_{L_p(\Omega)}} \ll 1 \qquad \text{and} \qquad \frac{\mu_d(\Omega_0)}{\mu_d(\Omega)} \ll 1. \tag{7.1}$$

Qualitatively, a localized function essentially “lives” on a small subset of the domain and takes small values on the remaining part. For instance, a Gaussian function $\exp(-x^2)$ on $\Omega = \mathbb{R}$ is L_p-localized for any $p \ge 1$ since one can choose $\Omega_0 = [-a, a]$ with large enough a so that the ratio of L_p-norms can be made arbitrarily small, while the ratio of lengths $\mu_1(\Omega_0)/\mu_1(\Omega)$ is strictly 0. In turn, when $\Omega = [-A, A]$, the localization character of $\exp(-x^2)$ becomes conventional and dependent on A. This example illustrates that the above inequalities do not provide a universal quantitative criterion to distinguish between localized and non-localized (or extended) functions.

In this Section, we will describe various kinds of localization for which some quantitative criteria can be formulated. We will also illustrate that the choice of the norm (i.e., p) may be important.

Another "definition" of localization was given by Felix *et al.* who combined L_2 and L_4 norms to define the "existence area" as [139]

$$S(u) = \frac{\|u\|^4_{L_2(\Omega)}}{\|u\|^4_{L_4(\Omega)}}. \tag{7.2}$$

A function u was called localized when its existence area $S(u)$ was much smaller than the area $\mu_2(\Omega)$ [139] (this definition trivially extends to higher dimensions). In fact, if a function is small in a subdomain, the fourth power diminishes it stronger than the second power. For instance, if $\Omega = (0,1)$ and u is 1 on the subinterval $\Omega_0 = (1/4, 1/2)$ and 0 otherwise, one has $\|u\|_{L_2(\Omega)} = \|u\|_{L_4(\Omega)} = 1/2$ so that $S(u) = 1/4$, i.e. the length of the subinterval Ω_0. Once again, the smallness of $S(u)/\mu_2(\Omega)$ is conventional. Note that a family of "existence areas" can be constructed by comparing L_p and L_q norms,

$$S_{p,q}(u) = \left(\frac{\|u\|_{L_p(\Omega)}}{\|u\|_{L_q(\Omega)}}\right)^{\frac{1}{\frac{1}{p}-\frac{1}{q}}}. \tag{7.3}$$

7.1. Bound quantum states in a potential. The notion of bound, trapped or localized quantum states is known for a long time [59, 333]. The simplest "canonical" example is the quantum harmonic oscillator, i.e. a particle of mass m in a harmonic potential of frequency ω which is described by the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2, \tag{7.4}$$

where $\hat{p} = -i\hbar\partial_x$ is the momentum operator, and $\hat{x} = x$ is the position operator ($\hbar$ being the Planck's constant). The eigenfunctions of this operator are well known:

$$\psi_n(x) = \sqrt{\frac{1}{2^n\ n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)H_n\big(\sqrt{m\omega/\hbar}x\big), \tag{7.5}$$

where $H_n(x)$ are the Hermite polynomials. All these functions are localized in a region around the minimum of the harmonic potential (here, $x = 0$), and rapidly decay outside this region. For this example, the definition (7.1) of localization is rigorous. In physical terms, the presence of a strong potential forbids the particle to travel far from the origin, the size of the localization region being $\sqrt{\hbar/(m\omega)}$. This so-called strong localization has been thoroughly investigated in physics and mathematical physics [3, 264, 265, 295, 333, 353, 357, 361].

7.2. Anderson localization. The previous example of a single quantum harmonic well is too idealized. A piece of matter contains an extremely large number of interacting atoms. Even if one focuses onto a single atom in an effective potential, the form of this potential may be so complicated that the study of the underlying eigenfunctions would in general be intractable. In 1958, Anderson considered a lattice model for a charge carrier in a random potential and proved the localization of eigenfunctions under certain conditions [6]. The localization of charge carriers means no electric current through the medium (insulating state), in contrast to metallic or

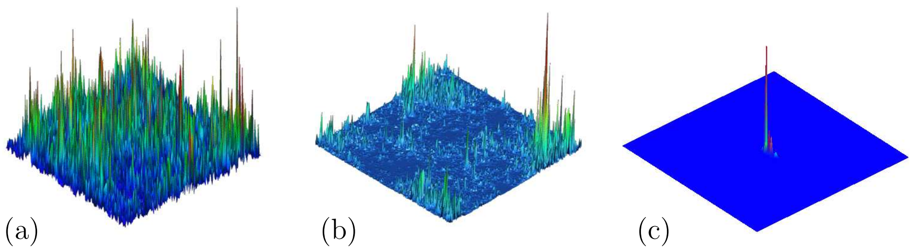

FIG. 7.1. *Illustration of the Anderson transition in a tight-binding model (or so-called SU(2) model) in the two-dimensional symplectic class [13, 14, 288, 289]. Three shown eigenfunctions (with the energy close to* 1*) were computed for three disorder strengths* W *that correspond to (a) metallic state (*$W < W_0$*), (b) critical state (*$W = W_0$*), and (c) insulating state (*$W > W_0$*),* $W_0 = 5.952$ *being the critical disorder strength. The latter eigenfunction is strongly localized that prohibits diffusion of charge carriers (i.e., no electric current). The eigenfunctions were computed and provided by Dr. Hideaki Obuse (unpublished earlier).*

conducting state when the charge carriers are not localized. The Anderson transition between insulating and conducting states is illustrated for the tight-binding model on Fig. 7.1. The shown eigenfunctions were computed for three disorder strengths W that correspond to metallic ($W < W_0$), critical ($W = W_0$), and insulating ($W > W_0$) states, $W_0 = 5.952$ being the critical desorder strength. The latter eigenfunction is strongly localized that prohibits diffusion of charge carriers (i.e., no electric current). The Anderson localization which explains the metal-insulator transitions in semiconductors, was thoroughly investigated during the last fifty years (see reviews [46, 134, 227, 246, 276, 379] for details and references). Similar localization phenomena were observed for microwaves with two-dimensional random scattering [111], for light in a disordered medium [398] and in disordered photonic crystals [347, 356], for matter waves in a controlled disorder [56] and in non-interacting Bose-Einstein condensate [338], and for ultrasound [195]. The multifractal structure of the eigenfunctions at the critical point (look at an example on Fig. 7.1b) has also been intensively investigated (see [134, 171] and references therein).

7.3. Trapping in infinite waveguides. In both previous cases, localization of eigenfunctions was related to an external potential. In particular, if the potential was not strong enough, Anderson localization could disappear (Fig. 7.1a). Is the presence of a potential necessary for localization? The formal answer is positive because the eigenstates of the Laplace operator in the whole space $\mathbb{R}^d$ are simply $e^{i(k\cdot x)}$ (parameterized by the vector k) which are all extended in $\mathbb{R}^d$. These waves are called "resonances" (not eigenfunctions) of the Laplace operator, as their L_2-norm is infinite.

The situation is different for the Laplace operator in a bounded domain with Dirichlet boundary condition. In quantum mechanics, such a boundary presents a "hard wall" that separates the interior of the domain with zero potential from the exterior of the domain with infinite potential. For instance, this "model" was employed by Crommie *et al.* to describe the confinement of electrons to quantum corrals on a metallic surface [109] (see also their figure 2 that shows the experimental spatial structure of the electron's wavefunction). Although the physical interpretation of a boundary through an infinite potential is instructive, we will use the mathematical terminology and speak about the eigenvalue problem for the Laplace operator in a bounded domain without potential.

For unbounded domains, the spectrum of the Laplace operator consists of two parts: (i) the discrete (or point-like) spectrum, with eigenfunctions of finite L_2 norm that are necessarily "trapped" or "localized" in a bounded region of the waveguide, and (ii) the continuous spectrum, with associated functions of infinite L_2 norm that are extended over the whole domain. The continuous spectrum may also contain embedded eigenvalues whose eigenfunctions have finite L_2 norm. A wave excited at the frequency of the trapped eigenmode remains in the localization region and does not propagate. In this case, the definition (7.1) of localization is again rigorous, as for any bounded subset Ω_0 of an unbounded domain Ω, one has $\mu_d(\Omega_0)/\mu_d(\Omega) = 0$, while the ratio of L_2 norms can be made arbitrarily small by expanding Ω_0.

This kind of localization in classical and quantum waveguides has been thoroughly investigated (see reviews [129, 258] and also references in [292]). In the seminal paper, Rellich proved the existence of a localized eigenfunction in a deformed infinite cylinder [334]. His results were significantly extended by Jones [206]. Ursell reported on the existence of trapped modes in surface water waves in channels [383–385], while Parker observed experimentally the trapped modes in locally perturbed acoustic waveguides [299, 300]. Exner and Seba considered an infinite bent strip of smooth curvature and showed the existence of trapped modes by reducing the problem to Schrödinger operator in the straight strip, with the potential depending on the curvature [135]. Goldstone and Jaffe gave the variational proof that the wave equation subject to Dirichlet boundary condition always has a localized eigenmode in an infinite tube of constant cross-section in any dimension, provided that the tube is not exactly straight [158]. This result was further extended by Chenaud *et al.* to arbitrary dimension [98]. The problem of localization in acoustic waveguides with Neumann boundary condition has also been investigated [131, 132]. For instance, Evans *et al.* considered a straight strip with an inclusion of arbitrary (but symmetric) shape [132] (see [118] for further extensions). Such an inclusion obstructed the propagation of waves and was shown to result in trapped modes. The effect of mixed Dirichlet, Neumann and Robin boundary conditions on the localization was also investigated (see [74, 125, 146, 292] and references therein). A mathematical analysis of guided water waves was developed by Bonnet-Ben Dhia and Joly [64]. Lower bounds for the eigenvalues below the cut-off frequency (for which the associated eigenfunctions are localized) were obtained by Ashbaugh and Exner for infinite thin tubes in two and three dimensions [15]. In addition, these authors derived an upper bound for the number of the trapped modes. More recently, Exner *et al.* considered the Laplacian in finite-length curved tubes of arbitrary cross-section, subject to Dirichlet boundary conditions on the cylindrical surface and Neumann conditions at the ends of the tube. They expressed a lower bound for the spectral threshold of the Laplacian through the lowest eigenvalue of the Dirichlet Laplacian in a torus determined by the geometry of the tube [137]. In a different work, Exner and co-worker investigated bound states and scattering in quantum waveguides coupled laterally through a boundary window [136].

Examples of waveguides with *numerous* localized states were reported in the literature. For instance, Avishai *et al.* demonstrated the existence of many localized states for a sharp "broken strip", i.e. a waveguide made of two channels of equal width intersecting at a small angle θ [23]. Carini and co-workers reported an experimental confirmation of this prediction and its further extensions [88, 89, 261]. Bulgakov *et al.* considered two straight strips of the same width which cross at an angle $\theta \in (0, \pi/2)$ and showed that, for small θ, the number of localized states is

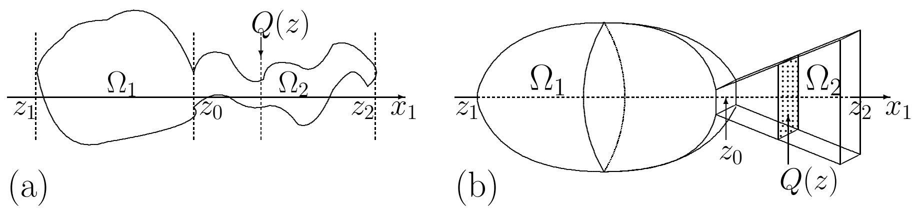

FIG. 7.2. *Two examples of a bounded domain Ω with a branch of variable cross-sectional profile. When the eigenvalue λ is smaller than the cut-off "frequency" μ, the associated eigenfunction exponentially decays in the branch Ω_2 and is thus mainly localized in Ω_1. Note that the branch itself may even be increasing.*

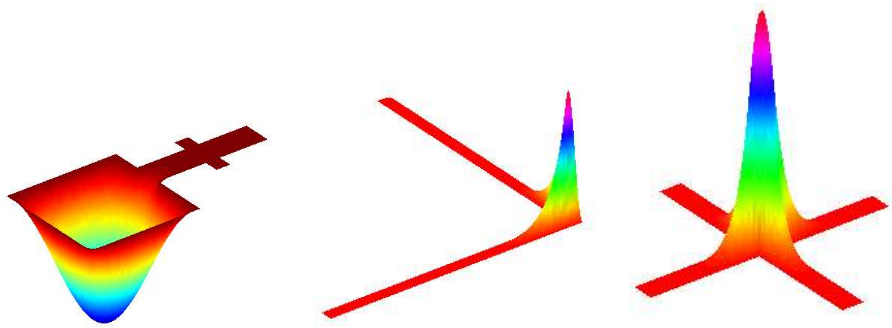

FIG. 7.3. *Examples of localized Dirichlet eigenfunctions with an exponential decay: square with a branch (from [119]), L-shape and crossing of two stripes (from [120]).*

greater than $(1 - 2^{-2/3})^{3/2}/\theta$ [73]. Even for the simple case of two strips crossed at the right angle $\theta = \pi/2$, Schult *et al.* showed the existence of two localized states, one lying below the cut-off frequency and the other being embedded into the continuous spectrum [355].

7.4. Exponential estimate for eigenfunctions. Qualitatively, an eigenmode is trapped when it cannot "squeeze" outside the localization region through narrow channels or branches of the waveguide. This happens when typical spatial variations of the eigenmode, which are in the order of $\pi\lambda^{-1/2}$, are larger than the size a of the narrow part, i.e. $\pi\lambda^{-1/2} \geq a$ or $\lambda \leq \pi^2/a^2$ [198]. This simplistic argument suggests that there exists a threshold value μ (which may eventually be 0), or so-called cut-off frequency, such that the eigenmodes with $\lambda \leq \mu$ are localized. Moreover, this qualitative geometrical interpretation is well adapted for both unbounded and bounded domains. While the former case of infinite waveguides was thoroughly investigated, the existence of trapped or localized eigenmodes in bounded domains has attracted less attention. Even the definition of localization remains conventional because all eigenfunctions in a bounded domain have finite L_2 norm.

This problem was studied by Delitsyn and co-workers for domains with branches of variable cross-sectional profiles [119]. More precisely, one considers a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3, ...$) with a piecewise smooth boundary $\partial\Omega$ and denote $Q(z) = \Omega \cap \{x \in \mathbb{R}^d \ : \ x_1 = z\}$ the cross-section of Ω at $x_1 = z \in \mathbb{R}$ by a hyperplane perpendicular to the coordinate axis x_1 (Fig. 7.2). Let

$$z_1 = \inf\{z \in \mathbb{R} \ : \ Q(z) \neq \emptyset\}, \qquad z_2 = \sup\{z \in \mathbb{R} \ : \ Q(z) \neq \emptyset\},$$

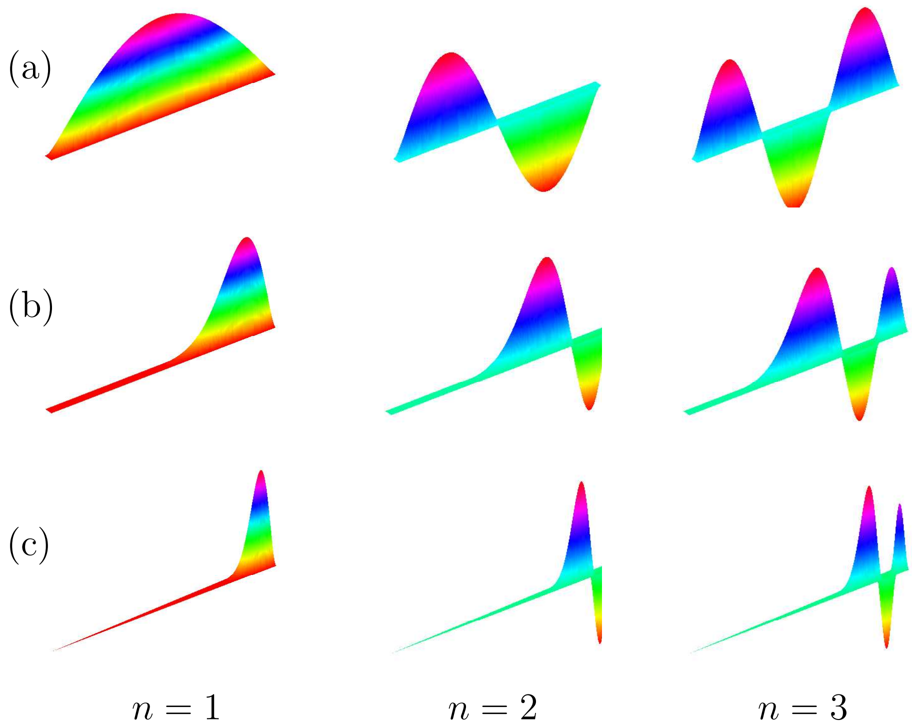

Fig. 7.4. *The first three Dirichlet eigenfunctions for three elongated domains: (a) rectangle of size* 25×1*, (b) right trapezoid with bases* 1 *and* 0.9 *and height* 25 *which is very close to the above rectangle, and right triangle with edges* 25 *and* 1 *(half of the rectangle). There is no localization for the first shape, while the first eigenfunctions for the second and third domains tend to be localized.*

and we fix some z_0 such that $z_1 < z_0 < z_2$. Let $\mu(z)$ be the first eigenvalue of the Laplace operator in $Q(z)$ (with Dirichlet boundary condition on $\partial Q(z)$), and $\mu = \inf\limits_{z \in (z_0, z_2)} \mu(z)$. Let u be a Dirichlet-Laplacian eigenfunction in Ω, and λ the associate eigenvalue. If $\lambda < \mu$, then

$$\|u\|_{L_2(Q(z))} \leq \|u\|_{L_2(Q(z_0))} \exp(-\beta\sqrt{\mu - \lambda}\,(z - z_0)) \quad (z \geq z_0), \tag{7.6}$$

with $\beta = \sqrt{2}$. Moreover, if $(e_1 \cdot n(x)) \geq 0$ for all $x \in \partial\Omega$ with $x_1 > z_0$, where e_1 is the unit vector $(1, 0, ..., 0)$ in the direction x_1, and $n(x)$ is the normal vector at $x \in \partial\Omega$ directed outwards the domain, then the above inequality holds with $\beta = 2$.

In this statement, a domain Ω is conventionally split into two subdomains, Ω_1 (with $x_1 < z_0$) and Ω_2 (with $x_1 > z_0$), by the hyperplane at $x_1 = z_0$ (the coordinate axis x_1 can be replaced by any straight line). Under the condition $\lambda < \mu$, the eigenfunction u exponentially decays in Ω_2 which is loosely called "branch". Note that the choice of the splitting hyperplane (i.e., z_0) determines the threshold μ.

The theorem formalizes the notion of the cut-off frequency μ for branches of variable cross-sectional profiles and provides a constructive way for its computation. For instance, if Ω_2 is a rectangular channel of width a, the first eigenvalue in all cross-sections $Q(z)$ is π^2/a^2 (independently of z) so that $\mu = \pi^2/a^2$, as expected. The exponential estimate quantifies the "difficulty" of penetration, or "squeezing", through the branch Ω_2 and ensures the localization of the eigenfunction u in Ω_1.

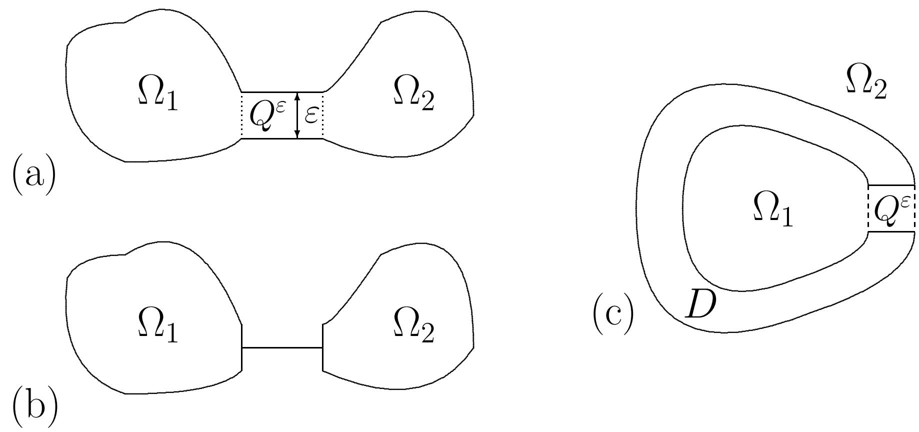

FIG. 7.5. **(a)** *A dumbbell domain Ω^ε is the union of two bounded domains Ω_1, Ω_2 and a narrow "connector" Q^ε of width ε.* **(b)** *In the limit $\varepsilon \to 0$, the connector degenerates to a curve (here, an interval) so that the subdomains Ω_1 and Ω_2 become disconnected.* **(c)** *In Beale's work, Ω^ε is composed of two components, a bounded domain Ω_1 and unbounded domain Ω_2, which are connected by a narrow channel Q^ε.*

Since the cut-off frequency μ is independent of the subdomain Ω_1, one can impose any boundary condition on $\partial\Omega_1$ (that still ensures the self-adjointness of the Laplace operator). In turn, the Dirichlet boundary condition on the boundary of the branch Ω_2 is relevant, although some extensions were discussed in [119]. It is worth noting that the theorem also applies to infinite branches Ω_2, under supplementary condition $\mu(z) \to \infty$ to ensure the existence of the discrete spectrum.

According to this theorem, the L_2-norm of an eigenfunction with $\lambda < \mu$ in $\Omega(z) = \Omega \cap \{x \in \mathbb{R}^d \ : \ x_1 > z\}$ can be made exponentially small provided that the branch Ω_2 is long enough. Taking $\Omega_0 = \Omega \setminus \Omega(z)$, the ratio of L_2-norms in Eq. (7.1) can be made arbitrarily small. However, the second ratio may not be necessarily small. In fact, its smallness depends on the shape of the domain Ω. This is once again a manifestation of the conventional character of localization in bounded domains.

Figure 7.3 presents several examples of localized Dirichlet eigenfunctions showing an exponential decay along the branches. Since an increase of branches diminishes the eigenvalue and thus further enhances the localization, the area of the localized region Ω_1 can be made arbitrarily small with respect to the total area (one can even consider infinite branches). Examples of an L-shape and a cross illustrate that the linear sizes of the localized region do not need to be large in comparison with the branch width (a sufficient condition for getting this kind of localization was proposed in [120]). It is worth noting that the separation into the localized region and branches is conventional. For instance, Fig. 7.4 shows several localized eigenfunctions for elongated triangle and trapezoid, for which there is no explicit separation.

7.5. Dumbbell domains. Yet another type of localization emerges for domains that can be split into two or several subdomains with narrow connections (of "width" ε) [330], a standard example being a dumbbell: $\Omega^\varepsilon = \Omega_1 \cup Q^\varepsilon \cup \Omega_2$ (Fig. 7.5a). The asymptotic behavior of eigenvalues and eigenfunctions in the limit $\varepsilon \to 0$ was thoroughly investigated for both Dirichlet and Neumann boundary conditions. We start by considering the Dirichlet boundary condition.

In the limiting case of zero width connections, the subdomains Ω_i $(i = 1, ..., N)$

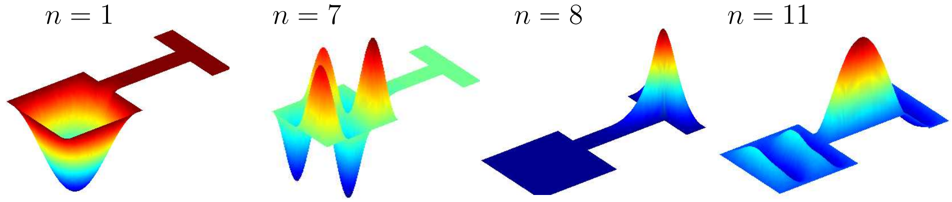

FIG. 7.6. *Several Dirichlet eigenfunctions for a dumbbell domain which is composed of two rectangles and connected by the third rectangle (from [119]). The 1st and 7th eigenfunctions are localized in the larger subdomain, the 8th eigenfunction is localized in the smaller subdomain, while the 11th eigenfunction is not localized at all. Note that the width of connection is not small (1/4 of the width of both subdomains).*

become disconnected, and the eigenvalue problem can be independently formulated for each subdomain. Let Λ_i be the set of eigenvalues for the subdomain Ω_i. Each Dirichlet eigenvalue λ^ε of the domain Ω^ε approaches to an eigenvalue λ^0 of one limiting subdomain $\Omega_i \subset \Omega^0$: $\lambda^0 \in \Lambda_i$ for certain i. Moreover, if

$$\Lambda_i \cap \Lambda_j = \emptyset \quad \forall\ i \neq j, \tag{7.7}$$

the space of eigenfunctions in the limiting (disconnected) domain Ω^0 is the direct product of spaces of eigenfunctions for each subdomain Ω_i (see [112] for discussion on convergence and related issues). This is a basis for what we will call "bottle-neck localization". In fact, each eigenfunction u_m^ε on the domain Ω^ε approaches an eigenfunction u_m^0 of the limiting domain Ω^0 which is fully localized in one subdomain Ω_i and zero in the others. For a small ε, the eigenfunction u_m^ε is therefore mainly localized in the corresponding i-th subdomain Ω_i, and is *almost* zero in the other subdomains. In other words, for any eigenfunction, one can take the width ε small enough to ensure that the L_2-norm of the eigenfunction in the subdomain Ω_i is arbitrarily close to that in the whole domain Ω^ε:

$$\forall\ m \geq 1 \quad \exists i \in \{1, ..., N\} \quad \forall\ \delta \in (0,1) \quad \exists\ \varepsilon > 0\ :\ \|u_m^\varepsilon\|_{L_2(\Omega_i)} > (1-\delta)\|u_m^\varepsilon\|_{L_2(\Omega^\varepsilon)}. \tag{7.8}$$

This behavior is exemplified for a dumbbell domain which is composed of two rectangles and connected by the third rectangle (Fig. 7.6). The 1st and 7th eigenfunctions are localized in the larger rectangle, the 8th eigenfunction is localized in the smaller rectangle, while the 11th eigenfunction is not localized at all. Note that the width of connection is not too small (1/4 of the width of both subdomains).

It is worth noting that, for a small fixed width ε and a small fixed threshold δ, there may be infinitely many high-frequency "non-localized" eigenfunctions, for which the above inequality is not satisfied. In other words, for a given connected domain with a narrow connection, one can only expect to observe a finite number of low-frequency localized eigenfunctions. We note that the condition (7.7) is important to ensure that limiting eigenfunctions are fully localized in their respective subdomains. Without this condition, a limiting eigenfunction may be a linear combination of eigenfunctions in different subdomains with the same eigenvalue that would destroy localization. Note that the asymptotic behavior of eigenfunctions at the "junction" was studied by Felli and Terracini [141].

For Neumann boundary condition, the situation is more complicated, as the eigenvalues and eigenfunctions may also approach the eigenvalues and eigenfunctions of the limiting connector (in the simplest case, the interval). Arrieta considered a planar dumbell domain Ω_ε consisted of two disjoint domains Ω_1 and Ω_2 joint by a channel Q^ε of variable profile $g(x)$: $Q^\varepsilon = \{x \in \mathbb{R}^2 \ : \ 0 < x_1 < 1, \ 0 < x_2 < \varepsilon g(x_1)\}$, where $g \in C^1(0,1)$ and $g(x_1) \geq 0$ for all $x_1 \in [0,1]$. In the limit $\varepsilon \to 0$, each eigenvalue of the Laplace operator in Ω^ε with Neumann boundary condition was shown to converge either to an eigenvalue μ_k of the Neumann-Laplace operator in $\Omega_1 \cup \Omega_2$, or to an eigenvalue ν_k of the Sturm-Liouville operator $-\frac{1}{g}(gu_x)_x$ acting on a function u on $(0,1)$, with Dirichlet boundary condition [10, 11]. The first-order term in the small ε-asymptotic expansion was obtained. The special case of cylindrical channels (of constant profile) in higher dimensions was studied by Jimbo [203] (see also results by Hempel *et al.* [185]). Jimbo and Morita studied an N-dumbell domain, i.e. a family of N pairwise disjoint domains joint by thin channels [204]. They proved that $\lambda_m^\varepsilon = C_m \varepsilon^{d-1} + o(\varepsilon^{d-1})$ as $\varepsilon \to 0$ for $m = 1, 2, \ldots, N$, while $\lambda_{N+1}^\varepsilon$ is uniformly bounded away from zero, where d is the dimension of the embedding space, and C_m are shape-dependent constants. Jimbo also analyzed the asymptotic behavior of the eigenvalues λ_m^ε with $m > N$ under the condition that the sets $\{\mu_k\}$ and $\{\nu_k\}$ do not intersect [205]. In particular, for an eigenvalue λ_m^ε that converges to an element of $\{\mu_k\}$, the asymptotic behavior is $\lambda_m^\varepsilon = \mu_k + C_m \varepsilon^{d-1} + o(\varepsilon^{d-1})$.

Brown and co-workers studied upper bounds for $|\lambda_m^\varepsilon - \lambda_m^0|$ and showed [69]:

(i) If $\lambda_m^0 \in \{\mu_k\} \setminus \{\nu_k\}$,

$$\begin{aligned} |\lambda_m^\varepsilon - \lambda_m^0| &\leq C |\ln \varepsilon|^{-1/2} \qquad (d = 2), \\ |\lambda_m^\varepsilon - \lambda_m^0| &\leq C \varepsilon^{(d-2)/d} \qquad (d \geq 3). \end{aligned}$$

(ii) If $\lambda_m^0 \in \{\nu_k\} \setminus \{\mu_k\}$,

$$\begin{aligned} |\lambda_m^\varepsilon - \lambda_m^0| &\leq C \varepsilon^{1/2} |\ln \varepsilon| \qquad (d = 2), \\ |\lambda_m^\varepsilon - \lambda_m^0| &\leq C \varepsilon^{1/2} \qquad (d \geq 3). \end{aligned}$$

For a dumbbell domain in $\mathbb{R}^d$ with a thin cylindrical channel of a smooth profile, Gadyl'shin obtained the complete small ε asymptotics of the Neumann-Laplace eigenvalues and eigenfunctions and explicit formulas for the first term of these asymptotics, including multiplicities [151–153].

More recently, Arrieta and Krejcirik considered the problem of spectral convergence from another point of view [12]. They showed that if $\Omega_0 \subset \Omega_\varepsilon$ are bounded domains and if the eigenvalues and eigenfunctions of the Laplace operator with Neumann boundary condition in Ω_ε converge to the ones in Ω_0, then necessarily $\mu_d(\Omega_\varepsilon \backslash \Omega_0) \to 0$ as $\varepsilon \to 0$, while it is not necessarily true that $\mathrm{dist}(\Omega_\varepsilon, \Omega_0) \to 0$. As a matter of fact, they constructed an example of a perturbation where the spectra behave continuously but $\mathrm{dist}(\Omega_\varepsilon, \Omega_0) \to \infty$ as $\varepsilon \to 0$.

A somewhat related problem of scattering frequencies of the wave equation associated to an exterior domain in $\mathbb{R}^3$ with an appropriate boundary condition was investigated by Beale [45] (for more general aspects of geometric scattering theory, see [274]). We recall that a scattering frequency $\sqrt{\lambda}$ of an unbounded domain Ω is a (complex) number for which there exists a nontrivial solution of $\Delta u + \lambda u = 0$ in Ω, subject to Dirichlet, Neumann, or Robin boundary condition and to an "outgoing" condition at infinity. In Beale's work, a bounded cavity Ω_1 was connected by a thin channel to the exterior (unbounded) space Ω_2. More specifically, he considered a bounded domain D such that its complement in $\mathbb{R}^3$ has a bounded component Ω_1 and an unbounded component Ω_2. After that, a thin "hole" Q^ε in D was made to connect both components (Fig. 7.5c). Beale showed that the joint domain $\Omega^\varepsilon = \Omega_1 \cup Q^\varepsilon \cup \Omega_2$

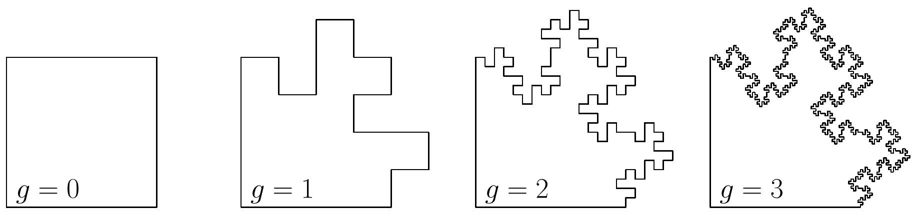

FIG. 7.7. *The unit square and three prefractal domains obtained iteratively one from the other (two sides of these domains are finite generations of the Von Koch curve of fractal dimension 3/2). These domains were intensively studied, both numerically and experimentally, by Sapoval and co-workers [133, 175, 180, 340, 341, 348–350].*

with Dirichlet boundary condition has a scattering frequency which is arbitrarily close either to an eigenfrequency (i.e., the square root of the eigenvalue) of the Laplace operator in Ω_1, or to a scattering frequency in Ω_2, provided the channel Q^ε is narrow enough. The same result was extended to Robin boundary condition of the form $\partial u/\partial n + hu = 0$ on $\partial\Omega^\varepsilon$, where h is a function on $\partial\Omega^\varepsilon$ with a positive lower bound. In both cases, the method in his proof relies on the fact that the lowest eigenvalue of the channel tends to infinity as the channel narrows. However, it is no longer true for Neumann boundary condition. In this case, with some restrictions on the shape of the channel, Beale proved that the scattering frequencies converge not only to the eigenfrequencies of Ω_1 and scattering frequencies of Ω_2 but also to the longitudinal frequencies of the channel. Similar results can be obtained in domains of space dimension other than 3.

7.6. Localization in irregularly-shaped domains. As we have seen, a narrow connection between subdomains could lead to localization. How narrow should it be? A rigorous answer to this question is only known for several "tractable" cases such as dumbbell-like or cylindrical domains (Sec. 7.5). In general, even the notion of "connection" is conventional. Sapoval and co-workers have formulated and studied the problem of localization in irregularly-shaped or fractal domains through numerical simulations and experiments [133, 139, 175, 180, 340, 341, 348–350]. In the first publication, they monitored the vibrations of a prefractal "drum" (i.e., a thin membrane with a fixed boundary) which was excited at different frequencies [348]. Tuning the frequency allowed them to directly visualize different Dirichlet eigenfunctions in a (prefractal) quadratic von Koch snowflake (an example is shown on Fig. 7.7). For this and similar domains, certain eigenfunctions were found to be localized in a small region of the domain, for both Dirichlet and Neumann boundary conditions (Fig. 7.8). This effect was first attributed to self-similar structure of the domain. However, similar effects were later observed through numerical simulations for non-fractal domains [139], as illustrated by Fig. 7.9. In the study of sound attenuation by noise-protective walls, Félix and co-workers have further extended the analysis to the union of two domains with different refraction indices which are separated by an irregular boundary [139]. Many eigenfunctions of the related second order elliptic operator were shown to be localized on this boundary (so-called "astride localization"). A rigorous mathematical theory of these important phenomena is still missing. Takeda *al.* observed experimentally the electromagnetic field at specific frequency to be confined in the central part of the third stage of three-dimensional fractals called the Menger sponge [377]. This localization was attributed to a singular photon density of states realized in the fractal structure.

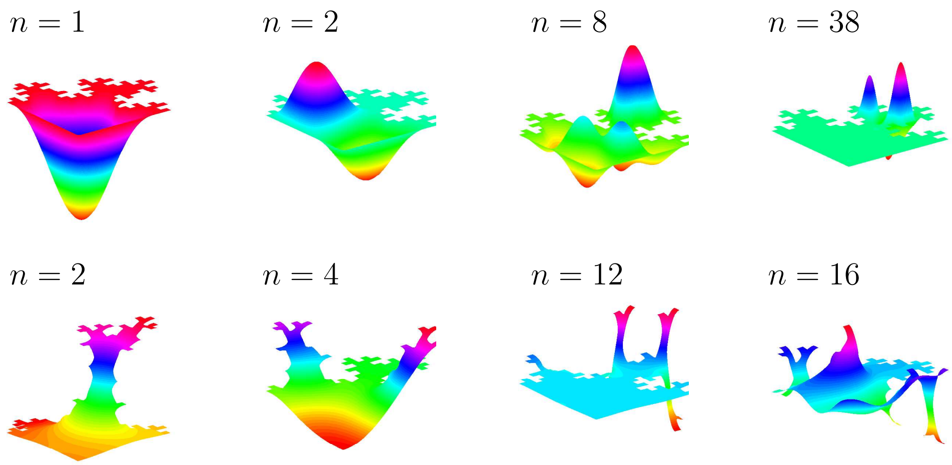

Fig. 7.8. *Several Dirichlet (top) and Neumann (bottom) eigenfunctions for the third domain on Fig. 7.7 ($g = 2$). The 38th Dirichlet and the 12th Neumann eigenfunctions are localized in a small subdomain (located in the upper right corner on Fig. 7.7), while the first/second Dirichlet and the 4th Neumann eigenfunctions are almost zero on this subdomain. Finally, the 8th Dirichlet and the second Neumann eigenfunctions are examples of eigenfunctions extended over the whole domain.*

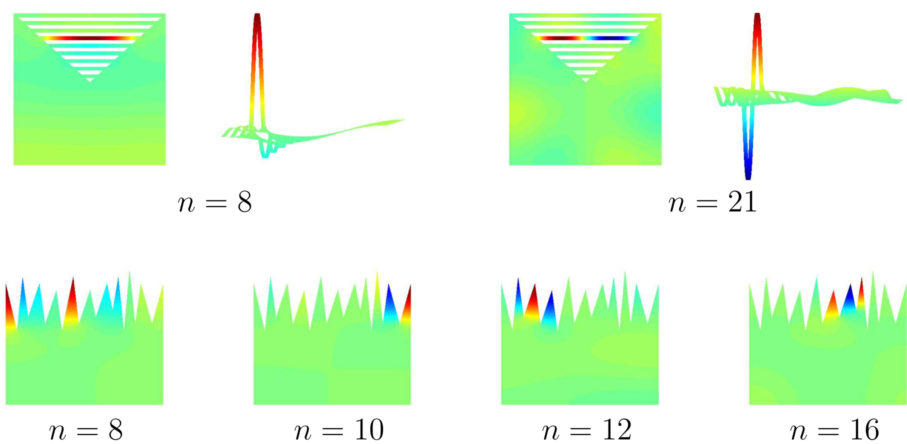

Fig. 7.9. *Examples of localized Neumann eigenfunctions in two domains adapted from [139]: square with many elongated holes (top) and random sawteeth (bottom). Colors represent the amplitude of eigenfunctions, from the most negative value (dark blue), through zero (green), to the largest positive value (dark red). One can notice that the eigenfunctions on the top are not negligible outside the localization region. This is yet another illustration for the conventional character of localization in bounded domains.*

A number of mathematical studies were devoted to the theory of partial differential equations on fractals in general and to localization of Laplacian eigenfunctions in particular (see [219, 374] and references therein). For instance, the spectral properties of the Laplace operator on Sierpinski gasket and its extensions were thoroughly investigated [33–35, 37, 60, 150, 359]. Barlow and Kigami studied the localized eigenfunctions of the Laplacian on a more general class of self-similar sets (so-called post critically finite self-similar sets, see [220, 221] for details). They related the asymptotic behavior of the eigenvalue counting function to the existence of localized eigenfunctions and established a number of sufficient conditions for the existence of a localized eigenfunction in terms of the symmetries of a set [36].

Berry and co-workers developed a new method to approximate the Neumann spectrum of a Laplacian on a planar fractal set Ω as a renormalized limit of the Neumann spectra of the standard Laplacian on a sequence of domains that approximate Ω from the outside [54]. They applied this method to compute the Neumann-Laplacian eigenfunctions in several domains, including a sawtooth domain, Sierpinski gasket and carpet, as well as nonsymmetric and random carpets and the octagasket. In particular, they gave a numerical evidence for the localized eigenfunctions for a sawtooth domain, in agreement with the earlier work by Félix *et al.* [139].

Heilman and Strichartz reported several numerical examples of localized Neumann-Laplacian eigenfunctions in two domains [181], one of them is illustrated on Fig. 7.10a. Each of these domains consists of two subdomains with a narrow but not too narrow connection. This is a kind of dumbbell shape with a connector of zero length. Heilman and Strichartz argued that one subdomain must possess an axis of symmetry for getting localized eigenfunctions. Since an anti-symmetric eigenfunction vanishes on the axis of symmetry, it is necessarily small near the bottle-neck that somehow "prevents" its extension to the other domain. Although the argument is plausible, we have to stress that such a symmetry is neither sufficient, nor necessary for localization. It is obviously not sufficient because even for symmetric domain, there exist plenty of extended eigenfunctions (including the trivial example of the ground eigenmode which is a constant over the whole domain). In order to illustrate that the reflection symmetry is not necessary, we plot on Fig. 7.10b,c examples of localized eigenfunctions for modified domains for which the symmetry is broken. Although rendering the upper domain less and less symmetric gradually reduces or even fully destroys localization (Fig. 7.10d), its "mechanism" remains poorly understood. We also note that methods of Sec. 7.4 are not applicable in this case because of Neumann boundary condition.

Lapidus and Pang studied the boundary behavior of the Dirichlet Laplacian eigenfunctions and their gradients on a class of planar domains with fractal boundary, including the triangular and square von Koch snowflakes and their polygonal approximations [237]. A numerical evidence for the boundary behavior of eigenfunctions was reported in [239], with numerous pictures of eigenfunctions. Later, Daubert and Lapidus considered more specifically the localization character of eigenfunctions in von Koch domains [114]. In particular, different "measures" of localization were discussed.

Note also that Filoche and Mayboroda studied the problem of localization for bi-Laplacian in rigid thin plates and discovered that clamping just one point inside such a plate not only perturbs its spectral properties, but essentially divides the plate into two independently vibrating regions [142].

7.7. High-frequency localization. A hundred years ago, Lord Rayleigh documented an interesting acoustical phenomenon in the whispering gallery under the dome of Saint Paul's Cathedral in London [331] (see also [328, 329]). A whisper of one person propagated along the curved wall to another person stood near the wall. Keller and Rubinow discussed the related "whispering gallery modes" and also "bouncing ball modes", and showed that these modes exist for a two-dimensional domain with arbitrary smooth convex curve as its boundary [217]. A semiclassical approximation of Laplacian eigenfunctions in convex domains was developed by Lazutkin [24, 242–245] (see also [9, 326, 327, 366]). Chen and co-workers analyzed Mathieu and modified

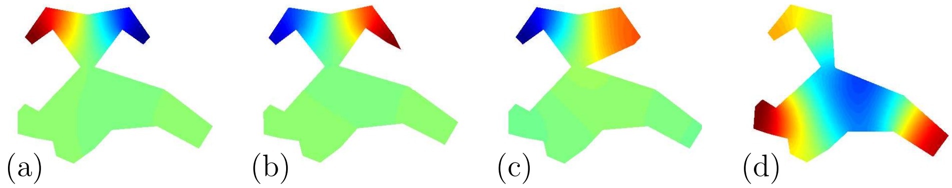

FIG. 7.10. *Neumann-Laplace eigenfunction u_4 in the original "cow" domain from [181] (a) and in three modified domains (b,c,d), in which the reflection symmetry of the upper subdomain is broken. The fourth eigenfunction is localized for the first three domains (a,b,c), while the last domain with the stronger modification shows no localization (d). Colors represent the amplitude of an eigenfunction, from the most negative value (dark blue), through zero (green), to the largest positive value (dark red).*

Mathieu functions and reported another type of localization named "focusing modes" [97]. All these eigenmodes become more and more localized in a small subdomain when the associated eigenvalue increases. This so-called high-frequency or high-energy limit was intensively studied for various domains, named also as quantum billiards [173, 183, 199, 352, 373]. In quantum mechanics, this limit is known as semi-classical approximation [49]. In optics, it corresponds to ray approximation of wave propagation, from which the properties of an optical, acoustical or quantum system can often be reduced to the study of rays in classical billiards. Jakobson *et al.* gave an overview of many results on geometric properties of the Laplacian eigenfunctions on Riemannian manifolds, with a special emphasis on high-frequency limit (weak star limits, the rate of growth of L_p norms, relationships between positive and negative parts of eigenfunctions, etc.) [199]. Bearing in mind this comprehensive review, we start by illustrating the high-frequency localization and the related problems in simple domains such as disk, ellipse and rectangle for which explicit estimates can be done. After that, some results for quantum billiards are summarized.

7.7.1. Whispering gallery and focusing modes. The disk is the simplest shape for illustrating the whispering gallery and focusing modes. The explicit form (3.9) of eigenfunctions allows one to get accurate estimates and bounds, as shown below. When the index k is fixed and n increases, the Bessel functions $J_n(\alpha_{nk}r/R)$ become strongly attenuated near the origin (as $J_n(z) \sim (z/2)^n/n!$ at small z) and essentially localized near the boundary, yielding whispering gallery modes. In turn, when n is fixed and k increases, the Bessel functions highly oscillate, the amplitude of oscillations decreasing towards to the boundary. In that case, the eigenfunctions are mainly localized at the origin, yielding focusing modes.

These qualitative arguments were rigorously formulated in [287]. For each eigenfunction u_{nk} on the unit disk Ω, one introduces the subdomain $\Omega_{nk} = \{x \in \mathbb{R}^2 \ : \ |x| < d_n/\alpha_{nk}\} \subset \Omega$, where $d_n = n - n^{2/3}$, and α_{nk} are, depending on boundary conditions, the positive zeros of either $J_n(z)$ (Dirichlet), or $J'_n(z)$ (Neumann) or $J'_n(z) + hJ_n(z)$ for some $h > 0$ (Robin), with $n = 0, 1, 2, ...$ denoting the order of Bessel function $J_n(z)$ and $k = 1, 2, 3, ...$ counting zeros. Then for any $p \geq 1$, there exists a universal constant $c_p > 0$ such that for any $k = 1, 2, 3, ...$ and any large enough n, the Laplacian eigenfunction u_{nk} for Dirichlet, Neumann or Robin boundary condition satisfies

$$\frac{\|u_{nk}\|_{L_p(\Omega_{nk})}}{\|u_{nk}\|_{L_p(\Omega)}} < c_p n^{\frac{1}{3}+\frac{2}{3p}} \exp(-n^{1/3}\ln(2)/3). \tag{7.9}$$

The definition of Ω_{nk} and the above estimate imply

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(\Omega_{nk})}}{\|u_{nk}\|_{L_p(\Omega)}} = 0, \qquad \lim_{n\to\infty} \frac{\mu_2(\Omega_{nk})}{\mu_2(\Omega)} = 1. \tag{7.10}$$

This theorem shows the existence of infinitely many Laplacian eigenmodes which are L_p-localized in a thin layer near the boundary $\partial\Omega$. In fact, for any prescribed thresholds for both ratios in (7.1), there exists n_0 such that for all $n > n_0$, the eigenfunctions u_{nk} are L_p-localized. These eigenfunctions are called "whispering gallery eigenmodes" and illustrated on Fig. 7.11.

We outline a peculiar relation between high-frequency and low-frequency localization. The explicit form (3.9) of Dirichlet Laplacian eigenfunctions u_{nk} leads to their simple nodal structure which is formed by $2n$ radial nodal lines and $k-1$ circular nodal lines. The radial nodal lines split the disk into $2n$ circular sectors with Dirichlet boundary conditions. As a consequence, whispering gallery eigenmodes in the disk and the underlying exponential estimate (7.9) turn out to be related to the exponential decay of eigenfunctions in domains with branches (Sec. 7.4), as illustrated on Fig. 7.4 for elongated triangles.

A simple consequence of the above theorem is that for any $p \geq 1$ and any open subset V compactly included in the unit disk Ω (i.e., $\bar{V} \cap \partial\Omega = \emptyset$), one has

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(V)}}{\|u_{nk}\|_{L_p(\Omega)}} = 0, \tag{7.11}$$

and

$$C_p(V) \equiv \inf_{nk} \left\{ \frac{\|u_{nk}\|_{L_p(V)}}{\|u_{nk}\|_{L_p(\Omega)}} \right\} = 0. \tag{7.12}$$

Qualitatively, for any subset V, there exists a sequence of eigenfunctions that progressively "escape" V.

The localization of focusing modes at the origin is revealed in the limit $k \to \infty$. For each $R \in (0,1)$, one defines an annulus $\Omega_R = \{x \in \mathbb{R}^2 \ : \ R < |x| < 1\} \subset \Omega$ of the unit disk Ω. Then, for any $n = 0, 1, 2, ...$, the Laplacian eigenfunction u_{nk} with Dirichlet, Neumann or Robin boundary condition satisfies

$$\lim_{k\to\infty} \frac{\|u_{nk}\|_{L_\infty(\Omega_R)}}{\|u_{nk}\|_{L_\infty(\Omega)}} = 0, \qquad \lim_{k\to\infty} \frac{\|u_{nk}\|_{L_2(\Omega_R)}}{\|u_{nk}\|_{L_2(\Omega)}} = \sqrt{1-R} > 0. \tag{7.13}$$

When the index k increases (with fixed n), the eigenfunctions u_{nk} become localized more and more near the origin [287]. These eigenfunctions are called "focusing eigenmodes" and illustrated on Fig. 7.12. The theorem illustrates that the definition of localization is sensitive to the norm: the above focusing modes are L_∞-localized, but they are not L_2-localized. Similar results for whispering gallery and focusing modes hold for a ball in three dimensions [287].

7.7.2. Bouncing ball modes. Filled ellipses and elliptical annuli are simple domains for illustrating bouncing ball modes. For fixed foci (i.e., a given parameter a in the elliptic coordinates in Eq. (3.15)), these domains are characterized by two radii, R_1 ($R_1 = 0$ for filled ellipses) and R_2, as $\Omega = \{(r,\theta) \ : \ R_1 < r < R_2, \ 0 \leq \theta < 2\pi\}$, while the eigenfunctions u_{nkl} were defined in Sec. 3.4. For each $\alpha \in \left(0, \frac{\pi}{2}\right)$, we consider an elliptical sector Ω_α inside an elliptical domain Ω (Fig. 3.1)

$$\Omega_\alpha = \{(r,\theta) \ : \ R_1 < r < R_2, \ \theta \in (\alpha, \pi - \alpha) \cup (\pi + \alpha, 2\pi - \alpha)\}.$$

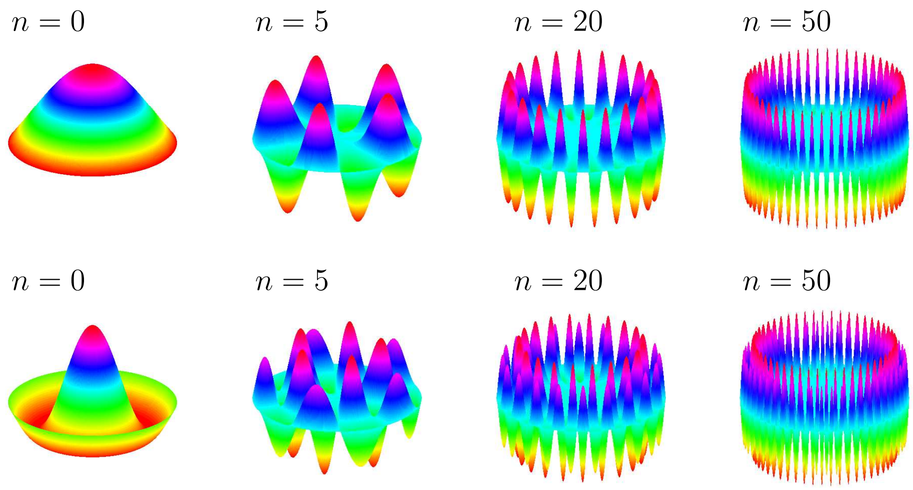

FIG. 7.11. *Formation of whispering gallery modes for the unit disk with Dirichlet boundary condition: for a fixed k ($k = 1$ for top figures and $k = 2$ for bottom figures), an increase of the index n leads to stronger localization of eigenfunctions near the boundary.*

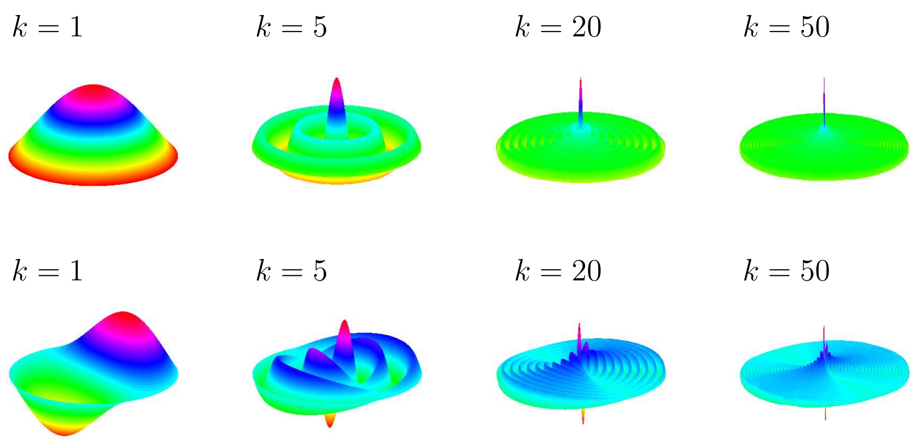

FIG. 7.12. *Formation of focusing modes for the unit disk: for a fixed n ($n = 0$ for top figures and $n = 1$ for bottom figures), an increase of the index k leads to stronger localization of eigenfunctions at the origin.*

For any $p \geq 1$, there exists $\Lambda_{\alpha,n} > 0$ such that for any eigenvalue $\lambda_{nkl} > \Lambda_{\alpha,n}$, the corresponding eigenfunction u_{nkl} satisfies [287]

$$\frac{\|u_{nkl}\|_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nkl}\|_{L_p(\Omega)}} < D_n \left(\frac{16\alpha}{\pi - \alpha/2}\right)^{1/p} \exp\left(-a\sqrt{\lambda_{nkl}}\left[\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \sin\alpha\right]\right), \tag{7.14}$$

where

$$D_n = 3\sqrt{\frac{1 + \sin\left(\frac{3\pi}{8} + \frac{\alpha}{4}\right)}{\left[\tan\left(\frac{\pi}{16} - \frac{\alpha}{8}\right)\right]^n}}.$$

Given that $\lambda_{nkl} \to \infty$ as k increases (for any fixed n and l), while the area of Ω_α can be made arbitrarily small by sending $\alpha \to \pi/2$, the estimate implies that there are

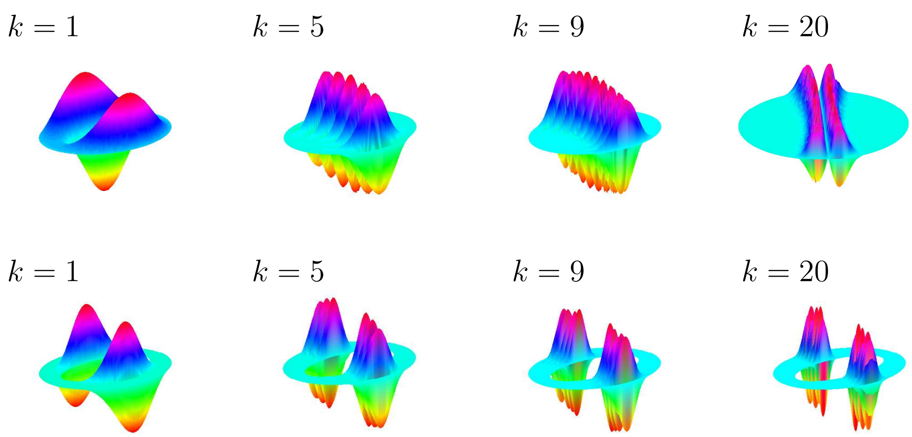

FIG. 7.13. *Formation of bouncing ball modes u_{nkl} in a filled ellipse of radius $R = 1$ (top) and an elliptical annulus of radii 0.5 and 1 (bottom), with the focal distance $a = 1$. For fixed $n = 1$ and $l = 1$, an increase of the index k leads to stronger localization of the eigenfunction near the vertical semi-axis (from [287]).*

infinitely many eigenfunctions u_{nkl} which are L_p-localized in the elliptical sector Ω_α:

$$\lim_{k\to\infty} \frac{\|u_{nkl}\|_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nkl}\|_{L_p(\Omega)}} = 0. \tag{7.15}$$

These eigenfunctions, which are localized near the minor axis, are called "bouncing ball modes" and illustrated on Fig. 7.13. The above estimate allows us to illustrate bouncing ball modes which emerge for any convex planar domain with smooth boundary [97, 217]. At the same time, the estimate is as well applicable to elliptical annuli, providing thus an example of bouncing ball modes for non-convex domains.

7.7.3. Domains without localization. The analysis of geometrical properties of eigenfunctions in rectangle-like domains $\Omega = (0, \ell_1) \times ... \times (0, \ell_d) \subset \mathbb{R}^d$ (with sizes $\ell_1 > 0$, ..., $\ell_d > 0$) may seem to be the simplest case because the eigenfunctions are expressed through sines (Dirichlet) and cosines (Neumann), as discussed in Sec. 3.1. The situation is indeed elementary when all eigenvalues are simple, i.e. $(\ell_i/\ell_j)^2$ are not rational numbers for all $i \neq j$. For any $p \geq 1$ and any open subset $V \subset \Omega$, one can prove that [287]

$$C_p(V) = \inf_{n_1,...,n_d} \left\{ \frac{\|u_{n_1,...,n_d}\|_{L_p(V)}}{\|u_{n_1,...,n_d}\|_{L_p(\Omega)}} \right\} > 0. \tag{7.16}$$

This property is in sharp contrast to Eq. (7.12) for eigenfunctions in the unit disk (or ball). The fact that $C_p(V) > 0$ for any open subset V means that there is no eigenfunction that could fully "avoid" any location inside the domain, i.e., there is no L_p-localized eigenfunction. Since the set of rational numbers has zero Lebesgue measure, there is no L_p-localized eigenfunctions in almost any randomly chosen rectangle-like domain.

When at least one ratio $(\ell_i/\ell_j)^2$ is rational, certain eigenvalues are degenerate, and the associated eigenfunctions are linear combinations of products of sines or cosines (see Sec. 3.1). Although the computation is still elementary for each eigenfunction, it is unknown whether the infimum $C_p(V)$ from Eq. (7.16) is strictly positive or not,

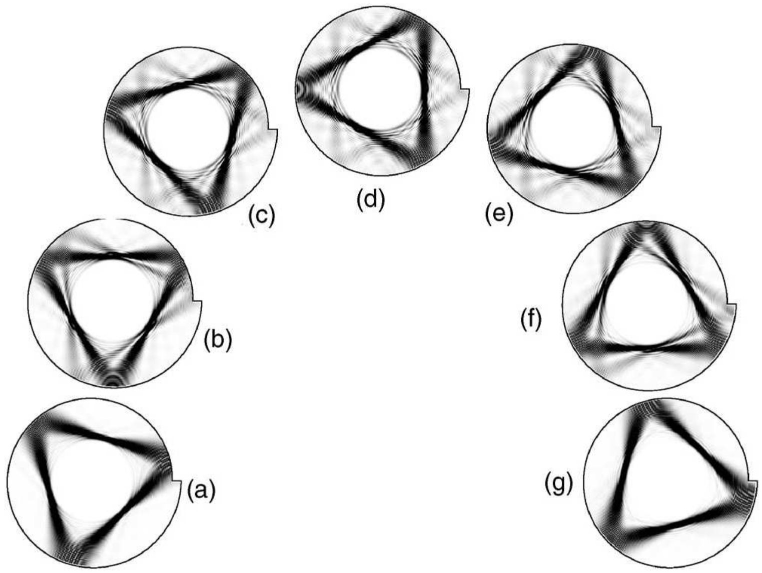

FIG. 7.14. *Several eigenstates with localization on period orbits for the spiral-shaped billiard with $\epsilon = 0.1$, from [260] (by Liu et al.).*

for arbitrary rectangle-like domain Ω and any open subset V. For instance, the most general known result for a rectangle $\Omega = (0, \ell_1) \times (0, \ell_2)$ states that $C_2(V) > 0$ for any $V \subset \Omega$ of the form $V = (0, \ell_1) \times \omega$, where ω is any open subset of $(0, \ell_2)$ [82]. Even for the unit square, the statement $C_p(V) > 0$ for any open subset V appears as an open problem. More generally, one may wonder whether $C_p(V)$ is strictly positive or not for any open subset V in polygonal (convex) domains.

7.7.4. Quantum billiards. The above examples of whispering gallery or bouncing ball modes illustrate that certain high-frequency eigenfunctions tend to be localized in specific regions of circular and elliptical domains. But what is the structure of a high-frequency eigenfunction in a general domain? What are these specific regions on which a sequence of eigenfunctions may be localized, and whether do they exist for a given domain? Answers to these and other related questions can be found by relating the high-frequency behavior of a quantum system (in our case, the structure of Laplacian eigenfunctions) to the classical dynamics in a billiard of the same shape [8, 100, 172, 257]. In particular, some orbits of a particle moving in a classical billiard may be reflected as "scars" in the spatial structure of eigenfunctions in the related quantum billiard [22, 53, 173, 174, 182, 183, 199, 210, 212, 351, 352, 372, 373]. This effect is illustrated on Fig. 7.14 by Liu and co-workers who investigated the localization of Dirichlet Laplacian eigenfunctions on classical period orbits in a spiral-shaped billiard [260] (see also [247]).

In the classical dynamics, one may distinguish the domains with regular, integrable and chaotic dynamics. In particular, for a bounded domain Ω with an ergodic billiard flow [362], Shnirelman's theorem (also known as quantum ergodicity theorem [105, 403, 404]) states that among the set of L_2-normalized Dirichlet (or Neumann) Laplacian eigenfunctions, there is a sequence u_{j_k} of density 1 (i.e., $\lim\limits_{k\to\infty} j_k/k = 1$),

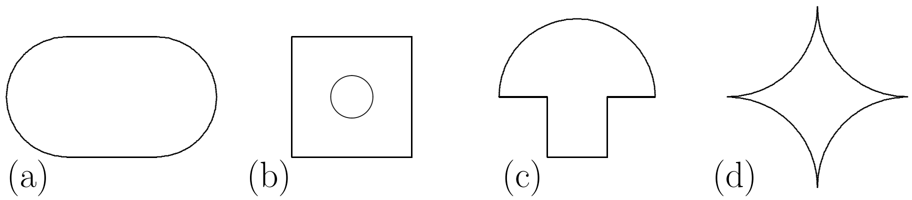

FIG. 7.15. *Examples of chaotic billiards: (a) Bunimovich stadium (union of a square and two half-disks) [75, 76, 102, 182, 380, 381], (b) Sinai's billiard [363, 364], (c) mushroom billiard [39, 77], and (d) hyperbolic billiard [2]. Many other examples are given in [76].*

such that for any open subset $V \subset \Omega$, one has [360]

$$\lim_{k\to\infty} \int\limits_V |u_{j_k}(x)|^2 dx = \frac{\mu_d(V)}{\mu_d(\Omega)}. \tag{7.17}$$

(this version of the theorem was formulated in [82]). Loosely speaking, $\{u_{j_k}\}$ is a sequence of non-localized eigenfunctions which become more and more uniformly distributed over the domain (see [82, 154, 199] for further discussion and references). At the same time, this theorem does not prevent the existence of localized eigenfunctions. How large the excluded subsequence of (localized) eigenfunctions may be? In the special case of arithmetic hyperbolic manifolds, Rudnick and Sarnak proved that there is no such excluded subsequence [339]. This statement is known as the quantum unique ergodicity (QUE). The validity of this statement for other dynamical systems (in particular, ergodic billiards) remains under investigation [38, 127, 178]. The related notion of weak quantum ergodicity was discussed by Kaplan and Heller [211]. A classification of eigenstates to regular and irregular ones was thoroughly discussed (see [322, 391] and references therein).

There were numerous studies of Laplacian eigenfunctions in chaotic domains such as Bunimovich stadium [75, 76, 102, 182, 291, 380, 381], Sinai's billiard [363, 364], mushroom billiard [39, 77] or hyperbolic billiard [2], illustrated on Fig. 7.15. Although the literature on quantum billiards is vast, we only mention selected works on the spatial structure of high-frequency eigenfunctions. McDonald and Kaufman studied the Bunimovich stadium billiard and reported a random structure of nodal lines of eigenfunctions and Wigner-type distribution for eigenvalue spacings [267, 268]. Bohigas and co-workers studied eigenvalue spacings for the Sinai's billiard and also obtained the Wigner-type distribution [62]. It means that eigenvalue spacings for these chaotic billiards obey the same distribution as that for random matrices from the Gaussian Orthogonal Ensemble. This is in a sharp contrast to regular billiards for which eigenvalue spacings generally follow a Poisson distribution. The problem of circular-sector and polygon billiards was studied (e.g., see [254–256, 337]).

Bäcker and co-workers analyzed the number of bouncing ball modes in a class of two-dimensional quantized billiards with two parallel walls [26]. Bunimovich introduced a family of simple billiards (called "mushrooms") that demonstrate a continuous transition from a completely chaotic system (stadium) to a completely integrable one (circle) [77]. Barnett and Betcke reported the first large-scale statistical study of very high-frequency eigenfunctions in these billiards [39]. Using nonstandard numerical techniques [38], Barnett also studied the rate of equidistribution for a uniformly hyperbolic, Sinai-type, planar Euclidean billiard with Dirichlet boundary condition, as illustrated on Fig. 7.16. This study brought a strong numerical evidence for the

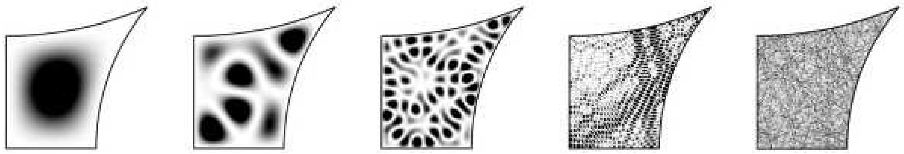

FIG. 7.16. *Illustration of spatial distribution of the Dirichlet eigenfunction* $|u_m|^2$ *(shown as density plots: larger values are darker) with* $m = 1, 10, 100, 1000$ *and* $m \approx 50000$ *[38] (by A. Barnett, with permission).*

QUE in this system. The spatial structure of high-frequency eigenfunctions shown on Fig. 7.16 looks somewhat random. This observation goes back to Berry who conjectured that high-frequency eigenfunctions in domains with ergodic flow should look locally like a random superposition of plane waves with a fixed wavenumber [50]. This analogy is nicely illustrated on Fig. 7.17 by Barnett [38]. O'Connor and co-workers analyzed the random pattern of ridges in a random superposition of plane waves [290].

Dietz and co-workers analyzed the number of nodal domains in a pseudointegrable barrier billiard [124]. Tomsovic and Heller reported a remarkable accuracy of the semiclassical approximation that relates the classical and quantum billiards [380, 381]. In some cases, eigenfunctions can therefore be constructed by purely semiclassical calculations. Li *et al.* studied the spatial distribution of eigenstates of a rippled billiard with sinusoidal walls [252]. For this type of ripple billiards, a Hamiltonian matrix can be found exactly in terms of elementary functions that greatly improves computation efficiency. They found both localized and extended eigenfunctions, as well as peculiar hexagon and circle-like pattern formations.

Prosen computed numerically very high-lying energy spectra for a generic chaotic 3D quantum billiard (a smooth deformation of a unit sphere) and analyzed Weyl's asymptotic formula and the nearest neighbor level spacing distribution. He found significant deviations from the Gaussian Orthogonal Ensemble statistics that were explained in terms of localization of eigenfunctions onto lower dimensional classically invariant manifolds [323]. He also found that the majority of eigenstates were more or less uniformly extended over the entire energy surface, except for a fraction of strongly localized scarred eigenstates [324]. An extensive study of 3D Sinai's billiard was reported by Primack and Smilansky [321]. Deviations from a semi-classical description were discussed by Tanner [378]. Casati and co-workers investigated how the interplay between quantum localization and the rich structure of the classical phase space influences the quantum dynamics, with applications to hydrogen atoms under microwave fields [91–94] (see also references therein).

A large number of physical experiments were performed with classical and quantum billiards. For instance, Gräf and co-workers measured more than thousand first eigenmodes in a quasi two-dimensional superconducting microwave stadium billiard with chaotic dynamics [161]. Sridhar and co-workers performed a series of experiments in microwave cavities in the shape of Sinai's billiard [368, 369]. In particular, they observed bouncing ball modes and modes with quasi-rectangular or quasi-circular symmetry which are associated with nonisolated periodic orbits (which avoid the central disk). Some scarring eigenstates, which are associated with isolated periodic orbits (which hit the central disk, see Fig. 7.15b), were also observed. Kudrolli *et al.* investigated the signatures of classical chaos and the role of periodic orbits in the eigenvalue spectra of two-dimensional billiards through experiments in microwave cavities

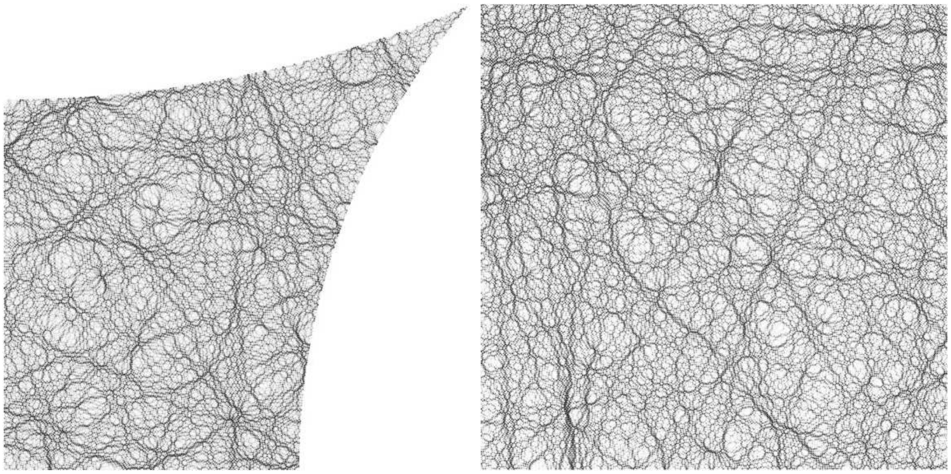

FIG. 7.17. **(Left)** *Density plot of a Dirichlet-Laplacian eigenfunction* $|u_m|^2$ *for* $m \approx 50000$ *with the eigenvalue* $\lambda_m \approx 10^6$. *There are about 225 wavelengths across the diagonal;* **(Right)** *Density plot of one sample from the ensemble of random plane waves with the same wavenumber and mean intensity, shown in a square region of space (with no boundary conditions) [38] (by A. Barnett, with permission).*

[230, 231]. The eigenvalue spectra were analyzed by using the nearest neighbor spacing distribution for short-range correlations and the spectral rigidity for longer-range correlations. The density correlation function was used for studying the spatial structure of eigenstates. The role of disorder was also investigated. Chinnery and Humphrey visualized experimentally acoustic resonances within a stadium-shaped cavity [102]. Bittner *et al.* performed double-slit experiments with regular and chaotic microwave billiards [58]. Chaotic resonators were also employed for getting specific properties of lasers (e.g., high-power directional emission or "Fresnel filtering") [157, 336].

8. Other points and concluding remarks. This review was focused on the geometrical properties of Laplacian eigenfunctions in Euclidean domains. We started from the basic properties of the Laplace operator and explicit representations of its eigenfunctions in simple domains. After that, the properties of eigenvalues and their relation to the shape of a domain were briefly summarized, including Weyl's asymptotic behavior, isoperimetric inequalities, and Kac's inverse spectral problem. The structure of nodal domains and various estimates for the norms of eigenfunctions were then presented. The main Section 7 was devoted to the spatial structure of eigenfunctions, with a special emphasis on their localization in small subsets of a domain. One of the major difficulties in the study of localization is that localization is a property of an individual eigenfunction. For the same domain, two consecutive eigenfunctions with very close eigenvalues may have drastically different geometrical structures (e.g., one is localized and the other is extended). One needs therefore fine analytical tools which would differently operate with localized and non-localized eigenfunctions. In the review, we distinguished two types of localization, for low-frequency and high-frequency eigenfunctions.

In the former case (that we also called bottleneck localization), an eigenfunction remains localized in a subset because of a geometric constraint that prohibits its extension to other parts of the domain. A standard example is a dumbbell (two domains connected by a narrow channel), for which an eigenfunction may be localized in one domain if its typical wavelength is larger than the width of the channel (meaning that an eigenfunction cannot "squeeze" through the channel). Such kind of "expulsion" from a channel is quite generic, as the analysis is applicable to domains with branches of variable cross-sectional profiles. It is important to note that a geometric constraint does not need to be strong (e.g., two domains may be separated by a cloud of point-like obstacles of zero measure). Another example is an elongated triangle, in which there is no "obstacles" at all. From a practical point of view, the low-frequency localization is important for the theory of quantum, optical and acoustical waveguides and microelectronic devices.

The high-frequency localization manifests in quantum billiards when a sequence of eigenfunctions tends to concentrate onto some orbits of the associated classical billiard. In this regime, the asymptotic properties of eigenvalues and eigenfunctions are strongly related to the underlying classical dynamics (e.g., regular, integrable or chaotic). For instance, the ergodic character of the classical system may be reflected in the spatial structure of eigenfunctions. Working on simple domains, we illustrated several kinds of localized eigenfunctions which emerge for a large class of domains. We also provided examples of rectangular domains without localization. Although a number of rigorous and numerical results were obtained (e.g., quantum ergodicity theorem for ergodic billiards), many questions about the spatial structure of high-frequency eigenfunctions remain open, even for very simple domains (e.g., a square).

Although the review is quite long and counts four hundred citations, it is far from being complete. As already mentioned, we focused on the Laplace operator in bounded Euclidean domains and mostly omitted technical details, in order to keep the review at a level accessible to scientists from various fields. Many other issues had to be omitted.

(i) Many important results for Laplacians on Riemannian manifolds or weighted graphs could not be included. In addition, we did not discuss the spectral properties of domains with "holes" [101, 224, 266, 345, 392], as well as their consequences for diffusion in domains with static traps [162, 215, 216, 286, 382].

(ii) There are important developments of numerical techniques for computing the Laplacian eigenbasis. In fact, standard finite difference or finite element methods rely on a regular or adapted discretization of a domain that reduces the continuous eigenvalue problem to a finite set of linear equations [25, 104, 110, 148, 169, 189, 232, 342]. Since finding the eigenbasis of the resulting matrix is still an expensive computational task, various hints and tricks are often implemented. For instance, for planar polygonal domains, one can exploit the behavior of eigenfunctions at corners through radial basis functions in polar coordinates and the integration of related Fourier-Bessel functions on subdomains [121, 126, 312]. Another "trick" is conformal mapping of planar polygonal domains onto the unit disk, for which the modified eigenvalue problem can be efficiently solved [28, 29]. Yet another approach known as the method of particular solutions was suggested by Fox and co-workers [145] and later progressively improved [40, 55]. The main idea is to consider various solutions of the eigenvalue equation for a given value of λ and to vary λ until a linear combination of such solutions would satisfy the boundary condition at a number of sample points along the boundary. One can also mention a stochastic method by Lejay and Maire for computing the principal eigenvalue [248]. The eigenvalue problem can also be reformulated in terms of boundary integral equations that reduces the dimensionality and allows for rapid computation of eigenvalues [262]. Kaufman and co-workers proposed a simple expansion method in which wave functions inside a two-dimensional quantum billiard are expressed in terms of an expansion of a complete set of orthonormal functions defined in a surrounding rectangle for which the Dirichlet boundary conditions apply, while approximating the billiard boundary by a potential energy step of a sufficiently large size [213].

(iii) we also did not discuss various applications of Laplacian eigenfunctions which nowadays range from pure and applied mathematics to physics, chemistry, biology and computer sciences. One can mention manifold parameterizations by eigenfunctions of the Laplacian and heat kernels [207], the use of Laplacian spectra as a diagnostic tool for network structure and dynamics [269], efficient image recognition and analysis [335, 343, 344], shape optimization and spectral partition problems [5, 85–87, 311, 367, 390], computation and analysis of diffusion-weighted NMR signals [164–166], etc.

Acknowledgments. The authors thank H. Obuse and A. Barnett for providing their images of eigenfunctions and allowing us to reproduce them. We acknowledge fruitful discussions with Y. G. Sinai and A. L. Delitsyn. We are grateful to B. Sapoval for many valuable discussions and for his passion to localization that strongly motivated our work.

Appendix: Lower estimate for the L_2-norm of Dirichlet-Laplacian eigenfunctions restricted onto subdomains. In this Appendix, we sketch the proof for the lower estimate (6.28), following and extending the ideas by Filoche and Mayboroda [143]. Although the results are formulated for the Laplace operator on domains with smooth boundaries, extensions to other elliptic operators or more general boundaries are possible.

THEOREM 8.1. *Let u be an eigenfunction of the Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary condition, and λ the associated eigenvalue. Let $D \subset \Omega$ is an open subdomain of Ω, and v the harmonic function in D with $v|_{\partial D} = u|_{\partial D}$ on a piecewise smooth boundary ∂D. Then the following inequality holds:*

$$\|u\|_{L_2(D)} \geq \frac{\lambda_1(D)}{\lambda + \lambda_1(D)} \|v\|_{L_2(D)}, \tag{8.1}$$

where $\lambda_1(D)$ is the first Dirichlet-Laplacian eigenvalue in D, and $d_D(\lambda)$ is the distance from λ to the spectrum of the Dirichlet-Laplace operator in D.

Proof. Following the proof by Filoche and Mayboroda, we consider the function $w = u - v$ which satisfies

$$-\Delta w = \lambda u \quad (x \in D), \qquad w = 0 \quad (x \in \partial D).$$

Let $\{\varphi_k^D\}$ denote the set of L_2-normalized eigenfunctions (with eigenvalues λ_k^D) of the Dirichlet-Laplace operator in D that form an orthonormal basis in $L_2(D)$. The function Δw can be expanded over this basis as

$$-\Delta w = \sum_k c_k \varphi_k^D,$$

where the coefficients c_k are

$$c_k = \int\limits_D (-\Delta w(x))\ \varphi_k^D(x) = \lambda_k^D \int\limits_D w(x) \varphi_k^D(x) dx.$$

One gets

$$(\lambda\|u\|_{L_2(D)})^2 = \|\Delta w\|^2_{L_2(D)} = \sum_k c_k^2 = \sum_k \left(\lambda_k^D \int\limits_D w(x)\varphi_k^D(x)dx\right)^2$$

$$\geq (\lambda_1^D)^2 \sum_k \left(\int\limits_D w(x)\varphi_k^D(x)dx\right)^2 \geq \left(\lambda_1^D \|w\|_{L_2(D)}\right)^2,$$

from which

$$\lambda\|u\|_{L_2(D)} \geq \lambda_1^D \|u-v\|_{L_2(D)}.$$

Adding $\lambda_1^D\|u\|_{L_2(D)}$ to both sides, one gets

$$(\lambda+\lambda_1^D)\|u\|_{L_2(D)} \geq \lambda_1^D\big(\|u-v\|_{L_2(D)} + \|u\|_{L_2(D)}\big) \geq \lambda_1^D\|v\|_{L_2(D)},$$

from which the inequality (8.1) follows. □

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