

RELATIVISTIC FORCES IN LAGRANGIAN MECHANICS

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ABSTRACT. We give a general definition of *relativistic force* in the context of Lagrangian mechanics. Once this is done we prove that the only relativistic forces which are linear on the velocities are those coming from differential 2-forms defined on the configuration space. In this sense, electromagnetic fields provide a mechanical system with the simplest type of relativistic forces.

June 8, 2012

1. INTRODUCTION

It remain in the folklore of Physics the Principles of Relativity (Special and General), which Einstein would not have needed make, if he had worked in our time, when it is assumed that the fundamental laws should be formulated in a geometric, intrinsic way.

Part of the folklore is also the inconsistency of the “Newton Mechanics” with the “Relativity”, and it is considered no necessary specify where the difference lies.

In this paper is given a precise definition of what a force relativistic is, and it is clarified how these forces are related to tensor fields. It is a bit surprising that the relativistic nature of a force has nothing to do with the metric of the space of configuration. Fundamental relativistic laws, as the Klein-Gordon equation, hold with any metric, and then also with or without limitation of the speed of interaction (see [1]).

2. THE GEOMETRY OF THE TANGENT BUNDLE AND THE SECOND ORDER EQUATIONS

Let M be a smooth manifold of dimension n , and TM be its tangent bundle. Each differential 1-form α on M can be considered as a function on TM , denoted by $\dot{\alpha}$, which assigns to each $v_a \in T_aM$ the value

$$\dot{\alpha}(v_a) = \langle \alpha_a, v_a \rangle$$

obtained by duality. In particular, a function $f \in \mathcal{C}^\infty(M)$ defines the function on TM associated to df that we denote in short by \dot{f} . This definition also applies to differential forms α on TM that are at each point the pull-back of a form on M . In the sequel we call these forms *horizontal forms*.

The map $f \mapsto \dot{f}$ from $\mathcal{C}^\infty(M)$ to $\mathcal{C}^\infty(TM)$ is a derivation of the ring $\mathcal{C}^\infty(M)$ taking values in the $\mathcal{C}^\infty(M)$ -module $\mathcal{C}^\infty(TM)$. We denote it by \dot{d} since it is essentially the differential. For each horizontal form α , we have $\dot{\alpha} = \langle \alpha, \dot{d} \rangle$ as functions on TM .

Using the vector space structure of the fibers of TM we can associate to each $v_a \in T_aM$ a tangent vector to T_aM at each point as the derivative along v_a in T_aM . Denoting by V_a this derivation, we have for $f \in \mathcal{C}^\infty(M)$ and a point $w_a \in T_aM$:

$$V_a(\dot{f})(w_a) = \lim_{t \rightarrow 0} \frac{\dot{f}(w_a + tv_a) - \dot{f}(w_a)}{t} = \dot{f}(v_a) = v_a(f) .$$

At each $w_a \in T_aM$, $V_a \in T_{w_a}(T_aM)$ is called the *vertical representative* of $v_a \in T_aM$ and v_a the *geometric representative* of V_a .

Definition 2.1. (Second Order Differential Equation). A vector field D on TM is a *second order differential equation* when its restriction (as derivation) to the subring $\mathcal{C}^\infty(M)$ of $\mathcal{C}^\infty(TM)$ is \dot{d} . This is equivalent to have $\pi_*(Dv_a) = v_a$ at each point $v_a \in T_aM$ (where $\pi: TM \rightarrow M$ denotes the canonical projection).

Definition 2.2. (Contact System). The *contact system* on TM is the Pfaff system in TM which consists of all the 1-forms annihilating all the second order differential equations. It will be denoted by Ω .

The forms in the contact system also annihilate the differences of second order differential equations, i.e. all vertical fields. Therefore, they are horizontal forms; each $\omega_{v_a} \in \Omega_{v_a}$ is the pull-back to $T_{v_a}^*TM$ of a form in T_a^*M . Now, a horizontal 1-form kills a second order differential equation if and only if it kills the field \dot{d} . Thus the contact system on TM consists of the horizontal 1-forms which annihilate \dot{d} : a horizontal form α is contact if and only if $\dot{\alpha} = 0$.

2.3. Local coordinate expressions.

We take local coordinates q^i , $i = 1, \dots, n$, in M and corresponding q^i, \dot{q}^i , in TM . We have, using Einstein summation convention,

$$\dot{d} = \dot{q}^i \frac{\partial}{\partial q^i} .$$

A vertical field has the expression

$$V = f^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} ,$$

and the one for a second order differential equation is

$$D = \dot{q}^i \frac{\partial}{\partial q^i} + f^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} .$$

Usually we will denote f^i by \ddot{q}^i understanding that it is a given function of the q 's and \dot{q} 's.

A local system of generators for the contact system Ω , out of the zero section, is given by

$$\dot{q}^i dq^j - \dot{q}^j dq^i \quad (i, j = 1, \dots, n) .$$

3. STRUCTURE OF A SECOND ORDER DIFFERENTIAL EQUATION RELATIVE TO A METRIC

Let T^*M be the cotangent bundle of M and $\pi: T^*M \rightarrow M$ the canonical projection. Recall that the *Liouville form* θ on T^*M is defined by $\theta_{\alpha_a} = \pi^*(\alpha_a)$ for $\alpha_a \in T_a^*M$. Abusing the notation we can write $\theta_{\alpha_a} = \alpha_a$.

The 2-form $\omega_2 = d\theta$ is the natural symplectic form associated to T^*M . In local coordinates (q^1, \dots, q^n) in M , and corresponding $(q^1, \dots, q^n, p_1, \dots, p_n)$ for T^*M , we have

$$\theta = p_i dq^i, \quad \omega_2 = dp_i \wedge dq^i .$$

Let T_2 be a (non-degenerate) pseudo-Riemannian metric in M . Then we have an isomorphism of vector fiber bundles

$$\begin{aligned} TM &\rightarrow T^*M \\ v_a &\mapsto i_{v_a} T_2 \end{aligned}$$

($i_{v_a} T_2$ is the inner contraction of v_a with T_2). Using the above isomorphism we can transport to TM all structures on T^*M . In particular, we work with the Liouville form θ and the symplectic form ω_2 transported in TM with the same notation.

From the definitions we have for the Liouville form in TM , at each $v_a \in T_a M$,

$$(3.1) \quad \theta_{v_a} = i_{v_a} T_2 ,$$

where the form of the right hand side is to be understood pulled-back from M to TM .

Definition 3.1. (*Kinetic Energy*). The function $T = \frac{1}{2} \dot{\theta}$ on TM is the kinetic energy associated to the metric T_2 . So, for each $v_a \in TM$, we have $T(v_a) = \frac{1}{2} \dot{\theta}(v_a) = \frac{1}{2} T_2(v_a, v_a)$.

Then, it can be proved the following main result

Theorem 3.2 ([2]). *The metric T_2 establishes a one-to-one correspondence between second order differential equations and horizontal 1-forms in TM .*

The second order differential equation D and the horizontal 1-form α that correspond to each other are related by

$$(3.2) \quad i_D \omega_2 + dT + \alpha = 0 .$$

The triple (M, T_2, α) will be called a *mechanical system*, where M is the *configuration space* which is provided with a pseudo-Riemannian metric T_2 (to which corresponds a *kinetic energy* T by (3.1)), and a *work-form* or *force form* α on TM . Finally, correspondence (3.2) will be called the *Newton Law*: under the influence of a force α the trajectories of the mechanical system satisfy (are the integral curves of) the second order differential equation D associated with α .

In particular, we consider the case of a mechanical system that is undisturbed by any force.

Definition 3.3. (Geodesic Field). The geodesic field of the metric T_2 is the second order differential equation, D_G , corresponding to $\alpha = 0$:

$$(3.3) \quad i_{D_G}\omega_2 + dT = 0 .$$

The projection to M of the curves solution of D_G in TM are the geodesics of T_2 . The geodesic field D_G is chosen as the origin in the affine bundle of second order differential equations. With this choice we establish a one-to-one correspondence between second order differential equations and vertical tangent fields.

$$D \longleftrightarrow W := D - D_G.$$

We define the covariant value of the second order differential equation D , denoted by D^∇ , as the field in TM taking values in TM corresponding canonically to $W = D - D_G$.

3.4. Local coordinate expressions.

Consider an open set of M with coordinates q^1, \dots, q^n and the corresponding open set in TM with coordinates $q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n$. If the expression in local coordinates of T_2 is

$$(3.4) \quad T_2 = g_{jk} dq^j dq^k$$

then the local equations for the isomorphism $TM \approx T^*M$ are

$$(3.5) \quad p_j = g_{jk} \dot{q}^k.$$

The Liouville form in TM is given by

$$(3.6) \quad \theta = g_{jk} \dot{q}^k dq^j$$

and for the kinetic energy we have, locally,

$$(3.7) \quad T = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j, \quad \text{so that} \quad p_j = \frac{\partial T}{\partial \dot{q}^j} .$$

Let the second order differential equation D be given by

$$(3.8) \quad D = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} ,$$

where the \ddot{q}^i 's are given function of q 's and \dot{q} 's.

Now

$$(3.9) \quad \alpha = -g_{lk}(\ddot{q}^l + \Gamma_{ij}^l \dot{q}^i \dot{q}^j) dq^k$$

is the horizontal 1-form related to D by formula (3.2). Equivalently,

$$(3.10) \quad \dot{q}^l = - \left(g^{lk} \alpha_k + \Gamma_{ij}^l \dot{q}^i \dot{q}^j \right)$$

In particular, for the geodesic field we have,

$$(3.11) \quad D_G = \dot{q}^i \frac{\partial}{\partial q^i} - \Gamma_{ij}^l \dot{q}^i \dot{q}^j \frac{\partial}{\partial \dot{q}^l}$$

and, finally, the covariant value of D is

$$(3.12) \quad D^\nabla = (\dot{q}^l + \Gamma_{ij}^l \dot{q}^i \dot{q}^j) \frac{\partial}{\partial \dot{q}^l} = -g^{lk} \alpha_k \frac{\partial}{\partial \dot{q}^l} .$$

4. RELATIVISTIC FORCES

For physicists, the motion of each point particle in Relativity (yet in the special one) is parametrized by the “proper time” of that particle, whose “infinitesimal element” ds is the so called length element associated with T_2 ; thus, in Special Relativity, it is written

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = (dx^0)^2 - \sum_1^3 (dx^i)^2$$

(by taking units such that the velocity of the light is $c = 1$).

For any metric, the length element is $\theta/\sqrt{\dot{\theta}}$, where θ is the Liouville form. In this way, the classical ds is the restriction of such a length element to the curve in TM describing the lifting of the corresponding parametrized curve in M . It turns out that length does not depend on the parametrization, and this is the reason for which it has sense to talking about the length of a curve.

If D is a second order differential equation on M , when we say that a curve solution can be parametrized by the length element ds we means that the proper parameter for such a curve solution of D is the specialization of $\theta/\sqrt{\dot{\theta}}$; that is to say, $\theta(D)/\sqrt{\dot{\theta}} = \dot{\theta}/\sqrt{\dot{\theta}} = 1$. Thus,

$$\dot{\theta} = 0 \quad \text{or} \quad \dot{\theta} = \pm 1$$

on such a curve solution of D .

Therefore, the second order differential equations on M which describe relativistic motions are vector fields D (on manifold TM) tangent to the hypersurfaces $\dot{\theta} = 0$, $\dot{\theta} = 1$, and $\dot{\theta} = -1$. Let us put $D = D_G + W$. From the Newton-Lagrange equation $i_D \omega_2 + dT + \alpha = 0$ we get

$$0 = i_D i_D \omega_2 + i_D dT + i_D \alpha = DT + \dot{\alpha}$$

on such submanifolds. As a consequence,

$$(4.1) \quad WT + \dot{\alpha} = 0$$

when $\dot{\theta} = 0, \pm 1$. On the other hand, field W is vertical and function T is homogeneous of second degree, so that

$$WT = \langle \theta, w \rangle,$$

where w is the geometric representative of W (i.e., $i_W \omega_2 = i_w T_2$). In local coordinates,

$$(4.2) \quad \langle \theta, w \rangle = g_{ij} \dot{q}^i w^j,$$

where $w = w^i \partial / \partial q^i$. If the w^i 's are homogeneous with respect to dotted coordinates \dot{q}^i , expression (4.2) vanishes for $\dot{\theta} = 1$ if and only vanishes for all $\dot{\theta}$.

Therefore, ‘relativistic’ fields $D = D_G + W$ such that W is homogeneous with respect to velocities, have to be tangent to each manifold $\dot{\theta} = k$, $k \in \mathbb{R}$. That is to say, $\dot{\theta} = 2T$ is a first integral of D and then, also of W . Since Equation (4.1) we derive $\dot{\alpha} = 0$. In other words, α is a contact differential form.

On the other hand, each second order differential equation D on M can be modified by means of a ‘relativistic constraint’ (analogous to the time constraints, see [2]) in order to be tangent to manifolds $\dot{\theta} = \text{constant}$. In the case of D being tangent to $\dot{\theta} = 0$, $\dot{\theta} = 1$, $\dot{\theta} = -1$, on these hypersurfaces does not suffer any modification. For all the above, it seems reasonable the following

Definition 4.1. A *relativistic field* on (M, T_2) is a second order differential equation D such that $DT = 0$, where T is the kinetic energy associated with T_2 .

The previous discussion gives us

Theorem 4.2. A second order differential equation D on (M, T_2) is a relativistic field if and only if the work form α associated by virtue of the Newton-Lagrange law ($i_D \omega_2 + dT + \alpha = 0$) belongs to the contact system Ω of M .

Corollary 4.3. The differential 1-forms α corresponding, as work forms, to relativistic fields, are the same, independently of the metric T_2 : they are those belonging to the contact system Ω of TM .

Corollary 4.4. A mechanical system (M, T_2, α) is relativistic if and only if, for each parametrized curve on M , solution of the system, the tangent field u has constant length (that is to say, $T_2(u, u) = \|u\|^2$ is constant along the curve).

Proof. We have $-\text{grad}(u) = D^\nabla = u^\nabla u$ along the given curve. By taking scalar product by u , it holds $T_2(u^\nabla u, u) = -\alpha(u) = -\dot{\alpha}(u)$; we can arbitrarily to fix a tangent vector in M as an initial condition for a second order differential equation; thus, we derive that $\dot{\alpha} = 0 \Leftrightarrow T_2(u^\nabla u, u) = 0$ for all curve solution of D . Then the result follows from the identity $T_2(u^\nabla u, u) = (1/2)u(T_2(u, u))$. \square

In the sense specified in Corollary 4.4, relativistic fields are the natural generalization of the geodesic field D_G , the one corresponding with $\alpha = 0$ (in the contact system).

At first glance, it is noteworthy that relativistic forces are characterized by corresponding to forms of work that are defined before any metric (the contact system of TM) and, on the other hand, given the metric, these forces are characterized by having as solutions curves whose tangent vector is of constant length, or, equivalently, by preserving the kinetic energy associated to the metric.

In local coordinates q^1, \dots, q^n for M , a local basis of the contact system in TM is the comprised by the forms $\dot{q}^i dq^j - \dot{q}^j dq^i$, ($i < j$). The 2-form $A_{ij} := dq^i \wedge dq^j$ holds

$$i_d A_{ij} = \dot{q}^i dq^j - \dot{q}^j dq^i.$$

Therefore, for each relativistic force, its 1-form of work α is $i_d \Phi_2$ where Φ_2 is a ‘‘horizontal’’ 2-form, linear combination of the A_{ij} ’ with coefficients in $\mathcal{C}^\infty(TM)$. That is to say, Φ_2 is a field on TM with values in $\Lambda_2 M$ (the fiber bundle of the differential 2-forms). Or, which is the same, Φ_2 is a section of $TM \times_M \Lambda_2 M \rightarrow TM$. For each contact 1-form α , there exists a tensor field Φ_2 with $i_d \Phi_2 = \alpha$; such a Φ_2 is not unique: it is undetermined up to a sum of forms $\phi^i \alpha_i \wedge \beta_i$ where the α_i, β_i are contact forms, and $\phi^i \in \mathcal{C}^\infty(TM)$.

Let (M, T_2, α) be a mechanical system whose equation of motion $D = D_G + W$ corresponds with a force (vertical tangent field in TM) W which depends linearly on the velocities. This means that, at the point $(a, v_a) \in T_a M$, the vector w_a is the transformed of v_a through a linear transformation $\Phi_a \in \text{Hom}(T_a M, T_a M)$. Letting the point a run along M we will have a tensor field Φ on M (a field of endomorphisms) such that $w = \Phi(\dot{d})$; then, $W = \Phi(V)$, where V is the tangent field associated with \dot{d} ($V = \dot{q}^i \partial / \partial \dot{q}^i$, in local coordinates). The work form α and the force W are related by means of the Newton-Lagrange equation $i_W \omega_2 + \alpha = 0$ or $i_w T_2 + \alpha = 0$.

If α is a relativistic force, then $\alpha = i_d \Phi_2$ for a differential 2-form on TM with values in $\Lambda^2 M$ (for short, a *horizontal 2-form*). Thus, $i_{\Phi(\dot{d})} T_2 + i_d \Phi_2 = 0$; by contracting with \dot{d} it gives us $T_2(\Phi(\dot{d}), \dot{d}) = 0$, which is equivalent to Φ being the field endomorphism associated with an alternate tensor field:

$$\Phi(\dot{d}) = -\text{grad} i_d \Phi_2.$$

Note that for any 2-form Φ_2 on M , the contact 2-form $i_d \Phi_2$ completely determines Φ_2 . As a consequence

Theorem 4.5. *The forces (vertical vector fields on TM) which are relativistic and linearly depend on the velocities are canonically associated with 2-forms on M , once the metric T_2 is given on M .*

ACKNOWLEDGEMENTS

This note is a result of a work in collaboration with my dear friend and colleague, Prof. R. Alonso.

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