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PRECISE LARGE DEVIATIONS FOR DEPENDENT REGULARLY VARYING SEQUENCES

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ABSTRACT. We study a precise large deviation principle for a stationary regularly varying sequence of random variables. This principle extends the classical results of A.V. Nagaev [44] and S.V. Nagaev [45] for iid regularly varying sequences. The proof uses an idea of Jakubowski [28, 29] in the context of central limit theorems with infinite variance stable limits. We illustrate the principle for stochastic volatility models, functions of a Markov chain satisfying a polynomial drift condition and solutions of linear and non-linear stochastic recurrence equations.

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1. Introduction

The aim of this paper is to study *precise large deviation probabilities* for sequences of dependent and heavy-tailed random variables. To make the notion of heavy tails precise, we assume that the stationary sequence (X_t) has regularly varying finite-dimensional distributions in the sense defined in Section 2.1. A particular consequence is that the distribution of a generic variable X of this sequence has regularly varying tails. This means that there exist $\alpha > 0$, $p, q \geqslant 0$ with $p + q = 1$ and a slowly varying function L such that

$$\frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} \sim p\,\frac{L(x)}{x^\alpha} \quad \text{and} \quad \frac{\mathbb{P}(X \leqslant -x)}{\mathbb{P}(|X| > x)} \sim q\,\frac{L(x)}{x^\alpha}\,, \quad x \to \infty. \tag{1.1}$$

The latter condition is often referred to as a *tail balance condition.*

In the case of an iid sequence satisfying (1.1) one can derive precise asymptotic bounds for the tails of the random walk (S_n) with step sequence (X_t) given by

$$S_0 = 0 \quad \text{and} \quad S_n = X_1 + \cdots + X_n\,, \quad n \geqslant 1\,.$$

We recall a classical result which can be found in the papers of A.V. and S.V. Nagaev [44, 45] and Cline and Hsing [13].

Theorem 1.1. *Assume that (X_i) is an iid sequence with a regularly varying distribution in the sense of (1.1). Then the following relations hold for $\alpha > 1$ and suitable sequences $b_n \uparrow \infty$:*

$$\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n - \mathbb{E}S_n > x)}{n\,\mathbb{P}(|X| > x)} - p \right| = 0 \tag{1.2}$$

and

$$\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n - \mathbb{E}S_n \leqslant -x)}{n\,\mathbb{P}(|X| > x)} - q \right| = 0\,. \tag{1.3}$$

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If $\alpha > 2$ one can choose $b_n = \sqrt{an\log n}$, where $a > \alpha - 2$, and for $\alpha \in (1,2]$, $b_n = n^{\delta+1/\alpha}$ for any $\delta > 0$. For $\alpha \leqslant 1$, (1.2) and (1.3) remain valid with $\mathbb{E}S_n$ replaced by 0 and one can choose $b_n = n^{\delta+1/\alpha}$ for any $\delta > 0$.

We call results of the type (1.2) and (1.3) a *precise large deviation principle* in contrast to the majority of results in large deviation theory where the logarithmic probabilities $n^{-1}\log \mathbb{P}(n^{-1}(Y_n - \mathbb{E}Y_n) \in A)$ are studied for sets A bounded away from zero and suitable sequences (Y_n) of random variables (not necessarily constituting a random walk) or even random elements taking values in some abstract spaces; see e.g. the monograph by Dembo and Zeitouni [17]. As a matter of fact, precise large deviation principles can be derived for iid heavy-tailed sequences more general than regularly varying ones, e.g. for the general class of random walks (S_n) with subexponential steps; see e.g. Cline and Hsing [13], Denisov et al. [18], Mogulskii [43] and the references cited therein. We also mention that Theorem 1.1 can be extended to iid regularly varying random vectors (see Section 2.1 for a definition) and an analog of Donsker's theorem for large deviations in Skorokhod space can be proved as well; see Hult et al. [26].

Theorem 1.1 serves as a benchmark result for the purposes of this paper. In this paper we extend Theorem 1.1 to suitable regularly varying stationary sequences (X_t). Various examples of precise large deviation principles have been derived in the literature. Under rather general dependence conditions on the regularly varying sequence (X_t) with index $\alpha < 2$, Davis and Hsing [14] and Jakubowski [28, 29] proved the existence of some sequences (b_n) such that $b_n^{-1}S_n \stackrel{P}{\to} 0$ and

$$\lim_{n\to\infty} \frac{\mathbb{P}(S_n > b_n)}{n\,\mathbb{P}(|X| > b_n)}\,. \tag{1.4}$$

They could in general not specify the order of magnitude of the sequences (b_n). The method of proof for these results could not be extended to the case $\alpha \geqslant 2$. Moreover, work of Lesigne and Volný [38] indicates that results of the type of Theorem 1.1 may fail for certain stationary ergodic martingale difference sequences. To be more precise, they proved that $\limsup_{n\to\infty} \mathbb{P}(S_n > n)/[n\,\mathbb{P}(|X| > n)] = \infty$ is possible for such sequences. Gantert [22] proved large deviation results of logarithmic type for stationary ergodic sequences (X_t) satisfying a geometric β-mixing condition. The latter condition ensures that the tail asymptotics do not differ from the iid case.

An analog of Theorem 1.1 for linear processes $X_t = \sum_{j=0}^\infty \psi_j Z_{t-j}$, $t \in \mathbb{Z}$, under suitable assumptions on the sequence of real numbers (ψ_j) (ensuring the existence of the infinite series) and assuming regular variation of the iid innovations (Z_t) was proved in Mikosch and Samorodnitsky [40]. The limiting constants p and q in (1.2) and (1.3), respectively, had to be replaced by quantities depending on p, q and the sequence (ψ_j). The region (b_n, ∞), where the large deviation principle holds, remains the same as for an iid regularly varying sequence.

Similar results were obtained in Konstantinides and Mikosch [34] for solutions to the stochastic recurrence equation $X_t = A_t X_{t-1} + B_t$, $t \in \mathbb{Z}$, with iid $((A_t, B_t))_{t\in\mathbb{Z}}$ with a generic element (A, B), $A, B \geqslant 0$ a.s., B regularly varying with index $\alpha > 0$ and $EA^\alpha < 1$. They showed that the limits (1.4) exist and are positive for sequences (b_n) comparable to those in Theorem 1.1; uniform results like in (1.2) and (1.3) were not achieved. For the same type of stochastic recurrence equation with B not necessarily positive, Buraczewski et al. [12] proved precise large deviation principles. The main difference to [34] is the assumption that (X_t) is regularly varying with some positive index α while (A_t, B_t) has moments of order $\alpha + \delta$ for some positive δ. In this case, the celebrated paper of Kesten [32], under appropriate conditions on the distribution of (A, B), yields that (X_t) is indeed regularly varying with index α; see also Goldie [23]. It is shown in [12] that the relation

$$\limsup_{n\to\infty} \sup_{x\geqslant b_n} \frac{\mathbb{P}(S_n > x)}{n\,\mathbb{P}(|X| > x)} < \infty$$

holds for suitable sequences $b_n \to \infty$ such that $b_n^{-1} S_n \stackrel{P}{\to} 0$. Again, the sequences (b_n) are close to those in Theorem 1.1. However, uniform relations of type (1.2) and (1.3) are not true in the unbounded regions (b_n,∞) but in bounded regions (b_n,c_n) such that $b_n\to\infty$ and $c_n = \mathrm{e}^{\,s_n}$ for $s_n\to\infty$ and $s_n = o(n)$.

In this paper, we will approach the problem of precise large deviations from a more general point of view. A key idea for this approach can be found in the papers of Jakubowski [28, 29], where this idea was used to prove central limit theory with infinite variance stable limits for the partial sums (S_n) of a general stationary sequence with regularly varying marginals; see also the recent paper Bartkiewicz et al. [3], where the same idea was exploited. The following inequality is crucial for proving the results of this paper: for every $k \geqslant 2$, some constant b_+,

$$\begin{aligned} &\Big|\frac{\mathbb{P}(S_n > x)}{n\,\mathbb{P}(|X|>x)} - b_+\Big| \\ (1.5)\qquad \leqslant\ &\Big|\frac{\mathbb{P}(S_n>x) - n\,(\mathbb{P}(S_{k+1}>x) - \mathbb{P}(S_k>x))}{n\,\mathbb{P}(|X|>x)}\Big| + \Big|\frac{\mathbb{P}(S_{k+1}>x) - \mathbb{P}(S_k>x)}{\mathbb{P}(|X|>x)} - b_+\Big|\,. \end{aligned}$$

Regular variation of (X_t) ensures that the second quantity in (1.5) is negligible, by first letting $x\to\infty$ and then $k\to\infty$. The first expression in (1.5) provides a link between the asymptotics of the tail $\mathbb{P}(S_n>x)$ for increasing values of n, $x \geqslant b_n$ and the regularly varying tails $\mathbb{P}(S_k>x)$ and $\mathbb{P}(S_{k+1}>x)$ for every fixed k. Thus the tail asymptotics of $\mathbb{P}(S_n>x)$ are derived from the known tail asymptotics for finite sums, again by first letting $n\to\infty$ and then $k\to\infty$.

This paper is organized as follows. In Section 2 we introduce some of the basic conditions and notions needed throughout the paper. These include regular variation of a stationary sequence and an anti-clustering condition. In Section 3 we prove the main result of this paper: Theorem 3.1 provides a general precise large deviation principle for regularly varying stationary sequences. Under regular variation and anti-clustering conditions we will show precise large deviation principles of the following type:

$$(1.6)\qquad \lim_{n\to\infty}\sup_{x\in\Lambda_n}\Big|\frac{\mathbb{P}(S_n>x)}{n\,\mathbb{P}(|X|>x)} - b_+\Big| = 0\,,$$

for some non-negative constant b_+ and a sequence of sets $\Lambda_n\subset(0,\infty)$ such that $b_n = \inf\Lambda_n\to\infty$. In Section 4 we will apply the large deviation principle (1.6) to a variety of important regularly varying time series models, including the stochastic volatility model, solutions to stochastic recurrence equations and functions of Markov chains. These are examples of rather different dependence structures, showing that the large deviation principle does not depend on a particular mixing condition or on the Markov property.

However, we give special emphasis to functions of a Markov chain satisfying a polynomial drift condition. Theorems 4.6 and 4.10 are our main results for Markov chains. Theorem 4.6 is obtained by a direct application of Theorem 3.1, exploiting a sophisticated exponential bound for partial sums of Markov chains due to Bertail and Clémencon [8]. Theorem 4.6 implies Theorem 4.10. It yields an intuitive interpretation of relation (1.6) in terms of the regeneration property of $(X_t)_{t=1,\dots,n}$. Given an atom A of the underlying chain, one can split the chain into a random number $N_A(n)$ of iid random cycles. Denoting the block sum of the X_t's over the ith cycle by $S_{A,i}$, it will be shown that the iid $S_{A,i}$'s inherit regular variation from X, and then we can apply the classical result of Theorem 1.1 to $\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x\Big)$. If $b_+>0$ the tails $\mathbb{P}_A(S_{A,1}>x)$ and $\mathbb{P}(|X|>x)$ are equivalent. There is a major difference between an iid sequence and the dependent sequence (X_t): if the first generation time τ_A is larger than n, it has significant influence on the region Λ_n, where

(1.6) holds. It turns out that one has for any $x \geqslant b_n$,

$$\frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(|X| > x)} \sim b_+ + \frac{\mathbb{P}(S_n > x, \tau_A > n)}{n\mathbb{P}(|X| > x)}\,,$$

and the second term is in general not negligible, leading to the fact that (1.6) may only be valid in a bounded region (b_n, c_n). Thus we found an explanation for the same observation we experienced in the case of a Markov chain given by a stochastic recurrence equation; see the discussion above.

2. Preliminaries

2.1. **Regular variation.** Throughout this paper we assume that (X_t) is stationary. Such a sequence is *regularly varying with index* $\alpha > 0$ if the finite-dimensional distributions of (X_t) have a jointly regularly varying distribution in the following sense: for every $d \geqslant 1$, there exists a non-null Radon measure μ_d on the Borel σ-field of $\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, (this means that μ_d is finite on sets bounded away from zero) such that

$$n\,\mathbb{P}(a_n^{-1}(X_1,\ldots,X_d) \in \cdot) \stackrel{v}{\to} \mu_d(\cdot)\,,$$

where $\stackrel{v}{\to}$ denotes vague convergence (see e.g. [31, 51]) and (a_n) satisfies $n\,\mathbb{P}(|X| > a_n) \sim 1$. The limiting measures have the property $\mu_d(tA) = t^{-\alpha}\mu_d(A)$, $t > 0$, for any Borel set A. We refer to α as the *index of regular variation* of (X_t) and its finite-dimensional distributions. We refer to Basrak and Segers [6] for an insightful description of regular variation for stationary processes.

In what follows, we refer to condition $\mathbf{RV}_\alpha$ if (X_t) satisfies the conditions above for some $\alpha > 0$ and a sequence of limiting measures (μ_d).

In Section 4 we will consider some prominent examples of regularly varying time series.

The regular variation property of (X_t) implies that the limits

$$b_+(k) = \lim_{x\to\infty} \frac{\mathbb{P}(S_k > x)}{\mathbb{P}(|X| > x)} = \lim_{n\to\infty} n\,\mathbb{P}(S_k > a_n), \qquad k \geqslant 1, \tag{2.1}$$

exist. These quantities play a crucial role in our investigations on large deviations; see for example Theorem 3.1. The limiting constants

$$b_-(k) = \lim_{x\to\infty} \frac{\mathbb{P}(S_k \leqslant -x)}{\mathbb{P}(|X| > x)} = \lim_{n\to\infty} n\,\mathbb{P}(S_k \leqslant -a_n), \qquad k \geqslant 1, \tag{2.2}$$

also exist by virtue of regular variation of (X_t).

In our main result Theorem 3.1 we require that the limit

$$b_+ = \lim_{k\to\infty} (b_+(k+1) - b_+(k))$$

exists; the existence of b_+ does not directly follow from regular variation of (X_t). In the examples of Section 4 we show that b_+ is easily calculated for some major time series models. If b_+ exists it is non-negative since it is the limit of a Cèsaro mean: $b_+ = \lim_{k\to\infty} k^{-1} b_+(k)$.

The constants b_+ and b_- (the latter constant is defined in the straightforward way) figure prominently in asymptotic results for the partial sums (S_n) with infinite variance stable limits. Indeed, the Lévy measure ν of the stable limit has representation $\nu(x,\infty) = b_+ x^{-\alpha}$ and $\nu(-\infty,-x) = b_- x^{-\alpha}$, $x > 0$; see Bartkiewicz et al. [3].

2.2. **Anti-clustering condition.** Assume that (X_t) satisfies the regular variation condition $\mathbf{RV}_\alpha$. For studying the limit theory for the extremes of dependent sequences it is common to assume *anti-clustering conditions*; see e.g. Leadbetter et al. [35], Leadbetter and Rootzén [36] and Embrechts et al. [20], Chapter 5. These conditions ensure that possible clusters of exceedances of high thresholds by the sequence (X_t) cannot be too large. In other words, "long-range dependencies of extremes" are avoided. Anti-clustering conditions are also needed for proving asymptotic theory for partial sums with infinite variance stable limits; see Davis and Hsing [14], Jakubowski [28, 29], Basrak and Segers [7], and Bartkiewicz et al. [3]. In the latter reference, the different conditions are discussed and compared. Davis and Hsing [14] and Jakubowski [28, 29] also proved large deviation results in the case $\alpha < 2$ under anti-clustering conditions.

We introduce the following anti-clustering condition which is close to those in the literature mentioned above.

Condition $\mathbf{AC}_\alpha$: There exist $\delta_k \downarrow 0$ as $k \to \infty$ and a sequence of sets $\Lambda_n \subset (0,\infty)$, $n = 1,2,\ldots$, with $b_n = \inf \Lambda_n$ such that $n\,\mathbb{P}(|X| > b_n) \to 0$ as $n \to \infty$ and

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} \delta_k^{-\alpha} \sum_{j=k}^{n} \mathbb{P}(|X_j| > x\delta_k \mid |X_0| > x\delta_k) = 0\,.$$

This condition is tailored for the purposes of our paper: the sets (Λ_n) with $\lim_{n\to\infty} b_n = \infty$ are those which appear in the precise large deviation results (1.6).

Condition $\mathbf{AC}_\alpha$ is easily verified for the examples of time series models in Section 4.

3. Main result

In this section we formulate and prove the main result on precise large deviation principles for regularly varying stationary sequences.

Theorem 3.1. *Assume that the stationary sequence (X_t) of real-valued random variables satisfies the following conditions.*

1. *The regular variation condition $\mathbf{RV}_\alpha$ for some $\alpha > 0$.*
2. *The anti-clustering condition $\mathbf{AC}_\alpha$ for a sequence $\delta_k = o(k^{-2})$, $k \to \infty$, and sets (Λ_n) such that $b_n = \inf \Lambda_n \to \infty$ as $n \to \infty$.*
3. *The limit $b_+ = \lim_{k\to\infty}(b_+(k+1) - b_+(k))$ exists, where the constants $(b_+(k))$ are defined in (2.1).*
4. *For the sequences (Λ_n), (δ_k) from $\mathbf{AC}_\alpha$ and a sequence (ε_k) satisfying $\varepsilon_k = o(k^{-1})$ and $(k+1)\delta_k \leqslant \varepsilon_k$,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} \frac{\mathbb{P}\Big(\sum_{i=1}^n X_i \mathbb{1}_{\{|X_i|\leqslant \delta_k\, x\}} > \varepsilon_k x\Big)}{n\,\mathbb{P}(|X| > x)} = 0. \tag{3.1}$$

Then the large deviation principle (1.6) *holds.*

The corresponding result for the left tails $\mathbb{P}(S_n \leqslant -x)$, $x > 0$, is obtained by replacing the variables X_t by $-X_t$, $t \in \mathbb{Z}$. Then one also needs to assume that the limit b_- exists which is defined correspondingly.

Remark 3.2. In the case $\alpha < 1$, (3.1) is satisfied for suitable choices of (δ_k) and (ε_k). Indeed, an application of Markov's inequality and Karamata's theorem (see Bingham at al. [9]) yields uniformly for $x > b_n$,

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^n X_i \mathbb{1}_{\{|X_i|\leqslant \delta_k\, x\}} > \varepsilon_k x\Big) &\leqslant (x\varepsilon_k)^{-1}\, n\, \mathbb{E}|X| \mathbb{1}_{\{|X|\leqslant \delta_k\, x\}}\\ &\sim \delta_k^{1-\alpha} \varepsilon_k^{-1}\, n\, \mathbb{P}(|X| > x)\,.\end{aligned}$$

Thus (3.1) is satisfied for $\Lambda_n = (b_n, \infty)$ if we choose e.g. $\delta_k = \mathrm{e}^{-k}$ and $\varepsilon_k = k^{-2}$.

Remark 3.3. Assume $\alpha \in (0,2)$ and (X_t) conditionally independent and symmetric given some σ-field $\mathcal{F}$. This condition is often satisfied in models of the type $X_t = \sigma_t\, Z_t$ with iid symmetric (Z_t), for example if (Z_t) and (σ_t) are independent; see the stochastic volatility model of Section 4.2.

Alternatively, if (σ_t) is predictable with respect to the filtration generated by the sequence (Z_t) then (X_t) is conditionally independent and symmetric. Prominent examples of this type are GARCH-type models, where (Z_t) is often assumed iid standard normal or student distributed. Indeed, first applying the Chebyshev inequality conditional on $\mathcal{F}$ and then taking expectations, we obtain by Karamata's theorem (see Bingham at al. [9]) uniformly for $x \in \Lambda_n = (b_n, \infty)$,

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^n X_i \mathbb{1}_{\{|X_i|\leqslant \delta_k x\}} > \varepsilon_k x\Big) &\leqslant (\varepsilon_k x)^{-2} n\, \mathbb{E}X^2 \mathbb{1}_{\{|X|\leqslant \delta_k x\}} \\ &\sim \delta_k^{2-\alpha}\varepsilon_k^{-2} n\,\mathbb{P}(|X|>x)\,.\end{aligned}$$

Thus (3.1) holds e.g. for $\delta_k = \mathrm{e}^{-k}$ and $\varepsilon_k = k^{-2}$.

Remark 3.4. Recall that (b_n) is chosen such that $n\,\mathbb{P}(|X| > b_n) \to 0$. For an iid (X_t), this condition is necessary for the weak law of large numbers $b_n^{-1} S_n \stackrel{P}{\to} 0$. Under this and some other mild conditions, we may assume without loss of generality that the random variables $(X_i \mathbb{1}_{\{|X_i|\leqslant \delta_k x\}})$ in (3.1) are mean corrected. Indeed, we will prove that

$$n \sup_{x\in\Lambda_n} x^{-1} |\mathbb{E}X \mathbb{1}_{\{|X|\leqslant x\}}| = o(1)\,, \quad n\to\infty\,. \tag{3.2}$$

This condition is trivial if X is symmetric.
The case $\alpha < 1$. By Karamata's theorem and the choice of (b_n),

$$n\,|\mathbb{E}X\mathbb{1}_{\{|X|\leqslant x\}}| \leqslant n\,\mathbb{E}|X|\mathbb{1}_{\{|X|\leqslant x\}} \sim c\,n\,x\,\mathbb{P}(|X|>x) \leqslant c\,x\,n\mathbb{P}(|X|>b_n) = o(x)\,.$$

Here and in what follows, we write c for any positive constants whose value is not of interest, for example, the same c may denote different constants in the same formula.
The case $\alpha = 1$. If $\mathbb{E}X = 0$ and $n = O(b_n)$ then $n\,|\mathbb{E}X\mathbb{1}_{\{|X|\leqslant x\}}| = o(n) = o(x)$. If $\mathbb{E}|X| = \infty$, $E|X|\mathbb{1}_{\{|X|\leqslant x\}}$ is a slowly varying function, and therefore for large n and any small $\epsilon > 0$, $n\,|EX\mathbb{1}_{\{|X|\leqslant x\}}| \leqslant n\,x^{\epsilon}$. If $b_n = n^{1+\delta}$ for some $\delta > 0$, choosing ϵ sufficiently small, we obtain $n\,|EX\mathbb{1}_{\{|X|\leqslant x\}}| = o(x)$.
The case $\alpha > 1$. By Karamata's theorem, since $EX = 0$ and by the choice of (b_n), as $n \to \infty$,

$$\begin{aligned}n\,|\mathbb{E}X\mathbb{1}_{\{|X|\leqslant x\}}| &= n\,|\mathbb{E}X\mathbb{1}_{\{|X|>x\}}| \leqslant n\,\mathbb{E}|X|\mathbb{1}_{\{|X|>x\}} \\ &\sim c\,n\,x\,\mathbb{P}(|X|>x) \leqslant c\,x\,[n\mathbb{P}(|X|>b_n)] = o(x)\,.\end{aligned}$$

Proof. We have for fixed $k \geqslant 2$,

$$\begin{aligned}&\sup_{x\in\Lambda_n}\Big|\frac{\mathbb{P}(S_n>x)}{n\,P(|X|>x)} - b_+\Big| \\ &\leqslant \sup_{x\in\Lambda_n}\Big|\frac{\mathbb{P}(S_n>x) - n\,(\mathbb{P}(S_{k+1}>x)-\mathbb{P}(S_k>x))}{n\,\mathbb{P}(|X|>x)}\Big| + \sup_{x\in\Lambda_n}\Big|\frac{\mathbb{P}(S_{k+1}>x)-\mathbb{P}(S_k>x)}{\mathbb{P}(|X|>x)} - b_+\Big| \\ &= I_{1,k} + I_{2,k}\,.\end{aligned}$$

By regular variation of (X_t), the limit

$$\lim_{n\to\infty} I_{2,k} = |(b_+(k+1) - b_+(k)) - b_+|$$

exists for every $k \geqslant 2$ and any sequence (Λ_n) such that $\inf \Lambda_n \to \infty$. By assumption, $\lim_{k\to\infty} |(b_+(k+1) - b_+(k) - b_+| = 0$. Therefore it suffices to study the asymptotic behavior of $I_{1,k}$.

For any $\delta > 0$ and $x > 0$, consider

$$\overline{X}_i = X_i \mathbb{1}_{\{|X_i|\leqslant x\delta\}} \quad \text{and} \quad \underline{X}_i = X_i \mathbb{1}_{\{|X_i|>x\delta\}}\,, \quad i=1,2,\ldots.$$

and for $n \geqslant 1$,

$$\overline{S}_n = \sum_{i=1}^n \overline{X}_i \quad \text{and} \quad \underline{S}_n = \sum_{i=1}^n \underline{X}_i \,.$$

Then, for any $\varepsilon \in (0,1)$ and $j \geqslant 1$,

$$\mathbb{P}(\underline{S}_j > (1+\varepsilon)\,x) - \mathbb{P}(-\overline{S}_j > \varepsilon\,x) \leqslant \mathbb{P}(S_j > x) \leqslant \mathbb{P}(\underline{S}_j > (1-\varepsilon)\,x) + \mathbb{P}(\overline{S}_j > \varepsilon\,x).$$

Multiple application of these inequalities yields

$$A_1 + A_2 + A_3 \leqslant \frac{\mathbb{P}(S_n > x) - n\,(\mathbb{P}(S_{k+1} > x) - \mathbb{P}(S_k > x))}{n\,\mathbb{P}(|X| > x)} \leqslant B_1 + B_2 + B_3\,,$$

where

$$\begin{aligned}
A_1 &= \frac{\mathbb{P}(\underline{S}_n > (1+\varepsilon)x) - n\,(\mathbb{P}(\underline{S}_{k+1} > (1+\varepsilon)x) - \mathbb{P}(\underline{S}_k > (1+\varepsilon)x))}{n\,\mathbb{P}(|X| > x)}\,,\\
A_2 &= \frac{-\mathbb{P}(-\overline{S}_n > \varepsilon x) - n\,(\mathbb{P}(\overline{S}_{k+1} > \varepsilon x) - \mathbb{P}(-\overline{S}_k > \varepsilon x))}{n\,\mathbb{P}(|X| > x)}\,,\\
A_3 &= \frac{\mathbb{P}(\underline{S}_{k+1} > (1+\varepsilon)x) - \mathbb{P}(\underline{S}_{k+1} > (1-\varepsilon)x)}{\mathbb{P}(|X| > x)}\,,\\
B_1 &= \frac{\mathbb{P}(\underline{S}_n > (1-\varepsilon)x) - n\,(\mathbb{P}(\underline{S}_{k+1} > (1-\varepsilon)x) - \mathbb{P}(\underline{S}_k > (1-\varepsilon)x))}{n\,\mathbb{P}(|X| > x)}\,,\\
B_2 &= \frac{\mathbb{P}(\overline{S}_n > \varepsilon x) + n\,(\mathbb{P}(-\overline{S}_{k+1} > \varepsilon x) + \mathbb{P}(\overline{S}_k > \varepsilon x))}{n\,\mathbb{P}(|X| > x)}\,,\\
B_3 &= \frac{\mathbb{P}(\underline{S}_{k+1} > (1-\varepsilon)x) - \mathbb{P}(\underline{S}_{k+1} > (1+\varepsilon)x)}{\mathbb{P}(|X| > x)}.
\end{aligned}$$

We will derive upper bounds for the B_i's. Lower bounds for the A_i's can be derived in the same way and are therefore omitted.

An application of Jakubowski [29], Lemma 3.2, to the stationary sequence $(\underline{X}_t)$ yields for fixed $k \geqslant 2$, $x, \delta, \varepsilon > 0$,

$$\begin{aligned}
|B_1| &\leqslant 3\,\frac{k\,\mathbb{P}(|X| > \delta\,x)}{n\,\mathbb{P}(|X| > x)} + 2\sum_{j=k}^n \frac{\mathbb{P}(|X_j| > \delta\,x, |X_0| > \delta\,x)}{\mathbb{P}(|X| > x)}\\
&= B_{11} + B_{12}\,.
\end{aligned}$$

In view of regular variation of X, $\mathbb{P}(|X| > \delta\,x)/\mathbb{P}(|X| > x) \to \delta^{-\alpha}$. Hence

$$\limsup_{n\to\infty} \sup_{x\in\Lambda_n} B_{11} = 0\,, \quad k \geqslant 2\,,$$

An application of $\mathbf{AC}_\alpha$ with $\delta = \delta_k$ yields that $\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} B_{12} = 0$. Hence

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} B_1 = 0\,.$$

Next consider B_2. In addition to the condition $\varepsilon = \varepsilon_k = o(k^{-1})$ assume that $(k+1)\delta_k \leqslant \varepsilon_k$. This choice is always possible since we also assume $\delta = \delta_k = o(k^{-2})$. Then $|\overline{S}_{k+1}| \leqslant \varepsilon\,x$, $\mathbb{P}(-\overline{S}_{k+1} > \varepsilon x) = \mathbb{P}(\overline{S}_k > \varepsilon x) = 0$ and B_2 degenerates to the expression $\mathbb{P}(\overline{S}_n > \varepsilon x)/(n\,\mathbb{P}(|X| > x))$. By assumption (3.1), this condition is asymptotically negligible.

Finally, consider B_3. Fix $k \geqslant 2$. In what follows, the constants $\varepsilon, \delta \in (0,1)$ will also depend on k. Consider the sets

$$A_{\gamma,\delta}(k) = \Big\{\mathbf{y} \in \mathbb{R}^k : \sum_{i=1}^k y_i \mathbb{1}_{\{|y_i|>\delta\}} > \gamma\Big\}, \quad \gamma\,,\delta > 0\,.$$

Observe that

$$\{\underline{S}_{k+1} > \gamma x\} = \{x^{-1}(X_1,\ldots,X_{k+1}) \in A_{\gamma,\delta}(k+1)\}\,,$$

the sets $A_{\gamma,\delta}(k)$ are bounded away from 0 and $A_{\gamma,\delta}(k) = \gamma A_{1,\delta/\gamma}(k)$. Condition $\mathbf{RV}_\alpha$ ensures the existence of the limit

$$\begin{aligned}
&\lim_{x\to\infty} B_3\\
&= \mu_{k+1}(A_{1-\varepsilon,\delta}) - \mu_{k+1}(A_{1+\varepsilon,\delta})\\
&= (1-\varepsilon)^{-\alpha}\mu_{k+1}(A_{1,\delta/(1-\varepsilon)}) - (1+\varepsilon)^{-\alpha}\mu_{k+1}(A_{1,\delta/(1+\varepsilon)})\\
&= ((1-\varepsilon)^{-\alpha} - (1+\varepsilon)^{-\alpha})\mu_{k+1}(A_{1,\delta/(1-\varepsilon)}) - (1+\varepsilon)^{-\alpha}(\mu_{k+1}(A_{1,\delta/(1+\varepsilon)}) - \mu_{k+1}(A_{1,\delta/(1-\varepsilon)}))\\
&= B_{31} + B_{32}\,.
\end{aligned}$$

By a Taylor expansion, $B_{31} \leqslant c\,\varepsilon\,\mu_{k+1}(A_{1,\delta/(1+\varepsilon)})$. We observe that

$$\begin{aligned}
&\Big\{\mathbf{y} \in \mathbb{R}^{k+1} : \sum_{i=1}^k y_i > 1 + k\delta/(1+\varepsilon)\Big\}\\
\subset\ & A_{1,\delta/(1+\varepsilon)} = \Big\{\mathbf{y} \in \mathbb{R}^{k+1} : \sum_{i=1}^k y_i > 1 + \sum_{i=1}^k y_i \mathbb{1}_{\{|y_i|\leqslant\delta/(1+\varepsilon)\}}\Big\}\\
\subset\ & \Big\{\mathbf{y} \in \mathbb{R}^{k+1} : \sum_{i=1}^k y_i > 1 - k\delta/(1+\varepsilon)\Big\}\,.
\end{aligned} \tag{3.3}$$

Assume that $\delta = \delta_k = o(k^{-1})$ as $k \to \infty$. Then for k sufficiently large,

$$B_{31} \leqslant c\,\varepsilon\,(1 - k\delta/(1+\varepsilon))^{-\alpha}\,\mu_{k+1}\Big(\Big\{\mathbf{y} \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} y_i > 1\Big\}\Big) \leqslant c\,\varepsilon\,b_+(k+1)\,.$$

Since we assume that b_+ exists a Cèsaro limit argument yields that $\lim_{k\to\infty} k^{-1}b_+(k+1) = b_+$. Now choose $\varepsilon = \varepsilon_k = o(k^{-1})$. Then $\lim_{k\to\infty} B_{31} = 0$. Similar arguments, using (3.3), yield

$$\begin{aligned}
B_{32} &\leqslant c\,b_+(k+1)\big((1 - k\delta/(1-\varepsilon))^{-\alpha} - (1 + k\delta/(1+\varepsilon))^{-\alpha}\big)\\
&\leqslant ck\,\delta\,b_+(k+1) = o(1)\,, \quad k \to \infty\,,
\end{aligned}$$

provided $\delta = \delta_k = o(k^{-2})$. Thus we proved that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} B_3 = 0\,.$$

This concludes the proof. □

4. Examples

In this section we want to apply Theorem 3.1 to a variety of time series models. Since there exists a calculus for multivariate regular variation (e.g. Resnick [51, 52], Hult and Lindskog [24, 25], Basrak and Segers [6]) it is not difficult to show the regular variation condition $\mathbf{RV}_\alpha$, the anti-clustering condition $\mathbf{AC}_\alpha$ and the existence of the limit $b_+ = \lim_{k\to\infty}(b_+(k+1) - b_+(k))$ for the examples below. However, it can take some efforts to prove condition (3.1). In the iid case, one would use exponential inequalities of Nagaev-Fuk or Prokhorov type; see e.g. the monograph Petrov [47] for an overview of such inequalities. In the case of dependent sequences (X_t) analogs of these inequalities exist, but their application is not always straightforward; see e.g. the case of Markov chains in Section 4.3 below.

4.1. **m_0-dependent sequences.** In this section we consider an m_0-dependent regularly varying sequence. A typical example of such a process is a moving average process of order $m_0 \geqslant 1$ (MA(m_0)) given by

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_{m_0} Z_{t-m_0}\,, \quad t \in \mathbb{Z}\,,$$

where (Z_t) is an iid regularly varying sequence with index $\alpha > 0$. Condition $\mathbf{RV}_\alpha$ is straightforward since (Z_t) is regularly varying with limiting measures concentrated on the axes. The regular variation of the finite-dimensional distributions of (X_t) is then an application of the continuous mapping theorem for regular variation; see Hult and Lindskog [24, 25]; cf. Hult et al. [26], Jessen and Mikosch [30].

A related example is given by a stochastic volatility model $X_t = \sigma_t \eta_t$, $t \in \mathbb{Z}$, where $(\log \sigma_t)$ constitutes an MA(m_0) process independent of the iid regularly varying sequence (η_t) with index α. If $E\sigma^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$ then (X_t) is regularly varying with index α; see e.g. Davis and Mikosch [15, 16]. By construction, (X_t) is m_0-dependent.

For m_0-dependent sequences the verification of the conditions of Theorem 3.1 is simple.

Proposition 4.1. *Consider an m_0-dependent stationary sequence (X_t) for some $m_0 \geqslant 1$. Assume that (X_t) satisfies $\mathbf{RV}_\alpha$ for some $\alpha > 0$ and $\mathbb{E}X = 0$ if $\mathbb{E}|X| < \infty$. Choose $b_n = n^{(1/\alpha)\vee 0.5+\delta}$ for any $\delta > 0$. Then Theorem 3.1 holds with $b_+ = b_+(m_0+1) - b_+(m_0)$ in the regions $\Lambda_n = (b_n, \infty)$.*

Proof. Condition $\mathbf{AC}_\alpha$ is trivially satisfied for any choice of constants $\delta_k \downarrow 0$ as $k \to \infty$ and any sets $\Lambda_n \subset (0, \infty)$ such that $n\mathbb{P}(|X| > b_n) \to 0$ as $n \to \infty$. Moreover, $b_+ = b_+(m_0+1) - b_+(m_0)$ follows from Bartkiewicz et al. [3].

It remains to prove that (3.1) holds. In view of the m_0-dependence of the sequence (X_t) it is possible to split the sum in (3.1) into two sums of independent subsums consisting of at most m_0 summands. More precisely, with the convention that $X_j = 0$ if $j > n$ we write

$$\begin{aligned}
\overline{S}_n &= \sum_{i=1}^n X_i \mathbb{1}_{\{|X_i| \leqslant \delta_k\, x\}} \\
&= \sum_{j=1, j \text{ even}}^{[n/m_0]} \sum_{i=m_0 j+1}^{m_0(j+1)} X_i \mathbb{1}_{\{|X_i| \leqslant \delta_k\, x\}} + \sum_{j=1, j \text{ odd}}^{[n/m_0]} \sum_{i=m_0 j+1}^{m_0(j+1)} X_i \mathbb{1}_{\{|X_i| \leqslant \delta_k\, x\}} \\
&= \overline{S}'_n + \overline{S}''_n.
\end{aligned}$$

Since

$$\mathbb{P}(\overline{S}_n \geqslant \varepsilon_k x) \leqslant \mathbb{P}(\overline{S}'_n \geqslant \varepsilon_k x/2) + \mathbb{P}(\overline{S}''_n \geqslant \varepsilon_k x/2)$$

we obtain an upper bound similar to (3.1) but with sums of at most $[n/2m_0]$ iid subsums. Therefore we may assume without loss of generality that the (X_t) in (3.1) are iid. In view of Remark 3.4 and the conditions above we may assume without loss of generality that the summands in (3.1) are mean corrected.

For $\alpha\in(0,2)$, an application of Chebyshev's inequality and Karamata's theorem yield the estimate

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^n (X_i\mathbb{1}_{\{|X_i|\leqslant\delta_k\,x\}}-\mathbb{E}X\mathbb{1}_{\{|X|\leqslant\delta_k\,x\}})>\varepsilon_k x\Big) &\leqslant n\,(\varepsilon_k x)^{-2}\mathbb{E}X^2\mathbb{1}_{\{|X|\leqslant\delta_k\,x\}}\\ &\sim \varepsilon_k^{-2}\delta_k^{2-\alpha}[n\,\mathbb{P}(|X|>x)]\,.\end{aligned}$$

Now choose e.g. $\delta_k=\mathrm{e}^{-k}$ and $\varepsilon_k=k^{-2}$. Then all assumptions on (ε_k) and (δ_k) in Theorem 3.1 are satisfied and $\lim_{k\to\infty}\varepsilon_k^{-2}\delta_k^{2-\alpha}=0$. Hence (3.1) is satisfied.

In the case $\alpha>2$, we use the Nagaev-Fuk inequality (cf. Petrov [47], p. 78, 2.6.5) for $p>\alpha$ and Karamata's theorem as $n\to\infty$, for $x\in\Lambda_n$:

$$\begin{aligned}&\mathbb{P}\Big(\sum_{i=1}^n (X_i\mathbb{1}_{\{|X_i|\leqslant\delta_k\,x\}}-\mathbb{E}X\mathbb{1}_{\{|X|\leqslant\delta_k\,x\}})>\varepsilon_k x\Big)\\ &\leqslant c\,(\varepsilon_k x)^{-p}n\,\mathbb{E}|X|^p\mathbb{1}_{\{|X|\leqslant x\delta_k\}}+\mathrm{e}^{-c(\varepsilon_k x)^2/n}\\ &\leqslant c\,\Big(\delta_k^{p-\alpha}\varepsilon_k^{-p}+\mathrm{e}^{-c(\varepsilon_k x)^2/n}/[n\,\mathbb{P}(|X|>x)]\Big)[n\,\mathbb{P}(|X|>x)]\,.\end{aligned}$$

Choosing $\delta_k=\mathrm{e}^{-k}$ and $\varepsilon_k=k^{-2}$, the requirements of Theorem 3.1 are satisfied and $\delta_k^{p-\alpha}\varepsilon_k^{-p}$ becomes arbitrarily small for large k. Moreover, $\sup_{x\in\Lambda_n}\mathrm{e}^{-c(\varepsilon_k x)^2/n}/[n\,\mathbb{P}(|X|>x)]\to 0$ by the choice of (b_n). This proves (3.1) for $\alpha>2$.

The boundary case $\alpha=2$ can be treated in a similar way by using another version of the Nagaev-Fuk inequality; see Petrov [47], p. 78, 2.6.4. We omit details. □

4.2. **Stochastic volatility model.** Consider a stationary sequence (σ_t) of non-negative random variables and assume that (Z_t) is an iid sequence which is independent of (σ_t). The stationary sequence

$$X_t=\sigma_t\,Z_t,\quad t\in\mathbb{Z},\tag{4.1}$$

is then called a *stochastic volatility model*. It is a standard model in financial time series analysis; see e.g. Andersen et al. [2].

The main result of this section is a large deviation principle for such models under various assumptions.

Theorem 4.2. *Consider a stochastic volatility model* (4.1) *such that Z is regularly varying with index $\alpha>0$, $\mathbb{E}Z=0$ for $\alpha>1$ and $\mathbb{E}\sigma_0^{2\alpha}<\infty$. Moreover, consider the following additional conditions:*

(1) *Z is symmetric.*
(2) *$\mathbb{E}\sigma_0^p<\infty$ for some $p>2\alpha$ and (σ_t) is strongly mixing with rate (α_j) such that $\alpha_j\leqslant cj^{-a}$ for some $a>1$.*

The large deviation principle (1.6) *holds with $b_+=\lim_{x\to\infty}\mathbb{P}(Z>x)/\mathbb{P}(|Z|>x)$ in the regions $\Lambda_n=(b_n,\infty)$ under the following conditions:*

- *$0<\alpha<1$: $b_n=n^{\varepsilon+1/\alpha}$ for any $\varepsilon>0$.*
- *$1<\alpha<2$: Assume* (1) *or* (2)*, $b_n=n^{\varepsilon+1/\alpha}$ for any $\varepsilon>0$.*
- *$\alpha>2$: Assume* (2) *for some $a>\max(1,(\alpha-2)p/(2p-\alpha))$, $b_n=\sqrt{n\log n}s_n$ for any sequence (s_n) such that $s_n\to\infty$.*

Remark 4.3. A Gaussian stationary process (Y_t) is strongly mixing under mild conditions; see Kolmogorov and Rozanov [33]. Ibragimov [27], Theorem 5, gave necessary and sufficient conditions for the relation $\alpha_n=O(n^{-a})$ for any choice of $a>0$. The conditions are in terms of the spectral density of (Y_t). It is also known that a linear Gaussian process $Y_t=\sum_{j=0}^\infty\psi_j\eta_{t-j}$, $t\in\mathbb{Z}$, with (η_t) iid standard normal and exponentially decaying coefficients (ψ_j) has an exponentially decaying mixing rate (α_j); see Pham and Tran [48]; cf. Doukhan [19]. For example, if (Y_t) is a causal Gaussian ARMA process the latter condition is satisfied.

Now assume $\log \sigma_t = Y_t$, $t \in \mathbb{Z}$, for a Gaussian stationary sequence (Y_t). This Gaussian model is chosen in the majority of the literature on stochastic volatility models; see e.g. Andersen et al. [2]. Then (σ_t) inherits strong mixing from (Y_t) with the same rate. Of course, $\mathbb{E}\sigma^p < \infty$ for all $p > 0$ and the large deviation principle holds for $\alpha_n = O(n^{-a})$ for any $a > 1$.

Remark 4.4. If (σ_t) is strongly mixing with rate (α_j), the corresponding stochastic volatility model (X_t) is strongly mixing with rate $(4\alpha_j)$; see e.g. Davis and Mikosch [15].

Proof. Condition $\mathbf{RV}_\alpha$ was verified for stochastic volatility models under the condition $\mathbb{E}\sigma^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$ in Davis and Mikosch [15]; see also [16]. The limit measures of the regularly varying finite-dimensional distributions are concentrated on the axes and therefore $b_+ = \lim_{x\to\infty} \mathbb{P}(Z > x)/\mathbb{P}(|Z| > x)$; see also Bartkiewicz et al. [3].

Next we verify condition $\mathbf{AC}_\alpha$. Fix any $\delta > 0$. We have

$$p_j(\delta) = \mathbb{P}(|X_j| > x\delta, |X_0| > \delta x) \leqslant \mathbb{P}(|Z_j Z_0|\sigma_j\sigma_0) > (\delta x)^2)\,.$$

The random variable $|Z_j Z_0|$ is regularly varying with index α; see Embrechts and Veraverbeke [21]. An application of Markov's and Hölder's inequalities yields for $\epsilon < 2\alpha$,

$$p_j(\delta) \leqslant (\delta x)^{-2\alpha+\epsilon}(\mathbb{E}|Z|^{\alpha-\epsilon/2})^2\, E|\sigma_j\sigma_0|^{\alpha-\epsilon/2} \leqslant (\delta x)^{-2\alpha+\epsilon}(\mathbb{E}|Z|^{\alpha-\epsilon/2})^2\, E|\sigma|^{2\alpha-\epsilon}\,.$$

We also have for any small $\epsilon > 0$ and large x, $P(|X| > \delta x) \geqslant (\delta x)^{-\alpha-\epsilon}$ in view of the regular variation of X. Therefore

$$\sup_{x>b_n} \delta^{-\alpha} \sum_{j=k}^n \mathbb{P}(|X_j| > x\delta \mid |X_0| > x\delta) \leqslant c\, n\, \delta^{-\alpha+2\epsilon} b_n^{-\alpha+2\epsilon}\,.$$

The right-hand side converges to zero if we choose $\alpha \leqslant 2$, $b_n = n^{\varepsilon+1/\alpha}$ for any $\varepsilon > 0$ or $\alpha > 2$, $b_n = \sqrt{n\log n} s_n$, $s_n \to \infty$ and ϵ sufficiently small. The choice of $\delta = \delta_k \to 0$ is arbitrary.

Next we prove condition (3.1).

The case $\alpha < 1$. Condition (3.1) is immediate from Remark 3.2 for $\delta_k = \mathrm{e}^{-k}$ and $\varepsilon_k = k^{-2}$.

The following decomposition will be useful in the case $\alpha > 1$:

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^n \sigma_i Z_i \mathbb{1}_{\{|\sigma_i Z_i| \leqslant \delta_k x\}} > \varepsilon_k x\Big)\\
&\quad\leqslant \mathbb{P}\Big(\sum_{i=1}^n [\sigma_i Z_i \mathbb{1}_{\{|\sigma_i Z_i| \leqslant \delta_k x\}} - \sigma_i \mathbb{E}(Z\mathbb{1}_{\{|\sigma_i Z| \leqslant \delta_k x\}} \mid \sigma_i)] > (\varepsilon_k/2)x\Big)\\
&\qquad+\mathbb{P}\Big(\sum_{i=1}^n \sigma_i \mathbb{E}(Z\mathbb{1}_{\{|\sigma_i Z| \leqslant \delta_k x\}} \mid \sigma_i) > (\varepsilon_k/2)x\Big) = I_1 + I_2\,.
\end{aligned}$$

Lemma 4.5. *Assume $\alpha > 1$, and either Z is symmetric or (σ_t) is strongly mixing with rate function (α_j) satisfying $\alpha_j \leqslant cj^{-a}$ for some $c > 0, a > 1$ and $E\sigma^p < \infty$ for some $p > 2\alpha$. Then*

$$\lim_{n\to\infty} \sup_{x>b_n} \frac{I_2}{n\,\mathbb{P}(|X| > x)} = 0\,.$$

Proof. In the case of symmetric Z, $I_2=0$. Thus we deal with the case of mixing (σ_t). First observe that for any $y>0$,

$$\begin{aligned} I_2 &\leqslant \mathbb{P}\Big(\sum_{i=1}^n \sigma_i \mathbb{1}_{\{\sigma_i\leqslant y\}}\mathbb{E}(Z\mathbb{1}_{\{|\sigma_i Z|\leqslant \delta_k\, x\}}\mid \sigma_i)>(\varepsilon_k/2)x\Big)+n\,\mathbb{P}(\sigma>y)\\ &= I_{21}+I_{22}\,. \end{aligned}$$

Clearly, since $\mathbb{E}\sigma^p<\infty$ for some $p>2\alpha$, we can find $y=y(x)=o(x)$, $y\to\infty$ as $x\to\infty$ such that

$$\sup_{x>b_n}\frac{I_{22}}{n\,\mathbb{P}(|X|>x)}=\sup_{x>b_n}\frac{\mathbb{P}(\sigma>y)}{\mathbb{P}(|X|>x)}=o(1)\,.$$

Indeed, we can choose $y=x^{0.5-\gamma}$ for any $\gamma>0$ close to zero. Write

$$\overline{\sigma}_i=\sigma_i\mathbb{1}_{\{\sigma_i\leqslant y\}}\mathbb{E}(Z\mathbb{1}_{\{|\sigma_i Z|\leqslant\delta_k\,x\}}\mid\sigma_i)\,,\quad i=1,2,\ldots\,,$$

and $\overline{S}_n=\sum_{i=1}^n\overline{\sigma}_i$. The Markov inequality yields

$$\begin{aligned} \mathbb{P}(\overline{S}_n>\varepsilon_k x) &\leqslant (\varepsilon_k x)^{-2}\mathbb{E}\overline{S}_n^2\\ &= (\varepsilon_k x)^{-2}\Big[n\mathbb{E}\overline{\sigma}^2+2\sum_{j=1}^{n-1}(n-j)\mathbb{E}(\overline{\sigma}_0\overline{\sigma}_j)\Big]=I_3+I_4\,. \end{aligned}$$

Then, since $\mathbb{E}Z=0$, by Karamata's theorem

$$\frac{I_3}{n\,\mathbb{P}(|X|>x)}\leqslant c\,\frac{x^{-2}[\mathbb{E}|Z|\mathbb{1}_{\{|Z|>\delta_k x/y\}}]^2}{\mathbb{P}(|X|>x)}\leqslant c\,\frac{y^{-2}[\mathbb{P}(|X>x/y)]^2}{\mathbb{P}(|X|>x)}\,.$$

The right-hand side is negligible uniformly for $x>b_n$. We also have

$$\begin{aligned} \frac{(n/x)^2\,(\mathbb{E}\overline{\sigma})^2}{n\mathbb{P}(|X|>x)} &= \frac{n(\mathbb{E}(X_1\mathbb{1}_{\{|X_1|>\delta_k x\,,\sigma_1\leqslant y\}}))^2}{x^2\mathbb{P}(|X|>x)}\\ &\leqslant \frac{n(E|X|\mathbb{1}_{\{|X|>\delta_k x\}})^2}{x^2\mathbb{P}(|X|>x)}\\ &\leqslant c\,n\,P(|X|>x)\leqslant n\,P(|X|>b_n)\to 0\,. \end{aligned}$$

Therefore we may assume without loss of generality that the random variables $\overline{\sigma}_j$ in I_4 are centered. Using a classical bound for the covariance of a strongly mixing sequence, the fact that $\mathbb{E}Z=0$ and Karamata's theorem, for $r,q>0$ such that $r^{-1}+2q^{-1}=1$, $1<r<a$,

$$\begin{aligned} |\mathrm{cov}(\overline{\sigma}_0,\overline{\sigma}_j)| &\leqslant c\,\alpha_j^{1/r}\,(\mathbb{E}\overline{\sigma}^q)^{2/q}\\ &\leqslant c\,\alpha_j^{1/r}y^2[\mathbb{E}(|Z|\mathbb{1}_{\{|Z|>\delta_k x/y\}})]^2\\ &\leqslant c\,\alpha_j^{1/r}x^2[\mathbb{P}(|Z|>x/y)]^2\,. \end{aligned}$$

Finally, we get the following bound

$$\sup_{x>b_n}\frac{I_4}{n\,\mathbb{P}(|X|>x)}\leqslant c\sum_{j=1}^{\infty}\alpha_j^{1/r}\sup_{x>b_n}\frac{[\mathbb{P}(|Z|>x/y)]^2}{\mathbb{P}(|X|>x)}\,.$$

The right-hand side converges to zero. This proves the lemma. □

The case $\alpha \in (1,2)$. In view of Lemma 4.5 it remains to bound I_1. Applying Chebyshev's inequality conditionally on (σ_i) we obtain

$$\begin{aligned} I_1 &\leqslant (\varepsilon_k x)^{-2}\mathbb{E}\Big[\sum_{i=1}^n \sigma_i^2 \operatorname{var}(Z\mathbb{1}_{\{|\sigma_i Z|\leqslant \delta_k x\}} \mid \sigma_i)\Big] \\ &\leqslant (\varepsilon_k x)^{-2} n\,\mathbb{E}(X^2\mathbb{1}_{\{|X|\leqslant \delta_k x\}})\,. \end{aligned}$$

Now an application of Karamata's theorem and regular variation of X yield

$$\sup_{x>b_n}\frac{I_1}{n\,\mathbb{P}(|X|>x)} \leqslant c\sup_{x>b_n}\frac{\delta_k^2}{\varepsilon_k^2}\frac{\mathbb{P}(|X|>x\delta_k)}{\mathbb{P}(|X|>x)} \sim c\,\frac{\delta_k^{2-\alpha}}{\varepsilon_k^2}\,.$$

Now choose (δ_k) and (ε_k) as in the case $\alpha<1$ to conclude that

$$\lim_{k\to\infty}\sup_{x>b_n}\frac{I_1}{n\,\mathbb{P}(|X|>x)} = 0\,.$$

The finishes the proof of (3.1) in the case $\alpha \in (1,2)$.

The case $\alpha > 2$. We again have to study I_1. Using the Nagaev-Fuk inequality (cf. Petrov [47], p. 78, 2.6.5) conditionally on (σ_t), we obtain for $p>\alpha$,

$$\begin{aligned} &\mathbb{P}\Big(\sum_{i=1}^n[\sigma_i Z_i\mathbb{1}_{\{|\sigma_i Z_i|\leqslant\delta_k x\}} - \sigma_i\,\mathbb{E}(Z_i\mathbb{1}_{\{|\sigma_i Z_i|\leqslant\delta_k x\}}\mid\sigma_i)] > (\varepsilon_k/2)x \mid (\sigma_i)\Big) \\ &\leqslant c\,(\varepsilon_k x)^{-p}\sum_{i=1}^n \sigma_i^p\mathbb{E}(|Z_i|^p\mathbb{1}_{\{|\sigma_i Z_i|\leqslant\delta_k x\}}\mid\sigma_i) + \mathrm{e}^{-c(\varepsilon_k x)^2/\sum_{i=1}^n\sigma_i^2}\,. \end{aligned}$$

The expectation of the first term is of the asymptotic order $c\delta_k^{p-\alpha}/\varepsilon_k^p$. The latter relation converges to zero for $\delta_k = \mathrm{e}^{-k}$ and $\varepsilon_k = k^{-2}$. Consider the expectation of the second term on the sets $\{\sum_{i=1}^n\sigma_i^2 > c(\varepsilon_k x)^2/(2\alpha\log x)\}$ and its complement to obtain the bound

$$\mathbb{E}(\mathrm{e}^{-c(\varepsilon_k x)^2/\sum_{i=1}^n\sigma_i^2}) \leqslant x^{-2\alpha} + \mathbb{P}\Big(\sum_{i=1}^n\sigma_i^2 > c(\varepsilon_k x)^2/(2\alpha\log x)\Big).$$

The first term is negligible with respect to $n\mathbb{P}(|X|>x)$. For the second one, note that $x^2/(n\log x)\geqslant c\,b_n^2/(n\log b_n)\to\infty$. Therefore we may assume without loss of generality that the σ_i^2's are mean corrected. Now use Rio [53], p. 87, (6.19a), under the mixing condition $\alpha_j \leqslant cj^{-a}$ to obtain for any $r\geqslant 1$:

$$\begin{aligned} &\mathbb{P}\Big(\sum_{i=1}^n(\sigma_i^2-\mathbb{E}\sigma^2) > c(\varepsilon_k x)^2/(2\alpha\log x)\Big) \\ &\leqslant c\,n^{r/2}(\log x)^r x^{-2r} + c\,n(\log(x)/x^2)^{(a+1)p/(a+p)}. \end{aligned}$$

The first term is negligible with respect to $n\,P(|X>x)$ for r sufficiently large. The second term is negligible as well if $2(a+1)p/(a+p)>\alpha$. The latter condition is satisfied by assumption. □

4.3. **Regularly varying functions of Markov chains.** In this section we assume that $X_t = h(\Phi_t)$, $t\in\mathbb{Z}$, is a measurable real-valued function of a stationary Markov chain (Φ_t) which possesses an atom A in some general space: The context is classical; see Nummelin [46] and Meyn and Tweedie [39] which will serve as our main references, and (Φ_t) can be seen as the enlargement of a Harris recurrent Markov chain. In Section 4.4 we will look at the example of a solution to a stochastic recurrence equation which constitutes such a Markov chain. We assume that the function h is such that (X_t) is regularly varying with index $\alpha>0$. Notice in particular that h is not the null function.

Throughout we will also assume the following *polynomial drift condition* for $p > 0$ which is inspired by Samur [54] who used a more general condition.

- $\mathbf{DC}_p$: There exist constants $\beta \in (0,1)$, $b > 0$ such that for any y,

$$\mathbb{E}(|h(\Phi_1)|^p \mid \Phi_0 = y) \leqslant \beta\,|h(y)|^p + b\,\mathbb{1}_A(y).$$

In this condition, we suppress the dependence of β, b, A on the value p. Note that $\mathbf{DC}_p$ implies geometric ergodicity of (Φ_t); see Meyn and Tweedie [39], p. 371. In what follows, we write τ_A for the first time the chain visits the set A, $\mathbb{P}_A$ denotes the probability measure of the Markov chain conditional on $\{\Phi_0 \in A\}$ and $\mathbb{E}_A$ is the corresponding expectation. We will also write $\mathbb{P}_x$ and $\mathbb{E}_x$ if $\{\Phi_0 = x\}$.

Here is the main result of this section.

Theorem 4.6. *Assume that (Φ_t) is a stationary Markov chain possessing an atom A and that h is a function such that $X_t = h(\Phi_t)$, $t \in \mathbb{Z}$, satisfies the conditions (1) − (3) of Theorem 3.1 for the regions $\Lambda_n = (b_n, c_n)$ specified below. Also assume $\mathbb{E}X = 0$ if $\mathbb{E}|X| < \infty$ and $\mathbf{DC}_p$ for all $p < \alpha$. Then the precise large deviation principle (1.6) holds under the following conditions:*

- $0 < \alpha < 1$*:* $\Lambda_n = (b_n, \infty)$ *for any sequence* (b_n) *satisfying* $n\mathbb{P}(|X| > b_n) \to 0$.
- $1 < \alpha$ *and* $\alpha \neq 2$*:* $\Lambda = (b_n, c_n)$ *for any sequence* (b_n) *satisfying* $b_n = n^{1/\alpha \vee 0.5+\delta}$ *for any* $\delta > 0$*, and* (c_n) *such that* $c_n > b_n$ *and*

$$\mathbb{P}(\tau_A > n) = o(n\,\mathbb{P}(|X| > c_n))\,. \tag{4.2}$$

Proof. We will apply Theorem 3.1. Since we assumed conditions (1)–(3) of this result it remains to verify (3.1).

The case $0 < \alpha < 1$. The proof follows from Remark 3.2.

The case $\alpha > 1$ *and* $\alpha \notin \mathbb{N}$. This case is more involved. We will prove it in a similar way as in the iid or m_0-dependent cases, by using moment and exponential inequalities tailored for regenerative split Markov chains. Without loss of generality we will only consider the strongly aperiodic case.

Notice that $\mathbf{DC}_p$ is satisfied for $p = [\alpha]$. For all integers $p < [\alpha]$, applying Jensen's inequality, we obtain

$$\begin{aligned}
\mathbb{E}(|X_1|^p \mid \Phi_0 = y) &\leqslant \Big(\mathbb{E}(|X_1|^{[\alpha]} \mid \Phi_0 = y)\Big)^{p/[\alpha]} \\
&\leqslant \Big(\beta\,|h(y)|^{[\alpha]} + b\,\mathbb{1}_A(y)\Big)^{p/[\alpha]} \leqslant \beta^{p/[\alpha]}\,|h(y)|^p + b^{p/[\alpha]}\,\mathbb{1}_A(y).
\end{aligned} \tag{4.3}$$

Thus $b > 0$, $\beta \in (0,1)$ and A in $\mathbf{DC}_p$ can be chosen the same as in $\mathbf{DC}_{[\alpha]}$.

Let $(\tau_A(j))_{j\geqslant 1}$ be the sequence of visiting times of the Markov chain to the set A, i.e. $\tau_A(1) = \tau_A$ and $\tau_A(j+1) = \min\{k > \tau_A(j) : \Phi_k \in A\}$. Notice that the sequence $(\tau_A(j+1) - \tau_A(j))_{j\geqslant 1}$ constitutes an iid sequence and $N_A(t) = \#\{j \geqslant 1 : \tau_A(j) \leqslant t\}$, $t \geqslant 0$, is a renewal process. The following inequality holds for any integrable function f on $\mathbb{R}$:

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^n f(X_i) > \varepsilon_k x\Big)\\
&= \mathbb{P}\Big(\sum_{i=1}^n f(X_i) > \varepsilon_k x\,, N_A(n)=0\Big) + \mathbb{P}\Big(\sum_{i=1}^n f(X_i) > \varepsilon_k x\,, N_A(n)=1\Big)\\
&\quad + \mathbb{P}\Big(\sum_{i=1}^n f(X_i) > \varepsilon_k x\,, N_A(n)\geqslant 2\Big)\\
&\leqslant \mathbb{P}(\tau_A > n) + 2\mathbb{P}\Big(\sum_{j=1}^{\tau_A} f(X_j) > \varepsilon_k x/3, \tau_A \leqslant n\Big)\\
&\quad + \mathbb{P}\Big(\sum_{j=1}^{N_A(n)-1}\ \sum_{t=\tau_A(j)+1}^{\tau_A(j+1)} f(X_j) > \varepsilon_k x/3\Big) + 2\mathbb{P}\Big(\sum_{i=\tau_A(N_A(n))+1}^{n} f(X_i) > \varepsilon_k x/3\Big)\\
&= I_1 + I_2 + I_3 + I_4\,.
\end{aligned}$$

We mentioned in Remark 3.4 that we may assume without loss of generality that the random variables $\overline{X}_i$, $i=1,2,\ldots$, are mean corrected. Now we choose $f(X_i) = \overline{X}_i - \mathbb{E}\overline{X}_i$ where

$$\overline{X}_i = X_i 1\!\!1_{\{|X_i|\leqslant \delta_k x\}}\,, \quad i=1,2,\ldots, \quad x>0\,.$$

Bounds for I_1, I_2, I_4. For I_4, we use the Markov inequality of order $k_0 = [\alpha]+1$ and the stationarity of (X_i)

$$\begin{aligned}
I_4 &\leqslant c\,(x\varepsilon_k)^{-k_0}\Big[\mathbb{E}\Big|\sum_{i=\tau_A(N_A(n))+1}^{n} \overline{X}_i\Big|^{k_0}\Big] + \mathbb{E}\tau_A^{k_0}\ [\mathbb{E}|X| 1\!\!1_{\{|X|>\delta_k x\}}]^{k_0}\Big]\\
&\leqslant c\,x^{-k_0}\Big[\mathbb{E}\Big(\sum_{i=\tau_A(N_A(n))+1}^{n} |\overline{X}_i|\Big)^{k_0} + [x\,\mathbb{P}(|X|>x)]^{k_0}\Big]\\
&\leqslant c\,x^{-k_0}\Big[\mathbb{E}_A\Big(\sum_{i=1}^{\tau_A} |\overline{X}_i|\Big)^{k_0} + [x\,\mathbb{P}(|X|>x)]^{k_0}\Big]\,.
\end{aligned}$$

Since for $\alpha > 1$, $k_0 \geqslant 2$, we use Proposition 4.7 given below to show that I_4 is negligible with respect to $n\mathbb{P}(|X|>x)$. As to I_2, we again use the Markov inequality:

$$\begin{aligned}
I_2 &\leqslant c\,(x\varepsilon_k)^{-k_0}\Big[\mathbb{E}\Big|1_{\{\tau_A\leqslant n\}}\sum_{i=1}^{\tau_A} \overline{X}_i\Big|^{k_0} + \mathbb{E}\tau_A^{k_0}\ [\mathbb{E}|X| 1\!\!1_{\{|X|>\delta_k x\}}]^{k_0}\Big]\\
&\leqslant c\,x^{-k_0}\Big[\mathbb{E}\Big(1_{\{\tau_A\leqslant n\}}\sum_{i=1}^{\tau_A} |\overline{X}_i|\Big)^{k_0} + [x\,\mathbb{P}(|X|>x)]^{k_0}\Big]\,.
\end{aligned}$$

We iteratively apply Lemma 4.8 given below to the first term in the right-hand side to obtain an estimate of I_2 proportional to

$$x^{-k_0}\mathbb{E}\Big(1_{\{\tau_A\leqslant n\}}\sum_{i=1}^{\tau_A} |\overline{X}_i|^{k_0}\Big) = x^{-k_0}\mathbb{E}\Big(\sum_{i=1}^{n} |\overline{X}_i|^{k_0} 1\!\!1_{\{\tau_A\geqslant i\}}\Big). \tag{4.4}$$

An application of Pitman's identity [50] yields

$$\begin{aligned}\mathbb{E}\Big(\sum_{i=1}^n |\overline{X}_i|^{k_0}\mathbb{1}_{\{\tau_A\geqslant i\}}\Big) &= \mathbb{P}(\Phi_0\in A)\,\mathbb{E}_A\Big(\sum_{k=0}^{\tau_A-1}\sum_{i=1}^n |\overline{X}_{k+i}|^{k_0}\mathbb{1}_{\{\tau_A\geqslant k+i\}}\Big)\\ &\leqslant n\mathbb{P}(\Phi_0\in A)\,\mathbb{E}_A\Big(\sum_{i=1}^{\tau_A}|\overline{X}_i|^{k_0}\Big).\end{aligned}$$

From a Wald-type identity, $I_2\leqslant cn(x\varepsilon_k)^{-k_0}\mathbb{E}|\overline{X}|^{k_0}$. Hence I_2 is negligible with respect to $n\mathbb{P}(|X|>x)$ by an application of Karamata's theorem.

Finally, I_1 is negligible with respect to $n\mathbb{P}(|X|>x)$ because we assume that $\mathbb{P}(\tau_A>n)=o(n\mathbb{P}(|X|>c_n))$.

Bounds for I_3. The following moment inequality is the key to the bound of I_3:

Proposition 4.7. *Assume that $(X_t)=(h(\Phi_t))$ for a real-valued measurable function h and a Markov chain (Φ_t) satisfying the drift condition $\mathbf{DC}_{k_0-1}$ for some integer $k_0\geqslant 2$. Then for $x>0$ and some constant $c>0$,*

$$\mathbb{E}_A\Big(\sum_{j=1}^{\tau_A}|\overline{X}_j|\Big)^{k_0}\leqslant c\,\mathbb{E}|\overline{X}|^{k_0}\,. \tag{4.5}$$

Proof. We can expand the left-hand side of (4.5) as follows

$$\mathbb{E}_A\Big(\sum_{j=1}^{\tau_A}|\overline{X}_j|\Big)^{k_0}=\sum_{k=1}^{k_0}\sum_{\sum_{i=1}^k s_i=k_0,s_i\geqslant1,i=1,\dots,k}\mathbb{E}_A\Big(\sum_{j_1=1}^{\tau_A}\sum_{j_2=j_1+1}^{\tau_A}\cdots\sum_{j_k=j_{k-1}+1}^{\tau_A}|\overline{X}_{j_i}|^{s_i}\Big)\,. \tag{4.6}$$

We will estimate the moments on the right-hand side by employing Lemma 4.8 below. For the cases $k_0=2,3$ such a result was proved by Samur [54] and we use the idea of the proof in [54] for our generalization. Before we formulate the basic moment estimate we need some notation: According to the proof of Theorem 14.2.3 of Meyn and Tweedie [39], there exists a constant $c(A)>0$ such that

$$\mathbb{E}_{\Phi_0}\Big(\sum_{k=1}^{\tau_A}1_A(X_k)\Big)\leqslant c(A)\quad\text{a.s.}$$

Lemma 4.8. *Assume $\mathbf{DC}_p$ and let f,g be non-negative measurable functions on $\mathbb{R}$ such that $f(x)\leqslant|y|^p$ and $g(y)=0$ for $|y|>\delta_k x$. Then for any $\ell\geqslant1$, $n\in\mathbb{N}\cup\{\infty\}$*

$$\mathbb{E}\Big(1_{\{\tau_A\leqslant n\}}\sum_{j=\ell}^{\tau_A}g(X_j)\sum_{i=j+1}^{\tau_A}f(X_i)\mid\mathcal{F}_\ell\Big)\leqslant\mathbb{E}\Big(1_{\{\tau_A\leqslant n\}}\sum_{j=\ell}^{\tau_A}g(X_j)[C\,|\overline{X}_j|^p+b\,c(A)]\mid\mathcal{F}_\ell\Big)\,, \tag{4.7}$$

where $\mathcal{F}_\ell=\sigma((\Phi_t)_{t\leqslant\ell})$.

Proof. As mentioned in Samur [54], $\{\tau_A\geqslant j\}\in\mathcal{F}_j$ for all j. Therefore

$$\begin{aligned}\mathbb{E}\Big(1_{\{\tau_A\leqslant n\}}\sum_{j=\ell}^{\tau_A}g(X_j)\sum_{i=j+1}^{\tau_A}f(X_i)\mid\mathcal{F}_\ell\Big) &= \sum_{j=\ell}^n\mathbb{E}\Big(\mathbb{1}_{\{\tau_A\geqslant j\}}g(X_j)\sum_{i=j+1}^{\tau_A}f(X_i)\mid\mathcal{F}_\ell\Big)\\ &= \sum_{j=\ell}^n\mathbb{E}\Big(\mathbb{1}_{\{\tau_A\geqslant j\}}g(X_j)\mathbb{E}\Big(\sum_{i=j+1}^{\tau_A}f(X_i)\mid\mathcal{F}_j\Big)\mid\mathcal{F}_\ell\Big)\\ &\leqslant \sum_{j=\ell}^n\mathbb{E}\Big(\mathbb{1}_{\{\tau_A\geqslant j\}}g(X_j)\,\mathbb{E}_{\Phi_j}\Big(\sum_{i=1}^{\tau_A}f(X_i)\Big)\mid\mathcal{F}_\ell\Big)\,.\end{aligned}$$

In the last inequality we used the stationarity of (Φ_t) and the strong Markov property. From Theorem 14.2.3 of Meyn and Tweedie [39] we obtain

$$\mathbb{E}_{\Phi_j}\Big(\sum_{i=1}^{\tau_A} f(X_i)\Big) \leqslant C\,|X_j|^p + b\,c(A).$$

Since g vanishes for $|y| \geqslant x$ the result for the truncated random variables $\overline{X}_j$ follows. This finishes the proof of Lemma 4.8. □

By (4.3) for $1 \leqslant p \leqslant k_0 - 1$, $\mathbf{DC}_p$ is satisfied for the same choice of (b, A). We can iteratively apply Lemma 4.8 to the expectations of the tetrahedral sums on the right-hand side of (4.6), starting with the tetrahedron with the largest index. In the last step of the iteration we are left with a sum of the type

$$\mathbb{E}_A\Big(\sum_{i=1}^{\tau_A} |\overline{X}_i|^{k_0}\Big) = \mathbb{E}|\overline{X}|^{k_0}\,\mathbb{E}_A(\tau_A)\,,$$

where we used Wald's identity for any bounded f on the right-hand side. Thus, each of the summands on the right-hand side of (4.6) can be bounded by the expression

$$\mathbb{E}_A(\tau_A)\,\mathbb{E}|\overline{X}|^{k_0}\,\sum_{j=0}^{k} C^{k-j}(b\,c(A))^j$$

and so the desired result follows. □

Bounds for I_3 in the case $1 < \alpha < 2$. By Markov's inequality of order 2,

$$\mathbb{P}\Big(\sum_{j=1}^{N_A(n)-1}\ \sum_{t=\tau_A(j)+1}^{\tau_A(j+1)} f(X_j) > \varepsilon_k x/3\Big) \leqslant c(\varepsilon x)^{-2}\mathbb{E}\Big(\sum_{j=1}^{N_A(n)-1}\ \sum_{t=\tau_A(j)+1}^{\tau_A(j+1)} f(X_j)\Big)^2.$$

From the regeneration scheme, we know that the cycles $(\sum_{t=\tau_A(j)+1}^{\tau_A(j+1)} f(X_j))$ are independent. Thus we can expand the expectation term and bound it by $n\mathbb{E}_A[S_A(f)^2]$. The desired result follows by an application of Proposition 4.7 with $k_0 = 2$ and Karamata's Theorem.

Bounds for I_3 in the case $\alpha > 2$ and $\alpha \notin \mathbb{N}$. The following inequality of Bertail and Clémencon [8] is the key to the bound of I_3 for $\alpha > 2$. It will be convenient to write $S_A(f) = \sum_{i=1}^{\tau_A} f(X_i)$.

Lemma 4.9. *Assume that $\sigma_A^2 = \mathbb{E}_A\tau_A^2 < \infty$ and $\sigma_f^2 = \mathbb{E}_A[(S_A(f))^2] < \infty$. Then for any x, sufficiently large n, $\mathbf{M} = (M_1, M_2) \in (0,\infty)^2$ with Euclidean norm $\|\mathbf{M}\|$,*

$$I_3 \leqslant c_0\,\|\mathbf{M}\|^2 \exp\Big\{-\frac{n(1+|\tilde\rho|)\tilde\sigma^2}{2\|\mathbf{M}\|^2}H\Big(\frac{\sqrt{2}\|\mathbf{M}\|\varepsilon_k x}{n(1+|\tilde\rho|\tilde\sigma\tilde\sigma_f}\Big)\Big\} \tag{4.8}$$

$$+(n-1)\mathbb{P}_A(|S_A(f)| > M_1) + (n-1)\mathbb{P}_A(\tau_A > M_2)\,, \tag{4.9}$$

where H is the Bennett function $H(x) = (1+x)\ln(1+x) - x$, $\tilde\sigma_f^2 = \mathrm{var}_A(S_A(f)\mathbb{1}_{\{|S_A(f)|\leqslant M_1\}})$, $\tilde\sigma_A^2 = \mathrm{var}_A(\tau_A\mathbb{1}_{\{|\tau_A|\leqslant M_2\}})$, $\tilde\rho = (\tilde\sigma_A\tilde\sigma_f)^{-1}\mathrm{cov}_A(S_A(f)\mathbb{1}_{\{|S_A(f)|\leqslant M_1\}}, \tau_A\mathbb{1}_{\{|\tau_A|\leqslant M_2\}})$, $\tilde\sigma^2 = \tilde\sigma_f^2\tilde\sigma_A^2/(\tilde\sigma_f^2 + \tilde\sigma_A^2)$, and some $c_0 > 0$.

Bertail and Clémencon [8] also assume that $\mathbb{E}_A S_A(f) = 0$. This condition is always satisfied in our situation since $\mathbb{E}f(X) = 0$; see Meyn and Tweedie [39], (17.23) in Theorem 17.3.1. Under our conditions, σ_A^2 is finite for any α and σ_f^2 is finite for $\alpha > 2$; see Proposition 4.7. One even has the stronger property: there exists a constant $\kappa > 0$ such that

$$\sup_{x\in A}\mathbb{E}_x \mathrm{e}^{\,\kappa\tau_A} < \infty\,, \tag{4.10}$$

see Meyn and Tweedie [39], (15.2) in Theorem 15.0.1. We will choose $M_1 = M_2 = \gamma_k\, x$ for some constants $\gamma_k > 0$. A careful study of the proof in [8] shows that $\tilde\rho, \tilde\sigma, \tilde\sigma_f$ are bounded for $\alpha > 2$. Then the exponential inequality (4.8) turns into

$$\begin{aligned} I_3 &\leqslant c\,(x\gamma_k)^2 \mathrm{e}^{\,-cn/(x\gamma_k)^2 H(cx^2\gamma_k\varepsilon_k/n)} + n\,\mathbb{P}_A(\tau_A > x\gamma_k) + n\,\mathbb{P}_A(|S_A(f)| > x\gamma_k)\\ &= I_{31} + I_{32} + I_{33}\,, \end{aligned}$$

for suitable constants $c > 0$. Choose $\gamma_k = o(\varepsilon_k)$. Then for k large, uniformly for $x \geqslant b_n$ such that $b_n/n^{\delta+0.5} \to \infty$ for some $\delta > 0$,

$$\frac{I_{31}}{n\,\mathbb{P}(|X| > x)} \leqslant c\,\frac{x^{2(1-c\varepsilon_k/\gamma_k)} n^{c\varepsilon_k/\gamma_k}}{n\,\mathbb{P}(|X| > x)} = o(1)\,, \quad n \to \infty\,.$$

As for I_{32}, it follows from (4.10) and Markov's inequality that

$$\frac{I_{32}}{n\,\mathbb{P}(|X| > x)} \leqslant c\,\frac{\mathrm{e}^{\,-\kappa x\gamma_k}}{\mathbb{P}(|X| > x)} = o(1)\,,$$

uniformly for $x \geqslant b_n$. Finally, Markov's inequality, an application of Proposition 4.7 to I_{33} with $k_0 = [\alpha] + 1$ and Karamata's theorem yield

$$\mathbb{P}_A(|S_A(f)| > x\gamma_k) \leqslant (x\gamma)^{-k_0}\mathbb{E}|S_A(f)|^{k_0} \leqslant c\,(x\gamma_k)^{-k_0}\mathbb{E}|\overline{X}|^{k_0} \sim c\,\delta_k^{k_0-\alpha}\gamma_k^{-k_0}\mathbb{P}(|X| > x)\,.$$

Choose $\delta_k = o(\gamma_k^{k_0/(k_0-\alpha)})$ as $k \to \infty$. This is always possible because we may choose $\varepsilon_k = k^{-2}$, $\delta_k = \mathrm{e}^{\,-k}$ and $\gamma_k = k^{-3}$ throughout the proof. Then we obtain

$$\lim_{k\to\infty}\sup_{x\geqslant b_n}\frac{I_{33}}{n\,\mathbb{P}(|X| > x)} \leqslant \lim_{k\to\infty} c\,\delta_k^{k_0-\alpha}\gamma_k^{-k_0} = 0\,.$$

Thus we proved for $\alpha > 2$

$$\limsup_{k\to\infty}\lim_{k\to\infty}\sup_{x\geqslant b_n}\frac{I_3}{n\,\mathbb{P}(|X| > x)} = 0\,.$$

The case $\alpha > 2$ and $\alpha \in \mathbb{N}$. In this case, let us fix $\alpha/(\alpha+1) < \beta < 1$ and consider the process $(|X_t|^\beta = |h(\Phi_t)|^\beta)$. It satisfies $\mathbf{DC}_\alpha$ and concavity of $x \to x^\beta$ as $\beta < 1$ implies that

$$\mathbb{E}_A\Big(\sum_{i=1}^{\tau_A}|\overline{X}_i|\Big)^{\beta k_0} \leqslant \mathbb{E}_A\Big(\sum_{i=1}^{\tau_A}|\overline{X}_i|^\beta\Big)^{k_0}.$$

We apply Proposition 4.7 to $(|X_t|^\beta)$ with $k_0 = \alpha+1$ and we obtain $\mathbb{E}_A|\overline{S}_A|^{\beta k_0} \leqslant \mathbb{E}|\overline{X}_1|^{\beta k_0}$. Noticing that $\beta k_0 > \alpha$, the use of Karamata's theorem as above yields that $\mathbb{E}|\overline{X}_1|^{\beta k_0}$ is negligible with respect to $n\mathbb{P}(|X| > x)$. Now we can follow the lines of the proof in the case of non-integer α. □

In what follows, we will use the notation of Theorem 4.6 and its proof. Our next goal is to give an intuitive interpretation of the large deviation principle of Theorem 4.6: we want to show that the large deviation probability $\mathbb{P}(S_n > x)$ is essentially determined by $\mathbb{P}(\max_{i=1,\dots,N_A(n)} S_{A,i} > x)$, where

$$S_{A,i} = \sum_{t=\tau_A(i)+1}^{\tau_A(i+1)} X_t\,, \quad i \in \mathbb{Z}\,,$$

and $(N_A(t))_{t\geqslant 0}$ is the renewal process generated from the iid sequence $(\tau_A(j+1)) - \tau_A(j))$. The sequence $(S_{A,i})$ constitutes an iid sequence. We write $\tau_A = \tau_A(1)$, $S_A = \sum_{i=1}^{\tau_A} X_i$ and $\lambda = (\mathbb{E}\tau_A)^{-1}$.

Theorem 4.10. *Assume that the conditions of Theorem 4.6 hold, $\alpha > 1$, $\alpha \neq 2$ and $b_+ > 0$. Then $\mathbb{P}_A(S_A > x) \sim \mathbb{E}(\tau_A) b_+ \mathbb{P}(|X| > x)$ and the precise large deviation principle for the function of Markov chain (X_t) can be written in the form*

$$\sup_{x \in \Lambda_n} \Big| \frac{\mathbb{P}(S_n > x)}{n\, \mathbb{P}_A(S_A > x)} - (\mathbb{E}\tau_A)^{-1} \Big| \to 0\,,$$

where $\Lambda_n = (b_n, c_n)$ is chosen as in Theorem 4.6.

Proof. Using the disjoint partition $\{N_A(n)=0\}, \{N_A(n)=1\}, \{N_A(n)\geqslant 2\}$, we obtain

$$\begin{aligned}\mathbb{P}(S_n > x) &= \mathbb{P}\Big(\sum_{i=1}^n X_i > x, \tau_A > n\Big) + \mathbb{P}\Big(\sum_{i=1}^{\tau_A(1)} X_i + \sum_{i=\tau_A(1)+1}^{n} X_i > x, \tau_A(2) > n \geqslant \tau_A(1)\Big)\\ &\quad + \mathbb{P}(S_n > x, N_A(n) \geqslant 2).\end{aligned}$$

Using the definitions of $(\tau_A(i))$ and $N_A(n)$, we obtain for small $\varepsilon \in (0,1)$

$$\begin{aligned}\mathbb{P}(S_n > x) &\leqslant \mathbb{P}(\tau_A > n) + 2\mathbb{P}(S_A > x\varepsilon/2, \tau_A \leqslant n) + \mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x(1-\varepsilon)\Big)\\ &\quad + 2\mathbb{P}\Big(\sum_{i=\tau_A(N_A(n))+1}^{n} X_i > x\varepsilon/2\Big) = J_1 + J_2 + J_3 + J_4.\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(S_n > x) &\geqslant \mathbb{P}\Big(S_A + \sum_{i=1}^{N_A(n)-1} S_{A,i} + \sum_{t=\tau_A(N_A(n))+1}^{n} X_t > x\,, N_A(n) \geqslant 2\Big)\\ &\geqslant \mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} \geqslant (1+\varepsilon)x, |S_A| \leqslant \varepsilon x/2, \Big|\sum_{t=\tau_A(N_A(n))+1}^{n} X_t\Big| \leqslant \varepsilon x/2\,, N_A(n) \geqslant 2\Big)\\ &\geqslant \mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} \geqslant (1+\varepsilon)x\Big) - \mathbb{P}(|S_A| > \varepsilon x/2)\\ &\quad - \mathbb{P}\Big(\Big|\sum_{t=\tau_A(N_A(n))+1}^{n} X_t\Big| > \varepsilon x/2\Big) - \mathbb{P}(N_A(n) \leqslant 2)\\ &= J_5 - J_6 - J_7 - J_8\,.\end{aligned}$$

Lemma 4.11. *Under the conditions of the theorem, for any small $\varepsilon > 0$, uniformly for $x \in \Lambda_n$,*

$$\frac{\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x(1+\varepsilon)\Big)}{n\mathbb{P}(|X| > x)} + o(1) \leqslant \frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(|X|>x)} \leqslant \frac{\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x(1-\varepsilon)\Big)}{n\mathbb{P}(|X| > x)} + o(1)\,.$$

Proof. By assumption, the probability $J_1 \leqslant \mathbb{P}(\tau_A > n)$ is negligible with respect to $n\mathbb{P}(|X| > x)$ on Λ_n.

By standard computations and using the same notation as in the proof of Theorem 4.6 we have

$$J_4/2 \leqslant \mathbb{P}\Big(\sum_{i=\tau_A(N_A(n))+1}^{n} \overline{X}_i > x\varepsilon/2\Big) + \mathbb{P}\Big(\cup_{i=\tau_A(N_A(n))+1}^{n} \{|X_i| > x\delta\}\Big).$$

The second term is estimated by

$$\mathbb{E}\Big(\sum_{i=\tau_A(N_A(n))+1}^{n} 1_{\{|X_i|>x\delta\}}\Big) \leqslant \mathbb{E}_A\Big(\sum_{i=1}^{\tau_A} 1_{\{|X_i|>x\delta\}}\Big) = \mathbb{E}(\tau_A)\,\mathbb{P}(|X|>x\delta).$$

The first term can be shown to be negligible with respect to $n\mathbb{P}(|X|>x)$ as in the proof of Theorem 4.6. So $J_4 = o(n\mathbb{P}(|X|>x))$.

The term J_2 can be treated in the same way as I_2 in the proof of Theorem 3.1. An application of Markov's inequality yields an estimate of the form $cx^{-k_0}\Big[\mathbb{E}\Big(1_{\{\tau_A\leqslant n\}}\sum_{i=1}^{\tau_A}|\overline{X}_i|^{k_0}\Big)+[n\,\mathbb{P}(|X|>x)]^{k_0}\Big]$. Using (4.4), Pitman's and Wald-type identities we obtain $J_2 \leqslant cn(x\varepsilon)^{-k_0}\mathbb{E}|\overline{X}|^{k_0}$. Hence J_2 is negligible with respect to $n\mathbb{P}(|X|>x)$ by an application of Karamata's theorem.

Collecting the bounds above, the upper bound in the lemma is proved.

As regards the lower bound, J_6 and J_7 are of the order $o(n\mathbb{P}(|X|>x))$ in view of the bounds for J_2 and J_4 in the proof above, respectively. Moreover,

$$J_8 = \mathbb{P}(N_A(n)\leqslant 2) \quad\leqslant\quad \mathbb{P}(\tau_A>n)+\mathbb{P}(\tau_A(2)>n)\leqslant 3\,\mathbb{P}(\tau_A>n/2)\,,$$

and the latter probability is negligible with respect to $n\mathbb{P}(|X|>x)$ as for J_1 above. □

Denote $\tilde\Lambda_n = (b_n, \mathrm{e}^{\,s_n})\cap\Lambda_n$ for some (s_n) such that $s_n/n\to 0$.

Lemma 4.12. *Under the conditions of the theorem, for any small $\xi,\varepsilon>0$, uniformly for $x\in\tilde\Lambda_n$,*

$$\frac{\lambda(1-\varepsilon)\mathbb{P}(S_A>x(1+\xi)(1+\varepsilon))}{\mathbb{P}(|X|>x)} + o\Big(\frac{\mathbb{P}_A(S_A>x)}{\mathbb{P}(|X|>x)}\Big)+o(1) \tag{4.11}$$

$$\leqslant \frac{\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1}S_{A,i}>x\Big)}{n\mathbb{P}(|X|>x)}$$

$$\leqslant \frac{\lambda\mathbb{P}_A(S_A>x(1-\xi))}{\mathbb{P}(|X|>x)} + o\Big(\frac{\mathbb{P}_A(S_A>x)}{\mathbb{P}(|X|>x)}\Big)+o(1)\,. \tag{4.12}$$

Proof. We have for $\delta>0$,

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1}S_{A,i}>x\Big) &= \mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1}S_{A,i}>x\,,|N_A(n)-1-n\,\lambda|>\delta n\Big)\\ &\quad+\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1}S_{A,i}>x\,,|N_A(n)-1-n\,\lambda|\leqslant\delta n\Big)\\ &= K_1+K_2\,.\end{aligned}$$

In view of (4.10), τ_A has exponential moment and therefore one can apply standard large deviation theory (e.g. Cramér's theorem; see Dembo and Zeitouni [17]) to obtain

$$K_1\leqslant\mathbb{P}(|N_A(n)-1-n\,\lambda|>\delta n)\leqslant\mathrm{e}^{\,-\gamma n}\,,$$

for some $\gamma=\gamma(\delta)>0$. In view of the definition of $\tilde\Lambda_n$, $K_1 = o(n\,\mathbb{P}(|X|>x))$ on $\tilde\Lambda_n$. We also have

$$\mathbb{P}\Big(\sum_{i=1}^{n\lambda}S_{A,i}-\max_{|m-n\lambda|\leqslant\delta n}\Big|\sum_{i=m}^{n\lambda}S_{A,i}\Big|>x\Big)\leqslant K_2\leqslant\mathbb{P}\Big(\sum_{i=1}^{n\lambda}S_{A,i}+\max_{|m-n\lambda|\leqslant\delta n}\Big|\sum_{i=m}^{n\lambda}S_{A,i}\Big|>x\Big)\,.$$

Here we define $\sum_{i=m}^{b}$ for any real value $b\geqslant m$, $m\in\mathbb{N}$, as $\sum_{i=m}^{[b]}$ and the sums $\sum_{i=b}^{m}$ are defined accordingly. Notice that $b_n^{-1}\sum_{i=1}^{n\lambda}S_{A,i}\stackrel{P}{\to}0$ from the fact that $n^{-1}N_A(n)\stackrel{\text{a.s.}}{\to}\lambda$. Then, for any

$\xi \in (0,1)$, a maximal inequality of Lévy-Ottaviani-Skorokhod type for sums of iid random variables (e.g. Petrov [47], Theorem 2.3 on p. 51) yields

$$\begin{aligned} K_2 &\leqslant \mathbb{P}\Big(\sum_{i=1}^{n\lambda} S_{A,i} > x\,(1-\xi)\Big) + \mathbb{P}\Big(\max_{|m-n\lambda|\leqslant \delta n}\Big|\sum_{i=m}^{n\lambda} S_{A,i}\Big| > x\xi\Big) \\ &\leqslant \mathbb{P}\Big(\sum_{i=1}^{n\lambda} S_{A,i} > x\,(1-\xi)\Big) + c\,\mathbb{P}\Big(\Big|\sum_{i=1}^{\delta n} S_{A,i}\Big| > 0.5\xi x\Big)\,. \end{aligned} \tag{4.13}$$

Similarly, using the independence of the random variables $(S_{A,i})$ and a maximal inequality,

$$K_2 \geqslant \mathbb{P}\Big(\sum_{i=1}^{\lambda n} S_{A,i} > x(1+\xi)\Big) - c\,\mathbb{P}\Big(\Big|\sum_{i=1}^{\delta n} S_{A,i}\Big| > 0.5\xi x\Big)\,, \tag{4.14}$$

where δ, ξ can be made arbitrarily small provided n is sufficiently large. Next we give bounds for the probabilities in (4.13) and (4.14). We have for any real $s > 0$ and $y > 0$,

$$\begin{aligned} \mathbb{P}(\sum_{i=1}^{sn} S_{A,i} > y) &\leqslant \sum_{i=1}^{sn} \mathbb{P}\Big(\sum_{k=1}^{sn} S_{A,k} > y, S_{A,i} > y, S_{A,j} \leqslant y, j \neq i\Big) \\ &\quad + \mathbb{P}(\cup_{k=1,j\neq k}^{sn}\{S_{A,k} > y, S_{A,j} > y\}) \\ &\leqslant sn\mathbb{P}(\sum_{k=1}^{sn} S_{A,k} > y, S_{A,1} > y, S_{A,j} \leqslant y, j \neq 1) + [n\mathbb{P}_A(S_A > y)]^2 \\ &\leqslant s\,n\mathbb{P}_A(S_A > y) + [s\,n(\mathbb{P}_A(S_A > y)]^2\,. \end{aligned}$$

Hence, because of the regular variation of X, uniformly for $x \in \tilde{\Lambda}_n$,

$$\begin{aligned} \frac{\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x\Big)}{n\mathbb{P}(|X| > x)} &\leqslant \frac{\lambda\mathbb{P}_A(S_A > x(1-\xi)) + c\delta\mathbb{P}_A(|S_A| > 0.5\xi x)}{\mathbb{P}(|X| > x)} + o\Big(\frac{\mathbb{P}_A(S_A > x)}{\mathbb{P}(|X| > x)}\Big) \\ &\leqslant \frac{\lambda\mathbb{P}_A(S_A > x(1-\xi))}{\mathbb{P}(|X| > x)} + o\Big(\frac{\mathbb{P}_A(S_A > x)}{\mathbb{P}(|X| > x)}\Big)\,. \end{aligned}$$

We obtain the last inequality, taking into account that the argument above can be applied to the left tail of S_A as well. This proves the upper bound (4.12).

On the other hand, for $s > 0$, sufficiently large n, small $\varepsilon > 0$ and $y \in \tilde{\Lambda}_n$,

$$\begin{aligned} \mathbb{P}(\sum_{i=1}^{sn} S_{A,i} > y) &\geqslant \mathbb{P}\Big(\cup_{i=1}^{sn}\Big\{\sum_{k\neq i} S_{A,k} \leqslant \varepsilon y, S_{A,i} > y(1+\varepsilon), S_{A,j} \leqslant y(1+\varepsilon), j \neq i\Big\}\Big) \\ &\geqslant sn\mathbb{P}\Big(\sum_{k=2}^{sn} S_{A,k} \leqslant \varepsilon y, S_{A,1} > y(1+\varepsilon), S_{A,j} \leqslant y(1+\varepsilon), j \neq 1\Big) \\ &\geqslant (1-\varepsilon)sn\mathbb{P}_A(S_A > y(1+\varepsilon))\,. \end{aligned}$$

We conclude from (4.14) that, uniformly for $x \in \tilde{\Lambda}_n$,

$$\frac{\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x\Big)}{n\mathbb{P}(|X| > x)} \geqslant (1-\varepsilon)\frac{\lambda\mathbb{P}(S_A > x(1+\xi)(1+\varepsilon)) - c\delta\mathbb{P}_A(|S_A| > 0.5\xi x(1+\varepsilon))}{\mathbb{P}(|X| > x)}\,.$$

Now, the lower bound (4.11) is proved in a similar fashion as above. □

In view of Lemmas 4.11 and 4.12, letting first $x \to \infty$ and then $\varepsilon \to 0$ and $\xi \to 0$ and using regular variation of X we obtain

$$\frac{b_+}{\lambda} = \lim_{x\to\infty} \frac{\mathbb{P}_A(S_A > x)}{\mathbb{P}(|X| > x)} \quad \text{uniformly on } \tilde{\Lambda}_n.$$

In particular this relation holds along the sequences $x_n = cb_n \in \tilde{\Lambda}_n$ satisfying $x_{n+1}/x_n \to 1$. A sequential version of regular variation then implies that $\mathbb{P}_A(S_A > x)$ is regularly varying; see Bingham et al. [9], Theorem 1.9.2. An application of Theorem 1.1 and Theorem 4.6 finishes the proof of the theorem. □

Remark 4.13. Regular variation of $\mathbb{P}_A(S_A > x)$ also implies the following:

$$\sup_{x>b_n} \Big| \frac{\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x\Big)}{n\,\mathbb{P}(|X| > x)} - b_+ \Big| \to 0\,.$$

For the region $x \in \tilde{\Lambda}_n$ this fact was proved above. Now assume that $x \geqslant \mathrm{e}^{\,s_n}$. We have by Theorem 1.1 for $\alpha > 1$, since $x \geqslant k$ for $k \leqslant n$, uniformly for $x \geqslant \mathrm{e}^{\,s_n}$,

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{N_A(n)-1} S_{A,i} > x\Big) &\sim \sum_{k=2}^{n} \mathbb{P}(N_A(n) = k)\, k\, \mathbb{P}_A(S_A > x)\\ &\sim \mathbb{P}_A(S_A > x)\, \mathbb{E}N_A(n) \sim n\,(\mathbb{E}\tau_A)^{-1}\mathbb{P}_A(S_A > x)\,.\end{aligned}$$

An inspection of the proof of Theorem 4.10 now shows why the precise large deviation principle for (X_n) might in general not hold in the region (c_n, ∞): the first and the last blocks in S_n are always negligible if $\tau_A \leqslant n$. Thus for any $x \geqslant b_n$ one has

$$\frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(|X| > x)} \sim b_+ + \frac{\mathbb{P}(S_n > x, \tau_A > n)}{n\mathbb{P}(|X| > x)} = b_+ + r(x). \tag{4.15}$$

In the region Λ_n, $r(x)$ is uniformly negligible because it is smaller than $\mathbb{P}(\tau_A > n)/(n\mathbb{P}(|X| > x))$. Therefore the precise large deviation result of Theorem 4.6 holds. However, $r(x)$ cannot be neglected in general. It may influence the very large deviations for $x > c_n$ in a complicated way: the Nummelin regeneration scheme cannot be used on $\{\tau_A > n\}$. Below two special examples of functions of Markov chains are given, where the specific dynamics of the models give some clue on the behavior of the second term.

Example 4.14. Consider the autoregressive process of order 1, $X_t = \varphi X_{t-1} + B_t$ for some constant $\varphi \in (-1, 1)$ and an iid sequence (B_t) such that B is regularly varying with index α and $\mathbb{E}B = 0$ if $\mathbb{E}|B| < \infty$. It is known from Mikosch and Samorodnitsky [40] that one can choose $\Lambda_n = (b_n, \infty)$ with (b_n) from Theorem 1.1 and

$$b_+ = (1 - |\varphi|^{\alpha})\Big(\frac{p}{(1-\varphi)_+^{\alpha}} + \frac{q}{(1-\varphi)_-^{\alpha}}\Big)\,,$$

where $p = 1 - q = \lim_{x\to\infty} \mathbb{P}(B > x)/\mathbb{P}(|B| > x)$. This result was derived without any further conditions on B. The same result follows from Theorem 4.6 under more restrictive conditions, e.g. if B has a non-singular distribution with respect to Lebesgue measure (see Alsmeyer [1]). Thus the remainder term $r(x)$ in (4.15) is uniformly negligible over (b_n, ∞).

4.4. **Solution to stochastic recurrence equations.** In this section, we consider a special class of stationary Markov chains (X_t) for which we can apply Theorem 4.6 by considering it as a function of its enlargement (Φ_t) possessing an atom. Let $((A_t, B_t))_{t\in\mathbb{Z}}$ be an iid sequence such that for a generic element (A, B) the following set of conditions $\mathbf{SRE}_\alpha$ holds:

- $A \geqslant 0$, $A \neq 0$ a.s., $B \neq 0$ a.s., and the distribution of (A, B) is non-singular with respect to the Lebesgue measure on $\mathbb{R}^2$.
- The Markov chain $X_t = \Psi_t(X_{t-1})$ is the unique solution to a stochastic recurrence equation with iid iterated functions Ψ_t satisfying the following additional conditions:
  - The Lipschitz coefficients L_t of the mapping Ψ_t satisfy $\mathbb{E}\log^+ L_t < \infty$.
  - The top Lyapunov exponent of (Ψ_t) is strictly negative.
  - For any t,

$$A_t\, X_{t-1} - |B_t| \leqslant X_t \leqslant A_t\, X_{t-1} + |B_t|\,. \tag{4.16}$$

- There exists an $\alpha > 0$ such that $\mathbb{E}A^\alpha = 1$, $\mathbb{E}A^{\alpha+\delta} < \infty$ and $\mathbb{E}|B|^{\alpha+\delta} < \infty$ for some $\delta > 0$.
- The conditional law of $\log A$, given $A \neq 0$, is non-arithmetic.
- The distribution of X is regularly varying with index $\alpha > 0$ in the following sense: There exist constants $c_\infty^+, c_\infty^- \geqslant 0$ such that $c_\infty^+ + c_\infty^- > 0$ and

$$\mathbb{P}(X > x) \sim c_\infty^+\, x^{-\alpha}\,, \quad \text{and} \quad \mathbb{P}(X \leqslant -x) \sim c_\infty^-\, x^{-\alpha} \quad \text{as } x \to \infty\,. \tag{4.17}$$

These conditions are motivated by the well studied affine case:

$$X_t = A_t\, X_{t-1} + B_t\,, \quad t \in \mathbb{Z}\,. \tag{4.18}$$

The stochastic recurrence equation (4.18) has attracted a lot of attention, starting with pioneering work of Kesten [32] who proved that (4.18) has a stationary solution (X_t) under mild conditions on the distribution of (A, B). This solution has a regularly varying marginal distribution with index $\alpha > 0$ solving the equation $\mathbb{E}A^\kappa = 1$, $\kappa > 0$. Kesten's theory was formulated for multivariate X_t's. In the one-dimensional case, Goldie [23] gave an alternative proof of the regular variation of X and he also determined the constants c_∞^- and c_∞^+. In particular, for $B \geqslant 0$ a.s. he showed that

$$c_\infty^+ = \frac{\mathbb{E}[(B_1 + A_1X_0)^\alpha - (A_1X_0)^\alpha]}{\alpha\mathbb{E}A^\alpha \log A}\,.$$

Buraczewski et al. [12] proved a precise large deviation principle (1.6) in the affine case (4.18) in the region $\Lambda_n = (b_n, c_n)$, where (b_n) is chosen as in Theorem 4.6 and $c_n = \mathrm{e}^{\,s_n}$ for any sequence (s_n) such that $s_n \to \infty$ and $s_n = o(n)$. The proof in [12] is rather technical and uses some deep analysis of the structure of the random walk (S_n) determined by the equation (4.18). In what follows, we will show that Theorem 3.1 can be used to establish the same results by using the Markov structure of the sequence (X_t). The proofs of this section will need less technical efforts than in [12] and give some insight into precise large deviation principles for classes of Markov chains larger than the affine case (4.18).

Goldie [23] already considered stochastic recurrence equations beyond affine structures. Some of his examples satisfy inequality (4.16):

Example 4.15. Consider the solution to the stochastic recurrence equation

$$X_t = \max(A_tX_{t-1}, B_t)\,, \quad t \in \mathbb{Z}. \tag{4.19}$$

It exists under the conditions $\mathbb{E}\log A < 0$, $\mathbb{E}\log^+ B < \infty$ and satisfies (4.16). Moreover, if $\mathbb{E}A^\alpha = 1$, $\mathbb{E}A^\alpha \log A < \infty$, the conditional law of $\log A$, given $A \neq 0$, is non-arithmetic and $\mathbb{E}(B^+)^\alpha < \infty$, then the unique solution to (4.19) satisfies relation (4.17); see Goldie [23], Theorem 5.2.

Example 4.16. Consider an iid sequence $((A_t, C_t, D_t))_{t\in\mathbb{Z}}$ with a generic element (A, C, D) such that $A \geqslant 0$ a.s. and C, D are real-valued. The solution to the equation

$$X_t = A_t \max(C_t, X_{t-1}) + D_t\,, \quad t \in \mathbb{R}\,,$$

was considered by Letac [37]. It exists under the conditions $\mathbb{E}\log A < 0$, $\mathbb{E}\log^+ C < \infty$, $\mathbb{E}\log^+ D < \infty$ and satisfies (4.16) if $D \geqslant 0$ a.s. Indeed, if we write $B_t = A_t\, C_t^+ + D_t$ then

$$|X_t - A_t\, X_{t-1}| \leqslant A_t\,(C_t - X_{t-1})_+ + D_t \leqslant A_t\, C_t^+ + D_t = B_t\,.$$

This example is also known to satisfy (4.17) (see Goldie [23], Theorem 6.2): if $A \geqslant 0$, $\mathbb{E}(AC^+)^\alpha < \infty$, $\mathbb{E}|B|^\alpha < \infty$ and A satisfies all conditions of the previous example then (4.17) holds.

Goldie [23] gave various other examples of stochastic recurrence equations satisfying (4.17). Recently, Mirek [42] considered multivariate analogs of not necessarily affine stochastic recurrence equations satisfying a condition of type (4.16) (adjusted to the multivariate case). He proved the regular variation of the marginal distribution and also gave examples supplementary to those in [23]. The use of (4.16) in his paper was also the motivation for us to include in this paper stochastic recurrence equations which do not necessarily satisfy (4.18).

In what follows, it will be convenient to write

$$\Pi_0 = 1 \quad \text{and} \quad \Pi_j = \prod_{i=1}^{j} A_1 \cdots A_j\,, \quad j \geqslant 1\,.$$

Theorem 4.17. *Assume that the stationary Markov chain (X_t) satisfies the condition $\mathbf{SRE}_\alpha$ for some $\alpha > 0$ and $\mathbb{E}X = 0$ if $\mathbb{E}|X| < \infty$. Then the precise large deviation principle* (1.6) *holds with*

$$b_+ = \mathbb{E}\Big[\Big(1 + \sum_{i=1}^{\infty} \Pi_i\Big)^\alpha - \Big(\sum_{i=1}^{\infty} \Pi_i\Big)^\alpha\Big] \tag{4.20}$$

in the regions $\Lambda_n = (b_n, c_n)$ given by

- $0 < \alpha < 1$*: $\Lambda_n = (b_n, \infty)$ for any (b_n) satisfying of $b_n/n^{1/\alpha} \to \infty$.*
- $1 < \alpha$ *and $\alpha \neq 2$: $\Lambda = (b_n, c_n)$ for any sequence (b_n) satisfying $b_n/n^{1/\alpha\vee 0.5+\delta} \to \infty$ for any $\delta > 0$, and $c_n = \mathrm{e}^{\gamma n}$ for sufficiently small $\gamma > 0$.*

Proof. The condition $\mathbf{RV}_\alpha$ follows from regular variation of the marginals. Indeed, iteration of (4.18) yields for fixed $d \geqslant 1$,

$$X_0\Pi_n + R_{n,1} \leqslant X_t \leqslant X_0\Pi_n + R_{n,2}\,, \quad n = 1, \ldots, d,$$

where $(R_{n,i})_{n=1,\ldots,d}$, $i = 1, 2$, is independent of X_0. Moreover, by the assumptions on (A, B), $\mathbb{E}|R_{n,i}|^{\alpha+\delta} < \infty$. Therefore

$$\mathbf{X}_d = (X_1, \ldots, X_d) = X_0(\Pi_1, \ldots, \Pi_d) + \mathbf{R}_d\,.$$

Since X_0 is assumed regularly varying with index α an application of a multivariate version of a result of Breiman [10] (see Basrak et. al [4]) shows that $X_0(\Pi_1, \ldots, \Pi_d)$ is regularly varying, and it follows from Lemma 3.12 in Jessen and Mikosch [30] and from $\mathbb{E}|\mathbf{R}_d|^{\alpha+\delta} < \infty$ for some $\delta > 0$ that $\mathbf{X}_d$ is regularly varying with index α. This also means that one can use the same calculations for $b_+(d)$ given in Bartkiewicz et al. [3] and hence the limit b_+ exists and is given by the expression (4.20). Notice that [3] derive the constant b_+ only for $\alpha \in (0, 2)$. However, the proofs in the cases $\alpha \in (1, 2)$ and $\alpha > 1$ are identical.

Next we verify condition $\mathbf{AC}_\alpha$ for the region (b_n, ∞) for any sequence (b_n) satisfying $b_n/n^{1/\alpha} \to \infty$ or, equivalently, $n\,\mathbb{P}(|X| > b_n) \to 0$. Write $\Pi_{ij} = A_i \cdots A_j$ for any $i, j \in \mathbb{Z}$ with the convention that

$\Pi_{ij} = 1$ if j, i. Iterating (4.16), we obtain

$$X_j \leqslant \Pi_j \, X_0 + \sum_{i=1}^j \Pi_{i+1,j} \, |B_i| \, , \quad j \geqslant 0 \, . \tag{4.21}$$

The second term in the right-hand side of (4.21) is independent of X_0. Hence for $\delta_k > 0$,

$$\begin{aligned} & \mathbb{P}(|X_j| > x\delta_k \mid |X_0| > x\delta_k) \\ \leqslant & \quad \mathbb{P}(\Pi_j |X_0| > x\delta_k/2 \mid |X_0| > x\delta_k) + \mathbb{P}\Big(\sum_{i=1}^j \Pi_{i+1,j} \, |B_i| > x\delta_k/2 \Big) \\ = & \quad I_1(x) + I_2(x) \, . \end{aligned}$$

Under condition $\mathbf{SRE}_\alpha$ it follows from Kesten [32] and Goldie [23] that

$$Q_j = \sum_{i=-\infty}^j \Pi_{i+1,j} \, |B_i| < \infty \, ,$$

and (Q_j) is the causal solution to the stochastic recurrence equation $Q_j = A_j Q_{j-1} + |B_j|$, $t \in \mathbb{Z}$, which according to the Kesten-Goldie theory is regularly varying with index α. Therefore

$$\sup_{x \geqslant b_n} n \, I_2(x) = n \, I_2(b_n) \to 0 \, , \quad n \to \infty \, ,$$

for every $\delta_k > 0$ and any sequence (b_n) such that $b_n/n^{1/\alpha} \to \infty$. We also have

$$I_1(x) \leqslant \frac{\mathbb{P}(\min(\Pi_j, 1) \, |X_0| > x\delta_k/2)}{\mathbb{P}(|X_0| > x\delta_k)} .$$

In view of (4.17) there exists a constant $c > 0$ such that $\mathbb{P}(X_0 > x) \leqslant c \, x^{-\alpha}$, $x > 0$.

Using this inequality conditionally on $(A_i)_{1 \leqslant i \leqslant j}$, we obtain

$$\mathbb{P}(\min(\Pi_j, 1) \, |X_0| > x\delta_k/2 \mid (A_i)_{1 \leqslant i \leqslant j}) \leqslant c \, (2 \min(\Pi_j, 1))^\alpha (x\delta_k)^{-\alpha} \, ,$$

and taking expectations,

$$I_1(x) \leqslant c \, \mathbb{E}(\min(\Pi_j, 1))^\alpha (x\delta_k)^{-\alpha} \, .$$

Since $\min(y^\alpha, 1) \leqslant y^{\alpha-\epsilon}$ for $y \geqslant 0$, $\epsilon \in (0, \alpha)$, fixed $\delta_k > 0$, and large n,

$$\sup_{x \geqslant b_n} \delta_k^{-\alpha} \sum_{j=k}^n \mathbb{P}(|X_j| > x\delta_k \mid |X_0| > x\delta_k) \leqslant c \, \delta_k^{-2\alpha} \sum_{j=k}^n (\mathbb{E}A^{\alpha-\epsilon})^j .$$

Since $\mathbb{E}A^{\alpha-\varepsilon} < 1$, the right-hand side is bounded by $c(\mathbb{E}A^{\alpha-\varepsilon})^k/\delta_k^{2\alpha}$. Thus $\mathbf{AC}_\alpha$ is satisfied for any choice of (b_n) with $b_n/n^{1/\alpha} \to \infty$ and (δ_k) such that $(\mathbb{E}A^{\alpha-\varepsilon})^k = o(\delta_k^{2\alpha})$ as $k \to \infty$. In particular, one can choose (δ_k) decaying to zero exponentially fast.

Our next goal is to verify (3.1).

The case $0 < \alpha < 1$. Condition (3.1) is immediate from Remark 3.2. We can choose (δ_k) decaying exponentially fast, as discussed above, and $\varepsilon_k = k^{-2}$.

The case $\alpha > 1$ *and* $\alpha \neq 2$. In this case the verification of (3.1) is much more involved. We will employ Theorem 4.6. According to this result, we need to verify that (X_t) is irreducible strongly aperiodic and that the Markov chain satisfies $\mathbf{DC}_p$ for $p < \alpha$. However, since $\mathbb{E}A^\alpha = 1$, by convexity of the function $f(x) = \mathbb{E}A^x$, $x > 0$, we have $f(p) < 1$ as $p < \alpha$. Writing $p = \beta k$ where $0 < \beta < 1$ and k is an integer then

$$\begin{aligned}\mathbb{E}(|X_1|^p - A^p|x|^p \mid X_0 = x) &\leqslant \mathbb{E}((A^\beta|x|^\beta + |B|^\beta)^k - (A^\beta|x|^\beta)^k \mid X_0 = x)\\ &= \sum_{j=0}^{k-1}\binom{k}{j}(|x|^\beta)^j\mathbb{E}[(A^\beta)^j(|B|^\beta)^{k-j}]\\ &\leqslant c(1+|x|^{p-\beta})\,.\end{aligned}$$

Hence $\mathbf{DC}_p$ is satisfied for any $p < \alpha$.

An application of a result of Alsmeyer [1] yields that the Markov chain (X_t) is aperiodic and irreducible. The aperiodicity and $\mathbb{P}$-irreducibility follow from Theorem 2.1 and Corollary 2.3 in [1] if and only if the transition kernel of the Markov chain has a component which is absolutely continuous with respect to Lebesgue measure. The latter condition is satisfied in view of the non-singularity of the distribution of (A, B) assumed in $\mathbf{SRE}_\alpha$ and since

$$\mathbb{P}_x(X > \varepsilon) \geqslant \mathbb{P}(A\,x - B > \varepsilon) \quad \text{and} \quad \mathbb{P}_x(X \leqslant -\varepsilon) \geqslant \mathbb{P}(A\,x + B \leqslant -\varepsilon) \quad \text{for any } \varepsilon > 0.$$

Thus all assumptions of Theorem 4.6 are satisfied and therefore its conclusion applies. □

4.5. **The** GARCH(1, 1) **model.** Consider the model (4.1) with the specification that (Z_t) is an iid symmetric sequence and

$$\sigma_t^2 = \alpha_0 + \sigma_{t-1}^2\,(\alpha_1 Z_{t-1}^2 + \beta_1) = \alpha_0 + \sigma_{t-1}^2 A_t\,, \tag{4.22}$$

where $\alpha_0, \alpha_1 > 0$ and $\beta_1 \geqslant 0$. This stochastic recurrence equation defines a GARCH(1, 1) process. The GARCH(1, 1) process has been used most frequently for applications in financial time series analysis; see Andersen et al. [2]. The theory of Section 4.4 can be applied to the affine stochastic recurrence equation (4.22). There exists a unique stationary solution to (4.22) under the assumption $\mathbb{E}\log A < 0$ and σ is regularly varying under mild conditions on the distribution of Z. We will now show a precise large deviation principle for the process (X_t)

Theorem 4.18. *Consider a* GARCH(1, 1) *process* (X_t) *given by* (4.1) *and* (4.22) *with* $\alpha_0, \alpha_1 > 0$, $\beta_1 \in [0, 1)$. *We assume that there exists an* $\alpha > 0$, $\alpha \neq 2$ *such that:*

- *Z is symmetric with* $\mathrm{var}(Z) = 1$, $\mathbb{E}|Z|^{\alpha+\delta} < \infty$ *for some* $\delta > 0$ *and the distribution of* Z^2 *is non-singular with respect to Lebesgue measure.*
- *There exists an* $\alpha > 0$ *such that* $\mathbb{E}A^{\alpha/2} = 1$.

Then the precise large deviation result (1.6) *holds in the region* $\Lambda_n = (n^{1/\alpha+\delta}, \infty)$ *if* $\alpha < 2$ *and* $\Lambda_n = (n^{1/2+\delta}, \mathrm{e}^{\gamma n})$ *for sufficiently small* $\gamma > 0$ *if* $\alpha > 2$ *with*

$$b_+ = \frac{\mathbb{E}[|Z_0 + A_1^{0.5}T_\infty|^\alpha - |A_1^{0.5}T_\infty|^\alpha]}{2\mathbb{E}|Z|^\alpha}\,,$$

and $T_\infty = \sum_{t=1}^\infty Z_t \prod_{i=1}^{t-1} A_{i+1}^{0.5}$.

Proof. We verify the conditions of Theorem 3.1. Since (σ_t^2) satisfies the affine stochastic recurrence equation (4.22) the assumptions on the distribution of A imply that the conditions of Goldie [23], Theorem 5.2, are satisfied and therefore σ satisfies the relation $\mathbb{P}(\sigma > x) \sim c_\infty x^{-\alpha}$ for some positive c_∞ as $x \to \infty$. Following the argument of the proof on top of p. 366 in Bartkiewicz et al. [3], we can show that for $d \geqslant 1$,

$$\frac{\mathbb{P}\Big(\Big|(X_1, \ldots, X_d) - \sigma_0(Z_1 A_1^{0.5}, \ldots, Z_d \Pi_d^{0.5})\Big| > x\Big)}{\mathbb{P}(|\sigma| > x)} \to 0\,.$$

Observing that $\mathbb{E}|Z_1 A_1^{0.5}|^{\alpha+\delta} < \infty$, it follows from Lemma 3.12 in Jessen and Mikosch [30] and from a generalization of Breiman's result in Basrak et al. [4] that $\mathbf{RV}_\alpha$ holds.

The constant b_+ was derived in [3] for $\alpha \in (0,2)$ but the proof generalizes to arbitrary $\alpha > 0$.

As to $\mathbf{AC}_\alpha$, it follows by the argument leading to (4.21) that

$$\sigma_j^2 = \Pi_j \sigma_0^2 + \alpha_0 \sum_{i=1}^{j} \Pi_{i+1,j}\,, \quad j \geqslant 0\,.$$

Then

$$\mathbb{P}(|X_j| > \delta_k x \mid |X_0| > \delta_k x) \leqslant \mathbb{P}\Big(\Pi_j Z_j^2 \sigma_0^2 > (\delta_k x)^2/2 \mid |X_0| > \delta_k x\Big) + \mathbb{P}\Big(Z_j^2 \alpha_0 \sum_{i=1}^{j} \Pi_{i+1,j} > (\delta_k x)^2/2\Big)\,,$$

and now one can follow the proof of $\mathbf{AC}_\alpha$ in Theorem 4.17. No conditions on (δ_k) are required so far and (b_n) is chosen such that $b_n/n^{1/\alpha} \to \infty$.

Next we verify (3.1).

The case $0 < \alpha < 2$. Here one can use Remark 3.3.

The case $\alpha > 2$. We apply Theorem 4.6 to $X_t = h(\Phi_t)$, $t \in \mathbb{Z}$, where the Markov chain (Φ_t) is an enlargement of the irreducible Markov chain (X_t, σ_t^2) possessing an atom A. □

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