

UNIQUE CONTINUATION FOR A QUASILINEAR ELLIPTIC EQUATION IN THE PLANE

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ABSTRACT. We consider planar solutions to certain quasilinear elliptic equations subject to the Dirichlet boundary conditions. We show that if the boundary data has finite number of relative maxima and minima then a solution has the unique continuation property. Our method is new and applicable in the plane.

1. INTRODUCTION AND PRELIMINARIES

In this paper we consider solutions of quasilinear second order elliptic partial differential equations of the form

$$\nabla \cdot \mathcal{A}(x, \nabla u) = \mathcal{B}(x, \nabla u), \quad (1.1)$$

where $\mathcal{A}: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\mathcal{B}: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are Carathéodory functions under certain structural conditions discussed in Section 1.2. A noteworthy example of such equations is the p -Laplace equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \quad (1.2)$$

where $1 < p < \infty$, which gives the Laplace equation when $p = 2$; we refer to [11].

The result of this note is the following. Let G be a bounded simply-connected Jordan domain in \mathbb{R}^2 . Suppose u is a solution to (1.1) subject to the Dirichlet boundary condition

$$u = g \text{ on } \partial G,$$

where $g \in W^{1,p}(G) \cap C(\overline{G})$. We assume minimal regularity conditions on ∂G so that every boundary point is regular, and hence $u \in C(\overline{G})$. If $g|_{\partial G}$ has finite number of relative maxima and minima, then u satisfies the unique continuation principle, i.e. if u vanishes in some open subset of G , then u vanishes identically in G .

For linear equations the study of unique continuation is rather complete [9]. It is known, however, that unique continuation does not hold in certain cases for solutions to the equation in (1.1). We mention a paper by Martio [13] in which he constructed some counterexamples in

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the case of $p = n$ in (1.3), $n \geq 3$, and $\mathcal{B} \equiv 0$. The planar case for (1.1) still remains an open problem, but in the present paper we provide a partial solution.

A typical two-dimensional phenomenon is the connection between quasiregular mappings and linear second order uniformly elliptic equations of the general form of two variables. Moreover, the unique continuation property can be derived from this intrinsic connection.

Unique continuation for nonlinear equations, such as (1.2), is still, to the best of our knowledge, an open problem. There are some results. The two-dimensional case of the p -Laplace equation was solved by Alessandrini [1] by considering the set of critical points of a two-dimensional solution to (1.2). We refer also to Bojarski–Iwaniec [7] and Manfredi [12] who observed that the complex gradient of a solution to (1.2) is quasiregular, and hence the unique continuation property follows for solutions in two variables.

We mention papers [4] and [5] in which the unique continuation property is achieved for solutions to certain generalizations of the p -Laplace equation in the plane as a consequence of studying the properties of critical points and level lines of such solutions; in [5], for instance, the authors consider planar solutions to the equation

$$\nabla \cdot (|A\nabla u \cdot \nabla u|^{(p-2)/2} A\nabla u) = 0$$

with $A = A(x)$ uniformly elliptic and Lipschitz continuous symmetric matrix.

In the present note, we consider two-dimensional solutions to more general nonlinear equations and provide a new approach to this problem in the plane. Our approach is based on the analysis of nodal domains, which are maximal connected components of the set

$$\{x \in G : u(x) \neq 0\},$$

and nodal lines

$$\overline{\{x \in G : u(x) = 0\}},$$

which are the boundaries of nodal domains. Our main tool is to couple the strong maximum principle and the Harnack inequality with some topological arguments; this argument applies in the situation in which there are finite number of nodal domains. Interestingly, the unique continuation property can be deduced from some features of the topology of planar sets.

The topological approach taken in the paper can be applied also to the nonlinear eigenvalue problem involving the p -Laplacian and to more general eigenvalue problems constituting the Fučík spectrum. We refer to a recent paper [10] for more details.

We want to refer to [2] since the framework and some ideas there are somewhat related to those taken in the present note. Finally, we refer to three recent papers on unique continuation. Alessandrini [3]

shows the (strong) unique continuation property for solutions to linear elliptic equations in two variables in divergence form, possibly non-self-adjoint and with lower order terms. Armstrong–Silvestre [6] show that a viscosity solution of a $C^{1,1}$ uniformly elliptic fully nonlinear equation satisfies the (strong) unique continuation property. In [8], on the other hand, the authors study the game p -Laplace equation on a tree and provide a characterization of the subsets of the tree that enjoy the unique continuation property.

1.1. Notation. Throughout, G is a bounded simply-connected Jordan domain of \mathbb{R}^2 . A domain is an open connected set in \mathbb{R}^2 . We write $B_r = B_r(x) = B(x, r)$ for concentric open balls of radii $r > 0$ centered at some $x \in G$. We denote the closure, interior, exterior, and boundary of E by \overline{E} , $\text{int}(E)$, $\text{ext}(E)$, and ∂E , respectively.

1.2. Structural assumptions. Let us specify the structure of \mathcal{A} and \mathcal{B} in (1.1); We shall assume that there are constants $0 < a_0 \leq a_1 < \infty$ and $0 < b_1 < \infty$ such that for all vectors h in \mathbb{R}^2 and almost every $x \in G$ the following structural assumptions apply

$$\left. \begin{aligned} \mathcal{A}(x, h) \cdot h &\geq a_0 |h|^p, \\ |\mathcal{A}(x, h)| &\leq a_1 |h|^{p-1}, \\ |\mathcal{B}(x, h)| &\leq b_1 |h|^{p-1} \end{aligned} \right\} \quad (1.3)$$

where $1 < p < \infty$. We do not assume the monotoneity or the homogeneity of the operator \mathcal{A} since we do not consider existence or uniqueness problems.

The structural conditions (1.3) result in Hölder continuity of a weak solution to (1.1), and moreover in the Harnack inequality and the strong maximum principle, we refer to Serrin [18].

We could also allow for the following structural conditions

$$\left. \begin{aligned} \mathcal{A}(x, h) \cdot h &\geq a_0 |h|^p, \\ |\mathcal{A}(x, h)| &\leq a_1 |h|^{p-1}, \\ |\mathcal{B}(x, h)| &\leq b_0 |h|^p + b_1 |h|^{p-1} \end{aligned} \right\} \quad (1.4)$$

where $0 < b_0 < \infty$ and $1 < p < \infty$. In this case local Hölder continuity and the Harnack inequality for a locally bounded weak solution of (1.1) follow from Trudinger [19].

We do not consider the case in which \mathcal{A} and \mathcal{B} may depend on u , or, for that matter, aim at the most general structure in (1.3) or (1.4). We will only need that solutions of (1.1) are continuous, and satisfy the Harnack inequality and the strong maximum principle.

1.3. Some elements of the plane topology. We recall a few facts about the topology of planar sets; a good reference is [16]. Let Ω be any domain in \mathbb{R}^2 . A Jordan arc is a point set which is homeomorphic with $[0, 1]$, whereas a Jordan curve is a point set which is homeomorphic with a circle. By Jordan's curve theorem a Jordan curve in \mathbb{R}^2 has

two complementary domains, and the curve is the boundary of each component. One of these two domains is bounded and this domain is called the interior of the Jordan curve. A domain whose boundary is a Jordan curve is called a Jordan domain.

As a related note, it is well known that the boundary of a bounded simply-connected domain in the plane is connected. In the plane a simply-connected domain Ω can be defined by the property that all points in the interior of any Jordan curve, which consists of points of Ω , are also points of Ω [15].

A Jordan arc with one end-point on $\partial\Omega$ and all its other points in Ω , is called an end-cut. If both end-points are in $\partial\Omega$, and the rest in Ω , a Jordan arc is said to be a cross-cut in Ω . A point $x \in \partial\Omega$ is said to be accessible from Ω if it is an end-point of an end-cut in Ω . Accessible boundary points of a planar domain are aplenty: The accessible points of $\partial\Omega$ are dense in $\partial\Omega$ [16, p. 162].

We recall a few facts about connected sets and ε -chains. If x and y are distinct points, then an ε -chain of points joining x and y is a finite sequence of points

$$x = a_1, a_2, \dots, a_k = y$$

such that $|a_i - a_{i+1}| \leq \varepsilon$, for $i = 1, \dots, k - 1$. A set of points is ε -connected if every pair of points in it can be joined by an ε -chain of points in the set. A compact set F in \mathbb{R}^2 is connected if and only if it is ε -connected for every $\varepsilon > 0$ [16, Theorem 5.1, p. 81]. If a connected set of points in \mathbb{R}^2 intersects both Ω and $\mathbb{R}^2 \setminus \Omega$ it intersects $\partial\Omega$ [16, Theorem 1.3, p. 73].

Lastly, we recall the following topological property [16, p. 159]. A subset E of \mathbb{R}^2 is said to be locally connected at any $x \in \mathbb{R}^2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that any two points of $B_\delta(x) \cap E$ are joined by a connected set in $B_\varepsilon(x) \cap E$. A set is uniformly locally connected, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that all pairs of points, x and y , for which $|x - y| < \delta$ can be joined by a connected subset of diameter less than ε . All convex domains and, more generally, Jordan domains are uniformly locally connected [16, Theorem 14.1, p. 161]. However, simply-connected domains are not necessarily locally connected.

2. UNIQUE CONTINUATION AND NODAL DOMAINS

We may interpret equation (1.1) in the weak sense. We recall that it follows from the structural assumptions (1.3) (or (1.4)) that a weak solution to (1.1) is Hölder continuous and satisfies the following Harnack inequality. We refer to Serrin [18]. The proof is based on the Moser iteration method [14].

Theorem 2.1 (Harnack's inequality). *Suppose u is a non-negative solution to (1.1) in $B_{3r} \subset G$. Then*

$$\sup_{B_r} u \leq C \inf_{B_r} u,$$

where $C = C(p, a_0, a_1, b_1)$.

Having the structure (1.4) in (1.1), Theorem 2.1 can be found in Trudinger [19]. Moreover, in this case we shall assume that a weak solution u is locally bounded.

We also point out the following important property, the strong maximum principle, which can be deduced from the Harnack inequality. We refer to a monograph by Pucci and Serrin [17] on maximum principles.

Theorem 2.2 (Strong maximum principle). *Suppose u is a non-constant solution to (1.1) in G . Then u cannot attain its maximum at an interior point of G .*

We shall make use of the fact that if u is a solution to (1.1), then $-u + c$, $c \in \mathbb{R}$, is also a solution to an equation similar to (1.1). Hence Harnack's inequality and the strong maximum principle apply to both u and $-u + c$.

Our main result is the following theorem. We assume minimal regularity conditions on ∂G so that every boundary point is regular, and hence $u \in C(\overline{G})$.

Theorem 2.3. *Suppose u is a solution to the equation (1.1) under structural conditions (1.3) (or (1.4), and u locally bounded) in a bounded simply-connected Jordan domain G of \mathbb{R}^2 subject to the Dirichlet condition*

$$u = g \quad \text{on} \quad \partial G,$$

where $g \in W^{1,p}(G) \cap C(\overline{G})$. *If $g|_{\partial G}$ has finite number of relative maxima and minima, then u has the unique continuation property: If u vanishes in some open subset of G , then u vanishes identically in G .*

We could also state the result as follows: If u is a constant in some open subset of G , then u is identically constant in G . In what follows, however, we stick to the classical formulation by dealing with a vanishing solution.

The crux of the proof is to study so-called nodal domains. A maximal connected component, i.e. one that is not a strict subset of any other connected set, of the set $\{x \in G : u(x) \neq 0\}$ is called, in what follows, a nodal domain. We denote these components by

$$N_i^+ = \{x \in G : u(x) > 0\}, \quad \text{and} \quad N_j^- = \{x \in G : u(x) < 0\},$$

where $i, j = 1, 2, \dots$. We remark that if u is, for instance, a solution to the p -Laplace equation (1.2) it is not known whether the number of nodal domains is finite.

In the proof of the following key lemma, we make use of the fact that G is a Jordan domain, and more precisely, G is uniformly locally connected at every $x \in \partial G$, we refer to Section 1.3.

Lemma 2.4. *Let the hypothesis of Theorem 2.3 be satisfied. Then the number of nodal domains, N_i^+ and N_j^- , is finite.*

Proof. We note first that u vanishes on all nodal lines in G , i.e. on $\partial N_i^+ \cap G$ and $\partial N_j^- \cap G$. Hence by the strong maximum principle each nodal line meets the boundary of G .

By the strong maximum principle the set $\partial N_i^+ \cap \partial G$ contains a global maximum of u in N_i^+ . We then show that such maximum point $x_0 \in N_i^+$ is also a relative maximum of g on ∂G : Let $x_0 \in \partial G$ be a maximum point of u on some fixed nodal domain N_i^+ . We shall then apply local connectedness of G at every boundary point $x \in \partial G$ (Section 1.3). We claim next that there exists $\delta_{x_0} > 0$ such that $B_\delta(x_0) \cap G$, for each $\delta < \delta_{x_0}$, contains only points of N_i^+ . But assume that this is not the case. Hence for each $\delta < \delta_{x_0}$ there exists $\tilde{x} \in B_\delta(x_0) \cap G$ such that \tilde{x} belongs to some other nodal domain than N_i^+ , say, N_j^- or $u(\tilde{x}) = 0$. Obviously, we need to consider only the former case.

We write $u(x_0) = \max_{x \in N_i^+} u(x) = \sigma > 0$. There exists a positive $\tilde{\delta} < \delta_{x_0}$ such that

$$u(x) > \frac{\sigma}{2}$$

for every $x \in \partial N_i^+ \cap \partial G \cap B_{\tilde{\delta}}(x_0)$ (it can be verified that such points exist since G is a Jordan domain). Moreover, since there exists a point $\tilde{x} \in B_{\tilde{\delta}}(x_0) \cap G$ such that $\tilde{x} \in N_j^-$ and G is locally connected at every $x \in \partial G$, there must exist also a point $\bar{x} \in B_{\tilde{\delta}}(x_0) \cap G$ so that $u(\bar{x}) = 0$. For small enough $\tilde{\delta}$ this is not possible since u is continuous and $u(x_0) > 0$.

We have therefore obtained that there exists a positive δ_{x_0} such that $B_\delta(x_0) \cap G$, $\delta < \delta_{x_0}$, contains only points of N_i^+ .

It follows that the inequality

$$u(x) \leq u(x_0)$$

is valid both for every $x \in B_\delta(x_0) \cap \partial N_i^+$ and for every $x \in B_\delta(x_0) \cap \partial G$ (in fact $B_\delta(x_0) \cap \partial N_i^+ = B_\delta(x_0) \cap \partial G$ as sets). Hence each maximum point $x_0 \in N_i^+$ constitutes a relative maximum of g on ∂G . An analogous reasoning applies for minima and relative minima on N_j^- and ∂G , respectively.

Since g is assumed to possess only finite number of relative maxima and minima on ∂G , the number of nodal domains must be finite. \square

Our idea in the proof of the preceding lemma has certain similarity to that of Lemma 1.1 in Alessandrini [2] where the number of interior critical points was considered to solution of linear equations.

The following proof resembles the argument presented in a recent paper by the authors, we refer to [10] for more details.

Proof of Theorem 2.3. We assume for contradiction that

- (A) u vanishes in a maximal open set $D \subset G$ but is not identically zero in G .

The maximal open set D is formed as follows: for every $x \in G$ for which there exists an open neighborhood such that $u \equiv 0$ on this neighborhood we denote by $B(x, r_x)$, $r_x = \sup \{t > 0 : u|_{\partial B(x,t)} \equiv 0\}$, the maximal open neighborhood of x where u vanishes identically. Then the maximal open set D is simply the union of all such neighborhoods. We pick a connected component of D , still denoted by D .

It is worth noting that (A) implies that the boundary data function g changes sign at least once on ∂G .

By Lemma 2.4, there exist positive M^+ , $M^- < \infty$ such that we may index the nodal domains $i = 1, \dots, M^+$ and $j = 1, \dots, M^-$.

Each nodal domain is simply-connected which can be seen as follows. Suppose that N_i^+ is not simply-connected for some $i \in \{1, \dots, M^+\}$. Then there exists a Jordan curve $\gamma \subset N_i^+$ with its interior S_γ and S_γ contains points which do not belong to the fixed nodal domain N_i^+ . Moreover, $S_\gamma \subset G$ since G is assumed to be simply-connected. It follows that the set $E = \{x \in S_\gamma \setminus N_i^+ : u(x) \leq 0 \text{ or } u(x) > 0\}$ is non-empty. If $\tilde{E} = \{x \in S_\gamma : u(x) < 0 \text{ or } u(x) > 0\}$ was empty, then $u(x) = 0$ for all $x \in E$, and $u(x) > 0$ for all $x \in S_\gamma \setminus E$. This is impossible by Harnack's inequality, Theorem 2.1. Hence N_i^+ is simply-connected.

We consider next the case in which $\tilde{E} = \{x \in S_\gamma : u(x) < 0 \text{ or } u(x) > 0\} \subset E$ is non-empty. It suffices to consider only the points at which $u < 0$ (the points at which $u > 0$ are handled in the same way); this set is still denoted by \tilde{E} . The set \tilde{E} is open and each component of \tilde{E} is a subset of some nodal domain N_j^- . This contradicts with the fact that each nodal line meets ∂G . Hence N_i^+ is simply-connected.

An analogous, symmetric, reasoning applies to N_j^- . Hence ∂N_i^+ and ∂N_j^- are connected as the boundaries of simply-connected domains, and thus continua, i.e. compact connected sets with at least two points, for each i and j .

Suppose next there exists a point $x \in \partial D \cap G$ and its neighborhood $B_\delta(x)$, $\delta > 0$, such that $\overline{B_\delta} \subset G$ and $B_\delta(x) \cap \text{ext}(D)$ contains only points of either N_i^+ or N_j^- for some i and j , i.e. points at which either $u > 0$ or $u < 0$. Assume, without loss of generality, that $B_\delta(x) \cap \text{ext}(D)$ contains points of N_i^+ only. Then $u \geq 0$ on $B_\delta(x)$ and by Harnack's inequality, Theorem 2.1, $u \equiv 0$ on $B_{\delta/2}(x)$, which contradicts the maximality of the set D , and hence also the antithesis (A). In this case our claim follows.

By the preceding reasoning it is sufficient to consider the following situation. For any $x \in \partial D \cap G$ and for any $\delta < \delta_0$, $\delta_0 > 0$, the neighborhood $B_\delta(x) \subset G$ contains points of the nodal domains N_i^+ and N_j^- for some indices i and for some indices j .

We point out that there exist a fixed index pair $(s, t) \in \{1, \dots, M^+\} \times \{1, \dots, M^-\}$ and $\delta_0 > 0$ such that each $B_\delta(x)$ contains points of N_s^+ and N_t^- , but there might be also points of other nodal domains in $B_\delta(x)$, for every $\delta < \delta_0$; this is a consequence of the fact that the number of nodal domains is finite in our case. We reason as follows: We consider a point $x \in \partial D \cap G$ and $B_\delta(x)$, $\delta < \delta_0$. We then select a decreasing sequence $\{\delta_i\}$ such that $\delta_i < \delta_0$ and $\lim_{i \rightarrow \infty} \delta_i = 0$. For each δ_i we may pick a pair of nodal domains, which we write

$$a_i := (N_{s(\delta_i)}^+, N_{t(\delta_i)}^-),$$

such that $B_{\delta_i}(x)$ contains points of both nodal domains. Since the number of all possible nodal domain pairs is finite, there exists a pair which appears infinitely many times in the sequence $\{a_i\}$. We may hence choose this fixed pair (N_s^+, N_t^-) , where $s(\delta_{i_j}) = s$ and $t(\delta_{i_j}) = t$, for some subsequence $\{\delta_{i_j}\}$ such that $\lim_{j \rightarrow \infty} \delta_{i_j} = 0$. It can be seen from this reasoning that the same pair occurs in any neighborhood $B_\delta(x)$, $\delta < \delta_0$.

We shall next base our reasoning on some topological arguments. We write $\partial D_A = \{x \in \partial D : x \text{ is accessible from } D\}$,

$$\partial N_{i,A}^+ = \{x \in \partial N_i^+ : x \text{ is accessible from } N_i^+\},$$

and correspondingly $\partial N_{j,A}^-$. By [16] accessible boundary points ∂D_A , $\partial N_{i,A}^+$, and $\partial N_{j,A}^-$ are dense in ∂D , ∂N_i^+ , and ∂N_j^- , respectively.

We will describe a selection process which gives a pair of points x_1 and x_2 such that $x_1, x_2 \in \partial D_A \cap G$ and that the associated spherical neighborhoods $B_\delta(x_1)$ and $B_\delta(x_2)$, $\delta < \delta_0$, contain points of the same nodal line ∂N_s^+ , $s \in \{1, \dots, M^+\}$ fixed. Moreover, it is assumed that $\overline{B}_\delta(x_1) \cap \overline{B}_\delta(x_2) = \emptyset$, and that $\overline{B}_\delta(x_1), \overline{B}_\delta(x_2) \subset G$. This procedure is as follows: We select a finite sequence of points $\{x_l\}$, each $x_l \in \partial D_A \cap G$. As pointed out earlier, for each x_l there exists $\delta_0 > 0$ such that the spherical neighborhood $B_\delta(x)$, $\delta < \delta_0$, contains points of N_s^+ and N_t^- for some $s \in \{1, \dots, M^+\}$ and $t \in \{1, \dots, M^-\}$. Since the number of all possible nodal domain pairs as described above is finite, after finite number of steps the sequence $\{x_l\}$ will contain a pair of points, denoted x_1 and x_2 , which have the aforementioned properties.

We then select $x_3 \in B_\delta(x_1) \cap \partial N_{s,A}^+$ and $x_4 \in B_\delta(x_2) \cap \partial N_{s,A}^+$.

We connect x_1 to x_2 by a cross-cut γ_D in D , and x_3 to x_4 by a cross-cut $\gamma_{N_s^+}$ in N_s^+ . We remark that x_3 , and analogously x_4 , is accessible in N_s^+ with a line segment (consult, e.g., Remark 3.3 in [10]). Also x_1 , and analogously x_2 , is accessible in D with a line segment. We fix such

line segments to access the points x_1, x_2, x_3 , and x_4 . In this way the line segments constitute part of the cross-cut γ_D and $\gamma_{N_s^+}$, respectively.

Since the boundary ∂N_s^+ is connected it is also ε -connected for every $\varepsilon > 0$. Hence for each $\varepsilon > 0$ the points x_1 and x_3 can be joined by an ε -chain $\{a_1, \dots, a_k\} \subset \partial N_s^+ \cap G$ such that

$$x_1 = a_1, a_2, \dots, a_{k-1}, a_k = x_3.$$

We consider a collection of open balls $\{B_{\frac{3}{2}\varepsilon}(a_i)\}_{i=1}^k$, $a_i \in \partial N_s^+ \cap G$, such that $B_{\frac{3}{2}\varepsilon}(a_i) \subset G$, and a domain U_ε^1 which is defined to be

$$U_\varepsilon^1 = \bigcup_{i=1}^k B_{\frac{3}{2}\varepsilon}(a_i).$$

Since U_ε^1 is a domain there exists a Jordan arc, $\gamma_{x_1 x_3}^\varepsilon$, connecting x_1 to x_3 in U_ε^1 . Correspondingly, the points x_2 and x_4 can be joined by an ε -chain in ∂N_s^+ and we obtain a domain U_ε^2 and a Jordan arc $\gamma_{x_2 x_4}^\varepsilon$ connecting x_2 to x_4 in U_ε^2 .

It is worth noting that we have selected $\gamma_{x_1 x_3}^\varepsilon$ and $\gamma_{x_2 x_4}^\varepsilon$ such that either of them does not intersect γ_D or $\gamma_{N_s^+}$, save the points x_1 and x_2 , and x_3 and x_4 , respectively. This is possible because of the line segment construction described above.

From the preceding Jordan arcs we obtain a Jordan curve Γ^ε , and by slight abuse of notation we write it as a product

$$\Gamma^\varepsilon = \gamma_{x_1 x_3}^\varepsilon \cdot \gamma_{N_s^+} \cdot \gamma_{x_2 x_4}^\varepsilon \cdot \gamma_D.$$

The Jordan curve Γ^ε divides the plane into two disjoint domains, and Γ^ε constitutes the boundary of both domains. We consider the bounded domain, denoted by T_ε , enclosed by Γ^ε . See Figure 1.

We next deal with the Jordan domain T_ε . There exists at least one point $y \in T_\varepsilon$ such that $u(y) < 0$, i.e. $y \in N_{j_0}^-$ for some fixed $j_0 \in \{1, \dots, M^-\}$. Assume that this is not the case: then $u(x) \geq 0$ for every $x \in T_\varepsilon$. As γ_D is part of the boundary T_ε contains also points of D , and hence u vanishes at such points. By Harnack's inequality, Theorem 2.1, $u \equiv 0$ in T_ε . This is, however, impossible since $\gamma_{N_s^+}$ is part of the boundary of T_ε , thus $u > 0$ on a sufficiently small neighborhood of a point in $\gamma_{N_s^+}$.

In an analogous way, it is possible to show that there exists a point $z \in N_{j_0}^- \cap (G \setminus \overline{T_\varepsilon})$. We stress that it is crucial that the selected points z and y belong to the same nodal domain $N_{j_0}^-$. It is worth noting here that by the strong maximum principle Γ^ε cannot enclose the nodal domain $N_{j_0}^-$ containing the point y , and therefore $G \setminus \overline{T_\varepsilon}$ must also contain points of $N_{j_0}^-$. We then connect z and y in $N_{j_0}^-$ by a Jordan arc γ_{zy} . Observe that $u(x) < 0$ for every $x \in \gamma_{zy}$.

The Jordan arc γ_{zy} as a connected set intersects Γ^ε at least at one point. We then distinguish the following four possible cases for the point of intersection: The point of intersection is contained in

- (1) γ_D ,
- (2) $\gamma_{N_s^+}$,
- (3) $\gamma_{x_1x_3}^\varepsilon$,
- (4) $\gamma_{x_2x_4}^\varepsilon$.

In the case (1) and (2) we have reached a contradiction as $u(x) = 0$ for every $x \in \gamma_D$ and $u(x) > 0$ for every $x \in \gamma_{N_s^+}$, respectively.

Consider the case (3) and case (4). We denote the point of intersection by x_ε for every $\varepsilon > 0$. We can select an appropriate subsequence $\{x_{\varepsilon_j}\}_{j=1}^\infty$, $\lim_{j \rightarrow \infty} \varepsilon_j = 0$, such that for each j either $x_{\varepsilon_j} \in U_{\varepsilon_j}^1$ or $x_{\varepsilon_j} \in U_{\varepsilon_j}^2$. We assume, without loss of generality, that $x_{\varepsilon_j} \in U_{\varepsilon_j}^1$. The sequence $\{x_{\varepsilon_j}\}$ is clearly bounded, and hence there exists a subsequence, still denoted $\{x_{\varepsilon_j}\}$, such that $\lim_{j \rightarrow \infty} x_{\varepsilon_j} = x_0$. Observe that each

$$x_{\varepsilon_j} \in B_{\frac{3}{2}\varepsilon_j}(a_m)$$

for some $a_m \in \partial N_s^+ \cap G$ in the ε_j -chain. We note that $u(a_m) = 0$. Moreover, if there existed δ_0 and a subsequence, still denoted $\{x_{\varepsilon_j}\}$, such that

$$|u(x_{\varepsilon_j})| \geq \delta_0 > 0$$

for every x_{ε_j} , this would contradict with uniform continuity of u (note that u is uniformly continuous on compact subsets of G). We hence have that

$$u(x_0) = \lim_{j \rightarrow \infty} u(x_{\varepsilon_j}) = 0.$$

In conclusion, we have reached a contradiction since $u(x_0) = 0$ but, on the other hand, $x_0 \in \gamma_{zy}$ and hence $u(x_0) < 0$.

All four cases (1)–(4) lead to a contradiction. Hence antithesis (A) is false, thus the claim follows. \square

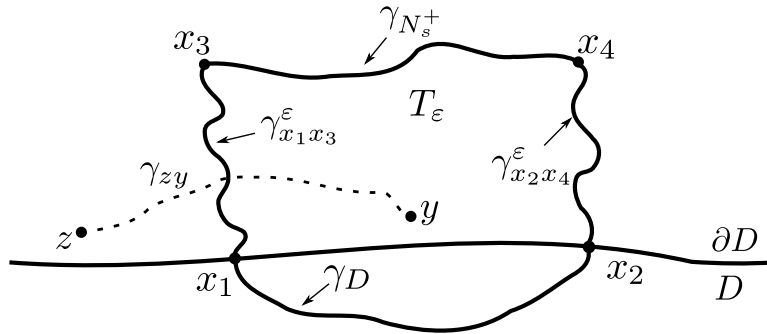


FIGURE 1. Jordan domain T_ε and Jordan curve γ_{zy} (dotted line) connecting z to y in $N_{j_0}^-$.

To conclude, a non-trivial solution of (1.1) in G under (1.3) or (1.4) and subject to the Dirichlet condition as described in Theorem 2.3 has only finitely many nodal domains and all nodal lines intersect ∂G . These two facts ensure that such solution does not vanish in an open subset of G .

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