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Convergence of approximate deconvolution models to the mean Magnetohydrodynamics Equations: Analysis of two models

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Abstract

We consider two Large Eddy Simulation (LES) models for the approximation of large scales of the equations of Magnetohydrodynamics (MHD in the sequel). We study two α-models, which are obtained adapting to the MHD the approach by Stolz and Adams with van Cittert approximate deconvolution operators. First, we prove existence and uniqueness of a regular weak solution for a system with filtering and deconvolution in both equations. Then we study the behavior of solutions as the deconvolution parameter goes to infinity. The main result of this paper is the convergence to a solution of the filtered MHD equations. In the final section we study also the problem with filtering acting only on the velocity equation.

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Key-words: Magnetohydrodynamics, Large Eddy Simulation, Deconvolution models.

1 Introduction

In this paper we study the equations of (double viscous) incompressible MHD

$$
\begin{aligned}
&\partial_t \mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) - \nabla\cdot(\mathbf{B}\otimes\mathbf{B}) + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f},\\
&\partial_t \mathbf{B} + \nabla\cdot(\mathbf{B}\otimes\mathbf{u}) - \nabla\cdot(\mathbf{u}\otimes\mathbf{B}) - \mu\Delta\mathbf{B} = 0,\\
&\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \qquad \mathbf{B}(0,\mathbf{x}) = \mathbf{B}_0(\mathbf{x}),
\end{aligned}
\tag{1.1}
$$

where $\nu > 0$ is the kinematic viscosity, while $\mu > 0$ is the magnetic diffusivity. The fields $\mathbf{u}$ and $\mathbf{B}$ are the velocity and the magnetic field respectively, while the scalar p is the pressure (rescaled by the density supposed constant here). We consider the problem in the three dimensional setting, and most of the technical difficulties are those known for the 3D Navier–Stokes equations (NSE). Examples of fluids which can be described by these equations (1.1) are for instance plasmas, liquid metals, and salt water or electrolytes. See Davidson [14] for an introduction to the topic. In this paper we aim to study the approximate deconvolution procedure (developed for turbulent flows by Stolz and Adams [37, 38, 1]) and especially its adaption to the MHD with the perspective of numerical simulations of turbulent incompressible flows, when coupled to a magnetic field.

In the recent years, the topic of MHD attracted the interest of many researchers and, for the study of the question of existence, uniqueness, regularity, and estimates on the number of degrees of freedom, we recall the following papers [8, 10, 20, 21, 22, 23, 24, 29].

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Approximate Deconvolution Models (ADM) for turbulent flows without magnetic effects were studied in [5, 16, 27, 28]. The problem of the limiting behavior of the models when the grid mesh size goes to zero is already under control [16, 25, 26, 32].On the other hand, the question of the limiting behavior of the solutions when the deconvolution parameter goes to infinity is a very recent topic, and is well-studied just for the NSE –without any coupling– in [5] (see also a short review in [3]).

In the context of MHD, the topic seems not explored yet, hence we adapt here the results of [5] to the equations with the magnetic field and we find also some interesting unexpected variant, related to the applications of two different filters. Especially the equation for the magnetic field turns out to behave much better than that for the velocity, hence it seems not to require filtering.

To briefly introduce the problem (the reader can find more details in the introduction of [5]), we recall that the main underlying idea of LES, see [4, 12, 34], is that of computing the "mean values" of the flow fields $\mathbf{u} = (u^1, u^2, u^3)$, $\mathbf{B} = (B^1, B^2, B^3)$, and p. In the spirit of the work started with Boussinesq [7] and then with Reynolds [33], this corresponds to find a suitable computational decomposition

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \qquad \mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}', \qquad \text{and} \qquad p = \overline{p} + p',$$

where the primed variables are fluctuations around the over-lined mean fields. In our context, the mean fields are defined by application of the inverse of a differential operator. By assuming that the averaging operation commutes with differential operators, one gets the filtered MHD equations

$$\begin{aligned} \partial_t \overline{\mathbf{u}} + \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) - \nabla \cdot \overline{(\mathbf{B} \otimes \mathbf{B})} + \nabla \overline{p} - \nu \Delta \overline{\mathbf{u}} &= \overline{\mathbf{f}}, \\ \partial_t \overline{\mathbf{B}} + \nabla \cdot \overline{(\mathbf{B} \otimes \mathbf{u})} - \nabla \cdot \overline{(\mathbf{u} \otimes \mathbf{B})} - \mu \Delta \overline{\mathbf{B}} &= 0, \\ \nabla \cdot \overline{\mathbf{u}} = \nabla \cdot \overline{\mathbf{B}} &= 0, \\ \overline{\mathbf{u}}(0, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}), \qquad \overline{\mathbf{B}}(0, \mathbf{x}) &= \overline{\mathbf{B}}_0(\mathbf{x}). \end{aligned} \tag{1.2}$$

This raises the question of the *interior closure problem*, that is the modeling of the tensors

$$\overline{(\mathbf{c} \otimes \mathbf{d})} \qquad \text{with either} \mathbf{c}, \mathbf{d} = \mathbf{u} \text{ or } \mathbf{B}$$

in terms of the filtered variables $(\overline{\mathbf{u}}, \overline{\mathbf{B}}, \overline{p})$.

From this point, there are many modeling options. The basic model is the sub-grid model (SGM) that introduces an eddy viscosity of the form $\nu_t = Ch(x)^2|\nabla \mathbf{u}|$, which may be deduced from Kolmogorov similarity theory (see [12]), where $h(x)$ denotes the local size of a computational grid, and C is a constant to be fixed from experiments. This model, that already appears in Prandtl work [31] with the mixing length ℓ instead of $h(x)$, was firstly used by Smagorinsky for numerical simulations [36]. This is a very good model, but introduces numerical instabilities in high-gradient regions, depending on the numerical scheme and potential CFL constraints.

Among all procedures to stabilize the SGM, the most popular was suggested by Bardina *et al.* [2], which reveals being a little bit too diffusive and underestimates some of the resolved scales, that are called "Sub Filter Scales" (SFS) (see for instance [11, 19]). Then the model needs to be "deconvolved" to reconstruct accurately the SFS. Hence, many options occur here, too. In the present paper we study the Approximate Deconvolution Model (ADM), introduced by Adams and Stolz [37, 1], who have successfully transferred image modeling procedures [6] to turbulence modeling.

From a simplified and naive mathematical viewpoint, this model, which uses similarity properties of turbulence, is defined by approximating the filtered bi-linear terms as follows:

$$\overline{(\mathbf{c} \otimes \mathbf{d})} \sim \overline{(D_N(\overline{\mathbf{c}}) \otimes D_N(\overline{\mathbf{d}})}\,.$$

Here the *filtering operators* G_i are defined thanks to the Helmholtz filter (cf. (2.1)–(2.2) below) by $G_1(\mathbf{u}) = \overline{\mathbf{u}}$, $G_2(\mathbf{B}) = \overline{\mathbf{B}}$, where $G_i := (\mathrm{I} - \alpha_i^2 \Delta)^{-1}$, $i = 1, 2$. Observe that we can then have two different filters corresponding to the equation for the velocity and for that of the magnetic field. There are two interesting values for the couple of parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$:

1. $\alpha_1 = \alpha_2 > 0$. In this case the approximate equations conserve Alfvén waves, see [22];
2. $\alpha_1 > 0$, $\alpha_2 = 0$, which means no filtering in the equation for $\mathbf{B}$.

The deconvolution operators D_{N_i} are defined through the van Cittert algorithm (2.10) and the initial value problem that we consider in the space periodic setting is:

$$
\begin{aligned}
&\partial_t \mathbf{w} + \nabla\cdot G_1\big(D_{N_1}(\mathbf{w})\otimes D_{N_1}(\mathbf{w})\big) - \nabla\cdot G_1\big(D_{N_2}(\mathbf{b})\otimes D_{N_2}(\mathbf{b})\big) + \nabla q - \nu\Delta\mathbf{w} = G_1\mathbf{f},\\
&\partial_t \mathbf{b} + \nabla\cdot G_2\big(D_{N_1}(\mathbf{w})\otimes D_{N_2}(\mathbf{b})\big) - \nabla\cdot G_2\big(D_{N_2}(\mathbf{b})\otimes D_{N_1}(\mathbf{w})\big) - \mu\Delta\mathbf{b} = 0,\\
&\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{b} = 0,\\
&\mathbf{w}(0,\mathbf{x}) = G_1\mathbf{u}_0(\mathbf{x}), \qquad \mathbf{b}(0,\mathbf{x}) = G_2\mathbf{B}_0(\mathbf{x}),\\
&\alpha_1 > 0, \qquad \alpha_2 \geq 0.
\end{aligned}
\tag{1.3}
$$

As usual, we observe that the equations (1.3) are not the equations (1.2) satisfied by $(\overline{\mathbf{u}},\overline{\mathbf{B}})$, but we are aimed at considering (1.3) as an approximation of (1.2), hence $\mathbf{w}\simeq G_1\mathbf{u}$ and $\mathbf{b}\simeq G_2\mathbf{B}$. This is mathematically sound since, at least *formally,*

$$
D_{N_i} \to A_i := \mathrm{I} - \alpha_i^2\Delta \quad \text{ in the limit } N_i\to+\infty,
$$

hence, again formally, (1.3) will become the filtered MHD equations (1.2). The existence and uniqueness issues have been also treated (even if without looking for estimates independent of N_i) in [22, 21] (for arbitrary deconvolution orders). What seems more challenging is to understand whether this convergence property is true or not, namely to show that as the approximation parameters N_i grow, then (as recently proved for the Navier–Stokes equations in [5])

$$
\mathbf{w}\to G_1\mathbf{u}, \qquad \mathbf{b}\to G_2\mathbf{B}, \qquad \text{and} \qquad q\to G_1 q.
$$

We prove that the model (1.3) converges, in some sense, to the averaged MHD equations (1.2), when the typical scales of filtration α_i remain fixed. Before analyzing such a convergence, we need to prove more precise existence results. To this end we follow the same approach from [5], which revisits the approach in [16] for the Navier–Stokes equations. To be more precise, the main result deals with $\alpha_1>0$ and $\alpha_2>0$. We first prove (cf. Thm. 3.1) existence and uniqueness of solutions $(\mathbf{w}_N,\mathbf{b}_B,q_N)$ of (1.3), with $N=(N_1,N_2)$, such that

$$
\begin{aligned}
&\mathbf{w}_N,\ \mathbf{b}_N \in L^2([0,T];H^2(\mathbb{T}_3)^3)\cap L^\infty([0,T];H^1(\mathbb{T}_3)^3),\\
&q_N \in L^2([0,T];W^{1,2}(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3)),
\end{aligned}
$$

and our main result is the following one.

Theorem 1.1. *Let $\alpha_1>0$ and $\alpha_2>0$; then, from the sequence $\{(\mathbf{w}_N,\mathbf{b}_N,q_N)\}_{N\in\mathbb{N}^2}$, one can extract a (diagonal) sub-sequence (still denoted $\{(\mathbf{w}_N,\mathbf{b}_N,q_N)\}_{N\in\mathbb{N}^2}$)*

$$
\begin{aligned}
&\begin{aligned}\mathbf{w}_N &\to \mathbf{w}\\ \mathbf{b}_N &\to \mathbf{b}\end{aligned} && \begin{cases}\textit{weakly in } L^2([0,T];H^2(\mathbb{T}_3)^3)\\ \textit{weakly}* \textit{ in } L^\infty([0,T];H^1(\mathbb{T}_3)^3),\end{cases}\\
&\begin{aligned}\mathbf{w}_N &\to \mathbf{w}\\ \mathbf{b}_N &\to \mathbf{b}\end{aligned} && \textit{strongly in } L^p([0,T];H^1(\mathbb{T}_3)^3), \quad \forall\, 1\leq p<+\infty,\\
&q_N \to q && \textit{weakly in } L^2([0,T];W^{1,2}(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3)),
\end{aligned}
$$

such that the system

$$
\begin{aligned}
\partial_t\mathbf{w} + \nabla\cdot G_1(A_1\mathbf{w}\otimes A_1\mathbf{w}) - \nabla\cdot G_1(A_2\mathbf{b}\otimes A_2\mathbf{b}) + \nabla q - \nu\Delta\mathbf{w} &= G_1\mathbf{f},\\
\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{b} &= 0,\\
\partial_t\mathbf{b} + \nabla\cdot G_2(A_2\mathbf{b}\otimes A_1\mathbf{w}) - \nabla\cdot G_2(A_1\mathbf{w}\otimes A_2\mathbf{b}) - \mu\Delta\mathbf{b} &= 0,\\
\mathbf{w}(0,\mathbf{x}) = G_1\mathbf{u}_0(\mathbf{x}), \qquad \mathbf{b}(0,\mathbf{x}) &= G_2\mathbf{B}_0(\mathbf{x}),
\end{aligned}
\tag{1.4}
$$

holds in the distributional sense. Moreover, the following energy inequality holds:

$$
\frac{1}{2}\frac{d}{dt}\big(\|A_1\mathbf{w}\|^2 + \|A_2\mathbf{b}\|^2\big) + \nu\|\nabla A_1\mathbf{w}\|^2 + \mu\|\nabla A_2\mathbf{b}\|^2 \leq \langle \mathbf{f}, A_1\mathbf{w}\rangle. \tag{1.5}
$$

As a consequence of Theorem 1.1, we deduce that the field $(\mathbf{u}, \mathbf{B}, p) = (A_1\mathbf{w}, A_2\mathbf{b}, A_1 q)$ is a dissipative (of Leray-Hopf's type) solution to the MHD Equations (1.1).

Remark 1.1. *Following the work carried out in [17] about ADM without coupling, we conjecture that the error modeling in the case of the approximate deconvolution MHD is of order $N^{-1/2}$.*

Remark 1.2. *The question of the boundary conditions is the great challenge in LES modeling, see [35] for a general review. This is why, either from physical or mathematical viewpoint, theoretical issues are raised in the case of periodic boundary conditions, although the reality of such boundary conditions may be controversial. In a paragraph in [38], the authors outline a possible numerical discrete algorithm for a deconvolution procedure, by the finite difference method. This is proposed in the case of homogeneous boundary conditions, but there is no mathematical analysis about this method, which still remains an open problem.*

Plan of the paper. In Sec. 2 we introduce the notation and the filtering operations. In Sec. 3-4 we consider the model with the double filtering with non-vanishing parameters α_i and then we study the limiting behavior as $N_i \to +\infty$. Finally, in Section 5, we treat the same problems in the case $\alpha_1 > 0$ and $\alpha_2 = 0$. Since most of the calculations are in the same spirit of those in [5], instead of proofs at full length we just point out the changes needed to adapt the proof valid for the NSE to the MHD equations.

2 Notation and Filter/Deconvolution operators

This section is devoted to the definition of the functional setting which we will use, and to the definition of the filter through the Helmholtz equation, with the related deconvolution operator. All the results are well-known and we refer to [5, 27, 28] for further details. We will use the customary Lebesgue L^p and Sobolev $W^{k,p}$ and $W^{s,2} = H^s$ spaces, in the periodic setting. Hence, we use Fourier series on the 3D torus $\mathbb{T}_3$. Let be given $L \in \mathbb{R}^\star_+ := \{x \in \mathbb{R} : \ x > 0\}$, and define $\Omega :=]0, L[^3 \subset \mathbb{R}^3$. We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis of $\mathbb{R}^3$, and by $\mathbf{x} := (x_1, x_2, x_3) \in \mathbb{R}^3$ the standard point in $\mathbb{R}^3$. We put $\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$ and $\mathbb{T}_3$ is the torus defined by $\mathbb{T}_3 := \left(\mathbb{R}^3/\mathcal{T}_3\right)$. We use $\|\cdot\|$ to denote the $L^2(\mathbb{T}_3)$-norm and associated operator norms. We always impose the zero mean condition on the fields that we consider and we define, for a general exponent $s \geq 0$,

$$\mathbf{H}_s := \left\{ \mathbf{w} : \mathbb{T}_3 \to \mathbb{R}^3, \ \mathbf{w} \in H^s(\mathbb{T}_3)^3, \quad \nabla \cdot \mathbf{w} = 0, \quad \int_{\mathbb{T}_3} \mathbf{w}\, d\mathbf{x} = \mathbf{0} \right\}.$$

For $\mathbf{w} \in \mathbf{H}_s$, we can expand the fields as $\mathbf{w}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{x}}$, where $\mathbf{k} \in \mathcal{T}_3^\star$ is the wave-number, and the Fourier coefficients are $\widehat{\mathbf{w}}_{\mathbf{k}} := \frac{1}{|\mathbb{T}_3|}\int_{\mathbb{T}_3} \mathbf{w}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$. The magnitude of $\mathbf{k}$ is defined by $k := |\mathbf{k}| = \{|k_1|^2 + |k_2|^2 + |k_3|^2\}^{\frac{1}{2}}$. We define the $\mathbf{H}_s$ norms by $\|\mathbf{w}\|_s^2 := \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}_{\mathbf{k}}|^2$, where of course $\|\mathbf{w}\|_0^2 := \|\mathbf{w}\|^2$. The inner products associated to these norms are $(\mathbf{w}, \mathbf{v})_{\mathbf{H}_s} := \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} \widehat{\mathbf{w}}_{\mathbf{k}} \cdot \overline{\widehat{\mathbf{v}}_{\mathbf{k}}}$, where $\overline{\widehat{\mathbf{v}}_{\mathbf{k}}}$ denotes the complex conjugate of $\widehat{\mathbf{v}}_{\mathbf{k}}$. To have real valued vector fields, we impose $\widehat{\mathbf{w}}_{-\mathbf{k}} = \overline{\widehat{\mathbf{w}}_{\mathbf{k}}}$ for any $\mathbf{k} \in \mathcal{T}_3^\star$ and for any field denoted by $\mathbf{w}$. It can be shown (see e.g. [15]) that when s is an integer, $\|\mathbf{w}\|_s^2 := \|\nabla^s \mathbf{w}\|^2$ and also, for general $s \in \mathbb{R}$, $(\mathbf{H}_s)' = \mathbf{H}_{-s}$.

We now recall the main properties of the Helmholtz filter. In the sequel, $\alpha > 0$ denotes a given fixed number and for $\mathbf{w} \in \mathbf{H}_s$ the field $\overline{\mathbf{w}}$ is the solution of the Stokes-like problem:

$$\begin{aligned} -\alpha^2 \Delta \overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla \pi &= \mathbf{w} \qquad \text{in } \mathbb{T}_3, \\ \nabla \cdot \overline{\mathbf{w}} &= 0 \qquad \text{in } \mathbb{T}_3, \\ \int_{\mathbb{T}_3} \pi\, d\mathbf{x} &= 0. \end{aligned} \tag{2.1}$$

For $\mathbf{w} \in \mathbf{H}_s$ this problem has a unique solution $(\overline{\mathbf{w}}, \pi) \in \mathbf{H}_{s+2} \times H^{s+1}(\mathbb{T}_3)$, whose velocity is denoted also by $\overline{\mathbf{w}} = G(\mathbf{w})$. Observe that, with a common abuse of notation, for a scalar function χ we still denote (this is a standard notation) by $\overline{\chi}$ the solution of the pure Helmholtz problem

$$A\overline{\chi} := -\alpha^2 \Delta \overline{\chi} + \overline{\chi} = \chi \quad \text{in } \mathbb{T}_3. \tag{2.2}$$

In particular, in the LES model (1.3) and in the filtered equations (1.2)–(1.4), the symbol "$\overline{}$" denotes the pure Helmholtz filter, applied component-wise to the various tensor fields.

We recall now a definition that we will use several times in the sequel.

Definition 2.1. *Let K be an operator acting on $\mathbf{H}_s$. Assume that $e^{-i\mathbf{k}\cdot\mathbf{x}}$ are eigen-vectors of K with corresponding eigenvalues $\widehat{K}_{\mathbf{k}}$. Then we shall say that $\widehat{K}_{\mathbf{k}}$ is the symbol of K.*

The deconvolution operator D_N is constructed thanks to the Van-Cittert algorithm by $D_N := \sum_{n=0}^{N}(\mathrm{I}-G)^n$. Starting from this formula, we can express the deconvolution operator in terms of Fourier series $D_N(\mathbf{w}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{D}_N(\mathbf{k})\widehat{\mathbf{w}}_{\mathbf{k}}e^{+i\mathbf{k}\cdot\mathbf{x}}$, where

$$\widehat{D}_N(\mathbf{k}) = \sum_{n=0}^{N}\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^n = (1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}}, \qquad \rho_{N,\mathbf{k}} = 1-\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^{N+1}. \tag{2.3}$$

The basic properties satisfied by $\widehat{D}_N$ that we will need are summarized in the following lemma.

Lemma 2.1. *For each $N\in\mathbb{N}$ the operator $D_N : \mathbf{H}_s \to \mathbf{H}_s$ is self-adjoint, it commutes with differentiation, and the following properties hold true:*

$$1 \le \widehat{D}_N(\mathbf{k}) \le N+1 \qquad \forall\,\mathbf{k}\in\mathcal{T}_3; \tag{2.4}$$

$$\widehat{D}_N(\mathbf{k}) \approx (N+1)\frac{1+\alpha^2|\mathbf{k}|^2}{\alpha^2|\mathbf{k}|^2} \qquad \textit{for large } |\mathbf{k}|; \tag{2.5}$$

$$\lim_{|\mathbf{k}|\to+\infty}\widehat{D}_N(\mathbf{k}) = N+1 \qquad \textit{for fixed } \alpha>0; \tag{2.6}$$

$$\widehat{D}_N(\mathbf{k}) \le (1+\alpha^2|\mathbf{k}|^2) \qquad \forall\,\mathbf{k}\in\mathcal{T}_3,\ \alpha>0; \tag{2.7}$$

$$\textit{the map } \mathbf{w}\mapsto D_N(\mathbf{w}) \textit{ is an isomorphism} \quad \textit{s.t. } \|D_N\|_{\mathbf{H}_s} = O(N+1) \quad \forall\, s\ge 0; \tag{2.8}$$

$$\lim_{N\to+\infty} D_N(\mathbf{w}) = A\mathbf{w} \textit{ in } \mathbf{H}_s \qquad \forall\, s\in\mathbb{R} \textit{ and } \mathbf{w}\in\mathbf{H}_{s+2}. \tag{2.9}$$

All these claims follow from direct inspection of the formula (2.3) and, in the sequel, we will also use the natural notations $G_i := A_i^{-1} := (\mathrm{I}-\alpha_i^2\Delta)^{-1}$ and

$$D_{N_i} := \sum_{n=0}^{N_i}(\mathrm{I}-G_i)^n, \qquad i=1,2. \tag{2.10}$$

3 Existence results

In order to be self-contained, we start by considering the initial value problem for the model (1.3). In this section, N_1, $N_2\in\mathbb{N}$ are fixed as well as $\alpha_1>0$, $\alpha_2>0$, and we assume that the data are such that

$$\mathbf{u}_0, \mathbf{B}_0 \in \mathbf{H}_0, \quad \mathbf{f}\in L^2([0,T]\times\mathbb{T}_3), \tag{3.1}$$

which naturally yields $G_1\mathbf{u}_0, G_2\mathbf{B}_0\in\mathbf{H}_2$, $G_1\mathbf{f}\in L^2([0,T];\mathbf{H}_2)$. We start by defining the notion of what we call a "regular weak" solution to this system.

Definition 3.1 ("Regular weak" solution)**.** *We say that the triple $(\mathbf{w},\mathbf{b},q)$ is a "regular weak" solution to system (1.3) if and only if the three following items are satisfied:*

1) Regularity:

$$\mathbf{w},\mathbf{b}\in L^2([0,T];\mathbf{H}_2)\cap C([0,T];\mathbf{H}_1), \tag{3.2}$$

$$\partial_t\mathbf{w},\partial_t\mathbf{b}\in L^2([0,T];\mathbf{H}_0) \tag{3.3}$$

$$q\in L^2([0,T];H^1(\mathbb{T}_3)), \tag{3.4}$$

2) Initial data:

$$\lim_{t\to0}\|\mathbf{w}(t,\cdot)-G_1\mathbf{u}_0\|_{\mathbf{H}_1} = 0, \qquad \lim_{t\to0}\|\mathbf{b}(t,\cdot)-G_2\mathbf{B}_0\|_{\mathbf{H}_1} = 0, \tag{3.5}$$

3) Weak Formulation: *For all* $\mathbf{v}, \mathbf{h} \in L^2([0,T]; H^1(\mathbb{T}_3)^3)$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{w} \cdot \mathbf{v} - \int_0^T \int_{\mathbb{T}_3} G_1(D_{N_1}(\mathbf{w}) \otimes D_{N_1}(\mathbf{w})) : \nabla \mathbf{v} \\
&\quad + \int_0^T \int_{\mathbb{T}_3} G_1(D_{N_2}(\mathbf{b}) \otimes D_{N_2}(\mathbf{b})) : \nabla \mathbf{v} + \int_0^T \int_{\mathbb{T}_3} \nabla q \cdot \mathbf{v} \\
&\quad + \nu \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{w} : \nabla \mathbf{v} = \int_0^T \int_{\mathbb{T}_3} G_1(\mathbf{f}) \cdot \mathbf{v},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{b} \cdot \mathbf{h} - \int_0^T \int_{\mathbb{T}_3} G_2(D_{N_2}(\mathbf{b}) \otimes D_{N_1}(\mathbf{w})) : \nabla \mathbf{h} \\
&\quad + \int_0^T \int_{\mathbb{T}_3} G_2(D_{N_1}(\mathbf{w}) \otimes D_{N_2}(\mathbf{b})) : \nabla \mathbf{h} + \mu \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{b} : \nabla \mathbf{h} = 0.
\end{aligned} \tag{3.7}$$

Observe that, for simplicity, we suppressed all $d\mathbf{x}$ and dt from the space-time integrals. With the same observations as in [5], one can easily check that all integrals involving $D_{N_1}\mathbf{w}$ and $D_{N_2}\mathbf{b}$ in (3.6)–(3.7) are finite under the regularity in (3.2)-(3.3). We now prove the following theorem, which is an adaption of the existence theorem in [5] and at the same time a slightly more precise form of the various existence theorems available in literature for doubly viscous MHD systems.

Theorem 3.1. *Assume that (3.1) holds,* $0 < \alpha_i \in \mathbb{R}$ *and* $N_i \in \mathbb{N}$, $i = 1, 2$, *are given and fixed. Then, problem* (1.3) *has a unique regular weak solution.*

In the proof we use the usual Galerkin method (see for instance the basics for incompressible fluids in [30]) with divergence-free finite dimensional approximate velocities and magnetic fields. We also point out that Theorem 3.1 greatly improves the corresponding existence result in [22] and it is not a simple restatement of those results. Some of the main original contributions are here the estimates, uniform in N, that will allow later on to pass to the limit when $N_i \to +\infty$.

Proof of Theorem 3.1. Let be given $m \in \mathbb{N}^\star$ and define $\mathbf{V}_m$ to be the following space of real valued trigonometric polynomial vector fields

$$\mathbf{V}_m := \{\mathbf{w} \in \mathbf{H}_1 : \quad \int_{\mathbb{T}_3} \mathbf{w}(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}} = \mathbf{0}, \quad \forall \mathbf{k} \text{ with } |\mathbf{k}| > m\}.$$

In order to use classical tools for systems of ordinary differential equations, we approximate the external force $\mathbf{f}$ with $\mathbf{f}_{1/m}$ by means of Friederichs mollifiers. Thanks to the Cauchy-Lipschitz Theorem, we can prove existence of $T_m > 0$ and of unique C^1 solutions $\mathbf{w}_m(t, \mathbf{x})$ and $\mathbf{b}_m(t, \mathbf{x})$ (belonging to $\mathbf{V}_m$ for all $t \in [0, T_m[$) to

$$\begin{aligned}
&\int_{\mathbb{T}_3} \partial_t \mathbf{w}_m \cdot \mathbf{v} - \int_{\mathbb{T}_3} G_1(D_{N_1}(\mathbf{w}_m) \otimes D_{N_1}(\mathbf{w}_m)) : \nabla \mathbf{v} \\
&\quad + \int_{\mathbb{T}_3} G_1(D_{N_2}(\mathbf{b}_m) \otimes D_{N_2}(\mathbf{b}_m)) : \nabla \mathbf{v} \\
&\quad + \nu \int_{\mathbb{T}_3} \nabla \mathbf{w}_m : \nabla \mathbf{v} = \int_{\mathbb{T}_3} G_1(\mathbf{f}_{1/m}) \cdot \mathbf{v},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\int_{\mathbb{T}_3} \partial_t \mathbf{b}_m \cdot \mathbf{h} - \int_{\mathbb{T}_3} G_2(D_{N_2}(\mathbf{b}_m) \otimes D_{N_1}(\mathbf{w}_m)) : \nabla \mathbf{h} \\
&\quad + \int_{\mathbb{T}_3} G_2(D_{N_1}(\mathbf{w}_m) \otimes D_{N_2}(\mathbf{b}_m)) : \nabla \mathbf{h} + \mu \int_{\mathbb{T}_3} \nabla \mathbf{b}_m : \nabla \mathbf{h} = 0\,,
\end{aligned} \tag{3.9}$$

for all $\mathbf{v}, \mathbf{h} \in L^2([0,T]; \mathbf{V}_m)$.

Remark 3.1. *Instead of* $(\mathbf{w}_m, \mathbf{b}_m)$, *a more precise and appropriate notation for the solution of the Galerkin system would be* $(\mathbf{w}_{m,N_1,N_2,\alpha_1,\alpha_2}, \mathbf{b}_{m,N_1,N_2,\alpha_1,\alpha_2})$. *We are asking for a simplification, since in this section* N_i *and* α_i *are fixed and the only relevant parameter is the Galerkin one* $m \in \mathbb{N}^\star$.

The natural and correct test functions to get *a priori* estimates are $A_1 D_{N_1}(\mathbf{w}_m)$ for the first equation and $A_2 D_{N_2}(\mathbf{b}_m)$ for the second one. Arguing as in [5], it is easily checked that both are in V_m. Since A_1, A_2 are self-adjoint and commute with differential operators, it holds:

$$\int_{\mathbb{T}_3} G_1(D_{N_1}(\mathbf{w}_m) \otimes D_{N_1}(\mathbf{w}_m)) : \nabla(A_1 D_{N_1}(\mathbf{w}_m))\, d\mathbf{x} = 0,$$
$$\int_{\mathbb{T}_3} G_2(D_{N_2}(\mathbf{b}_m) \otimes D_{N_2}(\mathbf{b}_m)) : \nabla(A_2 D_{N_2}(\mathbf{b}_m))\, d\mathbf{x} = 0.$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{T}_3} G_1(D_{N_2}(\mathbf{b}_m) \otimes D_{N_2}(\mathbf{b}_m)) : \nabla(A_1 D_{N_1}(\mathbf{w}_m))\, d\mathbf{x} \\
&\qquad - \int_{\mathbb{T}_3} G_2(D_{N_2}(\mathbf{b}_m) \otimes D_{N_1}(\mathbf{w}_m)) : \nabla(A_2 D_{N_2}(\mathbf{b}_m))\, d\mathbf{x} \\
&\qquad + \int_{\mathbb{T}_3} G_2(D_{N_1}(\mathbf{w}_m) \otimes D_{N_2}(\mathbf{b}_m)) : \nabla(A_2 D_{N_2}(\mathbf{b}_m))\, d\mathbf{x} \\
&= - \int_{\mathbb{T}_3} (D_{N_2}(\mathbf{b}_m) \cdot \nabla) D_{N_2}(\mathbf{b}_m) \cdot D_{N_1}(\mathbf{w}_m)\, d\mathbf{x} \\
&\qquad + \int_{\mathbb{T}_3} (D_{N_1}(\mathbf{w}_m) \cdot \nabla) D_{N_2}(\mathbf{b}_m) \cdot D_{N_2}(\mathbf{b}_m)\, d\mathbf{x} \\
&\qquad - \int_{\mathbb{T}_3} (D_{N_2}(\mathbf{b}_m) \cdot \nabla) D_{N_1}(\mathbf{w}_m) \cdot D_{N_2}(\mathbf{b}_m)\, d\mathbf{x} = 0\,.
\end{aligned}$$

Summing up the equations satisfied by $\mathbf{w}_m$ and $\mathbf{b}_m$, using standard integration by parts and Poincaré's inequality combined with Young's inequality, we obtain

$$\begin{aligned}
&\|A_1^{\frac12} D_{N_1}^{\frac12}(\mathbf{w}_m)(t,\cdot)\|^2 + \|A_2^{\frac12} D_{N_2}^{\frac12}(\mathbf{b}_m)(t,\cdot)\|^2 \\
&\qquad + \int_0^t \|\nabla A_1^{\frac12} D_{N_1}^{\frac12}(\mathbf{w}_m)\|^2\, d\tau + \int_0^t \|\nabla A_2^{\frac12} D_{N_2}^{\frac12}(\mathbf{b}_m)\|^2\, d\tau \\
&\qquad \le C(\|\mathbf{u}_0\|,\, \|\mathbf{B}_0\|,\, \nu^{-1}\|\mathbf{f}\|_{L^2([0,T];\mathbf{H}_{-1})}),
\end{aligned} \tag{3.10}$$

which shows that the natural quantities under control are $A_1^{\frac12} D_{N_1}^{\frac12}(\mathbf{w}_m)$ and $A_2^{\frac12} D_{N_2}^{\frac12}(\mathbf{b}_m)$.

Since we need to prove many *a priori* estimates, for the reader's convenience we organize the results in tables as (3.11). In the first column we have labeled the estimates, while the second column specifies the variable under concern. The third one explains the bound in terms of function spaces: The symbol of a space means that the considered sequence is bounded in such a space. Finally, the fourth column states the order in terms of α, m and N for each bound.

(3.11)

Label	Variable	bound	order
a)	$A_1^{\frac12} D_{N_1}^{\frac12}(\mathbf{w}_m)$, $A_2^{\frac12} D_{N_2}^{\frac12}(\mathbf{b}_m)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
b)	$D_{N_1}^{1/2}(\mathbf{w}_m)$, $D_{N_2}^{1/2}(\mathbf{b}_m)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
c)	$D_{N_1}^{1/2}(\mathbf{w}_m)$, $D_{N_2}^{1/2}(\mathbf{b}_m)$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
d)	$\mathbf{w}_m$, $\mathbf{b}_m$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
e)	$\mathbf{w}_m$, $\mathbf{b}_m$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
f)	$D_{N_1}(\mathbf{w}_m)$, $D_{N_2}(\mathbf{b}_m)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
g)	$D_{N_1}(\mathbf{w}_m)$, $D_{N_2}(\mathbf{b}_m)$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\frac{\sqrt{N_i+1}}{\alpha})$
h)	$\partial_t \mathbf{w}_m$, $\partial_t \mathbf{b}_m$	$L^2([0,T];\mathbf{H}_0)$	$O(\alpha^{-1})$.

In the previous table, $\alpha = \alpha_1$ for $\mathbf{w}_m$, $\alpha = \alpha_2$ for $\mathbf{b}_m$, while in *h)* we can take $\alpha := \min\{\alpha_1, \alpha_2\}$ for both $\mathbf{w}_m$ and $\mathbf{b}_m$.

Proof of (3.11-a) — This estimate follows directly from (3.10). Notice also that since the operator $A_i^{-\frac12} D_{N_i}^{\frac12}$ has for symbol $\rho_{N_i,\mathbf{k}}^{1/2} \le 1$, then $\|A_1^{\frac12} D_{N_1}^{\frac12} G_1 \mathbf{f}_{1/m}\| \le C\|\mathbf{f}\|$ and also

$$\|A_i^{\frac12} D_{N_i}^{\frac12} \mathbb{P}_m G_i \mathbf{a}\| = \|\mathbb{P}_m A_i^{\frac12} D_{N_i}^{\frac12} G_i \mathbf{a}\| \le \|A_i^{\frac12} D_{N_i}^{\frac12} \mathbf{a}\| \le \|\mathbf{a}\|,$$

which will be used with $\mathbf{a} = \mathbf{u}_0$, $\mathbf{B}_0$.

Proof of (3.11-b)-(3.11-c) — Let $\mathbf{v} \in \mathbf{H}_2$. Then, with obvious notations, one has

$$\|A_i^{\frac{1}{2}}\mathbf{v}\|^2 = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} (1+\alpha_i^2|\mathbf{k}|^2)|\widehat{\mathbf{v}}_{\mathbf{k}}|^2 = \|\mathbf{v}\|^2 + \alpha_i^2\|\nabla\mathbf{v}\|^2.$$

It suffices to apply this identity to $\mathbf{v} = D_{N_1}^{\frac{1}{2}}(\mathbf{w}_m)$, $D_{N_2}^{\frac{1}{2}}(\mathbf{b}_m)$ and to $\mathbf{v} = \partial_i D_{N_1}^{\frac{1}{2}}(\mathbf{w}_m)$, $\partial_i D_{N_1}^{\frac{1}{2}}(\mathbf{b}_m)$ $(i = 1, 2, 3)$ in (3.10) to get the claimed result.

Proof of (3.11-d)-(3.11-e)-(3.11-f) —These are direct consequence of (3.11-a)-(3.11-b)-(3.11-c) combined with (2.4).

Proof of (3.11-g) — This follows directly from (3.11-e), together with (2.4).

Remark 3.2. *One crucial point is that (3.11-g) is valid for each $N = (N_1, N_2)$, but the bound may grow with N_i.*

Proof of (3.11-h) –Let us take $\partial_t\mathbf{w}_m, \partial_t\mathbf{b}_m \in \mathbf{V}_m$ as test vector fields in (3.8). We get

$$\begin{aligned}
&\|\partial_t\mathbf{w}_m\|^2 + \int_{\mathbb{T}_3} G_1\big(\nabla\cdot[D_{N_1}(\mathbf{w}_m)\otimes D_{N_1}(\mathbf{w}_m)]\big)\cdot\partial_t\mathbf{w}_m \\
&\quad - \int_{\mathbb{T}_3} G_1\big(\nabla\cdot[D_{N_2}(\mathbf{b}_m)\otimes D_{N_2}(\mathbf{b}_m)]\big)\cdot\partial_t\mathbf{w}_m + \frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{w}_m\|^2 = \int_{\mathbb{T}_3} G_1\mathbf{f}_{1/m}\cdot\partial_t\mathbf{w}_m, \\
&\|\partial_t\mathbf{b}_m\|^2 + \int_{\mathbb{T}_3} G_2\big(\nabla\cdot[D_{N_2}(\mathbf{b}_m)\otimes D_{N_1}(\mathbf{w}_m)]\big)\cdot\partial_t\mathbf{b}_m \\
&\quad - \int_{\mathbb{T}_3} G_2\big(\nabla\cdot[D_{N_1}(\mathbf{w}_m)\otimes D_{N_2}(\mathbf{b}_m)]\big)\cdot\partial_t\mathbf{b}_m + \frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{b}_m\|^2 = 0\,.
\end{aligned}$$

To estimate the time derivative, we need bounds on the bi-linear terms

$$\begin{aligned}
\mathbf{A}_{N,m} &:= G_1\nabla\cdot\big(D_{N_1}(\mathbf{w}_m)\otimes D_{N_1}(\mathbf{w}_m)\big), \\
\mathbf{B}_{N,m} &:= G_1\nabla\cdot\big(D_{N_2}(\mathbf{b}_m)\otimes D_{N_2}(\mathbf{b}_m)\big), \\
\mathbf{C}_{N,m} &:= G_2\nabla\cdot\big(D_{N_1}(\mathbf{w}_m)\otimes D_{N_2}(\mathbf{b}_m)\big).
\end{aligned}$$

Even if we have two additional terms, this can be easily done as in [5] by observing that, by interpolation inequalities, both $D_{N_1}(\mathbf{w}_m)$ and $D_{N_2}(\mathbf{b}_m)$ belong to $L^4([0,T]; L^3(\mathbb{T}_3)^3)$. Therefore, by observing that the operator $(\nabla\cdot)\circ G_i$ has symbol corresponding to the inverse of one space derivative, it easily follows that $\mathbf{A}_{N,m}, \mathbf{B}_{N,m}, \mathbf{C}_{N,m} \in L^2([0,T]\times\mathbb{T}_3)^3$. Moreover, the bound is of order $O(\alpha_i^{-1})$ as well.

From the bounds proved in (3.11) and classical Aubin-Lions compactness tools, we can extract sub-sequences $\{\mathbf{w}_m, \mathbf{b}_m\}_{m\in\mathbb{N}}$ converging to $\mathbf{w}, \mathbf{b} \in L^\infty([0,T]; \mathbf{H}_1) \cap L^2([0,T]; \mathbf{H}_2)$ and such that

$$\begin{aligned}\mathbf{w}_m &\to \mathbf{w} \\ \mathbf{b}_m &\to \mathbf{b}\end{aligned} \qquad \text{weakly in } L^2([0,T]; \mathbf{H}_2), \tag{3.12}$$

$$\begin{aligned}\mathbf{w}_m &\to \mathbf{w} \\ \mathbf{b}_m &\to \mathbf{b}\end{aligned} \qquad \text{strongly in } L^p([0,T]; \mathbf{H}_1), \quad \forall\, p \in [1,\infty[, \tag{3.13}$$

$$\begin{aligned}\partial_t\mathbf{w}_m &\to \partial_t\mathbf{w} \\ \partial_t\mathbf{b}_m &\to \partial_t\mathbf{b}\end{aligned} \qquad \text{weakly in } L^2([0,T]; \mathbf{H}_0). \tag{3.14}$$

This already implies that $(\mathbf{w}, \mathbf{b})$ satisfies (3.2)-(3.3). From (3.13) and the continuity of D_{N_i} in $\mathbf{H}_s$, we get strong convergence of $D_{N_1}(\mathbf{w}_m)$, $D_{N_2}(\mathbf{b}_m)$ in $L^4([0,T]\times\mathbb{T}_3)$, hence the convergence of

the corresponding bi-linear products in $L^2([0,T]\times\mathbb{T}_3)$. This proves that for all $\mathbf{v},\mathbf{h}\in L^2([0,T];\mathbf{H}_1)$

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}_3}\partial_t\mathbf{w}\cdot\mathbf{v}-\int_0^T\int_{\mathbb{T}_3}G_1(D_{N_1}(\mathbf{w})\otimes D_{N_1}(\mathbf{w})):\nabla\mathbf{v}\\
&\quad+\int_0^T\int_{\mathbb{T}_3}G_1(D_{N_2}(\mathbf{b})\otimes D_{N_2}(\mathbf{b})):\nabla\mathbf{v}\\
&\quad+\nu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{w}:\nabla\mathbf{v}=\int_0^T\int_{\mathbb{T}_3}G_1(\mathbf{f})\cdot\mathbf{v},
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}_3}\partial_t\mathbf{b}\cdot\mathbf{h}-\int_0^T\int_{\mathbb{T}_3}G_2(D_{N_2}(\mathbf{b})\otimes D_{N_1}(\mathbf{w})):\nabla\mathbf{h}\\
&\quad+\int_0^T\int_{\mathbb{T}_3}G_2(D_{N_1}(\mathbf{w})\otimes D_{N_2}(\mathbf{b})):\nabla\mathbf{h}+\mu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{b}:\nabla\mathbf{h}=0.
\end{aligned}\tag{3.16}$$

To introduce the pressure, observe that taking the divergence of the equation for $\mathbf{w}$, we get

$$\Delta q=\nabla\cdot G_1\mathbf{f}+\nabla\cdot\mathcal{A}_N,\tag{3.17}$$

for $\mathcal{A}_N:=-G_1\big[\nabla\cdot\big(D_{N_1}(\mathbf{w})\otimes D_{N_1}(\mathbf{w})\big)-\nabla\cdot\big(D_{N_2}(\mathbf{b})\otimes D_{N_2}(\mathbf{b})\big)\big]$. A fairly standard application of De Rham's Theorem shows existence of q, and the regularity of $\mathcal{A}_N$ yields $q\in L^2([0,T];H^1(\mathbb{T}_3))$.

The meaning in which the initial data are taken is completely standard and we end the proof by showing uniqueness: Let $(\mathbf{w}_1,\mathbf{b}_1)$ and $(\mathbf{w}_2,\mathbf{b}_2)$ be two solutions corresponding to the same data $(\mathbf{u}_0,\mathbf{B}_0,\mathbf{f})$ and let us define, as usual, $\mathcal{W}:=\mathbf{w}_1-\mathbf{w}_2$ and $\mathcal{B}:=\mathbf{b}_1-\mathbf{b}_2$. By standard calculations (mimicking those employed in [5]), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\|A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\Big)+\nu\|\nabla A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\mu\|\nabla A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\\
&\quad=\int_{\mathbb{T}_3}(D_{N_2}(\mathcal{B})\cdot\nabla)D_{N_2}(\mathbf{b}_1)\cdot D_{N_1}(\mathcal{W})-\int_{\mathbb{T}_3}(D_{N_1}(\mathcal{W})\cdot\nabla)D_{N_1}(\mathbf{w}_1)\cdot D_{N_1}(\mathcal{W})\\
&\quad+\int_{\mathbb{T}_3}(D_{N_2}(\mathcal{B})\cdot\nabla)D_{N_1}(\mathbf{w}_1)\cdot D_{N_2}(\mathcal{B})-\int_{\mathbb{T}_3}(D_{N_1}(\mathcal{W})\cdot\nabla)D_{N_2}(\mathbf{b}_1)\cdot D_{N_2}(\mathcal{B}),\\
&\quad\le 2\|D_{N_2}(\mathcal{B})\|_{L^4}\|D_{N_1}\mathcal{W}\|_{L^4}\|\nabla D_{N_2}(\mathbf{b}_1)\|_{L^2}+\|D_{N_1}\mathcal{W}\|_{L^4}^2\|\nabla D_{N_1}(\mathbf{w}_1)\|_{L^2},\\
&\quad+\|D_{N_2}(\mathcal{B})\|_{L^4}^2\|\nabla D_{N_1}(\mathbf{w}_1)\|_{L^2}\\
&\quad\le 2\|D_{N_2}(\mathcal{B})\|^{1/4}\|D_{N_1}(\mathcal{W})\|^{1/4}\|\nabla D_{N_2}(\mathcal{B})\|^{3/4}\|\nabla D_{N_1}(\mathcal{W})\|^{3/4}\|\nabla D_{N_2}(\mathbf{b}_1)\|\\
&\quad+\|D_{N_1}(\mathcal{W})\|^{1/2}\|\nabla D_{N_1}(\mathcal{W})\|^{3/2}\|\nabla D_{N_1}(\mathbf{w}_1)\|+\|D_{N_2}(\mathcal{B})\|^{1/2}\|\nabla D_{N_2}(\mathcal{B})\|^{3/2}\|\nabla D_{N_1}(\mathbf{w}_1)\|\,.
\end{aligned}$$

By using $\|D_{N_i}\|=(N_i+1)$, the bound of $\mathbf{w}_1,\mathbf{b}_1$ in $L^\infty([0,T];\mathbf{H}_1)$, and Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\|A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\big)+\frac{\nu}{2}\|\nabla A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\frac{\mu}{2}\|\nabla A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\\
&\quad\le C(N_1+1)^4\Big(\sup_{t\ge0}\|\nabla\mathbf{w}_1\|^4\Big)\Big[\frac{1}{\nu^3}\|A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\frac{1}{\mu^3}\|A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\Big]\\
&\qquad+C(N_2+1)^4\Big(\sup_{t\ge0}\|\nabla\mathbf{b}_1\|^4\Big)\frac{1}{\nu^{3/2}\mu^{3/2}}\big[\|A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\|A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\big].
\end{aligned}$$

In particular, we get

$$\frac{1}{2}\frac{d}{dt}\big(\|A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\|A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\big)\le M\big[\|A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})\|^2+\|A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})\|^2\big],$$

where

$$M:=C\big(\max\{\frac{1}{\nu},\frac{1}{\mu}\}\big)^3\Big[(N_1+1)^4\Big(\sup_{t\ge0}\|\nabla\mathbf{w}_1\|^4\Big)+(N_2+1)^4\Big(\sup_{t\ge0}\|\nabla\mathbf{b}_1\|^4\Big)\Big].$$

Since the initial values $\mathcal{W}(0)=\mathcal{B}(0)$ are vanishing, we deduce from Gronwall's Lemma that $A_1^{\frac12}D_{N_1}^{\frac12}(\mathcal{W})=A_2^{\frac12}D_{N_2}^{\frac12}(\mathcal{B})=\mathbf{0}$ and we conclude that $\mathcal{W}=\mathcal{B}=\mathbf{0}$. □

Remark 3.3. *The same calculations show also that the following energy equality is satisfied*

$$\frac{1}{2}\frac{d}{dt}\big(\|A_1^{\frac{1}{2}}D_{N_1}^{\frac{1}{2}}(\mathbf{w})\|^2+\|A_2^{\frac{1}{2}}D_{N_2}^{\frac{1}{2}}(\mathbf{b})\|^2\big)+\nu\|\nabla A_1^{\frac{1}{2}}D_{N_1}^{\frac{1}{2}}(\mathbf{w})\|^2+\mu\|\nabla A_2^{\frac{1}{2}}D_{N_2}^{\frac{1}{2}}(\mathbf{b})\|^2$$
$$=(A_2^{\frac{1}{2}}D_{N_1}^{\frac{1}{2}}\big(G_1\mathbf{f}\big),A_1^{\frac{1}{2}}D_{N_1}^{\frac{1}{2}}(\mathbf{w})\big).$$

As we shall see in the sequel, it seems that it is not possible to pass to the limit $N\to+\infty$ directly in this "energy equality" and some work to obtain an "energy inequality" is needed.

4 Passing to the limit when $N\to\infty$

The aim of this section is the proof of the main result of the paper. For a given $N\in\mathbb{N}$, we denote by $(\mathbf{w}_N,\mathbf{b}_N,q_N)$ the unique "regular weak" solution to Problem 1.3, where $N=\min\{N_1,N_2\}\to+\infty$. For the sake of completeness and to avoid possible confusion between the Galerkin index m and the deconvolution index N, we write again the system:

$$\begin{aligned}
&\partial_t\mathbf{w}_N+\nabla\cdot G_1(D_{N_1}(\mathbf{w}_N)\otimes D_{N_1}(\mathbf{w}_N))-\nabla\cdot G_1(D_{N_2}(\mathbf{b}_N)\otimes D_{N_2}(\mathbf{b}_N))\\
&\qquad\qquad+\nabla q_N-\nu\Delta\mathbf{w}_N=G_1\mathbf{f}\qquad\text{in}\quad[0,T]\times\mathbb{T}_3,\\
&\partial_t\mathbf{b}_N+\nabla\cdot G_2(D_{N_2}(\mathbf{b}_N)\otimes D_{N_1}(\mathbf{w}_N))-\nabla\cdot G_2(D_{N_1}(\mathbf{w}_N)\otimes D_{N_2}(\mathbf{b}_N))\\
&\qquad\qquad-\mu\Delta\mathbf{b}_N=0\qquad\text{in}\quad[0,T]\times\mathbb{T}_3,\\
&\nabla\cdot\mathbf{w}_N=\nabla\cdot\mathbf{b}_N=0\qquad\text{in}\quad[0,T]\times\mathbb{T}_3,\\
&(\mathbf{w}_N,\mathbf{b}_N)(0,\mathbf{x})=(G_1\mathbf{u}_0,G_2\mathbf{B}_0)(\mathbf{x})\qquad\text{in}\quad\mathbb{T}_3.
\end{aligned}\tag{4.1}$$

More precisely, for all fixed scales $\alpha_1,\alpha_2>0$, we set

$$\mathbf{w}_N=\lim_{m\to+\infty}\mathbf{w}_{m,N_1,N_2,\alpha_1,\alpha_2}$$

and similarly for $\mathbf{b}_N$.

Proof of Thm. 1.1. We look for additional estimates, uniform in N, to get compactness properties about the sequences $\{D_{N_1}(\mathbf{w}_N),D_{N_2}(\mathbf{b}_N)\}_{N\in\mathbb{N}}$ and $\{\mathbf{w}_N,\mathbf{b}_N\}_{N\in\mathbb{N}}$. We then prove strong enough convergence results in order to pass to the limit in the equation (4.1), especially in the nonlinear terms. With the same notation of the previous section, we quote in the following table the estimates that we will use for passing to the limit. The Table (4.2) is organized as (3.11) and $\alpha=\min\{\alpha_1,\alpha_2\}$.

(4.2)

Label	Variable	bound	order
a	$\mathbf{w}_N$, $\mathbf{b}_N$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
b	$\mathbf{w}_N$, $\mathbf{b}_N$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
c	$D_{N_1}(\mathbf{w}_N)$, $D_{N_2}(\mathbf{b}_N)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
d	$\partial_t\mathbf{w}_N$, $\partial_t\mathbf{b}_N$	$L^2([0,T]\times\mathbb{T}_3)^3$	$O(\alpha^{-1})$
e	q_N	$L^2([0,T];H^1(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3))$	$O(\alpha^{-1})$
f	$\partial_tD_{N_1}(\mathbf{w}_N)$, $\partial_tD_{N_2}(\mathbf{b}_N)$	$L^{4/3}([0,T];\mathbf{H}_{-1})$	$O(1)$

Estimates (4.2-a), (4.2-b), (4.2-c), and (4.2-d) have already been obtained in the previous section. Therefore, we just have to check (4.2-e) and (4.2-f).

Proof of (4.2-e) — To obtain further regularity properties of the pressure we use again (3.17). We already know from the estimates proved in the previous section that $\mathcal{A}_N\in L^2([0,T]\times\mathbb{T}_3)^3$. Moreover, classical interpolation inequalities combined with (4.2-c) yield $D_{N_1}(\mathbf{w}_N),D_{N_2}(\mathbf{b}_N)\in L^{10/3}([0,T]\times\mathbb{T}_3)$. Therefore, $\mathcal{A}_N\in L^{5/3}([0,T];W^{1,5/3}(\mathbb{T}_3))$. Consequently, we obtain the claimed bound on q_N.

Proof of (4.2-f) — Let be given $\mathbf{v}, \mathbf{h} \in L^4([0,T];\mathbf{H}_1)$. We use $D_{N_1}(\mathbf{v}), D_{N_2}(\mathbf{h})$ as test functions. By using that $\partial_t \mathbf{w}$, $\partial_t \mathbf{b} \in L^2([0,T]\times\mathbb{T}_3)^3$, D_{N_i} commute with differential operators, G_i and D_{N_i} are self-adjoint, and classical integrations by parts, we get

$$\begin{aligned}(\partial_t \mathbf{w}_N,\, D_{N_1}(\mathbf{v})) &= (\partial_t D_{N_1}(\mathbf{w}_N), \mathbf{v})\\ &= \nu(\Delta \mathbf{w}_N, D_{N_1}(\mathbf{v})) + (D_{N_1}(\mathbf{w}_N)\otimes D_{N_1}(\mathbf{w}_N), G_1 D_{N_1}(\nabla \mathbf{v}))\\ &\quad - (D_{N_2}(\mathbf{b}_N)\otimes D_{N_2}(\mathbf{b}_N), G_1 D_{N_1}(\nabla \mathbf{v})) + (D_{N_1}(G_1\mathbf{f}), \mathbf{v}),\end{aligned}$$

$$\begin{aligned}(\partial_t \mathbf{b}_N,\, D_{N_2}(\mathbf{h})) &= (\partial_t D_{N_2}(\mathbf{b}_N), \mathbf{h})\\ &= \mu(\Delta \mathbf{b}_N, D_{N_2}(\mathbf{h})) + (D_{N_2}(\mathbf{b}_N)\otimes D_{N_1}(\mathbf{w}_N), G_2 D_{N_2}(\nabla \mathbf{h}))\\ &\quad - (D_{N_1}(\mathbf{w}_N)\otimes D_{N_2}(\mathbf{b}_N), G_2 D_{N_2}(\nabla \mathbf{h}))\,.\end{aligned}$$

We first observe that

$$\begin{aligned}|(\Delta \mathbf{w}_N, D_{N_1}(\mathbf{v}))| &= |(\nabla D_N(\mathbf{w}_N), \nabla \mathbf{v})| \le C_1(t)\|\mathbf{v}\|_1,\\ |(\Delta \mathbf{b}_N, D_{N_2}(\mathbf{h}))| &= |(\nabla D_{N_2}(\mathbf{b}_N), \nabla \mathbf{h})| \le C_1(t)\|\mathbf{h}\|_1,\end{aligned}$$

and that the $L^2([0,T];H^1(\mathbb{T}_3)^3)$ bound for $D_{N_1}(\mathbf{w}_N)$, $D_{N_2}(\mathbf{b}_N)$ implies that $C_1(t) \in L^2([0,T])$, uniformly with respect to $N \in \mathbb{N}$. Therefore, when we combine the latter estimates with the properties of D_{N_i} we get, uniformly in N,

$$\begin{aligned}&|(\partial_t D_{N_1}(\mathbf{w}_N), \mathbf{v})| + |(\partial_t D_{N_2}(\mathbf{b}_N), \mathbf{v})|\\ &\quad \le \big(\nu C_1(t) + C_2(t)\big)\|\mathbf{v}\|_1 + \big(\mu C_1(t) + C_2(t)\big)\|\mathbf{h}\|_1 + \|\mathbf{f}(t,\cdot)\| \in L^{4/3}(0,T).\end{aligned}$$

From the estimates (4.2) and classical rules of functional analysis, we can infer that there exist

$$\begin{aligned}&\mathbf{w}, \mathbf{b} \in L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2),\\ &\mathbf{z}_1, \mathbf{z}_2 \in L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1),\\ &q \in L^2([0,T];H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3))\end{aligned}$$

such that, up to sub-sequences,

$$\begin{aligned}
&\mathbf{w}_N \to \mathbf{w}\\ &\mathbf{b}_N \to \mathbf{b} && \begin{cases}\text{weakly in } L^2([0,T];\mathbf{H}_2),\\ \text{weakly}* \text{ in } L^\infty([0,T];\mathbf{H}_1),\\ \text{strongly in } L^p([0,T];\mathbf{H}_1) \qquad \forall\, p\in[1,\infty[,\end{cases}\\
&\partial_t \mathbf{w}_N \to \partial_t \mathbf{w}\\ &\partial_t \mathbf{b}_N \to \partial_t \mathbf{b} && \text{weakly in } L^2([0,T]\times\mathbb{T}_3),\\
&D_{N_1}(\mathbf{w}_N) \to \mathbf{z}_1\\ &D_{N_2}(\mathbf{b}_N) \to \mathbf{z}_2 && \begin{cases}\text{weakly in } L^2([0,T];\mathbf{H}_1),\\ \text{weakly}* \text{ in } L^\infty([0,T];\mathbf{H}_0),\\ \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^3 \qquad \forall\, p\in[1,10/3[,\end{cases}\\
&\partial_t D_{N_1}(\mathbf{w}_N) \to \partial_t \mathbf{z}_1\\ &\partial_t D_{N_2}(\mathbf{b}_N) \to \partial_t \mathbf{z}_2 && \text{weakly in } L^{4/3}([0,T];\mathbf{H}_{-1}),\\
&q_N \to q && \text{weakly in } L^2([0,T];H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3)).
\end{aligned} \tag{4.3}$$

We notice that

$$\begin{aligned}
&D_{N_1}(\mathbf{w}_N)\otimes D_{N_1}(\mathbf{w}_N) \longrightarrow \mathbf{z}_1\otimes\mathbf{z}_1 \quad \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^9 \quad \forall\, p\in[1,5/3[,\\
&D_{N_2}(\mathbf{b}_N)\otimes D_{N_2}(\mathbf{b}_N) \longrightarrow \mathbf{z}_2\otimes\mathbf{z}_2 \quad \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^9 \quad \forall\, p\in[1,5/3[,\\
&D_{N_1}(\mathbf{w}_N)\otimes D_{N_2}(\mathbf{b}_N) \longrightarrow \mathbf{z}_1\otimes\mathbf{z}_2 \quad \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^9 \quad \forall\, p\in[1,5/3[,
\end{aligned} \tag{4.4}$$

while all other terms in the equation pass easily to the limit as well. By using the same identification of the limit used in [5], we can easily check that $\mathbf{z}_1 = A_1\mathbf{w}$ and $\mathbf{z}_2 = A_2\mathbf{b}$, ending the proof. □

By using well established results on semicontinuity and adapting calculations well-known for the NSE, we can prove that the solution $(\mathbf{w}, \mathbf{b})$ satisfies an "energy inequality."

Proposition 4.1. *Let be given $\mathbf{u}_0, \mathbf{B}_0 \in \mathbf{H}_0$, $\mathbf{f} \in L^2([0,T];\mathbf{H}_0)$, and let $\{(\mathbf{w}_N, \mathbf{b}_N, q_N)\}_{N\in\mathbb{N}}$ be a (possibly relabelled) sequence of regular weak solutions converging to a weak solution $(\mathbf{w}, \mathbf{b}, q)$ of the filtered MHD equations. Then $(\mathbf{w}, \mathbf{b})$ satisfies the energy inequality (1.5) in the sense of distributions (see also [13, 18, 39]). This implies that $(\mathbf{w}, \mathbf{b})$ is the average of a weak (in the sense of Leray-Hopf) or dissipative solution $(\mathbf{u}, \mathbf{B})$ of the MHD equation (1.1). In fact, the energy inequality can also be read as*

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2 + \|\mathbf{B}\|^2) + \nu\|\nabla\mathbf{u}\|^2 + \mu\|\nabla\mathbf{B}\|^2 \le \langle \mathbf{f}, \mathbf{u}\rangle.$$

Proof. The proof is a straightforward adaption of the one in [5]. We start from the energy equality for the approximate model as in Remark 3.3 and we observe that the same arguments as before show also that

$$\left\{\begin{array}{c} D_{N_1}^{1/2}(\mathbf{w}_N) \to A_1^{1/2}(\mathbf{w}) \\ D_{N_2}^{1/2}(\mathbf{b}_N) \to A_2^{1/2}(\mathbf{b}) \end{array}\right\} \qquad \text{weakly in } L^2([0,T];\mathbf{H}_1).$$

Next, due to the assumptions on $\mathbf{f}$, we have $A_1^{-1/2}D_N^{1/2}\mathbf{f} \to \mathbf{f}$ strongly in $L^2([0,T];\mathbf{H}_0)$ and, since for all $N \in \mathbb{N}$ we have $\mathbf{w}_N(0) = G_1\mathbf{u}(0) \in \mathbf{H}_2$ and $\mathbf{b}_N(0) = G_2\mathbf{b}(0) \in \mathbf{H}_2$, we get

$$\begin{aligned}
&\frac{1}{2}(\|A_1^{1/2}D_{N_1}^{1/2}(\mathbf{w}_N)(0)\|^2 + \|A_2^{1/2}D_{N_2}^{1/2}(\mathbf{b}_N)(0)\|^2) + \int_0^t \left(A_1^{-1/2}D_{N_1}^{1/2}(\mathbf{f}), A_1^{1/2}D_{N_1}^{1/2}(\mathbf{w}_N)\right) ds \\
&\overset{N\to+\infty}{\longrightarrow} \frac{1}{2}(\|A_1\mathbf{w}(0)\|^2 + \|A_2\mathbf{b}(0)\|^2) + \int_0^t (\mathbf{f}, A_1\mathbf{w})\, ds.
\end{aligned}$$

Next, we use the elementary inequalities for lim inf and lim sup to infer that

$$\begin{aligned}
&\limsup_{N\to+\infty} \frac{1}{2}\big(\|A_1^{1/2}D_{N_1}^{1/2}(\mathbf{w}_N)(t)\|^2 + \|A_2^{1/2}D_{N_2}^{1/2}(\mathbf{b}_N)(t)\|^2\big) \\
&+ \liminf_{N\to+\infty}\left(\nu\int_0^t \|\nabla A_1^{1/2}D_{N_1}^{1/2}(\mathbf{w}_N)(s)\|^2\, ds + \mu\int_0^t \|\nabla A_2^{1/2}D_{N_2}^{1/2}(\mathbf{b}_N)(s)\|^2\, ds\right) \\
&\le \frac{1}{2}\big(\|A_1\mathbf{w}(0)\|^2 + \|A_2\mathbf{b}(0)\|^2\big) + \int_0^t (\mathbf{f}(s), A_1\mathbf{w}(s))\, ds.
\end{aligned}$$

By lower semicontinuity of the norm and identification of the weak limit, we get the thesis. □

5 Results for the second model

In this section, we consider the following LES model for MHD, which is based on filtering only the velocity equation (and on the use of deconvolution operators):

$$\begin{aligned}
&\partial_t\mathbf{w} + \nabla\cdot G_1\big(D_{N_1}(\mathbf{w}) \otimes D_{N_1}(\mathbf{w})\big) - \nabla\cdot G_1\big(\mathbf{B}\otimes\mathbf{B}\big) + \nabla q - \nu\Delta\mathbf{w} = G_1\mathbf{f}, \\
&\partial_t\mathbf{B} + \nabla\cdot\big(\mathbf{B}\otimes D_{N_1}(\mathbf{w})\big) - \nabla\cdot\big(D_{N_1}(\mathbf{w})\otimes\mathbf{B}\big) - \mu\Delta\mathbf{B} = 0, \\
&\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{B} = 0, \\
&\mathbf{w}(0,\mathbf{x}) = G_1\mathbf{u}_0(\mathbf{x}), \qquad \mathbf{B}(0,\mathbf{x}) = \mathbf{B}_0(\mathbf{x}),
\end{aligned} \tag{5.1}$$

and we will work with periodic boundary conditions. A similar model in the case without deconvolution has been also studied in [9].

Here we take $\alpha_2 = 0$, so that $\mathbf{b} = \mathbf{B}$ and $A_2 = G_2 = I$, and $N_2 = 0$, so that $D_{N_2}\mathbf{B} = I\,\mathbf{B} = \mathbf{B}$. We set for simplicity

$$\alpha = \alpha_1 > 0, \quad G = G_1, \quad A = A_1, \quad N = N_1.$$

The first aim of this section is to show the changes needed (w.r.t Thm. 3.1) to prove the existence of a unique solution to the system (5.1) for a given $N \in \mathbb{N}$, when we assume that the data are such that

$$\mathbf{u}_0 \in \mathbf{H}_0, \quad \mathbf{B}_0 \in \mathbf{H}_0, \quad \text{and} \quad \mathbf{f} \in L^2([0,T]\times\mathbb{T}_3), \tag{5.2}$$

which naturally yields $G_1\mathbf{u}_0 \in \mathbf{H}_2,\ G_1\mathbf{f} \in L^2([0,T];\mathbf{H}_2)$.

We start by defining the notion of what we call a "regular weak" solution to this system.

Definition 5.1 ("Regular weak" solution)**.** *We say that the triple $(\mathbf{w},\mathbf{B},q)$ is a "regular weak" solution to system (5.1) if and only if the three following items are satisfied:*

1) Regularity

$$\mathbf{w} \in L^2([0,T];\mathbf{H}_2)\cap C([0,T];\mathbf{H}_1), \quad \mathbf{B} \in L^2([0,T];\mathbf{H}_1)\cap C([0,T];\mathbf{H}_0), \tag{5.3}$$

$$\partial_t\mathbf{w} \in L^2([0,T];\mathbf{H}_0), \quad \partial_t\mathbf{B} \in L^2([0,T];\mathbf{H}_{-1}), \tag{5.4}$$

$$q \in L^2([0,T];H^1(\mathbb{T}_3)), \tag{5.5}$$

2) Initial data

$$\lim_{t\to 0}\|\mathbf{w}(t,\cdot)-G_1\mathbf{u}_0\|_{\mathbf{H}_1}=0, \qquad \lim_{t\to 0}\|\mathbf{B}(t,\cdot)-\mathbf{B}_0\|_{\mathbf{H}_0}=0, \tag{5.6}$$

3) Weak Formulation*: For all $\mathbf{v},\mathbf{h} \in L^2([0,T];H^1(\mathbb{T}_3)^3)$,*

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}_3}\partial_t\mathbf{w}\cdot\mathbf{v}-\int_0^T\int_{\mathbb{T}_3}G_1(D_{N_1}(\mathbf{w})\otimes D_{N_1}(\mathbf{w})):\nabla\mathbf{v}\\ &\quad+\int_0^T\int_{\mathbb{T}_3}G_1(\mathbf{B}\otimes\mathbf{B}):\nabla\mathbf{v}+\int_0^T\int_{\mathbb{T}_3}\nabla q\cdot\mathbf{v}\\ &\quad+\nu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{w}:\nabla\mathbf{v}=\int_0^T\int_{\mathbb{T}_3}(G_1\mathbf{f})\cdot\mathbf{v},\end{aligned} \tag{5.7}$$

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}_3}\partial_t\mathbf{B}\cdot\mathbf{h}-\int_0^T\int_{\mathbb{T}_3}(\mathbf{B}\otimes D_{N_1}(\mathbf{w})):\nabla\mathbf{h}\\ &\quad+\int_0^T\int_{\mathbb{T}_3}(D_{N_1}(\mathbf{w})\otimes\mathbf{B}):\nabla\mathbf{h}+\mu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{B}:\nabla\mathbf{h}=0.\end{aligned} \tag{5.8}$$

Remark 5.1. *Due to the certain symmetry in the equations, it turns out that $\mathbf{B}$ has the same regularity of $D_N\mathbf{w}$ (not that of $\mathbf{w}$).*

All terms in the weak formulation are well-defined. Indeed, the only term to be checked (which is different from the previous section) is the bi-linear one involving $\mathbf{B} \in L^4([0,T];L^3(\mathbb{T}_3))^3$ and $D_N(\mathbf{w}) \in L^\infty([0,T];L^6(\mathbb{T}_3))^3$. To this end, we observe that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}_3}(\mathbf{B}\otimes D_N(\mathbf{w})):\nabla\mathbf{h} &\le C\int_0^T\|\mathbf{B}(t)\|_{L^3}\|D_N(\mathbf{w})(t)\|_{L^6}\|\nabla\mathbf{h}(t)\|_{L^2}\\ &\le C_T\|\mathbf{B}\|_{L^4([0,T];L^3)}\|D_N(\mathbf{w})\|_{L^\infty([0,T];L^6)}\|\nabla\mathbf{h}\|_{L^2([0,T];L^2)}.\end{aligned}$$

We have now the following theorem showing that system (5.1) is well-posed.

Theorem 5.1. *Assume that (5.2) holds, $\alpha>0$ and $N\in\mathbb{N}$ are given. Then, Problem (5.1) has a unique regular weak solution satisfying the energy inequality*

$$\frac{d}{dt}\big(\|A^{\frac12}D_N^{\frac12}(\mathbf{w})\|^2+\|\mathbf{B}\|^2\big)+\nu\|\nabla A^{\frac12}D_N^{\frac12}(\mathbf{w})\|^2+\mu\|\nabla\mathbf{B}\|^2\le C(\|\mathbf{u}_0\|,\,\|\mathbf{B}_0\|,\,\nu^{-1}\|\mathbf{f}\|_{L^2([0,T];\mathbf{H}_{-1})}).$$

Proof. We use the same notation and tools from the previous section and the main result can be derived from the energy estimate. We just give some details on the estimates which are different from the previous case, since the reader can readily fill the missing details. We use $D_N(\mathbf{w}_m)$ in the first equation and $\mathbf{B}_m$ in the second one as test functions to obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\Big(\|A^{1/2}D_N^{1/2}(\mathbf{w}_m)\|^2+\|\mathbf{B}_m\|^2\Big)+\nu\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_m)\|^2+\mu\|\nabla\mathbf{B}_m\|^2\\ &\quad=\big(A^{1/2}D_N^{1/2}(G\mathbf{f}_{1/m}),A^{1/2}D_N^{1/2}(\mathbf{w}_m)\big).\end{aligned}$$

Then, by using the same tools employed in the previous section, we have the following estimates.

(5.9)

Label	Variable	Bound	Order
$a)$	$A^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{w}_m)$, $\mathbf{B}_m$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$b)$	$D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$c)$	$D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
$d)$	$\mathbf{w}_m$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$e)$	$\mathbf{w}_m$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
$f)$	$D_N(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$g)$	$D_N(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\frac{\sqrt{N+1}}{\alpha})$
$h)$	$\partial_t\mathbf{w}_m$	$L^2([0,T];\mathbf{H}_0)$	$O(\alpha^{-1})$.
$i)$	$\partial_t\mathbf{B}_m$	$L^2([0,T];\mathbf{H}_{-1})$	$O(\frac{(N+1)^{1/4}}{\alpha^{1/2}})$.

The estimates (5.9-a)–(5.9-h) are the exact analogous of the corresponding ones from (3.11). What it remains to be proved is just (5.9-i). Let be given $\mathbf{h}\in L^2([0,T];\mathbf{H}_1)$; then

$$(\partial_t\mathbf{B}_m,\mathbf{h}) = -\mu(\nabla\mathbf{B}_m,\nabla\mathbf{h}) + (\mathbf{B}_m\otimes D_N(\mathbf{w}_m),\nabla\mathbf{h}) - (D_N(\mathbf{w}_m)\otimes\mathbf{B}_m,\nabla\mathbf{h})\,.$$

Hence we obtain, by the usual Sobolev and convex interpolation inequalities,

$$\begin{aligned}|(\partial_t\mathbf{B}_m,\mathbf{h})| &\le \mu\|\nabla\mathbf{B}_m\|\,\|\nabla\mathbf{h}\| + 2\|\mathbf{B}_m\|_{L^6}\|D_N\mathbf{w}_m\|_{L^3}\|\nabla\mathbf{h}\|\\ &\le \|\nabla\mathbf{B}_m\|\big(\mu + C\|D_N\mathbf{w}_m\|^{1/2}\|\nabla D_N\mathbf{w}_m\|^{1/2}\big)\|\nabla\mathbf{h}\|.\end{aligned}$$

Next, by employing estimates (5.9-a)-d)-f)-g), we get

$$\begin{aligned}&\left|\int_0^T(\partial_t\mathbf{B}_m,\mathbf{h})\,dt\right|\\ &\le \|\mathbf{B}_m\|_{L^2([0,T];\mathbf{H}_1)}\big(\mu + \|D_N(\mathbf{w}_m)\|^{1/2}_{L^\infty([0,T];\mathbf{H}_0)}\|\nabla D_N(\mathbf{w}_m)\|^{1/2}_{L^\infty([0,T];\mathbf{H}_0)}\big)\|\nabla\mathbf{h}\|_{L^2([0,T];L^2)},\\ &\le C\big(\mu + \frac{(N+1)^{1/4}}{\alpha^{1/2}}\big)\|\nabla\mathbf{h}\|_{L^2([0,T];L^2)}.\end{aligned}$$

These estimates are enough to pass to the limit as $m\to+\infty$ and to show that the limit $(\mathbf{w},\mathbf{B})$ is a weak solution which satisfies

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}_3}\partial_t\mathbf{w}\cdot\mathbf{v} - \int_0^T\int_{\mathbb{T}_3}G(D_N(\mathbf{w})\otimes D_N(\mathbf{w})):\nabla\mathbf{v} + \int_0^T\int_{\mathbb{T}_3}G(\mathbf{B}\otimes\mathbf{B}):\nabla\mathbf{v}\\ &\quad+\nu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{w}:\nabla\mathbf{v} = \int_0^T\int_{\mathbb{T}_3}G(\mathbf{f})\cdot\mathbf{v},\end{aligned}\tag{5.10}$$

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}_3}\partial_t\mathbf{B}\cdot\mathbf{h} - \int_0^T\int_{\mathbb{T}_3}(\mathbf{B}\otimes D_N(\mathbf{w})):\nabla\mathbf{h}\\ &\quad+\int_0^T\int_{\mathbb{T}_3}(D_N(\mathbf{w})\otimes\mathbf{B}):\nabla\mathbf{h} + \mu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{B}:\nabla\mathbf{h} = 0.\end{aligned}\tag{5.11}$$

The introduction of the pressure follows exactly as in the previous section, while the uniqueness needs some minor adjustments. Let in fact $(\mathbf{w}_1,\mathbf{B}_1)$ and $(\mathbf{w}_2,\mathbf{B}_2)$ be two solutions corresponding to the same data $(\mathbf{u}_0,\mathbf{B}_0,\mathbf{f})$ and let us define as usual $\mathcal{W}:=\mathbf{w}_1-\mathbf{w}_2$ and $\mathcal{B}:=\mathbf{B}_1-\mathbf{B}_2$. We will use $AD_N(\mathcal{W})$ and $\mathcal{B}$ as test functions in the equations satisfied by $\mathcal{W}$ and $\mathcal{B}$, respectively. Observe that, by standard calculations, $AD_N(\mathcal{W})$ lives in $L^2([0,T]\times\mathbb{T}_3)^3$, while $\mathcal{B}\in L^2([0,T];\mathbf{H}_1)$. In order to justify the calculations — those for the velocity equation are analogous to the previous ones — first observe that, for any fixed order of deconvolution N,

$$\int_0^t\langle\partial_t\mathcal{B},\mathcal{B}\rangle_{\mathbf{H}_1,\mathbf{H}_{-1}} = \frac{1}{2}\big(\|\mathcal{B}(t)\|^2 - \|\mathcal{B}(0)\|^2\big),$$

since the duality is well-defined thanks to (5.9-a)-i). We formally write the distributional expression, keeping the time derivative, and we get the following equality (to be more precise, one should write directly the integral formula, after integration over $[0,t]$, but the reader can easily fill the details):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\|\mathcal{B}\|^2\big)+\nu\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\mu\|\nabla\mathcal{B}\|^2\\
&\qquad=-\big((D_N(\mathcal{W})\cdot\nabla)D_N(\mathbf{w}_1),\,D_N(\mathcal{W})\big)+\big((\mathcal{B}\cdot\nabla)\mathbf{B}_1,\,D_N(\mathcal{W})\big)\\
&\qquad\quad-\big((D_N(\mathcal{W})\cdot\nabla)\mathbf{B}_1,\,\mathcal{B}\big)+\big((\mathcal{B}\cdot\nabla)D_N(\mathbf{w}_1),\,\mathcal{B}\big)\\
&\qquad=:I_1+I_2+I_3+I_4\,.
\end{aligned}$$

Now, we need to estimate the four integrals in the right-hand side. The estimates are obtained by using the standard interpolation and Sobolev inequalities together with the properties of D_N. We have:

$$|I_1|\le\varepsilon\nu\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\frac{C_\varepsilon(N+1)^4\sup_{t\ge0}\|\nabla\mathbf{w}_1\|^4}{\nu^3}\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2\,,$$

$$\begin{aligned}
|I_2|&\le\|\mathcal{B}\|_{L^4}\|\nabla D_N(\mathcal{W})\|_{L^4}\|\mathbf{B}_1\|\\
&\le C\|\mathcal{B}\|^{1/4}\|\nabla\mathcal{B}\|^{3/4}\|\nabla D_N(\mathcal{W})\|^{1/4}\|\Delta D_N(\mathcal{W})\|^{3/4}\|\mathbf{B}_1\|\\
&\le C\frac{(N+1)^{1/2}}{\alpha}\|\mathcal{B}\|^{1/4}\|\nabla\mathcal{B}\|^{3/4}\alpha^{1/4}\|\nabla D_N^{1/2}(\mathcal{W})\|^{1/4}\alpha^{3/4}\|\Delta D_N^{1/2}(\mathcal{W})\|^{3/4}\|\mathbf{B}_1\|\\
&\le C\frac{(N+1)^{1/2}}{\alpha}\|\mathcal{B}\|^{1/4}\|\nabla\mathcal{B}\|^{3/4}\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^{1/4}\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^{3/4}\|\mathbf{B}_1\|\\
&\le\varepsilon\mu\|\nabla\mathcal{B}\|^2+\frac{C_\varepsilon(N+1)^{4/5}}{\mu^{3/5}\alpha^{8/5}}\|\mathcal{B}\|^{2/5}\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^{2/5}\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^{6/5}\|\mathbf{B}_1\|^{8/5}\\
&\le\varepsilon\mu\|\nabla\mathcal{B}\|^2+\varepsilon\nu\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\frac{C_\varepsilon(N+1)^2}{\mu^{3/2}\nu^{3/2}\alpha^4}\|\mathcal{B}\|\,\|A^{1/2}D_N^{1/2}(\mathcal{W})\|\,\|\mathbf{B}_1\|^4\\
&\le\varepsilon\mu\|\nabla\mathcal{B}\|^2+\varepsilon\nu\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\frac{C_\varepsilon(N+1)^2}{\mu^{3/2}\nu^{3/2}\alpha^4}\|\mathbf{B}_1\|^4\big(\|\mathcal{B}\|^2+\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2\big).
\end{aligned}$$

$$\begin{aligned}
|I_3|&\le\|D_N(\mathcal{W})\|_{L^\infty}\|\nabla\mathcal{B}\|\,\|\mathbf{B}_1\|\\
&\le C\|\nabla D_N(\mathcal{W})\|^{1/2}\|\Delta D_N(\mathcal{W})\|^{1/2}\|\nabla\mathcal{B}\|\,\|\mathbf{B}_1\|\\
&\le C\frac{(N+1)^{1/2}}{\alpha}\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^{1/2}\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^{1/2}\|\nabla\mathcal{B}\|\,\|\mathbf{B}_1\|\\
&\le\varepsilon\mu\|\nabla\mathcal{B}\|^2+\frac{C_\varepsilon(N+1)}{\mu\alpha^2}\|A^{1/2}D_N^{1/2}(\mathcal{W})\|\,\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|\,\|\mathbf{B}_1\|^2\\
&\le\varepsilon\mu\|\nabla\mathcal{B}\|^2+\varepsilon\nu\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\frac{C_\varepsilon(N+1)^2}{\mu^2\nu\alpha^4}\|\mathbf{B}_1\|^4\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2\,,
\end{aligned}$$

$$\begin{aligned}
|I_4|&\le\|\mathcal{B}\|_{L^4}^2\|\nabla D_N(\mathcal{W})\|\le C\|\mathcal{B}\|^{1/2}\|\nabla\mathcal{B}\|^{3/2}(N+1)\|\nabla\mathbf{w}_1\|\\
&\le\varepsilon\mu\|\nabla\mathcal{B}\|^2+\frac{C_\varepsilon(N+1)^4}{\mu^3}\|\nabla\mathbf{w}_1\|^4\|\mathcal{B}\|^2\,.
\end{aligned}$$

We then set $\varepsilon=1/6$ and, by collecting all the estimates, we finally obtain

$$\frac{d}{dt}\big(\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\|\mathcal{B}\|^2\big)+\nu\|\nabla A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\mu\|\nabla\mathcal{B}\|^2\le CM\big(\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\|\mathcal{B}\|^2\big),$$

where

$$M\doteq(N+1)^4\max\sup_{t\ge0}\Big\{\frac{\|\nabla\mathbf{w}_1(t)\|^4}{\nu^3},\frac{\|\nabla\mathbf{w}_1(t)\|^4}{\mu^3},\frac{\|\mathbf{B}_1(t)\|^4}{\mu^{3/2}\nu^{3/2}\alpha^4},\frac{\|\mathbf{B}_1(t)\|^4}{\mu^2\nu\alpha^4}\Big\}.$$

An application of the Gronwall's lemma proves (for any fixed N) that $\|A^{1/2}D_N^{1/2}(\mathcal{W})\|^2+\|\mathcal{B}\|^2=0$; hence, by using the properties of A and D_N exploited before, we finally get $\mathcal{W}=\mathcal{B}\equiv\mathbf{0}$. □

We can now pass to the problem of the convergence as $N \to +\infty$, proving the counterpart of Theorem 1.1.

Theorem 5.2. *From the sequence $\{(\mathbf{w}_N, \mathbf{B}_N, q_N)\}_{N\in\mathbb{N}}$, one can extract a sub-sequence (still denoted $\{(\mathbf{w}_N, \mathbf{B}_N, q_N)\}_{N\in\mathbb{N}}$) such that*

$$\mathbf{w}_N \longrightarrow \mathbf{w} \quad \begin{cases} \textit{weakly in } L^2([0,T];\mathbf{H}_2), \\ \textit{weakly* in } L^\infty([0,T];\mathbf{H}_1), \\ \textit{strongly in } L^p([0,T];\mathbf{H}_1) \quad \forall\, p \in [1,\infty[, \end{cases}$$

$$\mathbf{B}_N \longrightarrow \mathbf{B} \quad \begin{cases} \textit{weakly in } L^2([0,T];\mathbf{H}_1), \\ \textit{weakly* in } L^\infty([0,T];\mathbf{H}_0), \\ \textit{strongly in } L^p([0,T]\times\mathbb{T}_3)^3 \quad \forall\, p \in [1,10/3[, \end{cases}$$

$$q_N \longrightarrow q \qquad \textit{weakly in } L^2([0,T];H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3)),$$

and such that the system

$$\begin{aligned} \partial_t \mathbf{w} + \nabla\cdot G(A\mathbf{w}\otimes A\mathbf{w}) - \nabla\cdot G(\mathbf{B}\otimes\mathbf{B}) + \nabla q - \nu\Delta\mathbf{w} &= \mathbf{f}, \\ \nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{B} &= 0, \\ \partial_t\mathbf{B} + \nabla\cdot(\mathbf{B}\otimes A\mathbf{w}) - \nabla\cdot(A\mathbf{w}\otimes\mathbf{B}) - \mu\Delta\mathbf{b} &= 0, \\ \mathbf{w}(0,\mathbf{x}) = G\mathbf{u}_0(\mathbf{x}), \qquad \mathbf{B}(0,\mathbf{x}) &= \mathbf{B}_0(\mathbf{x}) \end{aligned} \tag{5.12}$$

holds in distributional sense and the following energy inequality is satisfied:

$$\frac{1}{2}\frac{d}{dt}(\|A\mathbf{w}\|^2 + \|\mathbf{b}\|^2) + \nu\|\nabla A\mathbf{w}\|^2 + \mu\|\nabla\mathbf{b}\|^2 \le (\mathbf{f}, A\mathbf{w}).$$

Proof. This result is based on the following estimates and from compactness results.

(5.13)

Label	Variable	Bound	Order
a)	$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
b)	$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
c)	$D_N(\mathbf{w}_N)$, $\mathbf{B}_N$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
d)	q_N	$L^2([0,T];H^1(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3))$	$O(\alpha^{-1})$
e)	$\partial_t\mathbf{w}_N$	$L^2([0,T];\mathbf{H}_0)$	$O(\alpha^{-1})$
f)	$\partial_t D_N(\mathbf{w}_N), \partial_t\mathbf{B}_N$	$L^{4/3}([0,T];\mathbf{H}_{-1})$	$O(1)$

The only new bound here is represented by the one for $\partial_t\mathbf{B}_N$ from (5.13-f). In fact, by the usual interpolation inequalities, we get

$$\begin{aligned} |(\partial_t\mathbf{B}_N,\mathbf{h})| &\le \mu\|\nabla\mathbf{B}_N\|\,\|\nabla\mathbf{h}\| + 2\|\mathbf{B}_N\|_{L^4}\|D_N(\mathbf{w}_N)\|_{L^4}\|\nabla\mathbf{h}\| \\ &\le \big(\mu\|\nabla\mathbf{B}_N\| + C\|\mathbf{B}_N\|^{1/4}\|\nabla\mathbf{B}_N\|^{3/4}\|D_N(\mathbf{w}_N)\|^{1/4}\|\nabla D_N(\mathbf{w}_N)\|^{3/4}\big)\|\nabla\mathbf{h}\|. \end{aligned}$$

Next, by employing estimate (5.13-c), we get

$$|(\partial_t\mathbf{B}_N,\mathbf{h})| \le \big(\mu\|\nabla\mathbf{B}_N\| + C\|\nabla\mathbf{B}_N\|^{3/4}\|\nabla D_N(\mathbf{w}_N)\|^{3/4}\big)\|\nabla\mathbf{h}\|,$$

and since both $\nabla\mathbf{B}_N$, $\nabla D_N(\mathbf{w}) \in L^2([0,T];L^2(\mathbb{T}_3)^9)$, we can show that

$$\begin{aligned} \left|\int_0^T (\partial_t\mathbf{B}_N,\mathbf{h})\,dt\right| &\le \mu\|\nabla\mathbf{B}_N\|_{L^2([0,T];L^2)}\|\nabla\mathbf{h}\|_{L^2([0,T];L^2)} \\ &\quad + C\|\nabla\mathbf{B}_N\|^{3/4}_{L^2([0,T];L^2)}\|\nabla D_N(\mathbf{w})_N\|^{3/4}_{L^2([0,T];L^2)}\|\nabla\mathbf{h}\|_{L^4([0,T];L^2)}\,, \end{aligned}$$

thus proving that $\partial_t\mathbf{B}_N \in L^{4/3}([0,T];\mathbf{H}_{-1})$, independently of N.

The limit $N \to +\infty$ can be studied as in the previous section. In addition to the same estimates proved before, from the bound on the time derivative of $\mathbf{B}$ we obtain that $\mathbf{B}_N \to \mathbf{B}$ in $L^p([0,T];\mathbf{H}_0)$, $\forall\, p \in [1,\infty[$, and reasoning as in (4.4) we get

$$\begin{aligned} D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N) &\longrightarrow A\mathbf{w}\otimes A\mathbf{w} \quad \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^9 \quad \forall\, p\in[1,5/3[, \\ \mathbf{B}_N\otimes\mathbf{B}_N &\longrightarrow \mathbf{B}\otimes\mathbf{B} \quad \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^9 \quad \forall\, p\in[1,5/3[, \\ D_N(\mathbf{w}_N)\otimes\mathbf{B}_N &\longrightarrow A\mathbf{w}\otimes\mathbf{B} \quad \text{strongly in } L^p([0,T]\times\mathbb{T}_3)^9 \quad \forall\, p\in[1,5/3[. \end{aligned}$$

Finally, the proof of the energy inequality follows the same steps as before. □

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