Intersection cuts from multiple rows: a disjunctive programming approach

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Abstract

We address the issue of generating cutting planes for mixed integer programs from multiple rows of the simplex tableau with the tools of disjunctive programming. A cut from q rows of the simplex tableau is an intersection cuts from a q-dimensional parametric cross-polytope, which can also be viewed as a disjunctive cut from a 2^q-term disjunction. We define the disjunctive hull of the q-row problem, describe its relation to the integer hull, and show how to generate its facets. For the case of binary basic variables, we derive cuts from the stronger disjunctions whose terms are equations. We give cut strengthening procedures using the integrality of the nonbasic variables for both the integer and the binary case. Finally, we discuss some computational experiments.

1 Introduction: intersection cuts and disjunctive programming

In the last few years a considerable effort has been devoted to generating valid cuts for mixed integer programs from multiple rows of the simplex tableau, with a focus on cuts from two rows. This research was pioneered by the 2007 paper of Andersen, Louveaux, Weismantel and Wolsey [1], followed by Borozan and Cornuéjols [13], Cornuéjols and Margot [17], Dey and Wolsey [19] and many others ([11, 12, 18, 20]; for a recent survey see [15]).

All of these papers view and derive the multiple-row cuts as intersection cuts, a concept introduced in [2], i.e. cuts obtained by intersecting the extreme rays of the cone defined by a basic linear programming solution with the boundary of a convex set whose interior contains no feasible integer point. Intersection cuts are equivalent to disjunctive cuts, and in this paper we apply the tools of disjunctive programming to the study of cuts from multiple rows of the simplex tableau. Two early versions of

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this paper were presented at the 2009 Spring Meeting of the AMS in San Francisco [5] and at the 20^{th} ISMP in Chicago [6].

The structure of our paper is as follows. In the remainder of this section we outline the connection of intersection cuts with disjunctive programming. In section 2 we introduce the concept of disjunctive hull associated with q rows of the simplex tableau and examine the relation between the disjunctive hull and the integer hull. We then give a geometric interpretation of cuts from q rows of the simplex tableau as cuts from a q-dimensional parametric cross-polytope (section 3), followed by a theorem relating the facets of the disjunctive hull to those of the integer hull (section 4). In section 5 we specialize these results to the case of $q = 2$. The next section (6) discusses the strengthening of our cuts when some of the nonbasic variables are integer-constrained. Section 7 deals with the 0-1 case, when the stronger disjunction whose terms are equations can be used to derive stronger cuts. Finally, section 8 describes some computational experiments.

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Suppose a Mixed Integer Program is given in the form of q rows of the simplex tableau

$$x = \bar{x} + \sum_{j \in J} r^j s_j, \quad x \in \mathbb{Z}^q_+, \ s \in \mathbb{R}^n_+ \tag{1.1}$$

where $\bar{x}$ is a basic feasible solution to LP, the linear programming relaxation of a MIP, and we are interested in generating an inequality that cuts off $\bar{x}$ but no feasible integer point.

Theorem 1.1. *(Balas [2]). Let $T \subseteq \mathbb{R}^q$ be a closed convex set whose interior contains $\bar{x}$ but no feasible integer point. For $j \in J$, let $s^*_j := \max\{s_j : \bar{x} + r^j s_j \in T\}$. Then the inequality $\alpha s \geq 1$, where $\alpha_j = \frac{1}{s^*_j}$, $j \in J$, cuts off $\bar{x}$ but no feasible integer point.*

The inequality $\alpha s \geq 1$ is known as an *intersection cut.* Theorem 1.1 is illustrated

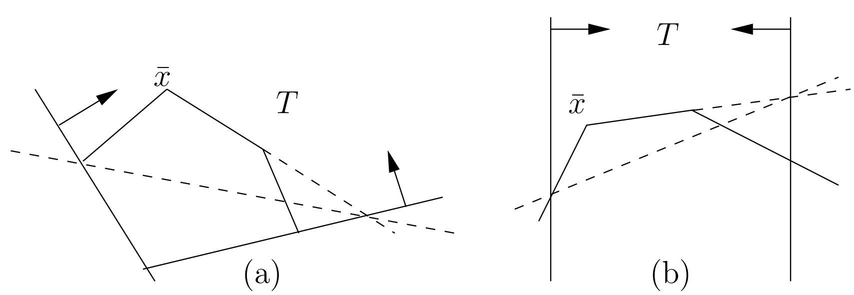


Figure 1: Two intersection cuts

by Figure 1. In both cases (a) and (b) the convex set T consists of the intersection of two halfspaces, but in (b) the two halfspaces are defined by hyperplanes parallel to one of the coordinate axes, and so their intersection defines an infinite strip. The intersection cut from this latter set T is the Gomory Mixed Integer cut (GMI) [21].

This particular class of intersection cuts, the GMI cuts, has played a crucial role in making mixed integer programs practically solvable. These cuts are derived from a convex set of the form $\lfloor \bar{x}_i \rfloor \leq x_i \leq \lceil \bar{x}_i \rceil$, where $x_i = \bar{x}_i + \sum_{j \in J} r^i_j s_j$ is one of the rows of an optimal simplex tableau and $\lfloor \bar{x}_i \rfloor < \bar{x}_i < \lceil \bar{x}_i \rceil$. More generally, cuts obtained from a convex set of the form $\pi_0 \leq \pi x \leq \pi_0 + 1$, where (π, π_0) is an integer vector with $\gcd(\pi) = 1$, are known in the literature as split cuts [16]. It is then natural to ask the question whether intersection cuts derived simultaneously from several rows of a simplex tableau have some properties that distinguish them from split cuts. It was this question that has led to the investigation of intersection cuts from maximal lattice-free convex sets by [1, 13] and others.

We propose a different approach to the same problem, which promises some computational advantages. The approach is that of Disjunctive Programming, a natural outgrowth of the study of intersection cuts. To see the connection, consider an intersection cut from a polyhedral set with the required properties, of the form $T := \{x : d^i x \leq d^i_0,\ i = 1, \ldots, m\}$. Clearly, the requirement that $\operatorname{int} T$ should contain no feasible integer point, can be rephrased as the requirement that every feasible integer point should satisfy at least one of the weak complements of the inequalities defining T, i.e. should satisfy the disjunction

$$\bigvee_{i=1}^{m} (d^i x \geq d^i_0). \tag{1.2}$$

Therefore an intersection cut from T can be viewed as a disjunctive cut from (1.2). While these two cuts are essentially the same, the disjunctive point of view opens up new perspectives. Thus, suppose that in addition to (1.2), all feasible solutions have to satisfy the inequalities $Ax \geq b$. Then one way to proceed is to generate all valid cutting planes from (1.2) and append these to $Ax \geq b$. The resulting system will be

$$P := \left\{ x \in \mathbb{R}^n : (Ax \geq b) \cap \ \operatorname{conv} \left(\bigvee_{i=1}^{m} \left(d^i x \geq d^i_0 \right) \right) \right\}.$$

But another way to proceed is to introduce $Ax \geq b$ into each term of the disjunction (1.2), i.e. replace (1.2) with

$$\bigvee_{i=1}^{m} \left(\begin{array}{c} Ax \geq b \\ d^i x \geq d^i_0 \end{array} \right), \tag{1.3}$$

and take the convex hull of this union of polyhedra:

$$Q := \ \operatorname{conv} \left(\bigvee_{i=1}^{m} \left(\begin{array}{c} Ax \geq b \\ d^i x \geq d^i_0 \end{array} \right) \right)$$

Now it is not hard to see that $Q \subseteq P$, and in fact Q is in most cases a much tighter constraint set than P. We illustrate the difference on a 2-term disjunction. Given an arbitrary Mixed Integer Program, let (π, π_0) be an integer vector with a component π_j for every integer-constrained variable. Then the disjunctive cut derived from

$$\pi x \leq \pi_0 \ \vee \ \pi x \geq \pi_0 + 1 \tag{1.4}$$

is equivalent to the intersection cut derived from the convex set

$$\pi_0 \leq \pi x \leq \pi_0 + 1,$$

illustrated in Figure 1. On the other hand, the disjunction

$$\begin{pmatrix} Ax & \geq & b \\ \pi x & \leq & \pi_0 \end{pmatrix} \quad \vee \quad \begin{pmatrix} Ax & \geq & b \\ \pi x & \geq & \pi_0 + 1 \end{pmatrix} \tag{1.5}$$

gives rise to an entire family of cuts, whose members are determined by the multipliers u, v associated with $Ax \geq b$ in the two terms of this more general disjunction

$$(\pi - uA)x \leq \pi_0 - ub \ \vee \ (\pi + vA)x \geq \pi_0 + vb + 1 \tag{1.6}$$

Cuts derived from a disjunction of the form (1.4) are called split cuts, a term that reflects the fact that (1.4) splits the space into two disjoint half-spaces. Cook, Kannan and Schrijver [16] who coined this term also extended it to the much larger family of cuts derived from disjunctions of the form (1.6).

Disjunctive sets of the form (1.3) or (1.5) represent unions of polyhedra, and the study of optimization over unions of polyhedra is known as Disjunctive Programming. Its two basic results are a compact representation of the convex hull of a union of polyhedra in a higher dimensional space, and the sequential convexifiability of facial disjunctive sets [4, 3]. The application of disjunctive programming to mixed 0-1 programs has become known as the lift-and-project method [7]. Here we apply this approach to the study of intersection cuts from multiple rows of the simplex tableau.

2 Integer and disjunctive hulls

Consider again a system defined by q rows of the simplex tableau, this time without the integrality constraints:

$$x = f + \sum_{j \in J} r^j s_j, \ s_j \geq 0, \ j \in J, \tag{2.1}$$

where f, $r^j \in \mathbb{R}^q$, $j \in J := \{1, \ldots, n\}$, and assume $0 < f_i < 1$, $i \in Q := \{1, \ldots, q\}$. This assumption can be made without loss of generality since setting $x'_i = x_i - \lfloor f_i \rfloor$ and $f'_i = f_i - \lfloor f_i \rfloor$, $i \in Q$, we have that x'_i, f'_i, $i \in Q$ satisfy the assumption. The set

$$P_L := \{(x, s) \in \mathbb{R}^q \times \mathbb{R}^n : (x, s) \text{ satisfies } (2.1)\} \tag{2.2}$$

is the polyhedral cone with apex at $(x, s) = (f, 0)$ defined by the constraints that are tight for this particular basic solution. Imposing the integrality constraints on the basic components we get the mixed integer set

$$P_I := \{(x, s) \in P_L : x_i \text{ integer}, i \in Q\}, \tag{2.3}$$

whose convex hull, $\operatorname{conv} P_I$, is Gomory's corner polyhedron [22], or the *integer hull* of the MIP over the cone P_L. The main objective of the papers mentioned in the introduction was to study the structure of P_I for small q, with a view of characterizing the facets of $\operatorname{conv} P_I$ and minimal valid inequalities for P_I.

Consider now the following disjunctive relaxation of P_I, obtained by replacing the integrality constraints on x_i with the simple disjunctions $x_i \leq 0 \ \vee \ x_i \geq 1$, $i \in Q$:

$$P_D := \{(x,s) \in P_L : x_i \leq 0 \ \vee \ x_i \geq 1, \ i \in Q\}. \tag{2.4}$$

Like P_I, P_D is a nonconvex set. Its convex hull, $\operatorname{conv} P_D$, which we call the simple *disjunctive hull*, is a weaker relaxation of P_I than $\operatorname{conv} P_I$, i.e. $\operatorname{conv} P_D \supseteq \operatorname{conv} P_I$, but it is easier to handle, since it is the convex hull of the union of 2^q polyhedra. Thus one can apply disjunctive programming and lift-and-project techniques to generate facets of $\operatorname{conv} P_D$ at a computational cost that for small q seems acceptable. In this context, the crucial question is of course, how much weaker is the relaxation $\operatorname{conv} P_D$ than $\operatorname{conv} P_I$? We will pose this question in a more specific form that will enable us to give it a practically useful answer: when is it that a facet defining inequality for $\operatorname{conv} P_D$ is also facet defining for $\operatorname{conv} P_I$? In other words, which facets of the (simple) disjunctive hull are also facets of the integer hull? Before addressing this question, however, we will take a side-step, by introducing a third kind of hull. If we strengthen the disjunctive relaxation of P_I by replacing the inequalities in the disjunctions $x_i \leq 0 \ \vee \ x_i \geq 1$, $i \in Q$, with equations, we get the set

$$P_D^{=} := \{(x,s) \in P_L : x_i = 0 \ \vee \ x_i = 1, \ i \in Q\}, \tag{2.5}$$

whose convex hull, $\operatorname{conv} P_D^{=}$, we call *the 0-1 disjunctive hull*. For a general mixed integer program, the 0-1 Disjunctive Hull is not a valid relaxation, in that it may cut off nonbinary feasible integer points. Indeed, we have

$$\operatorname{conv} P_D \supseteq \operatorname{conv} P_I \supseteq \operatorname{conv} P_D^{=},$$

where both inclusions are strict and are valid in the context of mixed integer 0-1 programs only, since all the non-0-1 integer points that it cuts off are infeasible. Hence $\operatorname{conv} P_D^{=}$ is equivalent to the convex hull of $P_I \cap \{x : x_i \leq 1, \ i \in Q\}$, or the integer hull of P_I reinforced with the bounds on the x_i. However, as we will see later on, finding facets of $\operatorname{conv} P_D^{=}$ requires roughly the same computational effort as finding facets of $\operatorname{conv} P_D$.

The upshot of this is that for the important class of mixed integer 0-1 programs, all facet defining inequalities of $\operatorname{conv} P_D^{=}$ are facet defining for the integer hull. Furthermore, from the sequential convexification theorem of disjunctive programming, all such inequalities are of split rank $\leq q$, i.e. they can be obtained by applying a split cut generating procedure at most q times recursively.

The set P_D of (2.4) is the collection of those points $(x,s) \in \mathbb{R}^q \times \mathbb{R}^n$ satisfying (2.1) and $x_i \leq 0 \ \vee \ x_i \geq 1$, $i \in Q$. Put in disjunctive normal form, this last constraint set becomes

$$\begin{pmatrix} x_1 \leq 0 \\ x_2 \leq 0 \\ \vdots \\ x_q \leq 0 \end{pmatrix} \vee \begin{pmatrix} x_1 \geq 1 \\ x_2 \leq 0 \\ \vdots \\ x_q \leq 0 \end{pmatrix} \vee \cdots \vee \begin{pmatrix} x_1 \geq 1 \\ x_2 \geq 1 \\ \vdots \\ x_q \geq 1 \end{pmatrix} \tag{2.6}$$

Each term of (2.6) defines an orthant-cone with apex at a vertex of the q-dimensional unit cube. These 2^q orthant-cones are illustrated for $q = 2$ in Figure 2.

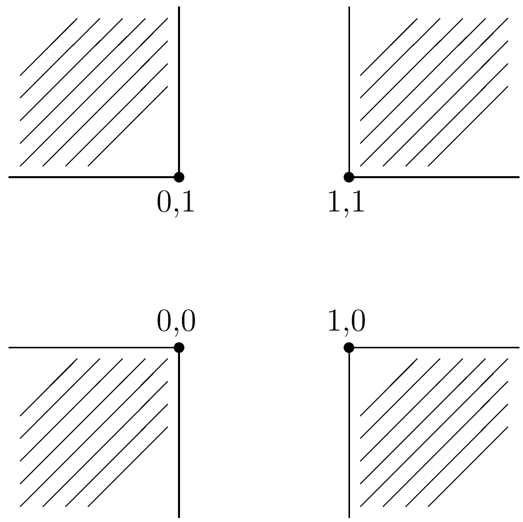


Figure 2: Orthant-cones for the case $q = 2$

Using (2.1) to eliminate the x-components and denoting by r^i the i-th row of the $q \times n$ matrix $R = (r^j)_{j=1}^n$, (2.6) can be represented in $\mathbb{R}^n$ as $s \geq 0$ and

$$\begin{pmatrix} -r^1 s & \geq & f_1 \\ -r^2 s & \geq & f_2 \\ & \vdots & \\ -r^q s & \geq & f_q \end{pmatrix} \vee \begin{pmatrix} r^1 s & \geq & 1 - f_1 \\ -r^2 s & \geq & f_2 \\ & \vdots & \\ -r^q s & \geq & f_q \end{pmatrix} \vee \cdots \vee \begin{pmatrix} r^1 s & \geq & 1 - f_1 \\ r^2 s & \geq & 1 - f_2 \\ & \vdots & \\ r^q s & \geq & 1 - f_q \end{pmatrix} \tag{2.7}$$

If $P_i^{(n)} \subseteq \mathbb{R}^n$ denotes the polyhedron defined by the i-th term of this disjunction plus the constraints $s \geq 0$, then P_D can be defined in n-space as $P_D^{(n)} = \cup_{i=1}^t P_i^{(n)}$ where $t = 2^q$. Furthermore, we have the following:

Theorem 2.1. $\operatorname{conv} P_D^{(n)}$ *is the set of those* $s \in \mathbb{R}^n$ *satisfying* $s \geq 0$ *and all the inequalities* $\alpha s \geq 1$ *whose coefficient vectors* $\alpha \in \mathbb{R}^n$ *satisfy the system*

$$\begin{array}{llllr}
\alpha + r^1 u_{11} + \cdots + r^q u_{1q} & & & & \geq 0 \\
\alpha & - r^1 u_{21} + \cdots + r^q u_{2q} & & & \geq 0 \\
\vdots & & \ddots & & \vdots \\
\alpha & & & - r^1 u_{t1} - \cdots - r^q u_{tq} & \geq 0 \\
\quad f_1 u_{11} + \cdots + f_q u_{1q} & & & & \geq 1 \\
 & (1-f_1) u_{21} + \cdots + f_q u_{2q} & & & \geq 1 \\
 & & \ddots & & \vdots \\
 & & & (1-f_1) u_{t1} + \cdots + (1-f_q) u_{tq} & \geq 1
\end{array} \tag{2.8}$$

for some $u_{ik} \geq 0$, $i = 1, \ldots, t = 2^q, k = 1, \ldots, q$.

Proof. Applying the basic theorem of Disjunctive Programming to $\operatorname{conv} P_D^{(n)}$ we introduce auxiliary variables $s^i \in \mathbb{R}^n$, $z_i \in \mathbb{R}$, $i = 1, \ldots, t = 2^q$, and obtain the higher-dimensional representation

$$
\begin{array}{ccccccccccc}
s & -s^1 & -s^2 & \dots & -s^t & & & & & = & 0 \\
 & -r^1 s^1 & & & & -f_1 z_1 & & & & \geq & 0 \\
 & -r^2 s^1 & & & & -f_2 z_1 & & & & \geq & 0 \\
 & \vdots & & & & \vdots & & & & \vdots & \\
 & -r^q s^1 & & & & -f_q z_1 & & & & \geq & 0 \\
 & & r^1 s^2 & & & & -(1-f_1)z_2 & & & \geq & 0 \\
 & & -r^2 s^2 & & & & -f_2 z_2 & & & \geq & 0 \\
 & & \vdots & & & & \vdots & & & \vdots & \\
 & & -r^q s^2 & & & & -f_q z_2 & & & \geq & 0 \\
 & & & \ddots & & & & \ddots & & \vdots & \\
 & & & & -r^1 s^t & & & & -(1-f_1)z_t & \geq & 0 \\
 & & & & -r^2 s^t & & & & -(1-f_2)z_t & \geq & 0 \\
 & & & & \vdots & & & & \vdots & \vdots & \\
 & & & & -r^q s^t & & & & -(1-f_q)z_t & \geq & 0 \\
 & & & & & z_1 & +z_2 & +\cdots & +z_t & = & 1
\end{array}
\tag{2.9}
$$

$$
s^i \geq 0,\ i = 1, \dots, t; \quad z_i \geq 0,\ i = 1, \dots, t
$$

Projecting this system onto the s-space with multipliers α; $u_{11}, \dots, u_{1q}$; $u_{21}, \dots, u_{2q}$; $\dots$; $u_{t1}, \dots, u_{tq}$, we obtain

$$
\begin{array}{lcccccccc}
\alpha & + \; r^1 u_{11} & + \cdots + & r^q u_{1q} & & & & \geq & 0 \\
\vdots & & & & \ddots & & & \vdots & \\
\alpha & & & & & -r^1 u_{t1} & - \cdots - \quad -r^q u_{tq} & \geq & 0 \\
 & -\beta + f_1 u_{11} & + \cdots + & f_q u_{1q} & & & & \geq & 0 \\
 & \vdots & & & \ddots & & & \vdots & \\
 & -\beta & & & & +(1-f_1)u_t f_1 & + \cdots + \quad (1-f_q)u_{tq} & \geq & 0
\end{array}
\tag{2.10}
$$

$$
u_{ik} \geq 0,\ i = 1, \dots, t,\ k = 1, \dots, q
$$

Applying the normalization $\beta = 1$ (clearly $\beta = -1$ does not yield any cuts since it makes (2.10) unbounded) we obtain the representation given in the theorem. $\square$

In order to restate the system (2.8) in a more concise form, for each $i \in \{1, \dots, t\}$ we partition the index set $Q := \{1, \dots, q\}$ into

$Q_i^+ := \{k \in Q : u_{ik}$ has coefficient vector $r_k\}$
$Q_i^- := \{k \in Q : u_{ik}$ has coefficient vector $-r_k\}$,

with $Q_i^+ \cup Q_i^- = Q$, $i = 1, \ldots, t = 2^k$. Then (2.8) can be restated as

$$\begin{array}{rclr} \alpha + \sum\left(r^k u_{ik} : k \in Q_i^+\right) - \sum\left(r^k u_{ik} : k \in Q_i^-\right) & \geq & 0 & \\ \sum(f_k u_{ik} : k \in Q_i^+) + \sum((1-f_k)u_{ik} : k \in Q_i^-) & \geq & 1, \quad i = 1, \ldots, t & (2.8') \\ & & & \\ \end{array}$$

$$u_{ik} \geq 0,\ i = 1, \ldots, t = 2^q,\ k = 1, \ldots, q$$

The system (2.8′) has several interesting properties described in the next few propositions.

Proposition 2.2. *For any $p \in \mathbb{R}^n$, $p > 0$, all optimal basic solutions to the cut generating linear program*

$$\min\{p\alpha : (\alpha, u) \text{ satisfies (2.8}')\} \qquad (\text{CGLP})_Q$$

are of the form

$$\alpha_j = \max\{\alpha_j^1, \ldots, \alpha_j^t\}, \tag{2.11}$$

where

$$\alpha_j^i := -\sum(r_j^k u_{ik} : k \in Q_i^+) + \sum(r_j^k u_{ik} : k \in Q_i^-), \tag{2.12}$$

$i = 1, \ldots, t = 2^q$*, with the u_{ik} satisfying (2.8′).*

Proof. The constraints of (2.8′) require

$$\alpha_j \geq \alpha_j^i, \quad i = 1, \ldots, t,\ j = 1, \ldots, n$$

Suppose there is an optimal solution to $(\text{CGLP})_Q$ such that $\alpha_{j_*} > \max\{\alpha_{j_*}^i : i = 1, \ldots, t\}$ for some $j_* \in \{1, \ldots, n\}$. Then setting α_{j_*} equal to the maximum on the righthand side, and leaving α_j unchanged for all $j \neq j_*$ yields a better solution, contrary to the assumption. □

Proposition 2.3. *In any valid inequality $\alpha s \geq 1$ for* $\operatorname{conv} P_D^{(n)}$, $\alpha_j \geq 0$, $j = 1, \ldots, n$.

Proof. From (2.11), $\alpha_j \geq \alpha_j^i$ for all $i = 1, \ldots, 2^q$, and in view of the presence of all sign patterns of $r_j^k u_{ik}$ in the expressions (2.12), there is always an index $i \in \{i, \ldots, 2^q\}$ with $\alpha_j^i \geq 0$. □

Proposition 2.4. *For any basic solution (α, u) to $(CGLP)_Q$ that satisfies as strict inequality some of the nonhomogeneous constraints of (2.8′), there exists a basic solution $(\bar{\alpha}, u)$, with $\bar{\alpha} = \alpha$, that satisfies at equality all the nonhomogeneous constraints of $(CGLP)_Q$.*

Proof. Let (α, u) be a basic solution to $(\text{CGLP})_Q$ that satisfies as strict inequality some of the nonhomogeneous constraints of (2.8′). W.l.o.g., assume that

$$f_1 u_{11} + \cdots + f_q u_{1q} - \theta = 1$$

is one of those constraints with the surplus variable θ positive in the solution (α, u). We will show that there exists a solution $(\bar{\alpha}, \bar{u})$, with $\bar{\alpha} = \alpha$ and $\bar{u}_{ik} = u_{ik}$ for all $i \neq 1$ and all k, such that

$$f_1 \bar{u}_{11} + \cdots + f_q \bar{u}_{1q} = 1.$$

Applying this argument recursively then proves the Proposition.

Fix all variables of $(\text{CGLP})_Q$ except for $u_{11}, \ldots, u_{1q}$, at their values in the current solution. The fixing includes all the surplus variables except those in the $n+1$ rows containing $u_{11}, \ldots, u_{1q}$. This leaves the following constraint set in the free variables:

$$\begin{aligned} -r_j^1 u_{11} - \cdots - r_j^q u_{1q} + t_j &= \bar{\alpha}_j \qquad j = 1, \ldots, n \\ f_1 u_{11} + \cdots + f_q u_{1q} - \theta &= 1 \\ u_{11}, \ldots, u_{1q} \geq 0,\ t_j \geq 0,\ j &= 1, \ldots, n,\ \theta \geq 0 \end{aligned} \tag{2.13}$$

Here θ, t_j represent the surplus variables of the respective constraints. We claim that this system has a solution with $\theta = 0$. To see this, consider the linear program

$$\min\{\theta : u_{ik}, t_j \text{ and } \theta \text{ satisfy (2.13)}\}$$

and its dual,

$$\max \lambda_0 + \sum_{j=1}^{n} \bar{\alpha}_j \lambda_j$$

subject to

$$\begin{aligned} f_1 \lambda_0 - \sum_{j=1}^{n} r_j^1 \lambda_j &\leq 0 \\ \vdots \qquad\qquad & \quad \vdots \\ f_q \lambda_0 - \sum_{j=1}^{n} r_j^q \lambda_j &\leq 0 \\ -\lambda_0 \qquad\qquad &\leq 1 \\ \lambda_j &\leq 0, \quad j = 1, \ldots, n \end{aligned}$$

Since $\bar{\alpha}_j \geq 0$, $j = 1, \ldots, n$, it is not hard to see that the dual linear program has an optimal solution $\lambda_0 = 0$, $\lambda_j = 0$, $j = 1, \ldots, n$ and hence the primal has an optimal solution with $\theta = 0$. $\square$

The obvious and important consequence of Proposition 2.4 is that for all practical purposes we can replace all 2^q nonhomogeneous inequalities in the constraint set $(2.8')$ of $(\text{CGLP})_Q$ with equations. In view of Proposition 2.2, it then follows that we may restrict our attention to basic feasible solutions that satisfy at equality $n + 2^q$ out of the $n \times 2^q + 2^q$ inequalities of $(2.8')$ other than the nonnegativity constraints.

At this point we introduce the characterization of $\text{conv}\, P_D^=$, the 0-1 disjunctive hull defined by (2.5), closely related to that of $\text{conv}\, P_D$. Just as in the case of P_D, we denote by $P_D^{=(n)}$ the union of polyhedra in $\mathbb{R}^n$ representing the disjunction (2.7) in which all the inequalities have been replaced by equations. The following Theorem is the analog of Theorem 2.1 for this case.

Theorem 2.5. $\text{conv}\, P_D^{=(n)}$ *is the set of those* $s \in \mathbb{R}^n$ *satisfying* $s \geq 0$ *and all inequalities*

$\alpha s \geq \beta$ whose coefficients satisfy the system

$$\begin{array}{llllll}
\alpha + r^1u_{11} + \cdots + r^1u_{1q} & & & \geq & 0 & \\
\vdots & \ddots & & \vdots & & \\
\alpha & & -r^1u_{t1} - \cdots - r^qu_{tq} & \geq & 0 & \\
\quad -\beta + f_1u_{11} + \cdots + f_qu_{1q} & & & = & 0 & (2.14)\\
\vdots & \ddots & & \vdots & & \\
\quad -\beta & & +(1-f_1)u_{t1} + \cdots + (1-f_q)u_{tq} & = & 0 &
\end{array}$$

for some u_{ik}, $i = 1, \dots, t = 2^q$, $k = 1, \dots, q$.

Proof. The proof of Theorem 2.1 goes through with the following modifications. Since the inequalities in the disjunction (2.7) are all replaced with equations, the inequalities in the system (2.9), other than the nonnegativity constraints, also become equations. As a consequence, the variables u_{ik} of the projected system (2.10) become unrestricted in sign. The remaining difference between (2.14) and (2.8) is the fact that in (2.14) the last 2^q constraints are equations rather than inequalities. This is due to the fact that Proposition 2.4 applies here too. In other words, if we denote by (2.14″) the system obtained from (2.14) by replacing the equations containing β with inequalities $\geq$, then for any basic solution (α, u) to $(\mathrm{CGLP})_Q$ that satisfies as strict inequalities some of the constraints (2.14″) containing β, there exists a basic solution $(\bar{\alpha}, u)$, with $\bar{\alpha} = \alpha$, that satisfies at equality all the constraints containing β. The proof is essentially the same as that of Proposition 2.4.

Thus the two basic differences between the systems describing $\operatorname{conv} P_D^{(n)}$ and $\operatorname{conv} P_D^{=(n)}$ are that (a) the latter also contains inequalities of the form $\alpha x \leq 1$ (corresponding to $\beta < 0$), and (b) the coefficients α_j of the latter can be of any sign. □

We now return to the simple disjunctive hull, $\operatorname{conv} P_D$, and describe its vertices.

Proposition 2.6. *Every vertex of $\operatorname{conv} P_D^{(n)}$ is a vertex of some $P_i^{(n)}$, $i \in \{1, \dots, 2^q\}$.*

Proof. Let v be a vertex of $\operatorname{conv} P_D^{(n)}$. If $v \in P_i^{(n)}$ for some $i \in \{1, \dots, t = 2^q\}$, then v must be a vertex of $P_i^{(n)}$, or else it could be expressed as a convex combination of points in $P_i^{(n)}$, hence of $P_D^{(n)}$. On the other hand, if $v \notin \cup P_i^{(n)}$ but $v \in \operatorname{conv} P_i^{(n)}$, then v is a convex combination of points in $\cup_{i=1}^t P_i^{(n)}$, hence of $\operatorname{conv} P_D^{(n)}$, a contradiction. □

Next we describe the vertices of $P_i^{(n)}$, $i \in \{1, \dots, 2^q\}$. We will call a vertex of $\operatorname{conv} P_D^{(n)}$ (of $P_i^{(n)}$) integer if it defines an integer x through (2.1); in other words if $f_i + r^i s$ is integer for $i = 1, \dots, q$. All other vertices will be called fractional.

For any particular $i_* \in \{1, \dots, 2^q\}$,

$$P_{i_*}^{(n)} := \{s \in \mathbb{R}_+^n : r_h s \leq -f_h,\ h \in Q_{i_*},\ r_h s \geq 1 - f_h,\ h \in Q \setminus Q_{i_*}\}$$

where $(Q_{i_*}, Q \setminus Q_{i_*})$ is the partition of Q that defines i_*.

Proposition 2.7. *$P_{i_*}^{(n)}$ can have three kinds of vertices, distinguished by the corresponding x-vectors that belong to one of these types:*

(a) 0-1 vertices: $x_h = 0$, $h \in Q_{i_*}$ and $x_h = 1$, $h \in Q \setminus Q_{i_*}$.

(b) *non-binary integer vertices:* $x_h \in \mathbb{Z}_-$, $h \in Q_{i_*}$, $x_h \in \mathbb{Z}_+$, $h \in Q \setminus Q_{i_*}$ *(here* $\mathbb{Z}_-$ *and* $\mathbb{Z}_+$ *stand for the nonpositive and nonnegative integers respectively).*

(c) *fractional vertices:* $x_h \leq 0$, $h \in Q_{i_*}$, $x_h \geq 1$, $h \in Q \setminus Q_{i_*}$, *with at least one inequality strict.*

Proof. The three cases become exhaustive if the following fourth one is added: (d) fractional vertices with $0 < x_h < 1$ for some $h \in Q$. But this case clearly violates at least one of the constraints defining $P_{i_*}^{(n)}$. □

Note that $P_{i_*}^{(n)}$ can have several distinct vertices with the same associated x-vector, corresponding to basic solutions with the same x-component. Note also that if a component x_h of a vertex is fractional, then $x_h < 0$ or $x_h > 1$.

The next theorem characterizes the facets of the simple disjunctive hull.

Theorem 2.8. *The inequality* $\bar{\alpha}s \geq 1$ *defines a facet of* $\operatorname{conv} P_D^{(n)}$ *if and only if there exists an objective function of the linear program (CGLP)*$_Q$ *of Proposition 2.2 with* $p > 0$ *such that all optimal solutions* (α, u) *have* $\alpha = \bar{\alpha}$.

Proof outline. This is a special case of Theorem 4.6 of [3]. The inequality $\bar{\alpha}x \geq 1$ defines a facet of $\operatorname{conv} P_D^{(n)}$ if and only if $\bar{\alpha}$ is a vertex of the polar of $\operatorname{conv} P_D^{(n)}$, which is the projection of (2.8) onto the α-space. But $\bar{\alpha}$ is a vertex of this polar if and only if there exists an objective function vector $p > 0$ such that $p\alpha$ attains its unique minimum at $\bar{\alpha}$. □

If the system (2.4) defining P_L is of full row rank q, then the dimension of $\operatorname{conv} P_D$ is n, since there are $q + n$ variables and q independent equations. The dimension of $\operatorname{conv} P_D^{(n)}$ is also n, so the facets of $\operatorname{conv} P_D^{(n)}$ are of dimension $n - 1$.

From a computational standpoint, the most important feature of (CGLP)$_Q$ is that the facets of the n-dimensional $\operatorname{conv} P_D^{(n)}$ can be generated by solving a smaller CGLP in a subspace of at most $t = 2^q$ variables s_j, and lifting the resulting inequality into the full space. The idea of generating cuts in a subspace of the original higher dimensional cut generating linear program and then lifting them to the full space goes back to [7, 10], where lift-and-project cuts were generated from a 2-term disjunction by working in the subspace of the fractional variables of the LP solution. Here we are working with a 2^q-term disjunction, and are considering a different subspace, suggested by the structure of the problem at hand, but the lifting procedure is essentially the same as the one used in [7, 8].

Since our cuts are derived from a disjunction with 2^q terms, if we want to create a subproblem in which all terms are represented, we need 2^q out of the n variables α_j of our (CGLP)$_Q$. Furthermore, the 2^q vectors r^j corresponding to these α_j have to span the subspace $\mathbb{R}^q$ of the x-variables. Solving the (CGLP)$_Q$ in this subspace yields 2^q values α_j and $q \times 2^q$ associated multipliers u_{ik}, $i = 1, \ldots, 2^q$, $k = 1, \ldots, q$; and these multipliers can then be used to compute the remaining components of α, given by the expressions (2.11) and (2.12). The significance of this is that the computational cost of generating facets of $\operatorname{conv} P_D$ grows only linearly with n. Of course this cost grows exponentially with q, but the approach discussed here is being considered for small q.

The choice of the subspace is intimately related to the question of deciding which facets of the disjunctive hull are also facets of the integer hull. The best way to address this question and that of the subspace to be chosen, is to first interpret the inequalities defining the disjunctive hull as intersection cuts.

3 Geometric interpretation: Cuts from the q-dimensional parametric cross-polytope

Consider the q-dimensional unit cube centered at $(0,\ldots,0)$, $K_q := \{x \in \mathbb{R}^q : -\frac{1}{2} \le x_j \le \frac{1}{2},\ j \in Q\}$. Its polar, $K_q^o := \{x \in \mathbb{R}^q : xy \le 1,\ \forall x \in K\}$, is known to be the q-dimensional octahedron or cross-polytope; which, when scaled so as to circumscribe the unit cube, is the outer polar of K_q:

$$K_q^* = \{x \in \mathbb{R}^q : |x| \le \tfrac{1}{2}q\},$$

where $|x| = \sum(|x_j| : j = 1,\ldots,q\}$. Equivalently, $|x| \le \frac{1}{2}q$ can be written as the system

$$\begin{array}{rcccccl} -x_1 & - & \cdots & - & x_q & \le & \frac{1}{2}q \\ x_1 & - & \cdots & - & x_q & \le & \frac{1}{2}q \\ & & & & \vdots & & \\ x_1 & + & \cdots & + & x_q & \le & \frac{1}{2}q \end{array} \tag{3.1}$$

of $t = 2^q$ inequalities in q variables.

Moving the center of the coordinate system to $(\frac{1}{2},\cdots,\frac{1}{2})$ changes the righthand side coefficient of the i-th inequality in (3.1) from $\frac{1}{2}q$ to a value equal to the sum of positive coefficients on the lefthand side of the inequality. Indeed, if q^+ and q^- denotes the number of positive and negative coefficients, then $\frac{1}{2}q + \frac{1}{2}q^+ - \frac{1}{2}q^- = q^+$.

Next we introduce the parameters v_{ik}, $i = 1,\ldots,t = 2^q$, $k = 1,\ldots,q$, to obtain the system

$$\begin{array}{rcccccl} -v_{11}x_1 & - & \cdots & - & v_{1q}x_q & \le & 0 \\ v_{21}x_1 & - & \cdots & - & v_{2q}x_q & \le & v_{21} \\ -v_{31}x_1 & + & \cdots & - & v_{3q}x_q & \le & v_{31} \\ \vdots & & & & & \vdots & \\ v_{t1}x_1 & + & \cdots & + & v_{tq}x_q & \le & v_{t1} + \ldots + v_{tq} \end{array} \tag{3.2}$$
$$v_{ik} \ge 0,\ i = 1,\ldots,t = 2^q,\ k = 1,\ldots,q.$$

Note that the constraints of (3.2) are of the form

$$\sum_{k \in \widetilde{Q}_i^+} v_{ik}x_k - \sum_{k \in \widetilde{Q}_i^-} v_{ik}x_k \le \sum_{k \in \widetilde{Q}_i^+} v_{ik},$$

where $\widetilde{Q}_i^+$ and $\widetilde{Q}_i^-$ are the sets of indices for which the coefficient of x_k is $+v_{ik}$ and $-v_{ik}$, respectively. Note also that all inequalities that have the same number of coefficients with the plus sign have the same righthand side, equal to the sum of these coefficients.

The system (3.2) is homogeneous in the parameters v_{ik}, so every one of its inequalities can be normalized. Since we are looking for a connection with the system (2.8) defining $(\mathrm{CGLP})_Q$, we will use the normalization given by this system and Proposition 2.4, i.e.

$$\begin{array}{rcccrcl} f_1 v_{11} & + & \cdots & + & f_q v_{1q} & = & 1 \\ (1-f_1) v_{21} & + & \cdots & + & f_q v_{2q} & = & 1 \\ \cdots & & & & \cdots & & \\ (1-f_1) v_{t1} & + & \cdots & + & (1-f_q) v_{tq} & = & 1 \end{array} \tag{3.3}$$

Note that these normalization constraints are of the general form

$$\sum_{h \in \widetilde{Q}_i^+} (1-f_k) v_{ik} + \sum_{h \in \widetilde{Q}_i^-} f_k v_{ik} = 1.$$

Let $\widetilde{K}^*(v)$ denote the parametric cross-polytope defined by (3.2) and (3.3). It is not hard to see that for any fixed set of v_{ik}, (3.2) defines a convex polyhedron in x-space that contains in its boundary all $x \in \mathbb{R}^q$ such that $x_k \in \{0,1\}$, $k \in Q$, hence is suitable for generating intersection cuts. Furthermore, letting $\widetilde{K}^{*(n)}(v)$ be the expression for $\widetilde{K}^*(v)$ in the space of the s-variables, obtained by substituting $f + Rs$ for x into (3.2), we have

Theorem 3.1. *For any values of the parameters v_{ik} satisfying (3.2) and (3.3), the intersection cut $\widetilde{\alpha} s \geq 1$ from $\widetilde{K}^{*(n)}(v)$ has coefficients $\widetilde{\alpha}_j = \frac{1}{s_j^*}$, where*

$$s_j^* = \max\{s_j : f + r^j s_j \in K^{*(n)}(v)\}. \tag{3.4}$$

Proof. This is a special case of Theorem 1.1. □

In order to compare the intersection cut $\widetilde{\alpha} s \geq 1$ with the cut $\alpha s \geq 1$ from the q-term disjunction (2.7), we have to restate (3.4) in terms of the system of inequalities defining $\widetilde{K}^{*(n)}(v)$. This means that $f + r^j s_j^*$ has to be expressed as the intersection point of the ray $f + r^j s_j$, $s_j \geq 0$, with the first facet of $K^{*(n)}(v)$ encountered. This yields

$$s_j^* = \min\{s_j^1, \ldots, s_j^t\}, \tag{3.5}$$

where the s_j^i are obtained by substituting $f_k + \sum_{h=1}^n r_j^k s_h$ for x_k, $k = 1, \ldots, q$ into the i-th inequality of (3.2), and setting $s_h = 0$ for all $h \neq j$:

$$s_j^i = \max\left\{ s_j : \left(\sum_{k \in \widetilde{Q}_i^+} v_{ik} r_j^k - \sum_{k \in \widetilde{Q}_i^-} v_{ik} r_j^k \right) s_j \leq \sum_{k \in \widetilde{Q}_i^+} v_{ik}(1-f_k) + \sum_{k \in \widetilde{Q}_i^-} v_{ik} f_k \right\},$$

$i = 1, \ldots, t = 2^q$.

Clearly, this maximum is bounded whenever the coefficient of s_j is positive, in which case, if we normalize by setting $\sum_{k \in \widetilde{Q}_i^+} v_{ik}(1-f_k) + \sum_{k \in \widetilde{Q}_i^-} v_{ik} f_k = 1$, we obtain

$$s_j^i = \left(\sum_{k \in \widetilde{Q}_i^+} v_{ik} r_j^k - \sum_{k \in \widetilde{Q}_i^-} v_{ik} r_j^k \right)^{-1}. \tag{3.6}$$

Comparing (3.5) and (3.6) to the expressions (2.11) and (2.12) for the coefficient α_j of the lift-and-project cut $\alpha s \geq 1$ of Proposition 2.2, we find that setting $v_{ik} = u_{ik}$ for all i, k, as well as $\widetilde{Q}_i^+ = Q_i^-$ and $\widetilde{Q}_i^- = Q_i^+$, we obtain $\widetilde{\alpha}_j = \alpha_j$.

This proves

Corollary 3.2. *The intersection cut $\widetilde{\alpha} s \geq 1$ from the parametric octahedron $\widetilde{K}^{*(n)}(v)$ is the same as the lift-and-project cut $\alpha s \geq 1$ corresponding to the (CGLP)$_Q$ solution (α, u), with $v_{ik} = u_{ik}$, $i = 1, \ldots, t$, $k = 1, \ldots, q$.*

4 Facets of the disjunctive hull and the integer hull

Consider again the disjunctive relaxation of P_I

$$P_D = \{(x, s) \in \mathbb{R}^q \times \mathbb{R}^n : x = f + Rs,\ s \geq 0,\ x_i \leq 0\ \vee\ x_i \geq 1,\ i \in Q\}$$

introduced at the beginning of section 2, where $x, f \in \mathbb{R}^q$, $R \in \mathbb{R}^{q \times n}$, and $Q := \{1, \ldots, q\}$. For $i = 1, \ldots, t = 2^q$, let p^i be the vertex of K_q, the q-dimensional unit cube, defined by $p_k^i = 0$, $i \in Q_i^+$, $p_k^i = 1$, $i \in Q_i^-$.

Next we give a sufficient condition for an inequality $\alpha s \geq 1$ valid for P_D to define a facet of $\operatorname{conv} P_I$, which for small q leads to an efficient procedure for generating inequalities that are facet defining for $\operatorname{conv} P_I$.

The dimension of $P_I^{(n)}$ being $n \geq 2^q$, $\alpha s \geq 1$ defines a facet of $\operatorname{conv} P_I^{(n)}$ if there exists a subspace $\mathbb{R}^{2^q}$ of $\mathbb{R}^n$ such that the restriction of $\alpha s \geq 1$ to this subspace defines a facet of $\operatorname{conv} P_I^{(2^q)}$. If this is the case, then the inequality in question can be lifted to the full space to yield a facet of $\operatorname{conv} P_I^{(n)}$ by using the u-components of the solution (α, u) to the CGLP in the subspace to compute the missing coefficients α_j.

Theorem 4.1. *Let $\alpha s \geq 1$ be a valid inequality for P_D corresponding to a basic solution (α, u) of (CGLP)$_Q$, and let p^i, $i = 1, \ldots, 2^q$, be the vertices of K_q. Suppose for each p^i, $i = 1, \ldots, 2^q$, there exists a subset $J_i \subset J$ containing the indices of q linearly independent rays $r^{j_1}, \ldots, r^{j_q}$, and a vector $\lambda \in \mathbb{R}_+^q$, satisfying*

$$p^i - f = \sum_{j=j_1}^{j_q} \tfrac{1}{\alpha_j} r^j \lambda_j, \quad \sum_{j=j_1}^{j_q} \lambda_j = 1. \tag{4.1}$$

Then the inequality $\sum_{j \in J} \alpha_j s_j \geq 1$ defines a facet of $\operatorname{conv} P_I^{(|J|)}$, and its lifting based on the u-components of the solution (α, u) defines a facet of $\operatorname{conv} P_I^{(n)}$.

Proof. Suppose the subset of 2^q rays indexed by J satisfies the requirements of the Theorem. Then for every $i = 1, \ldots, 2^q$, the vertex p^i of K^q satisfies

$$p^i = \sum_{j=j_1}^{j_q} (f - \tfrac{1}{\alpha_j} r^j) \lambda_j, \quad \sum_{j=j_1}^{j_q} \lambda_j = 1$$

for some $\lambda_j \geq 0$, $j = j_1, \ldots, j_q$, i.e. p^i can be expressed as a convex combination of the q points $f + \frac{1}{\alpha_j} r^j$, $j = j_1, \ldots, j_q$. But $f + \frac{1}{\alpha_j} r^j = f + r^j s_j^*$ is the intersection point of the ray $f + r^j s_j$ with $\operatorname{bd} \widetilde{K}_q^*$, hence each of these points satisfies $\alpha s = 1$ and consequently so does p^i. Since $\sum\limits_{j \in J} \alpha_j s_j \geq 1$ is satisfied at equality by 2^q integer points of $\operatorname{conv} P_I^{(|J|)}$, it defines a facet of the latter. Furthermore, lifting the remaining coefficients α_j of the inequality by using the u-components of (α, u) yields a facet defining inequality for $\operatorname{conv} P_I^{(n)}$. □

The sufficient condition of Theorem 4.1 is not necessary. There are two kinds of situations not satisfying the above condition, in which a valid inequality $\alpha s \geq 1$ for P_D may define a facet of $\operatorname{conv} P_I$. The first one involves an inequality $\alpha s \geq 1$ such that although (4.1) is not satisfied for all 2^q vertices p^i of K^q, nevertheless $\operatorname{conv} P_D$ has 2^q vertices whose x-components p^i satisfy (4.1), i.e. $\operatorname{conv} P_D$ has multiple vertices with the same x-component. The second situation involves facet defining split cuts.

5 The two-row case

We now restrict our attention to the case $q = 2$, i.e. we consider two rows from a simplex tableau of a MIP problem with the variables x_1, x_2 and $s_j, j \in J$:

$$
\begin{aligned}
P_L = \{(x,s) \in \mathbb{R}^{2+|J|} : \quad & x_1 \;\; = f_1 + \textstyle\sum_{j \in J} r_j^1 s_j \\
& x_2 \;\; = f_2 + \textstyle\sum_{j \in J} r_j^2 s_j \\
& s_j \geq 0 \quad j \in J \;\}.
\end{aligned}
\tag{5.1}
$$

where x_1, x_2 are basic variables required to be integers and $s_j, j \in J$ are non-basic. This is the case studied by Anderson, Louveaux, Weismantel and Wolsey [1]. Let $P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^{|J|} : (x,s) \in P_L\}$, and $0 < f_1, f_2 < 1$. The column vectors $r_j, j \in J$, represent the extreme rays of the cone in $\mathbb{R}^{|J|}$ with apex at (f_1, f_2).

We will say that a ray r_j in (5.1) hits an orthant-cone $Q_i, i \in \{1, \ldots, 4\}$ if there exists $\lambda_0 > 0$ such that $f + \lambda r_j \in Q_i$ for all $\lambda \geq \lambda_0$.

For the case of 2 rows the disjunction (2.7) becomes

$$
\begin{pmatrix} -r^1 s \geq f_1 \\ -r^2 s \geq f_2 \end{pmatrix} \vee \begin{pmatrix} r^1 s \geq 1 - f_1 \\ -r^2 s \geq f_2 \end{pmatrix} \vee \begin{pmatrix} r^1 s \geq 1 - f_1 \\ r^2 s \geq 1 - f_2 \end{pmatrix} \vee \begin{pmatrix} -r^1 s \geq f_1 \\ r^2 s \geq 1 - f_2 \end{pmatrix}
\tag{5.2}
$$

with $s \geq 0$, and the system (2.8) of Theorem 2.1 becomes

$$
\begin{array}{llll}
\alpha & +r^1 v_1 & +r^2 w_1 & \geq 0 \\
\alpha & -r^1 v_2 & +r^2 w_2 & \geq 0 \\
\alpha & -r^1 v_3 & -r^2 w_3 & \geq 0 \\
\alpha & +r^1 v_4 & -r^2 w_4 & \geq 0 \\
 & +f_1 v_1 & +f_2 w_1 & = 1 \\
 & +(1-f_1) v_2 & +f_2 w_2 & = 1 \\
 & +(1-f_1) v_3 & +(1-f_2) w_3 & = 1 \\
 & +f_1 v_4 & +(1-f_2) w_4 & = 1 \\
\multicolumn{4}{l}{v_i, w_i \geq 0 \quad i \in \{1 \ldots 4\}.}
\end{array}
\tag{5.3}
$$

where v_i, w_i, $i = 1, \ldots, 4$ stand for u_{i1}, u_{i2}, $i = 1, \ldots, t = 2^q$ (since $q = 2$, $t = 2^q = 4$).

By Proposition 2.2 the cuts generated by the CGLP with constraint set (5.3) and objective function $\min p\alpha$ for some $p > 0$ have the form $\alpha s \geq 1$, where

$$\alpha_j = \max\{\alpha_j^1, \alpha_j^2, \alpha_j^3, \alpha_j^4\}$$

with

$$\begin{array}{lll} \alpha_j^1 & = -r_j^1 v_1 & -r_j^2 w_1 \\ \alpha_j^2 & = +r_j^1 v_2 & -r_j^2 w_2 \\ \alpha_j^3 & = +r_j^1 v_3 & +r_j^2 w_3 \\ \alpha_j^4 & = -r_j^1 v_4 & +r_j^2 w_4. \end{array} \tag{5.4}$$

As discussed in Section 3, a cut produced by the CGLP can be viewed as an intersection cut derived from a parametric cross-polytope or octahedron. For given v, w, we call the polyhedron

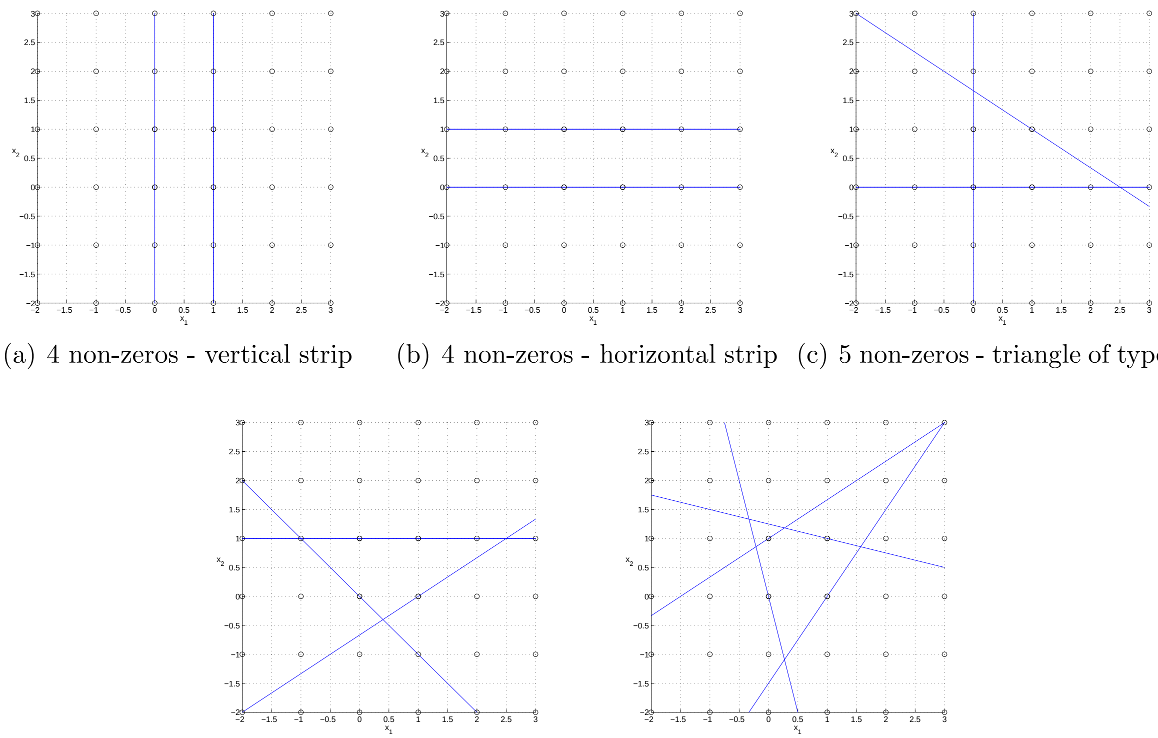
$$P_{\mathrm{octa}}(v,w) = \{(x_1,x_2) \in \mathbb{R}^2 : \begin{array}{ll} -v_1x_1 - w_1x_2 \leq 0 \ ; & \\ +v_2x_1 - w_2x_2 \leq v_2 \ ; & \\ +v_3x_1 + w_3x_2 \leq v_3 + w_3 \ ; & \\ -v_4x_1 + w_4x_2 \leq w_4 & \} \end{array}$$

the (v,w)-*parametric octahedron.*

If $v_i = 0$ or $w_i = 0$ for some $i \in \{1,\ldots,4\}$ the i-th facet of P_{octa} is parallel to one of the coordinate axes. If $v_i, w_i > 0$ then the i-th facet of P_{octa} is *tilted* (note that since we use the normalization $\beta = 1$, v_i and w_i cannot both be 0). Varying the parameters v, w, the (v,w)-parametric octahedron produces different configurations according to the non-zero components of v, w. Depending on the values taken by the parameters, $P_{\mathrm{octa}}(v,w)$ may be a quadrilateral (i.e. a full-fledged octahedron in $\mathbb{R}^2$), a triangle, or an infinite strip. In the rest of the section we refer to these configurations using the short reference indicated in parenthesis. It can easily be verified that the value-configurations of the parameters v_i, w_i which give rise to maximal convex sets are the following:

- (S) If exactly 4 components of (v,w) are positive, P_{octa} is the vertical strip $\{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ if $v_i > 0, i = 1,\ldots,4$; or the horizontal strip $\{x \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\}$ if $w_i > 0, i = 1,\ldots,4$ (see figure 3(a), 3(b)).
- (T_A) If exactly 5 components of (v,w) are positive, P_{octa} is a triangle with 1 tilted face (type A) (by "tilted" we mean a face that is not parallel to any of the two axes). Figure 3(c) illustrates the case with $v_1, w_2, v_3, w_3, v_4 > 0; w_1, v_2, w_4 = 0$. When in addition $v_i = w_i$ for some $i \in \{1,\ldots,4\}$ P_{octa} becomes a triangle with vertices $(0,0); (2,0); (0,2)$ or one of the other three configurations symmetric to this one. This corresponds to what is called a triangle of type 1 in [19]. In the general case T_A corresponds to a triangle of type 2 in [19].
- (T_B) If exactly 6 components of (v,w) are positive, P_{octa} is a triangle with 2 tilted faces (type B). Figure 3(d) illustrates the case with $v_1, w_1, v_2, w_2, w_3, w_4 > 0; v_3, v_4 = 0$. This configuration corresponds to a triangle of type 2 in [19].
- (Q) If all 8 components of (v,w) are positive, P_{octa} is a quadrilateral. See Figure 3(e).

The case with 7 components of (v, w) positive does not correspond to a maximal parametric octahedron, therefore we do not need to consider it. Suppose all the components are positive except for v_1 which is 0. The facet of P_{octa} corresponding to $(0, 0)$ is horizontal and goes through the point $(1, 0)$. Is not hard to see that setting $v_2 = 0$ we enlarge the set defined by the parametric octahedron.



(a) 4 non-zeros - vertical strip (b) 4 non-zeros - horizontal strip (c) 5 non-zeros - triangle of type A

(d) 6 non-zeros - triangle of type B (e) 8 non-zeros - quadrilateral

Figure 3: Configurations of the parametric octahedron for the MIP case

For a cut $\sum_{j \in J} \alpha_j s_j \geq 1$ Andersen et al. [1] introduce the set

$$L_\alpha = \left\{ x \in \mathbb{R}^2 : (x, s) \in P_L \wedge \sum_{j \in J} \alpha_j s_j \leq 1 \right\}. \tag{5.5}$$

Clearly, $L_\alpha \subseteq P_{\text{octa}}(v, w)$, and the inclusion is often strict.
Example.

In [1], Andersen et al. considered the two rows instance

$$\begin{aligned} x_1 &= \tfrac{1}{4} \quad +2s_1 \quad +1s_2 \quad -3s_3 \quad \quad \quad \quad +1s_5 \\ x_2 &= \tfrac{1}{2} \quad +1s_1 \quad +1s_2 \quad +2s_3 \quad -1s_4 \quad -2s_5, \\ & x_1, x_2 \in \mathbb{Z}, \quad s \geq 0 \end{aligned} \tag{5.6}$$

We present the complete description of the disjunctive hull for (5.6). In order to do so we generated the CGLP of (5.6) using the normalization constraint $\beta = 1$ and we considered all feasible bases. The CGLP produces 5 different facets. For each of these we show the configuration of the parametric octahedron that yields the corresponding cut in terms of the v, w variables:

1. Cut (T_B): $2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1$
   $v_1 = 2;\ v_2 = \frac{8}{7};\ v_3 = 0;\ v_4 = 0$
   $w_1 = 1;\ w_2 = \frac{2}{7};\ w_3 = 2;\ w_4 = 2$
2. Cut (T_B): $\frac{8}{3}s_1 + \frac{4}{3}s_2 + \frac{44}{9}s_3 + \frac{8}{9}s_4 + \frac{4}{3}s_5 \geq 1$
   $v_1 = \frac{20}{9};\ v_2 = \frac{4}{3};\ v_3 = \frac{4}{3};\ v_4 = \frac{4}{9}$
   $w_1 = \frac{8}{9};\ w_2 = 0;\ w_3 = 0;\ w_4 = \frac{16}{9}$
3. Cut (T_A): $\frac{8}{3}s_1 + 2s_2 + 4s_3 + s_4 + \frac{4}{3}s_5 \geq 1$
   $v_1 = 2;\ v_2 = \frac{4}{3};\ v_3 = 0;\ v_4 = 0$
   $w_1 = 1;\ w_2 = 0;\ w_3 = 2;\ w_4 = 2$
4. Cut (S): $\frac{8}{3}s_1 + \frac{4}{3}s_2 + 12s_3 + \frac{4}{3}s_5 \geq 1$
   $v_1 = 4;\ v_2 = \frac{4}{3};\ v_3 = \frac{4}{3};\ v_4 = 4$
   $w_1 = 0;\ w_2 = 0;\ w_3 = 0;\ w_4 = 0$
5. Cut (T_B): $2s_1 + 2s_2 + \frac{68}{7}s_3 + \frac{2}{7}s_4 + \frac{12}{7}s_5 \geq 1$
   $v_1 = \frac{24}{7};\ v_2 = \frac{8}{7};\ v_3 = 0;\ v_4 = 0$
   $w_1 = \frac{2}{7};\ w_2 = \frac{2}{7};\ w_3 = 2;\ w_4 = 2$

Of the 5 facets of P_D, 3 are facets for P_I: cuts 1, 2 and 4. Note that cut 4 is a split cut and can be derived using only the tableau row corresponding to the variable x_2. Cut 3 and 5 are facets of P_D by Theorem 2.8.

The condition given in Theorem 4.1 for an inequality $\alpha x \geq 1$, facet defining for the disjunctive hull, to also define a facet of the integer hull specializes for the case $q = 2$ to the following. For each of the four vertices p^i of K, p^i must lie on the line segment between two intersection points of rays r^j with the boundary of P_{octa}. As discussed in Section 3, the inequalities $\alpha x \geq 1$ can be generated in a subspace of $\leq 2^q = 4$ variables, and then lifted into the full space by using the multipliers (v_i, w_i), $i = 1, \ldots, 4$. In [25] two algorithms were implemented for generating facets of the integer hull from P_{octa}, one for the case of a quadrilateral, the other for the case of triangles, both of them linear in $|J|$, the number of rays.

Recently Dash et al. [18] have generalized the approach of [5, 6], by considering more general 4-term disjunctions that give rise to what they call cross cuts and crooked cross cuts. They relate the closures of their cuts with the split closure and show, among others, that any 2 dimensional lattice free cut can be obtained as a crooked cross cut.

6 Cut Strengthening

Given a facet $\alpha s \geq 1$ of the disjunctive hull, if some non-basic variable s_j is required to be integral in the original problem formulation, then the cut can be strengthened. Let J_1 be the index set of the integer-constrained variables s_j, and let $J_2 = J \setminus J_1$.

Lemma 6.1. *If the disjunction*

$$\begin{pmatrix} -r^1 s \geq f_1 \\ -r^2 s \geq f_2 \end{pmatrix} \vee \begin{pmatrix} r^1 s \geq 1 - f_1 \\ -r^2 s \geq f_2 \end{pmatrix} \vee \begin{pmatrix} r^1 s \geq 1 - f_1 \\ r^2 s \geq 1 - f_2 \end{pmatrix} \vee \begin{pmatrix} -r^1 s \geq f_1 \\ r^2 s \geq 1 - f_2 \end{pmatrix} \quad (6.1)$$

where $s \geq 0$ and $s_j \in \mathbb{Z}$, $j \in J_1 \subseteq J$, is valid for P_I, then so is the disjunction obtained from (6.1) by replacing some or all r^i_j, $i = 1, 2$, $j \in J_1$, with $r^i_j - m^i_j$, for any $m^i_j \in \mathbb{Z}$, $i = 1, 2$, $j \in J_1$.

Proof. Suppose there exists $i_* \in \{1, 2\}$ and $j_* \in J_1$ such that replacing $r^{i_*}_{j_*}$ with $r^{i_*}_{j_*} - \bar{m}^{i_*}_{j_*}$, where $\bar{m}^{i_*}_{j_*} \in \mathbb{Z}$, violates (6.1). Then there exists a solution $(x, s) \in P_I$ with $x \in \mathbb{Z}^2$ such that

$$(-(r^{i_*}_{j_*} - \bar{m}^{i_*}_{j_*})s_{j_*} - \sum_{j \in J \setminus \{j_*\}} r^{i_*}_j s_j < f_{i_*}) \quad \wedge \quad ((r^{i_*}_{j_*} - \bar{m}^{i_*}_{j_*})s_{j_*} + \sum_{j \in J \setminus \{j_*\}} r^{i_*}_j s_j < 1 - f_{i_*})$$

holds. Rewriting this expression so as to bring together the terms in $\bar{m}^{i_*}_{j_*}$ we get

$$\sum_{j \in J} r^{i_*}_j s_j + f_{i_*} - 1 < \bar{m}^{i_*}_{j_*} s_{j_*} < \sum_{j \in J} r^{i_*}_j s_j + f_{i_*}$$

or

$$-1 < \bar{m}^{i_*}_{j_*} < 0$$

contrary to the fact that both $\bar{m}^{i_*}_{j_*}$ and s_{j_*} are integer. $\square$

Theorem 6.2. *Given $(\bar{v}, \bar{w}) \geq 0$ defining a parametric octahedron, the cut $\alpha s \geq 1$ can be strengthened to $\bar{\alpha} s \geq 1$ with coefficients $\bar{\alpha}_j, j \in J_1$ given by the 3-variable mixed integer program*

$$\begin{array}{llcl} \min & \alpha_j & & \\ & \alpha_j \quad -\bar{v}_1 m^1_j - \bar{w}_1 m^2_j & \geq & -r^1_j \bar{v}_1 - r^2_j \bar{w}_1 \\ & \alpha_j \quad +\bar{v}_2 m^1_j - \bar{w}_2 m^2_j & \geq & +r^1_j \bar{v}_2 - r^2_j \bar{w}_2 \\ & \alpha_j \quad +\bar{v}_3 m^1_j + \bar{w}_3 m^2_j & \geq & +r^1_j \bar{v}_3 + r^2_j \bar{w}_3 \\ & \alpha_j \quad -\bar{v}_4 m^1_j + \bar{w}_4 m^2_j & \geq & -r^1_j \bar{v}_4 + r^2_j \bar{w}_4 \\ & m^1_j, m^2_j \in \mathbb{Z}. & & \end{array} \tag{6.2}$$

The coefficients for $j \in J_2$ remain unchanged at $\bar{\alpha}_j = \alpha_j$ as in Proposition 2.2.

Proof. Validity of $\bar{\alpha} s \geq 1$ follows from Lemma 6.1. $\square$

Theorem 6.3. *The mixed integer program (6.2) has an optimal solution $(\bar{\alpha}_j, \bar{m}^1_j, \bar{m}^2_j)$ satisfying $\bar{m}^i_j \in \{\lfloor \bar{r}^i_j \rfloor, \lceil \bar{r}^i_j \rceil\}$, $i = 1, 2$.*

Proof. Let $(\widetilde{\alpha}_j, \widetilde{m}^i_j, \widetilde{m}^2_j)$ be an optimal solution to the problem obtained from (6.2) by adding the constraint $m^i_j \in \{\lfloor \bar{r}^i_j \rfloor, \lceil \bar{r}^i_j \rceil\}$. We will show that this solution cannot be improved by replacing $\widetilde{m}^1_j, \widetilde{m}^2_j$ with any other pair of integers.

Consider the linear programming relaxation of (6.2), which asks for minimizing the maximum of four linear functions. This is a piece-wise linear convex programming problem whose minimum is attained for $m^i_j = r^i_j$, $i = 1, 2$, yielding $\alpha_j = \alpha^1_j = \ldots\ldots = \alpha^4_j = 0$. From the convexity of the objective function $\alpha(m^1_j, m^2_j)$ it follows that the integer optimum occurs at one of the points $(m^1_j, m^2_j) \in \{(\lfloor r^1_j \rfloor, \lfloor r^2_j \rfloor),\ (\lfloor r^1_j \rfloor, \lceil r^2_j \rceil),$

$(\lceil r_j^1\rceil, \lfloor r_j^2\rfloor)$, $(\lceil r_j^1\rceil, \lceil r_j^2\rceil)\}$. For suppose the optimum were to occur at some other point, say $(\widehat{m}_j^1, \widehat{m}_j^2)$, where $\widehat{m}_j^1 = \lceil r_j^1\rceil$ and $\widehat{m}_j^2 = \lceil r_j^2\rceil + d_j$ for some $d_j > 0$. Then

$$
\begin{aligned}
\alpha(\lceil r_j^1\rceil, \lceil r_j^2\rceil + d_j) &< \alpha(\lceil r_j^1\rceil, \lceil r_j^2\rceil), \\
\alpha(\lceil r_j^1\rceil, r_j^2) &< \alpha(\lceil r_j^1\rceil, \lceil r_j^2\rceil),
\end{aligned}
$$

hence

$$\alpha(\lceil r_j^1\rceil, \lceil r_j^2\rceil) > \lambda\alpha(\lceil r_j^1\rceil, r_j^2) + (1-\lambda)\alpha(\lceil r_j^1\rceil, \lceil r_j^2\rceil + d_j) \text{ for } 0 \le \lambda \le 1,$$

i.e. the value of the minimum at a point which lies on the line between $(\lceil r_j^1\rceil, r_j^2)$ and $(\lceil r_j^1\rceil, \lceil r_j^2\rceil + d_j)$ is larger than a convex combination of the values of the minimum at the endpoints of the line, contrary to the assumption that $\alpha(m_j^1, m_j^2)$ is a convex function. □

The operation of replacing r_j^i by $r_j^i - m_j^i$ for some $m_j^i \in \mathbb{Z}$, $i = 1, 2$, in the expression for α, is called the modularization of r_j^i, or more generally, the modularization of the cut $\alpha x \ge 1$. Using $m_j^i \in \{\lfloor r_j^i\rfloor, \lceil r_j^i\rceil\}$ is called the standard modularization. It can be shown (see below) that the mixed integer program (6.2) attains its optimum for a standard modularization.

Lemma 6.4. *There exists a standard modularization $\bar{r}$ of the ray r such that*

$$0 \le f_i + \bar{r}^i \le 1, \qquad i \in \{1, 2\} \tag{6.3}$$

i.e. the point $(f + \bar{r})$ belongs to K.

Proof. If $f_i + r^i - \lfloor r^i\rfloor \le 1$ then let $m^i = \lfloor r^i\rfloor$. Note that the condition $f_i + r^i - \lfloor r^i\rfloor \ge 0$ follows since $0 \le f_i \le 1$ and $r^i - \lfloor r^i\rfloor \ge 0$. Otherwise $(f_i + r^i - \lfloor r^i\rfloor > 1)$ let $m^i = \lceil r^i\rceil$ and from $f_i \le 1$ and $r^i - \lfloor r^i\rfloor \le 1$ we get $0 \le f_i + r^i - \lfloor r^i\rfloor - 1 = f_i + r^i - \lceil r^i\rceil \le 1$.

For $k = 1, \ldots, 4$, let $\bar{\alpha}_j^k$ be obtained from α_j^k of (5.4) by substituting $\bar{r}_j^i$ for r_j^i, $i = 1, 2$. One can show that each $\bar{\alpha}_j^k$ is the convex combination of one of the expressions $\frac{-\bar{r}_j^1}{f_1}$ or $\frac{\bar{r}_j^1}{1-f_1}$ with one of the expressions $\frac{-\bar{r}_j^2}{f_2}$ or $\frac{\bar{r}_j^2}{1-f_2}$. To be specific, we have

Lemma 6.5.

$$
\begin{aligned}
\bar{\alpha}_j^1 &= \lambda_1 \frac{-\bar{r}_j^1}{f_1} + (1-\lambda_1)\frac{-\bar{r}_j^2}{f_2} \quad , \qquad \textit{with } \lambda_1 = \bar{v}_1 f_1 \\
\bar{\alpha}_j^2 &= \lambda_2 \frac{\bar{r}_j^1}{1-f_1} + (1-\lambda_2)\frac{-\bar{r}_j^2}{f_2} \quad , \qquad \textit{with } \lambda_2 = \bar{v}_2(1-f_1) \\
\bar{\alpha}_j^3 &= \lambda_3 \frac{\bar{r}_j^1}{1-f_1} + (1-\lambda_3)\frac{\bar{r}_j^2}{1-f_2} \quad , \qquad \textit{with } \lambda_3 = \bar{v}_3(1-f_1) \\
\bar{\alpha}_j^4 &= \lambda_4 \frac{-\bar{r}_j^1}{f_1} + (1-\lambda_4)\frac{\bar{r}_j^2}{1-f_2} \quad , \qquad \textit{with } \lambda_4 = \bar{v}_4 f_1
\end{aligned}
$$

Proof. By substituting for the λ_k, $k = 1, \ldots, 4$, we get the corresponding expressions for $\bar{\alpha}_j^k$. □

Theorem 6.6. *The strengthened cut $\bar{\alpha}s \ge 1$ satisfies $0 \le \bar{\alpha}_j \le 1$, $j \in J_1$.*

Proof. Since $\bar{v}_k, \bar{w}_k \geq 0$ for all k, we have $\bar{\alpha}_j^k \geq 0$ for at least one of the four k, hence $\bar{\alpha}_j \geq 0$. Let $(\bar{\alpha}, \bar{m}^1, \bar{m}^2)$ be an optimal solution to (6.2). Let $\bar{r}^i = r^i - \bar{m}^i$, $i = 1, 2$, where $\bar{m}^i \in \{\lfloor r^i \rfloor, \lceil r^i \rceil\}$ $i = 1, 2$. There are four cases:
Case 1. $\bar{m}^i = \lfloor r^i \rfloor, i = 1, 2$. Then $\bar{r}^i = r^i - \bar{m}^i \geq 0$, $i = 1, 2$, and

$\bar{\alpha}^1 = -\bar{r}^1\bar{v}_1 - \bar{r}^2\bar{w}_1 \leq 0.$

$\bar{\alpha}^2 = \bar{r}^1\bar{v}_2 - \bar{r}^2\bar{w}_2 \leq \bar{r}^1/(1 - f_1)$ (from (5.3)). From (6.3), $\bar{r}^1/(1 - f_1) \leq 1$, hence $\bar{\alpha}^2 \leq 1$.

$\bar{\alpha}^3 = \bar{r}^1\bar{v}_3 + \bar{r}^2\bar{w}_3 = \lambda_3\ \bar{r}^1/(1 - f_1) + (1 - \lambda_3)\bar{r}^2/(1 - f_2)$, with $\lambda_3 = \bar{v}_3(1 - f_1)$ (from Lemma 6.5). But from Lemma 6.4, $\bar{r}^i/(1 - f_i) \leq 1$, $i = 1, 2$, hence $\bar{\alpha}^3 \leq 1$.

$\bar{\alpha}^4 = -\bar{r}^1\bar{v}_4 + \bar{r}^2\bar{w}_4 \leq \bar{r}^2/(1 - f_2) \leq 1$ (from 6.3), hence $\bar{\alpha}^4 \leq 1$.

The remaining three cases, namely $(\bar{m}^1, \bar{m}^2) = (\lceil \bar{r}^1 \rceil, \lfloor \bar{r}^2 \rfloor)$, $(\bar{m}^1, \bar{m}^2) = (\lfloor \bar{r}^1 \rfloor, \lceil \bar{r}^2 \rceil)$, and $\bar{m}^i = (\lceil \bar{r}^1 \rceil, \lceil \bar{r}^2 \rceil)$, $i = 1, 2$, are similar. □

A way to further strengthen these cuts consists in the following three-step procedure:

1. Apply standard modularization to each of the two rows from which the cut is generated (i.e. replace the ray r_j^i by $r_j^i - \lfloor r_j^i \rfloor$ if $r_j^i > 0$ and by $r_j^i - \lceil r_j^i \rceil$ if $r_j^i < 0$, $i = 1, 2$, $j \in J_1$).
2. Generate a cut $\alpha x \geq 1$ from the two modularized rows.
3. Modularize the resulting cut to obtain the strengthened cut $\bar{\alpha} x \geq 1$.

Yet another way to use the integrality of the variables s_j, $j \in J_1$, is to apply the monoidal cut strengthening procedure of [9]. For cuts generated from a disjunction of the form (6.1), this procedure involves the use of lower bounds on the expressions on the lefthand side of each inequality. While these bounds are readily available and quite tight in the case when $x_1, x_2 \in \{0, 1\}$, they can be weak in the general case of $x_1, x_2 \in \mathbb{Z}$. We therefore defer the dicussion of this procedure until the section on the 0-1 disjunctive hull.

7 The 0-1 Disjunctive Hull

We now consider the 0-1 disjunctive hull $P_D^=$ for $q = 2$, i.e. we work with $P_{01} = \{(x, s) \in \{0, 1\}^2 \times \mathbb{R}^{|J|} : (x, s) \in P_L\}$ where P_L is given in (5.1). The CGLP that produces the facets of $P_D^=$ is the linear program with the constraint set of Theorem 2.5. In addition to the four configurations of the parametric octahedron for the MIP CGLP given in Section 5, when v, w are unrestricted in sign some additional configurations are possible: (a) triangles with each face containing exactly one vertex of K, which we call triangles of type C (T_C); and (b) cones, designated as (C).

Note that our triangles of type C are similar to the class of triangles of type 3 for cuts for mixed integer programs described in [19]. The difference between these classes is that on the one hand, the three integer points contained in the faces of triangles of type 3 defined in [19] need not be vertices of K; on the other hand, our triangles of type C may also contain (non-0-1) integer points, positive or negative, in their interior. The presence among the parametric octahedra of unbounded ones, namely cones, implies that the cuts $\alpha s \geq 1$ of this class may have coefficients $\alpha_j < 0$.

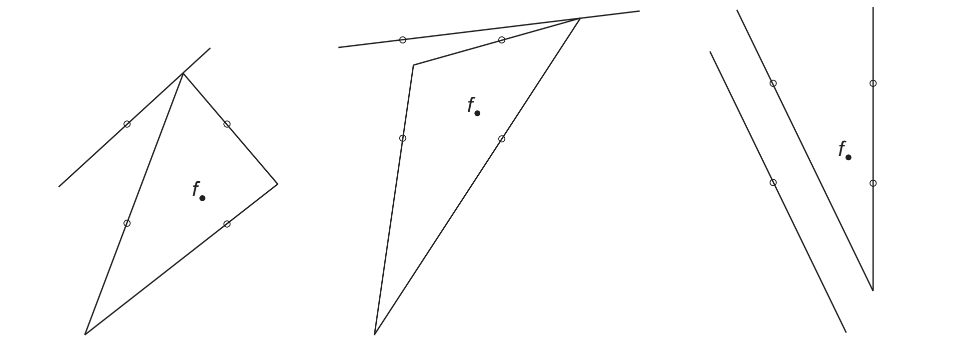
As we did in Section 5, we give a classification of the parametric cross-polytopes that correspond to disjunctive hull facets for the 0-1 case (i.e. facets of $P_D^=$). Let $k_1 \in \{1, \dots, 4\}$ be the index of any vertex of K. We denote by k_2, k_3, k_4 the indices of the vertices of K that follow k_1 in counter-clockwise order. The following configurations, in addition to those for facets of P_D, are exhaustive when considering every value for $k_1 \in \{1, \dots, 4\}$ (mod 4) and swapping v_i with w_i. In each case, the shape of P_{octa} is determined by a strict subset of the four pairs (v_i, w_i), the remaining pairs being inactive.

- (T_{C1}) $v_{k_1} > 0, w_{k_1} < 0$; $v_{k_2}, w_{k_2} > 0$; $v_{k_3} > 0, w_{k_3} > 0$, $(v_{k_4}, w_{k_4} > 0)$. P_{octa} is a triangle of type C with all its vertices outside the cube K. The face corresponding to k_4 is inactive. See Figure 4(a).
- (T_{C2}) $v_{k_1} > 0, w_{k_1} < 0$; $v_{k_2}, w_{k_2} > 0$; $v_{k_3} < 0, w_{k_3} > 0$, $(v_{k_4}, w_{k_4} > 0)$. P_{octa} is a triangle of type C with one vertex in the cube K. The face corresponding to k_4 is inactive. See Figure 4(b).
- (C_A) $v_{k_2}, v_{k_3} > 0$; $w_{k_2} = w_{k_3} = 0$, $v_{k_4} > 0, w_{k_4} < 0$, $(v_{k_1}, w_{k_1} > 0)$. P_{octa} is a cone with one face containing two adjacent vertices of K, the other face containing one vertex of K. The face corresponding to k_1 is inactive. See Figure 4(c).
- (C_B) $v_{k_1} < 0, w_{k_1} > 0$; $v_{k_3} > 0, w_{k_3} < 0$, $v_{k_4}, w_{k_4} > 0$, $(v_{k_2}, w_{k_2} > 0)$. P_{octa} is a cone with one face containing two nonadjacent vertices of K, the other face containing one vertex of K. The face corresponding to k_2 is inactive. See Figure 4(d).
- (C_C) $v_{k_1} > 0, w_{k_1} < 0$; $v_{k_2}, w_{k_2} > 0$; $(v_{k_3} < 0, w_{k_3} > 0)$, $(v_{k_4}, w_{k_4} > 0)$. P_{octa} is a cone with each face containing one vertex of K. The faces corresponding to k_3 and k_4 are inactive. See Figure 4(e).
- (C_{CT}) $v_{k_1} > 0, w_{k_1} < 0$; $v_{k_2}, w_{k_2} > 0$; $v_{k_3}, w_{k_3} > 0$; $(v_{k_4}, w_{k_4} > 0)$. P_{octa} is a truncated cone with each face containing one vertex of K. The face corresponding to k_4 is inactive.
- (S) $v_{k_1} < 0, w_{k_1} > 0$; $v_{k_3} > 0, w_{k_3} < 0$, $(v_{k_2}, w_{k_2} > 0, v_{k_4}, w_{k_4} > 0)$. P_{octa} is a tilted strip, each side of which contains one vertex of K. The faces corresponding to the remaining two vertices are inactive. See Figure 4(g).
- (S_T) $v_{k_1}, w_{k_1} > 0$; $v_{k_2}, w_{k_2} > 0$; $v_{k_3} > 0$, $w_{k_3} < 0$; $(v_{k_4}, w_{k_4} > 0)$. P_{octa} is a truncated (tilted) strip, each side of which contains a vertex of K. The face corresponding to k_4 is inactive.

Example Consider the Andersen et al. [1] instance, amended with the condition $x_i \in \{0, 1\}, i \in \{1, 2\}$:

$$
\begin{array}{llllllll}
x_1 & = \frac{1}{4} & +2s_1 & +1s_2 & -3s_3 & & +1s_5 \\
x_2 & = \frac{1}{2} & +1s_1 & +1s_2 & +2s_3 & -1s_4 & -2s_5 \\
\multicolumn{8}{l}{x_1, x_2 \in \{0, 1\}, \quad s \geq 0.}
\end{array}
\tag{7.1}
$$

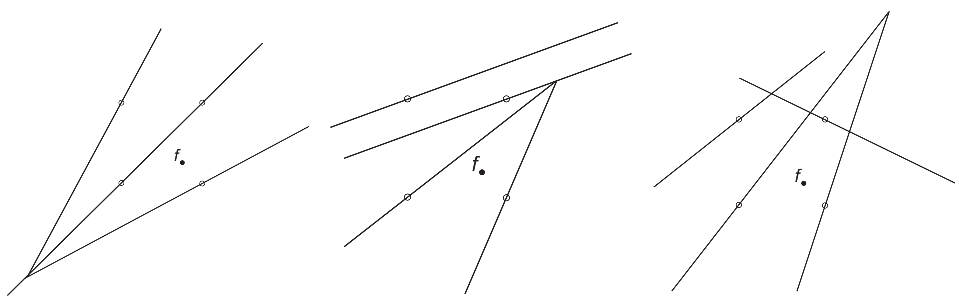
In section 5 we listed the 5 cuts defining the facets of the disjunctive hull for this example, without the 0-1 condition. Using the stronger disjunction expressing the 0-1 condition we obtain the following 12 cuts that define the facets of conv $P_D^=$.



(a) triangle of type C with all vertices outside K

(b) triangle of type C with one vertex inside K

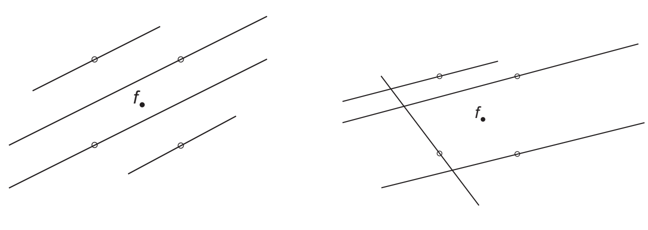
(c) cone with one face containing two adjacent vertices of K



(d) cone with one face containing two nonadjacent vertices of K

(e) cone with each face containing one vertex of K

(f) truncated cone with each face containing one vertex of K



(g) tilted strip

(h) truncated tilted strip

Figure 4: Additional configurations of the parametric octahedron for the 0-1 case

1. Cut (type S): $2.667s_1 + 1.333s_2 + 12s_3 + 0s_4 + 1.333s_5 \geq 1$
   $v_1 = 4;\ v_2 = 1.333;\ v_3 = 1.333;\ v_4 = 4$
   $w_1 = 0;\ w_2 = 0;\ w_3 = 0;\ w_4 = 0$
2. Cut (type T_B): $2.667s_1 + 1.333s_2 + 4.889s_3 + 0.8889s_4 + 1.333s_5 \geq 1$
   $v_1 = 2.222;\ v_2 = 1.333;\ v_3 = 1.333;\ v_4 = 0.4444$
   $w_1 = 0.8889;\ w_2 = 0;\ w_3 = 0;\ w_4 = 1.778$
3. Cut (type T_B): $2s_1 + 2s_2 + 4s_3 + 1s_4 + 1.714s_5 \geq 1$
   $v_1 = 2;\ v_2 = 1.143;\ v_3 = 0;\ v_4 = 0$
   $w_1 = 1;\ w_2 = 0.2857;\ w_3 = 2;\ w_4 = 2$
4. Cut (type T_{C1}): $2.947s_1 + 1.053s_2 + 5.263s_3 + 0.8421s_4 + 3.579s_5 \geq 1$
   $v_1 = 2.316;\ v_2 = 0.7719;\ v_3 = 1.895;\ v_4 = 0.6316$
   $w_1 = 0.8421;\ w_2 = 0.8421;\ w_3 = -0.8421;\ w_4 = 1.684$
5. Cut (type T_{C1}): $1.63s_1 + 2.37s_2 + 8.444s_3 + 0.4444s_4 + 1.926s_5 \geq 1$
   $v_1 = 3.111;\ v_2 = 1.037;\ v_3 = -0.7407;\ v_4 = 2.222$
   $w_1 = 0.4444;\ w_2 = 0.4444;\ w_3 = 3.111;\ w_4 = 0.8889$
6. Cut (type T_{C2}): $4.364s_1 + 2.545s_2 + 3.273s_3 + 1.091s_4 + 0.3636s_5 \geq 1$
   $v_1 = 1.818;\ v_2 = 1.818;\ v_3 = 1.818;\ v_4 = -0.3636$
   $w_1 = 1.091;\ w_2 = -0.7273;\ w_3 = 0.7273;\ w_4 = 2.182$
7. Cut (type T_{C2}): $3.765s_1 + 3.059s_2 + 2.588s_3 + 1.176s_4 + 0.7059s_5 \geq 1$
   $v_1 = 1.647;\ v_2 = 1.647;\ v_3 = 0.7059;\ v_4 = -0.7059$
   $w_1 = 1.176;\ w_2 = -0.4706;\ w_3 = 2.353;\ w_4 = 2.353$
8. Cut (type C_A): $12s_1 + 8s_2 + 12s_3 + 0s_4 - 4s_5 \geq 1$
   $v_1 = 4;\ v_2 = 4;\ v_3 = 4;\ v_4 = 4$
   $w_1 = 0;\ w_2 = -4;\ w_3 = 4;\ w_4 = 0$
9. Cut (type C_B): $32s_1 + 20s_2 - 20s_3 + 4s_4 + 12s_5 \geq 1$
   $v_1 = -4;\ v_2 = 4;\ v_3 = 4;\ v_4 = -12$
   $w_1 = 4;\ w_2 = 4;\ w_3 = -4;\ w_4 = 8$
10. Cut (type C_B): $12s_1 + 8s_2 + 44s_3 - 4s_4 - 4s_5 \geq 1$
    $v_1 = 12;\ v_2 = 4;\ v_3 = 4;\ v_4 = -4$
    $w_1 = -4;\ w_2 = -4;\ w_3 = 4;\ w_4 = 4$
11. Cut (type C_C): $-2s_1 + 6s_2 + 52s_3 + 2s_4 + 4s_5 \geq 1$
    $v_1 = 0;\ v_2 = 0;\ v_3 = -8;\ v_4 = 8$
    $w_1 = 2;\ w_2 = 2;\ w_3 = 14;\ w_4 = -2$
12. Cut (type C_B): $8s_1 - 4s_2 + 12s_3 + 16s_4 + 44s_5 \geq 1$
    $v_1 = 4;\ v_2 = -9.333;\ v_3 = 12;\ v_4 = 4$
    $w_1 = 0;\ w_2 = 16;\ w_3 = -16;\ w_4 = 0$

The above list of 12 cuts includes 3 of the 5 cuts defining facets of $\operatorname{conv} P_D$, namely 1, 2 and 4, which appear on our list in position 3, 2 and 1, respectively. The remaining 2 facets of $\operatorname{conv} P_D$, given by cuts 3 and 5, are redundant for $\operatorname{conv} P_D^{=}$; namely, cut 3 is a convex combination of cuts 2, 3, 6 and 7 on our list, while cut 5 is a convex combination of cuts 1 and 5 on our list.

The number of facets of $\operatorname{conv} P_D^{=}$ substantially exceeds the number of facets of $\operatorname{conv} P_D$. In order to assess the impact of the two sets of cuts, we computed the average integrality gap for 1,000 randomly generated objective functions. Adding the 5 cuts valid for the 2-row MIP reduces this gap by 77%; while adding the additional cuts valid for the 0-1 case reduces 100% of the gap.

Next we discuss the strengthening of valid cuts for $P_D^=$ when some variables s_j are integer-constrained. Let J_1 be the index set of such variables.

First of all, we observe that the standard modularization procedure described in Theorem 6.2 for strengthening cuts for P_D is not valid in the case of cuts for $P_D^=$. Indeed, Lemma 6.1 which underlies the correctness of the procedure in the case of P_D, is no longer valid in the case of $P_D^=$: if the disjunction (6.1) is modified by replacing every inequality with equality, then it is no longer equivalent to the disjunction obtained by replacing r_j^i with $r_j^i - m_j^i$. Instead, we will use a different modularization, known in the literature under the name of monoidal strengthening [9].

Consider a disjunction of the form

$$\bigvee_{k \in Q} (A^k x \geq a_0^k), \quad A^k = (a_j^k),\ j \in J, \quad a_j^k \in \mathbb{R}^m,\ j \in J \cup \{0\}, \tag{7.2}$$

and the valid cut $\alpha x \geq 1$, where

$$\alpha_j = \max_{k \in Q} \{\theta^k a_j^k / \theta^k a_0^k\} \tag{7.3}$$

for some $\theta^k \in \mathbb{R}_+^m$, $k \in Q$.

Suppose now that for each $A^k x$, $k \in Q$, a lower bound $b_0^k \leq a_0^k$ is known, i.e. $A^k x \geq b_0^k$, $k \in Q$.

Theorem 7.1. *Let $M := \{m \in \mathbb{Z}^{|Q|} : \sum_{k \in Q} m^k \geq 0\}$. If $x_j \in \mathbb{Z}$, $j \in J_1$, then the cut $\alpha x \geq 1$ can be strengthened to $\bar{\alpha} x \geq 1$, where*

$$\bar{\alpha}_j = \min_{m \in M} \max_{k \in Q} \left\{ \left(\theta^k a_j^k + m_j^k \theta^k (a_0^k - b_0^k) \right) / \theta^k a_0^k \right\} \qquad j \in J_1 \tag{7.4}$$

and $\bar{\alpha}_j = \alpha_j$ for $j \in J \setminus J_1$.

Proof. See [9] or [4]. □

We will now apply this Theorem to our case, first with the disjunction (6.1), then with the stronger disjunction defining $P_D^=$. Let $\alpha x \geq 1$ be an inequality implied by the disjunction (6.1), i.e. a valid inequality for P_D, and let's assume that $x_i \in \{0, 1\}$, $i = 1, 2$. It is not hard to see that a lower bound on the lefthand side of each of the 8 inequalities that occur in (6.1) is obtained by subtracting 1 from the righthand side. This means that if a_0^k denotes the righthand side and b_0^k the lower bound on the lefthand side of the k-th term, then $a_0^k - b_0^k = \binom{1}{1}$.

Now the cut from the disjunction (6.1) is $\alpha s \geq 1$, where

$$\alpha_j = \max_{k \in \{1,\ldots,4\}} \{\alpha_j^k\},$$

and

$$\begin{aligned} \alpha_j^1 &= -r_j^1 \bar{v}_1 - r_j^2 \bar{w}_1, \\ \alpha_j^2 &= r_j^1 \bar{v}_2 - r_j^2 \bar{w}_2, \\ \alpha_j^3 &= r_j^1 \bar{v}_3 + r_j^2 \bar{w}_3, \\ \alpha_j^4 &= -r_j^1 \bar{v}_4 + r_j^2 \bar{w}_4. \end{aligned} \tag{7.5}$$

To apply the theorem to this case, notice that $\theta^k = (\bar{v}_k, \bar{w}_k)$ and $\theta^k a_j^k = \alpha_j^k$, $\theta^k a_0^k = 1$ for $k = 1, \ldots, 4$.

Corollary 7.2. *Let $x_1, x_2 \in \{0,1\}$, and $M = \{m \in \mathbb{Z}^4 : \sum_{k=1}^4 m^k \geq 0\}$. Then $\bar{\alpha}s \geq 1$ is a valid cut for P_D, with $\bar{\alpha}_j = \max_{k\in\{1,\ldots,r\}}\{\bar{\alpha}_j^k\}$, and*

$$\bar{\alpha}_j^k \quad = \quad \begin{cases} \min\limits_{m_j^k \in M} \max\limits_{k\in\{1,\ldots,4\}} \{\alpha_j^k + m_j^k(\bar{v}_k + \bar{w}_k)\} & j \in J_1 \\ \alpha_j^k & j \in J \setminus J_1 \end{cases}$$

Proof. Denoting

$$\begin{aligned} a_0^1 &= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, & a_0^2 &= \begin{pmatrix} 1-f_1 \\ f_2 \end{pmatrix}, & a_0^3 &= \begin{pmatrix} 1-f_1 \\ 1-f_2 \end{pmatrix}, & a_0^4 &= \begin{pmatrix} f_1 \\ 1-f_2 \end{pmatrix}, \\ b_0^1 &= \begin{pmatrix} f_1-1 \\ f_2-1 \end{pmatrix}, & b_0^2 &= \begin{pmatrix} -f_1 \\ f_2-1 \end{pmatrix}, & b_0^3 &= \begin{pmatrix} -f_1 \\ -f_2 \end{pmatrix}, & b_0^4 &= \begin{pmatrix} f_1-1 \\ -f_2 \end{pmatrix}, \end{aligned} \tag{7.6}$$

it is easy to see that for $k = 1, \ldots, 4$,

$$(\bar{v}_k, \bar{w}_k)(a_0^k - b_0^k) = \bar{v}_k + \bar{w}_k.$$

$\square$

We now turn to strengthening a valid inequality for $P_D^=$, the set defined by the disjunction ($6.1^=$), obtained from (6.1) by replacing each inequality with equality. In this case the cut from ($6.1^=$) is $\widetilde{\alpha}x \geq 1$, where $\widetilde{\alpha}_j = \max_{k\in\{1,\ldots,4\}} \widetilde{\alpha}_j^k$ and the $\widetilde{\alpha}_j^k$ are given by the same expressions (7.5) as α_j^k, with the important difference that the parameters $(\bar{v}_k, \bar{w}_k)$ are unrestricted in sign. However, from the normalization constraints (5.3) it follows that for any $k \in \{1, \ldots, 4\}$, at most one member of the pair $(\bar{v}_k, \bar{w}_k)$ can be negative.

In order to derive the lower bounds b_0^k required by Theorem 7.1, the best way is to represent each equation of ($6.1^=$) as a pair of inequalities; i.e. the first term of ($6.1^=$) is restated as

$$\begin{pmatrix} -r^1 s & \geq & f_1 \\ r^1 s & \geq & -f_1 \\ -r^2 s & \geq & f_2 \\ r^2 s & \geq & -f_2 \end{pmatrix} \tag{7.7}$$

and so on. Denoting the corresponding parameters or multipliers by v'_k, v''_k, w'_k, w''_k for $k = 1, \ldots, 4$, we see that since at most one of the pairs of inequalities corresponding to an equation can be active in any given solution, at most one member of each pair (v'_k, v''_k) can be positive, and the same holds for each pair (w'_k, w''_k). Furthermore, it becomes clear that if in the equality formulation ($6.1^=$) a parameter, say v_1, takes on a negative value $\bar{v}_1 < 0$ in a solution, this corresponds to the fact that the member of the pair of inequalities corresponding to the equation associated with v_1 that is active, is the one with $\leq$, i.e. with the inequality reversed.

Corollary 7.3. *Let* $M = \{m \in \mathbb{Z}^4 : \sum_{k=1}^4 m^k \geq 0\}$*. Then* $\hat{\alpha} s \geq 1$ *is a valid cut for* $P_D^=$*, with* $\hat{\alpha}_j = \max_{k\in\{1,\dots,4\}}\{\hat{\alpha}_j^k\}$*, and*

$$\hat{\alpha}_j^k = \begin{cases} \min\limits_{m_j^k \in M} \max\limits_{k\in\{1,\dots,r\}} \{\tilde{\alpha}_j^k + m_j^k(\bar{v}_k^+ + \bar{w}_k^+)\}, & j \in J_1 \\ \tilde{\alpha}_j^k & j \in J \setminus J_1 \end{cases}$$

where $\bar{v}_k^+ = \max\{\bar{v}_k, 0\}$ *and* $\bar{w}_k^+ = \max\{\bar{w}_k, 0\}$*.*

Proof. If, using the inequality formulation (7.7) of the disjunction (6.1$^=$), we denote the righthand sides of the four terms by

$$\tilde{a}_0^1 = \begin{pmatrix} f_1 \\ -f_1 \\ f_2 \\ -f_2 \end{pmatrix}, \quad \tilde{a}_0^2 = \begin{pmatrix} 1-f_1 \\ f_1-1 \\ f_2 \\ -f_2 \end{pmatrix}, \quad \tilde{a}_0^3 = \begin{pmatrix} 1-f_1 \\ f_1-1 \\ 1-f_2 \\ f_2-1 \end{pmatrix}, \quad \tilde{a}_0^4 = \begin{pmatrix} f_1 \\ -f_2 \\ 1-f_2 \\ f_2-1 \end{pmatrix}, \tag{7.8}$$

then the lower bounds on the expressions on the lefthand sides of the inequalities are no longer equal to the righthand side minus 1. Instead, we have the following situation:

$$\tilde{b}_0^1 = \begin{pmatrix} f_1-1 \\ -f_1 \\ f_2-1 \\ -f_2 \end{pmatrix}, \quad \tilde{b}_0^2 = \begin{pmatrix} -f_1 \\ f_1-1 \\ f_2-1 \\ -f_2 \end{pmatrix}, \quad \tilde{b}_0^3 = \begin{pmatrix} -f_1 \\ f_1-1 \\ -f_2 \\ f_2-1 \end{pmatrix}, \quad \tilde{b}_0^4 = \begin{pmatrix} f_1-1 \\ -f_1 \\ -f_2 \\ f_2-1 \end{pmatrix}. \tag{7.9}$$

As a consequence,

$$\tilde{a}_0^k - \tilde{b}_0^k = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{for } k = 1, \dots, 4$$

Thus, if we denote $\bar{v}_k^+ := \max\{\bar{v}_k, 0\}$, $\bar{w}_k^+ = \max\{\bar{w}_k, 0\}$, we have $(\bar{v}_k^+, \bar{w}_k^+)(a_0^k, b_0^k) = (\bar{v}_k^+, \bar{w}_k^+)$, $k = 1, \dots, 4$, and the expression for $\hat{\alpha}_j^k$ follows. □

Finding the optimal $m_j^k \in M$ requires a small (single digit) number of comparisons. While [9] and [4] give simple procedures for the case of a general disjunction, the optimal m_j^k of Corollary 7.3 for a given $j \in J_1$ can be found as follows:

- Start with $m_j^k = 0$ for all k and apply the
  Iterative Step.
  - Find $\alpha_j^{\max} = \max_k \alpha_j^k$, $\alpha_j^{\min} = \min_k \alpha_j^k$ and let $m_j^{\max}, m_j^{\min}$ be the corresponding values of m_j^k.
  - Set $m_j^{\max} = m_j^{\max} - t$, $m_j^{\min} = m_j^{\min} + t$, where t is the smallest positive integer for which the identity of $\alpha_j^{\max}$ changes.
  - If the value of $\max_k \alpha_j^k$ has not been reduced, stop with $m_j^{\max} = m_j^{\max} - t + 1$, $m_j^{\min} = m_j^{\min} + t - 1$, and m_j^k unchanged for $k \neq \max, \min$. Otherwise repeat.

In the case where P_{octa} is a triangle with each face containing exactly one vertex of K, the term of the disjunction ($6.1^=$) corresponding to the vertex of K left outside the triangle plays no role in defining the cut, hence it can be dropped and the strengthening becomes simpler. This is even more true of the case of a cone, where only two terms of the disjunction are active. A particularly simple case is that of a "fixed shape" cone with apex at a vertex of K, and one face containing a side of K, the other face containing the diagonal of K. There are eight such cones, and every fractional $(f_1, f_2) \neq (\frac{1}{2}, \frac{1}{2})$ (i.e. not lying on the diagonal of K) is strictly contained in four of them (see figure 6 in the next section).

We will illustrate the monoidal strengthening procedure on the conic cuts obtainable from these disjunctions. Here is a couple of them:

1. $(-x_2 \geq 0) \vee (-x_1 + x_2 \geq 0)$
2. $(x_2 \geq 1) \vee (-x_1 - x_2 \geq -1)$

or, after substituting $f_i + r^i s$ for x_i, $i = 1, 2$,

1. $(-r^2 s \geq f_2) \vee ((-r^1 + r^2)s \geq f_1 - f_2)$
2. $(r^2 s \geq 1 - f_2) \vee ((-r^1 - r^2)s \geq f_1 + f_2 - 1)$.

Each disjunction violated by the point (f_1, f_2) has positive righthand sides and gives rise to a valid cut $\alpha s \geq 1$, with coefficients α_j shown below, obtained by using multipliers normalized to yield a cut with a righthand side of 1:

1. $\max \left\{ \frac{-r_j^2}{f_2}, \frac{-r_j^1 + r_j^2}{f_1 - f_2} \right\}$
2. $\max \left\{ \frac{r_j^2}{1 - f_2}, \frac{-r_j^1 - r_j^2}{f_1 + f_2 - 1} \right\}$

To apply the strengthening procedure, we note that for each of the 16 terms of the above 8 disjunctions, the lower bound on the lefthand side of the inequality is just 1 unit less than the righthand side, hence the difference between the latter and the former is exactly 1. Further, the weights $(\bar{v}_k, \bar{w}_k)$ are normalized so that $\bar{v}_k + \bar{w}_k = 1$, $k = 1, 2$. The resulting strengthened coefficients for the above illustration are

1. $\min_{m_j^k \in M} \max \left\{ \frac{-r_j^1 + m_j^1}{f_2}, \frac{r_j^1 - r_j^2 + m_j^2}{f_1 - f_2} \right\}$
2. $\min_{m_j^k \in M} \max \left\{ \frac{r_j^2 + m_j^1}{1 - f_2}, \frac{-r_j^1 - r_j^2 + m_j^2}{f_1 + f_2 - 1} \right\}$

8 Computational Experiments

In this section we present computational experiments with cuts derived from fixed configurations of the parametric octahedron. We assess the strength of the cuts by analyzing the gap closed on instances from MIPLIB3_C_V2 [23] when used in combination with standard Gomory cuts. MIPLIB3_C_V2 is a collection of 68 instances by Margot which are slight variations of the standard MIPLIB3 [24] and for which the validity of a candidate solution can be checked in finite precision arithmetic. We restricted the collection to a subset of 41 instances. The considered instances are such that they contain at least 2 binary variables fractional in the optimal LP solution and the cut generation procedure on each round takes less than 3600 seconds.

We generated the following two families of cuts

- Cuts from 4 Triangles T_A (shown in Figure 5) whose vertices, expressed in terms of their x_1, x_2 coordinates, are:
  - $(0,0); (2,0); (0,2)$
  - $(-1,0); (1,0); (1,2)$
  - $(0,-1); (2,1); (0,1)$
  - $(1,-1); (1,1); (-1,1)$
- Cuts from 4 of the 8 cones of type C_A (shown in Figure 6):
  - apex at $(0,0)$ and rays $(1,0), (1,1)$
  - apex at $(0,0)$ and rays $(0,1), (1,1)$
  - apex at $(0,1)$ and rays $(1,0), (1,-1)$
  - apex at $(0,1)$ and rays $(0,-1), (1,-1)$
  - apex at $(1,1)$ and rays $(-1,0), (-1,-1)$
  - apex at $(1,1)$ and rays $(0,-1), (-1,-1)$
  - apex at $(1,0)$ and rays $(-1,0), (-1,1)$
  - apex at $(1,0)$ and rays $(0,1), (-1,1)$

The reason we only used 4 of these 8 cones is that every (f_1, f_2)-pair is contained in 4 of these 8 cones.

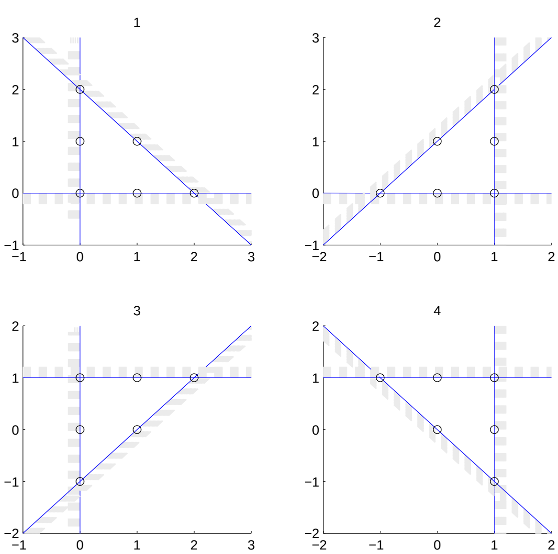


Figure 5: Fixed shape Triangles T_A

For each instance, we first solved the linear programming relaxation and generated a round of Gomory mixed integer (GMI) cuts, a round being one cut from every row of the optimal simplex tableau associated with a binary basic variable with a fractional

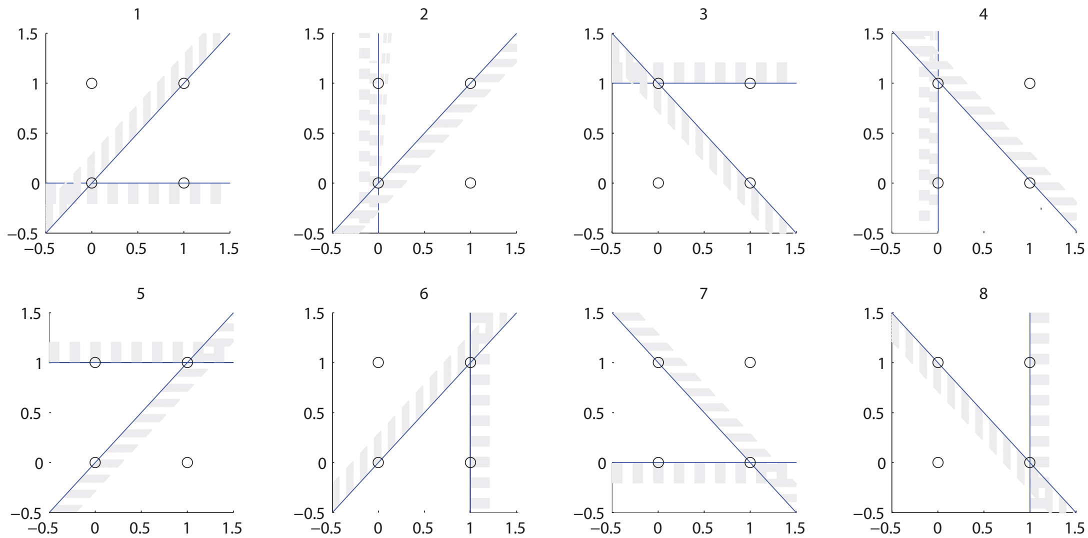


Figure 6: Fixed shape cones C_A

value. We then generated from each pair of rows with at least one fractional binary basic variable either (a) all cuts from the 4 triangles T_A, or (b) all cuts from 4 of the 8 cones C_A or both, and strengthened them via standard modularization (in case (a)) or monoidal strengthening (in case (b)).

We call this cut generating cycle a *round*. At the end of each round, we reoptimized the resulting linear program and removed all cuts that were not tight at the optimum. We generated up to 5 rounds of cuts for each instance. A statement of our routine follows.

Cut Generating Procedure(r, f)

Solve LP relaxation P
for $k \leftarrow 1$ **up to** 5
  do
    Initialize cut collection $C \leftarrow$ empty
    for each binary basic x_i fractional in the current solution
      do
        Compute Gomory cut G_i
        $C \leftarrow C \cup G_i$
    for each binary basic pair x_i, x_j with at least one fractional in the current solution
      do
        if GenerateTriangles==true
          then generate the cuts $T^1_{ij}, \ldots, T^4_{ij}$ from each of the 4 triangles of T_A that contain the fractional solution in their interior
            if StrengthenCuts==true
              then strengthen the cuts $T^1_{ij}, \ldots, T^4_{ij}$ via standard modularization
            $C \leftarrow C \cup T^1_{ij}, \ldots, T^4_{ij}$
        if GenerateCones==true
          then generate the cuts $K^1_{ij}, \ldots, K^8_{ij}$ from each of the 8 cones C_A that contain the fractional solution in their interior
            if StrengthenCuts==true
              then strengthen the cuts $K^1_{ij}, \ldots, K^8_{ij}$ via monoidal cut strengthening
            $C \leftarrow C \cup K_1, \ldots, K^8_{ij}$
    Resolve P and get new solution $\bar{x}^k$ with value $\overline{opt}^k$
    Remove from P the cuts in C that are not tight at $\bar{x}^k$

The Gomory mixed integer (GMI) cut generator we used is the *CglGomory* routine of the Cgl package of COIN-OR [14]. Tables 8.1–8.4 summarize the results of our experiments with these cuts. Table 8.1 shows the outcome of applying all three types of cuts in the above described manner, with strengthening, for one round. Column 1 lists the 41 test instances mentioned above. Column 2 shows the percentage of the integrality gap closed by one round of GMI cuts, while the next two columns show the number of cuts generated and added to the LP relaxation, along with the number of cuts deleted after reoptimization as nonbinding. The next three columns show the same data (i.e. percentage of gap closed and number of cuts added, respectively deleted) after generating a cut from each of the 4 triangles T_A associated with every pair of basic 0-1 variables with at least one fractional member, and a cut from each of the 8 cones associated with every such pair, provided the cone contains such a pair in its interior. Finally, the last column shows the percentage improvement in the integrality gap closed by all three types of cuts versus the GMI cuts alone.

As the table shows, the integrality gap closed, which is 19.49% in the case of the GMI cuts, reaches 29.06% when the two remaining types of cuts are added, an increase of 49.14%. The number of triangle cuts and conical cuts generated is of course much larger than that of GMI cuts. While the latter is bounded by the number of basic

0-1 variables fractional at the optimum, in case of the other two types of cuts this number gets multiplied by 8 times the number of basic 0-1 variables, fractional or not. From the table it is clear that after reoptimization few of the added cuts remain active (about 2%), while the rest get removed. The table also reveals marked differences in the impact of the 2-row cuts on different instances, from 0 impact in about 40% of the instances, to a more than 7-fold increase of the gap closed in the highest-impact case.

Tables 8.2–8.3 show the effect of using only triangle cuts or only conic cuts on top of the GMI cuts. Clearly, the joint effect of using both types of cuts is substantially stronger than is the case with a single type.

Finally, Table 8.4 shows the effect of generating both types of 2-row cuts on top of GMI cuts, as in Table 8.1, but this time for 5 rounds instead of just 1. The improvement in gap closing keeps growing after every round. At the end of the 5 rounds, the gap closed is 38.78%, roughly twice as large as the 19.49% gap closure obtained by 1 round of GMI cuts.

Table 8.1: GMI cuts + Triangle cuts + Conic cuts, all strengthened, 1 round

	GMI			GMI+ T_A + C_A, strengthened			
Instance	Gap closed %	Cuts added #	Cuts deleted #	Gap closed %	Cuts added #	Cuts deleted #	Improvement %
air03	100	36	10	100	12744	12667	0.00
cap6000	41.65	2	1	41.65	1360	1359	0.00
danoint	0.26	24	13	0.26	24	13	0.00
dcmulti	45.75	49	12	45.86	392	362	0.24
egout	21.84	16	0	60.91	3461	3423	178.89
enigma	100	6	5	100	341	340	0.00
fiber	53.32	39	24	64.97	22944	22881	21.85
fixnet3	6.62	6	0	56.19	2925	2827	748.79
fixnet4	4.79	6	0	13.02	2829	2731	171.82
fixnet6	3.98	6	0	13.03	2502	2388	227.39
khb05250	74.91	19	0	84.21	1435	1408	12.41
l152lav	0	0	0	26.68	16774	16684	0.00
lseu	55.94	12	7	56.59	567	562	1.16
markshare1	0	6	3	0	124	110	0.00
markshare2	0	7	3	0	173	147	0.00
mas74	6.52	9	0	7.57	511	485	16.10
mas76	6.36	9	1	7.7	433	412	21.07
misc03	8.62	20	17	8.62	2239	2231	0.00
misc06	26.17	8	0	26.17	8	0	0.00
misc07	0	28	25	0.72	4177	4171	0.00
mod008	20.1	4	1	20.27	96	90	0.85
mod010	0	0	0	99.26	14211	14070	0.00
mod011	11.44	8	1	32.81	855	171	186.80
modglob	13.32	16	2	16.26	137	40	22.07
p0033	12.6	5	1	57.04	174	168	352.70
p0201	16.89	20	13	19.31	1933	1929	14.33
p0282	3.47	24	17	6.2	2837	2829	78.67
p0548	3.06	19	2	18.53	5584	5521	505.56
p2756	0.21	7	1	0.56	908	884	166.67
pk1	0	15	5	0	22	12	0.00
pp08a	54.3	50	0	65.29	7825	7732	20.24
pp08aCUTS	32.83	40	0	41.18	2579	2513	25.43
qiu	0.33	36	24	0.33	36	24	0.00
rentacar	0	2	0	0	7	5	0.00
rgn	3.15	17	9	3.15	1341	1331	0.00
set1ch	30.36	125	1	44.51	28347	28111	46.61
stein27	0	21	17	0	1257	1254	0.00
stein45	0	35	28	0	3584	3575	0.00
swath	8.18	10	0	11.79	6917	6895	44.13
vpm1	20.73	12	0	23.94	1073	1047	15.48
vpm2	11.25	27	6	16.96	4345	4296	50.76
Average	19.49	19.54	6.07	29.06	3903.2	3846.29	49.14

Table 8.2: GMI cuts + Triangle cuts, strengthened, 1 round

	GMI			GMI + T_A, strengthened			
Instance	Gap closed %	Cuts added #	Cuts deleted #	Gap closed %	Cuts added #	Cuts deleted #	Improvement %
air03	100	36	10	100	5646	5569	0.00
cap6000	41.65	2	1	41.65	364	363	0.00
danoint	0.26	24	13	0.26	24	13	0.00
dcmulti	45.75	49	12	45.75	119	86	0.00
egout	21.84	16	0	60.9	2026	1990	178.85
enigma	100	6	5	100	143	142	0.00
fiber	53.32	39	24	59.77	9927	9877	12.10
fixnet3	6.62	6	0	47.01	1571	1479	610.12
fixnet4	4.79	6	0	12.49	1523	1426	160.75
fixnet6	3.98	6	0	12.03	1352	1240	202.26
khb05250	74.91	19	0	84.21	722	698	12.41
l152lav	0	0	0	13.42	7568	7483	0.00
lseu	55.94	12	7	55.94	291	286	0.00
markshare1	0	6	3	0	66	56	0.00
markshare2	0	7	3	0	91	69	0.00
mas74	6.52	9	0	7.44	261	236	14.11
mas76	6.36	9	1	7.12	225	203	11.95
misc03	8.62	20	17	8.62	1031	1028	0.00
misc06	26.17	8	0	26.17	8	0	0.00
misc07	0	28	25	0	1933	1930	0.00
mod008	20.1	4	1	20.11	48	43	0.05
mod010	0	0	0	93.23	5873	5735	0.00
mod011	11.44	8	1	32.53	464	103	184.35
modglob	13.32	16	2	15.75	107	12	18.24
p0033	12.6	5	1	57.04	85	80	352.70
p0201	16.89	20	13	19.31	1235	1230	14.33
p0282	3.47	24	17	5.38	1416	1409	55.04
p0548	3.06	19	2	17.37	2958	2917	467.65
p2756	0.21	7	1	0.56	546	523	166.67
pk1	0	15	5	0	18	8	0.00
pp08a	54.3	50	0	64.89	3862	3774	19.50
pp08aCUTS	32.83	40	0	40.9	1132	1058	24.58
qiu	0.33	36	24	0.33	36	24	0.00
rentacar	0	2	0	0	4	2	0.00
rgn	3.15	17	9	3.15	697	689	0.00
set1ch	30.36	125	1	44.3	12412	12154	45.92
stein27	0	21	17	0	846	843	0.00
stein45	0	35	28	0	2395	2386	0.00
swath	8.18	10	0	11.79	3358	3338	44.13
vpm1	20.73	12	0	21.94	480	457	5.84
vpm2	11.25	27	6	16.02	2133	2094	42.40
Average	19.49	19.54	6.07	27.98	1829.17	1781.78	43.61

Table 8.3: GMI cuts + Conic cuts, strengthened, 1 round

	GMI			GMI + C_A, strengthened			
Instance	Gap closed %	Cuts added #	Cuts deleted #	Gap closed %	Cuts added #	Cuts deleted #	Improvement %
air03	100	36	10	100	7134	7057	0.00
cap6000	41.65	2	1	41.65	998	997	0.00
danoint	0.26	24	13	0.26	24	13	0.00
dcmulti	45.75	49	12	45.86	322	292	0.24
egout	21.84	16	0	35.15	1451	1422	60.94
enigma	100	6	5	100	204	203	0.00
fiber	53.32	39	24	62.94	13056	12970	18.04
fixnet3	6.62	6	0	49.3	1360	1257	644.71
fixnet4	4.79	6	0	10.41	1312	1210	117.33
fixnet6	3.98	6	0	11.5	1156	1050	188.94
khb05250	74.91	19	0	77.59	732	704	3.58
l152lav	0	0	0	26.68	9206	9118	0.00
lseu	55.94	12	7	56.59	288	283	1.16
markshare1	0	6	3	0	64	55	0.00
markshare2	0	7	3	0	89	73	0.00
mas74	6.52	9	0	7.15	259	237	9.66
mas76	6.36	9	1	7.62	217	190	19.81
misc03	8.62	20	17	8.62	1228	1220	0.00
misc06	26.17	8	0	26.17	8	0	0.00
misc07	0	28	25	0.72	2272	2266	0.00
mod008	20.1	4	1	20.27	52	46	0.85
mod010	0	0	0	97.91	8338	8198	0.00
mod011	11.44	8	1	27.3	399	99	138.64
modglob	13.32	16	2	13.94	46	2	4.65
p0033	12.6	5	1	24.59	94	89	95.16
p0201	16.89	20	13	16.89	718	711	0.00
p0282	3.47	24	17	5.4	1445	1436	55.62
p0548	3.06	19	2	6.53	2645	2599	113.40
p2756	0.21	7	1	0.21	369	358	0.00
pk1	0	15	5	0	19	9	0.00
pp08a	54.3	50	0	57.01	4013	3927	4.99
pp08aCUTS	32.83	40	0	34.23	1487	1409	4.26
qiu	0.33	36	24	0.33	36	24	0.00
rentacar	0	2	0	0	5	3	0.00
rgn	3.15	17	9	3.15	661	650	0.00
set1ch	30.36	125	1	37.43	16060	15780	23.29
stein27	0	21	17	0	432	428	0.00
stein45	0	35	28	0	1224	1214	0.00
swath	8.18	10	0	9.89	3569	3537	20.90
vpm1	20.73	12	0	22.73	605	575	9.65
vpm2	11.25	27	6	15.25	2239	2199	35.56
Average	19.49	19.54	6.07	25.88	2093.56	2046.59	32.83

Table 8.4: GMI cuts + Triangle cuts + Conic cuts, all strengthened, 5 rounds

round	GMI Gap Closed %	GMI + T_A + C_A, strengthened Gap Closed %	Improvement %
1	19.49	29.06	49.14
2	24.94	34.01	36.38
3	27.87	36.70	31.67
4	29.57	37.95	28.31
5	30.48	38.78	27.24

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