An improved upper bound for the error in the zero-counting formulae for Dirichlet L-functions and Dedekind zeta-functions

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Abstract

This paper contains new explicit upper bounds for the number of zeroes of Dirichlet L-functions and Dedekind zeta-functions in rectangles.

1 Introduction and Results

This paper pertains to the functions $N(T,\chi)$ and $N_K(T)$, respectively the number of zeroes $\rho = \beta + i\gamma$ of $L(s,\chi)$ and of $\zeta_K(s)$ in the region $0 < \beta < 1$ and $|\gamma| \le T$. The purpose of this paper is to prove the following two theorems.

Theorem 1. *Let $T \ge 1$ and χ be a primitive nonprincipal character modulo k. Then*

$$\left| N(T,\chi) - \frac{T}{\pi}\log\frac{kT}{2\pi e}\right| \le 0.318\log kT + 6.534. \tag{1.1}$$

In addition, if the right side of (1.1) is written as $C_1 \log kT + C_2$, one may use the values of C_1 and C_2 contained in Table 1.

Theorem 2. *Let $T \ge 1$ and K be a number field with degree $n_K = [K:\mathbb{Q}]$ and absolute discriminant d_K. Then*

$$\left| N_K(T) - \frac{T}{\pi}\log\left\{ d_K \left(\frac{T}{2\pi e}\right)^{n_K}\right\}\right| \le 0.319\left\{\log d_K + n_K \log T\right\} + 6.026 n_K + 3.659. \tag{1.2}$$

In addition, if the right side of (1.2) is written as $D_1\{\log d_K + n_K \log T\} + D_2 n_K + D_3$, one may use the values of D_1, D_2 and D_3 contained in Table 2.

Theorem 1 improves on a result due to McCurley [3, Thm 2.1]; Theorem 2 improves on a result due to Kadiri and Ng [2, Thm 1]. The values of C_1 and D_1 given above are less than half of the corresponding values in [3] and [2]. The improvement is due to Backlund's trick — explained in §3 — and some minor optimisation.

Explicit expressions for C_1 and C_2 and for D_1, D_2 and D_3 are contained in (4.13) and (4.14) and in (5.11) and (5.12). These contain a parameter η which, when varied, gives rise to Tables 1 and 2. The values in the right sides of (1.1) and (1.2) correspond to $\eta = \frac{1}{4}$ in the tables. Note that some minor improvement in the lower order terms is possible if $T \geq T_0 > 1$; Tables 1 and 2 give this improvement when $T \geq 10$.

Table 1: C_1 and C_2 in Theorem 1 and in [3] for various values of η

η	McCurley [3]		When $T \geq 1$		When $T \geq 10$
	C_1	C_2	C_1	C_2	C_2
0.05	0.506	16.989	0.248	9.339	8.660
0.10	0.552	13.202	0.265	8.015	7.311
0.15	0.597	11.067	0.282	7.280	6.549
0.20	0.643	9.606	0.300	6.778	6.021
0.25	0.689	8.509	0.317	6.401	5.616
0.30	0.735	7.641	0.334	6.101	5.288
0.35	0.781	6.929	0.351	5.852	5.011
0.40	0.827	6.330	0.369	5.640	4.770
0.45	0.873	5.817	0.386	5.456	4.556
0.50	0.919	5.370	0.403	5.294	4.363

Table 2: D_1, D_2 and D_3 in Theorem 2 and in [2] for various values of η

η	Kadiri and Ng [2]			When $T \geq 1$			When $T \geq 10$	
	D_1	D_2	D_3	D_1	D_2	D_3	D_2	D_3
0.05	0.506	16.95	7.663	0.248	9.270	3.047	8.637	2.110
0.10	0.552	13.163	7.663	0.265	7.947	3.209	7.288	2.172
0.15	0.597	11.029	7.663	0.282	7.211	3.379	6.526	2.239
0.20	0.643	9.567	7.663	0.300	6.710	3.556	5.997	2.313
0.25	0.689	8.471	7.663	0.317	6.333	3.742	5.593	2.394
0.30	0.735	7.603	7.663	0.334	6.032	3.934	5.265	2.481
0.35	0.781	6.891	7.663	0.351	5.784	4.135	4.987	2.575
0.40	0.827	6.292	7.663	0.369	5.572	4.344	4.746	2.678
0.45	0.873	5.778	7.663	0.386	5.388	4.562	4.532	2.789
0.50	0.919	5.331	7.663	0.403	5.225	4.789	4.339	2.911

Explicit estimation of the error terms of the zero-counting function for $L(s, \chi)$ is done in §2. Backlund's trick is modified to suit Dirichlet L-functions in §3. Theorem 1 is proved in §4. Theorem 2 is proved in §5.

2 Estimating $N(T,\chi)$

Let χ be a primitive nonprincipal character modulo k, and let $L(s,\chi)$ be the Dirichlet L-series attached to χ. Let $a = (1-\chi(-1))/2$ so that a is 0 or 1 according as χ is an even or an odd character. Then the function

$$\xi(s,\chi) = \left(\frac{k}{\pi}\right)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi), \tag{2.1}$$

is entire and satisfies the functional equation

$$\xi(1-s,\overline{\chi}) = \frac{i^a k^{1/2}}{\tau(\chi)}\xi(s,\chi), \tag{2.2}$$

where $\tau(\chi) = \sum_{n=1}^{k} \chi(n)\exp(2\pi i n/k)$.

Let $N(T,\chi)$ denote the number of zeroes $\rho = \beta + i\gamma$ of $L(s,\chi)$ for which $0 < \beta < 1$ and $|\gamma| \le T$. For any $\sigma_1 > 1$ form the rectangle R having vertices at $\sigma_1 \pm iT$ and $1-\sigma_1 \pm iT$, and let $\mathcal{C}$ denote the portion of the rectangle in the region $\sigma \ge \frac{1}{2}$. From Cauchy's theorem and (2.2) one deduces that

$$N(T,\chi) = \frac{1}{\pi}\Delta_{\mathcal{C}} \arg \xi(s,\chi).$$

Thus

$$\begin{aligned} N(T,\chi) &= \frac{1}{\pi}\left\{\Delta_{\mathcal{C}} \arg\left(\frac{k}{\pi}\right)^{(s+a)/2} + \Delta_{\mathcal{C}} \arg\Gamma\left(\frac{s+a}{2}\right) + \Delta_{\mathcal{C}} \arg L(s,\chi)\right\} \\ &= \frac{T}{\pi}\log\frac{k}{\pi} + \frac{2}{\pi}\Im\log\Gamma\left(\frac{1}{4}+\frac{a}{2}+i\frac{T}{2}\right) + \frac{1}{\pi}\Delta_{\mathcal{C}} \arg L(s,\chi). \end{aligned} \tag{2.3}$$

To evaluate the second term on the right-side of (2.3) one needs an explicit version of Stirling's formula. Such a version is provided in [4, p. 294], to wit

$$\log\Gamma(z) = (z-\frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi + \frac{\theta}{6|z|}, \tag{2.4}$$

which is valid for $|\arg z| \le \frac{\pi}{2}$, and in which θ denotes a complex number satisfying $|\theta| \le 1$. Using (2.4) one obtains

$$\begin{aligned} \Im\log\Gamma\left(\frac{1}{4}+\frac{a}{2}+i\frac{T}{2}\right) =& \frac{T}{2}\log\frac{T}{2e} + \frac{T}{4}\log\left(1+\frac{(2a+1)^2}{4T^2}\right) \\ &+ \frac{2a-1}{4}\tan^{-1}\left(\frac{2T}{2a+1}\right) + \frac{\theta}{3|\frac{1}{2}+a+iT|}. \end{aligned} \tag{2.5}$$

Denote the last three terms in (2.5) by $g(a,T)$. Using elementary calculus one can show that $|g(0,T)| \le g(1,T)$ and that $g(1,T)$ is decreasing for $T \ge 1$. This, together with (2.3) and (2.5), shows that

$$\left|N(T,\chi) - \frac{T}{\pi}\log\frac{kT}{2\pi e}\right| \le \frac{1}{\pi}\left|\Delta_{\mathcal{C}} \arg L(s,\chi)\right| + \frac{2}{\pi}g(1,T). \tag{2.6}$$

All that remains is to estimate $\Delta_{\mathcal{C}} \arg L(s,\chi)$. Write $\mathcal{C}$ as the union of three straight lines, viz. let $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, where $\mathcal{C}_1$ connects $\frac{1}{2} - iT$ to $\sigma_1 - iT$; $\mathcal{C}_2$ connects $\sigma_1 - iT$ to $\sigma_1 + iT$; and $\mathcal{C}_3$ connects $\sigma_1 + iT$ to $\frac{1}{2} + iT$. Since $L(\overline{s},\chi) = \overline{L(s,\overline{\chi})}$ a bound on $\mathcal{C}_3$ will serve as a bound on $\mathcal{C}_1$. Estimating the contribution along $\mathcal{C}_2$ poses no difficulty since

$$|\arg L(\sigma_1 + it,\chi)| \leq |\log L(\sigma_1 + it,\chi)| \leq \log \zeta(\sigma_1).$$

To estimate $\Delta_{\mathcal{C}_3} \arg L(s,\chi)$ write

$$f(s) = \frac{1}{2}\{L(s+iT,\chi)^N + L(s-iT,\overline{\chi})^N\}, \tag{2.7}$$

for some positive integer N, to be determined later. Thus $f(\sigma) = \Re L(\sigma + iT,\chi)^N$. Suppose that there are n zeroes of $\Re L(\sigma+iT,\chi)^N$ for $\sigma \in \mathcal{C}_3$. These zeroes partition the segment into $n+1$ intervals. On each interval $\arg L(\sigma + iT,\chi)^N$ can increase by at most π. Thus

$$|\Delta_{\mathcal{C}_3} \arg L(s,\chi)| = \frac{1}{N}|\Delta_{\mathcal{C}_3} \arg L(s,\chi)^N| \leq \frac{(n+1)\pi}{N},$$

whence (2.6) may be written as

$$\left|N(T,\chi) - \frac{T}{\pi}\log\frac{kT}{2\pi e}\right| \leq \frac{2}{\pi}\left\{\log\zeta(\sigma_1) + g(1,T)\right\} + \frac{2(n+1)}{N}. \tag{2.8}$$

One may estimate n with Jensen's Formula.

Lemma 1 (Jensen's Formula)**.** *Let $f(z)$ be holomorphic for $|z-a| \leq R$ and non-vanishing at $z = a$. Let the zeroes of $f(z)$ inside the circle be z_k, where $k = 1,2,\ldots,n$, and let $|z_k - a| = r_k$. Then*

$$\log\frac{R^n}{|r_1 r_2 \cdots r_n|} = \frac{1}{2\pi}\int_0^{2\pi} \log f(a + Re^{i\phi})\, d\phi - \log|f(a)|. \tag{2.9}$$

3 Backlund's Trick

Backlund's trick is to use the functional equation to show that if there are zeroes of $\Re L(\sigma+iT,\chi)^N$ on the line $\sigma \in [\frac{1}{2},\sigma_1]$, then there are zeroes on the line $\sigma \in [1-\sigma_1, \frac{1}{2}]$. This device was introduced by Backlund in [1] for the Riemann zeta-function.

For a complex-valued function $f(s)$, define $\Delta_{\pm} \arg f(s)$ to be the change in argument of $f(s)$ as σ varies from $\frac{1}{2}$ to $\frac{1}{2} \pm \delta$, where $\delta > 0$. Following Backlund's approach one can easily prove a requisite lemma for L-functions.

Lemma 2. ***(i)*** *Let N be a positive integer and let $T \geq T_0 \geq 1$. Suppose that*

$$|\Delta_{+} \arg L(s,\chi) + \Delta_{-} \arg L(s,\chi)| < E,$$

where $E = E(\delta, T_0)$. If there are n zeroes of $\Re L(\sigma+iT,\chi)^N$ for $\sigma \in [\frac{1}{2}, \sigma_1]$, then there are at least $n-1-[NE/\pi]$ zeroes in $\sigma \in [1-\sigma_1, \frac{1}{2}]$

(ii) *Denote the zeroes in $[\frac{1}{2}, \sigma_1]$ by $\rho_\nu = a_\nu + iT$ where $\frac{1}{2} \le a_n \le a_{n-1} \le \cdots \le \sigma_1$, and the zeroes in $[1-\sigma_1, \frac{1}{2}]$ as $\rho'_\nu = a'_\nu + iT$ where $1-\sigma_1 \le a'_1 \le a'_2 \le \cdots \le \frac{1}{2}$. Then*

$$a_\nu \ge 1 - a'_\nu, \quad \text{for } \nu = 1, 2, \ldots, n-1-[NE/\pi], \tag{3.1}$$

and, if $\sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})$, then

$$\prod_{\nu=1}^{n} |1+\eta-a_\nu| \prod_{\nu=1}^{n-1-[NE/\pi]} |1+\eta-a'_\nu| \le (\tfrac{1}{2}+\eta)^{2n-1-[NE/\pi]}. \tag{3.2}$$

Proof. Suppose that there exists an $n \ge 1$ for which

$$n\pi \le |\Delta_{\mathcal{C}_3} \arg L(s,\chi)^N| < (n+1)\pi. \tag{3.3}$$

Thus $\arg L(s,\chi)^N$ must increase as σ varies from σ_1 to $\frac{1}{2}$. This increase may only occur if σ has passed over a zero of $\Re L(s,\chi)^N$. In particular as σ moves along $\mathcal{C}_3$

$$|\Delta \arg L(s,\chi)^N| \ge \pi, 2\pi, \ldots, n\pi.$$

Let $\rho_\nu = a_\nu + it$ denote zeroes of $\Re L(s,\chi)^N$ the passing over of which forces

$$|\Delta \arg L(s,\chi)| \ge \nu\pi.$$

It follows that there must be n such points, and that $\frac{1}{2} \le a_n \le a_{n-1} \le \ldots \le a_2 \le a_1 \le \sigma_1$. Also if $\delta \ge a_\nu$ then

$$|\Delta_+ \arg L(s,\chi)^N| \ge (n-\nu)\pi. \tag{3.4}$$

By the hypothesis in Lemma 2 (i),

$$|\Delta_+ \arg L(s,\chi)^N + \Delta_- \arg L(s,\chi)^N| < NE. \tag{3.5}$$

When $\delta \ge a_\nu$, (3.4) and (3.5) show that

$$|\Delta_- \arg L(s,\chi)^N| \ge (n-\nu-NE/\pi)\pi,$$

for $1 \le \nu \le n-1-[NE/\pi]$. The increase in the argument is only possible if there are zeros of $\Re L(s,\chi)^N$ in the segment $\sigma \in [1-\sigma_1, \frac{1}{2}]$. Label these zeroes $\rho'_\nu = a'_\nu + it$, whence $|a'_\nu - \frac{1}{2}| \le |a_\nu - \frac{1}{2}|$ for $1 \le \nu \le n-1-[NE/\pi]$ and so (3.1) follows. This produces a positive number of zeroes in $[1-\sigma_1, \frac{1}{2}]$ provided that $n \ge 2 + [NE/\pi]$. If this is not satisfied then (2.8) becomes

$$\left| N(T,\chi) - \frac{T}{\pi} \log \frac{kT}{2\pi e} \right| \le \frac{2}{\pi} \left\{ \log \zeta(\sigma_1) + g(1,T) + E \right\} + \frac{6}{N}. \tag{3.6}$$

This bound also holds if there is no $n \ge 1$ for which (3.3) holds.

For zeroes ρ_ν lying to the left of $1+\eta$ one has

$$|1+\eta-a_\nu||1+\eta-a'_\nu| \le (1+\eta-a_\nu)(\eta+a_\nu),$$

by (3.1). This is a decreasing function for $a_\nu \in [\frac{1}{2}, 1+\eta]$ and so, for these zeroes

$$|1+\eta-a_\nu||1+\eta-a'_\nu| \le (\frac{1}{2}+\eta)^2. \tag{3.7}$$

For zeroes lying to the right of $1+\eta$ one has

$$|1+\eta-a_\nu||1+\eta-a'_\nu| \le (a_\nu-1-\eta)(\eta+a_\nu).$$

This is increasing with a_n and so, for these zeroes

$$|1+\eta-a_\nu||1+\eta-a'_\nu| \le \sigma_1^2-\sigma_1-\eta(1+\eta). \tag{3.8}$$

The bounds in (3.7) and (3.8) are equal[1] when $\sigma_1 = \frac{1}{2}+\sqrt{2}(\eta+\frac{1}{2})$. Thus (3.2) holds for $\sigma_1 = \frac{1}{2}+\sqrt{2}(\eta+\frac{1}{2})$. For the unpaired zeroes one may use the bound $|1+\eta-a_\nu| \le \frac{1}{2}+\eta$, whence (3.2) follows. □

3.1 Calculation of E in Lemma 2 (i)

From (2.1) and (2.2) it follows that

$$\Delta_+ \arg \xi(s,\chi) = -\Delta_- \arg \xi(s,\chi).$$

Since $\arg(\pi/k)^{-\frac{s+a}{2}} = -\frac{t}{2}\log(\pi/k)$ then $\Delta_\pm(\pi/k)^{-\frac{s+a}{2}} = 0$, whence

$$|\Delta_+ \arg L(s,\chi) + \Delta_- \arg L(s,\chi)| = |\Delta_+ \arg \Gamma(\tfrac{s+a}{2}) + \Delta_- \arg \Gamma(\tfrac{s+a}{2})|.$$

Using (2.4) one may write

$$\left|\Delta_+ \arg \Gamma\left(\frac{s+a}{2}\right) + \Delta_- \arg \Gamma\left(\frac{s+a}{2}\right)\right| \le G(a,\delta,t), \tag{3.9}$$

where

$$\begin{aligned} G(a,\delta,t) =& \frac{1}{2}(a-\frac{1}{2}+\delta)\tan^{-1}\frac{a+\frac{1}{2}+\delta}{t} + \frac{1}{2}(a-\frac{1}{2}-\delta)\tan^{-1}\frac{a+\frac{1}{2}-\delta}{t} \\ & -(a-\frac{1}{2})\tan^{-1}\frac{a+\frac{1}{2}}{t} - \frac{t}{4}\log\left[1+\frac{2\delta^2\{t^2-(\frac{1}{2}+a)^2\}+\delta^4}{\left\{t^2+(\frac{1}{2}+a)^2\right\}^2}\right] \\ & +\frac{1}{3}\left\{\frac{1}{|\frac{1}{2}+\delta+a+it|}+\frac{1}{|\frac{1}{2}-\delta+a+it|}+\frac{2}{|\frac{1}{2}+a+it|}\right\}. \end{aligned} \tag{3.10}$$

One can show that $G(a,\delta,t)$ is decreasing in t and that $G(1,\delta,t) \le G(0,\delta,t)$. Therefore, since, in Lemma 2 (i), one takes $\sigma_1 = \frac{1}{2}+\sqrt{2}(\frac{1}{2}+\eta)$ it follows that $\delta = \sqrt{2}(\frac{1}{2}+\eta)$, whence one may take

$$E = G(0, \sqrt{2}(\tfrac{1}{2}+\eta), t_0), \tag{3.11}$$

for $t \ge t_0$.

[1]McCurley does not use Backlund's trick. Accordingly, his upper bounds in place of (3.7) and (3.8) are $\frac{1}{2}+\eta$ and $\sigma_1-1-\eta$. These are equal at $\sigma_1 = \frac{3}{2}+2\eta$, which is his choice of σ_1.

4 Proof of Theorem 1

In Lemma 1, take $a = 1 + \eta$, $f(z)$ as in (2.7), and $R = r(\frac{1}{2} + \eta)$, where $r > 1$. Suppose that there are n zeroes of $\Re L(\sigma + iT, \chi)^N$ for $\sigma \in [\frac{1}{2}, \sigma_1]$, where $\sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})$.

4.1 Applying Backlund's Trick

If $1+\eta-r(\frac{1}{2}+\eta) \leq 1-\sigma_1$ then all of the $2n-1-[NE/\pi]$ zeroes of $\Re L(\sigma+iT, \chi)^N$ are included in the contour. Thus the left side of (2.9) is

$$\begin{aligned}&\log \frac{\{r(\frac{1}{2}+\eta)\}^{2n-1-[NE/\pi]}}{|1+\eta-a_1|\cdots|1+\eta-a_n||1+\eta-a'_1|\cdots|1+\eta-a'_{n-1-[NE/\pi]}|} \\ &\geq (2n-1-[NE/\pi])\log r,\end{aligned} \tag{4.1}$$

by (3.2). If the contour does not enclose all of the $2n-1-[NE/\pi]$ zeroes of $\Re L(\sigma+iT,\chi)^N$, then the following argument, thoughtfully provided by Professor D.R. Heath-Brown, allows one still to make a saving.

To a zero at $x + it$, with $\frac{1}{2} \leq x \leq 1 + \eta$ one may associate a zero at $x' + it$ where, by (3.1), $1 - x \leq x' \leq \frac{1}{2}$. Thus, for an intermediate radius, zeroes to the right of $\frac{1}{2}$ yet still close to $\frac{1}{2}$ will have their pairs included in the contour. Let X satisfy $1 + \eta - (\frac{1}{2} + \eta)/r < X < \min\{1 + \eta, r(\frac{1}{2} + \eta) - \eta\}$. Since $r > 1$, this guarantees that $X > \frac{1}{2}$. For a zero at $x + it$ consider two cases: $x \geq X$ and $x < X$.

In the former, there is no guarantee that the paired zero $x' + it$ is included in the contour. Thus the zero at $x + it$ is counted in Jensen's formula with weight

$$\log \frac{r(\frac{1}{2}+\eta)}{1+\eta-x} \geq \log \frac{r(\frac{1}{2}+\eta)}{1+\eta-X}. \tag{4.2}$$

Now, when $x < X$, the paired zero at x' is included in the contour, since $1+\eta-r(\frac{1}{2}+\eta) < 1-X < 1-x \leq x'$. Thus, in Jensen's formula, the contribution is

$$\begin{aligned}\log \frac{r(\frac{1}{2}+\eta)}{1+\eta-x} + \log \frac{r(\frac{1}{2}+\eta)}{1+\eta-x'} &\geq \log \frac{r(\frac{1}{2}+\eta)}{1+\eta-x} + \log \frac{r(\frac{1}{2}+\eta)}{\eta+x} \\ &= \log \frac{r^2(\frac{1}{2}+\eta)^2}{(1+\eta-x)(\eta+x)}.\end{aligned} \tag{4.3}$$

The function appearing in the denominator of (4.3) is decreasing for $x \geq \frac{1}{2}$. Thus the zeroes at $x + it$ and $x' + it$ contribute at least $2\log r$.

Suppose now that there are n zeroes in $[\frac{1}{2}, \sigma_1]$, and that there are k zeroes the real parts of which are at least X. The contribution of all the zeroes ensnared by the integral in Jensen's formula is at least

$$k \log \frac{r(\frac{1}{2}+\eta)}{1+\eta-X} + 2(n-k)\log r = k \log \frac{(\frac{1}{2}+\eta)}{r(1+\eta-X)} + 2n\log r \geq 2n \log r,$$

which implies (4.1)

4.2 Remainder of Proof

To apply Jensen's formula it is necessary to show that $f(1+\eta)$ is non-zero: this is easy to do upon invoking an observation due to Rosser [6]. Write $L(1+\eta+iT,\chi) = Ke^{i\psi}$, where $K > 0$. Choose a sequence of N's tending to infinity for which $N\psi$ tends to zero modulo 2π. Thus

$$\frac{f(1+\eta)}{|L(1+\eta+iT,\chi)|^N} \to 1. \tag{4.4}$$

Since χ is a primitive nonprincipal character then $f(s)$ is holomorphic on the circle. It follows from (2.9) and (4.1) that

$$n \leq \frac{1}{4\pi\log r} J - \frac{1}{2\log r}\log|f(1+\eta)| + \frac{1}{2} + \frac{NE}{2\pi}, \tag{4.5}$$

where

$$J = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|f(1+\eta+r(\frac{1}{2}+\eta)e^{i\phi})|\, d\phi.$$

Write $J = J_1 + J_2$ where the respective ranges of integration of J_1 and J_2 are $\phi \in [-\pi/2, \pi/2]$ and $\phi \in [\pi/2, 3\pi/2]$. For $\sigma > 1$

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |L(s,\chi)| \leq \zeta(\sigma), \tag{4.6}$$

which shows that

$$J_1 \leq N \int_{-\pi/2}^{\pi/2} \log\zeta(1+\eta+r(\tfrac{1}{2}+\eta)\cos\phi)\, d\phi. \tag{4.7}$$

On J_2 use

$$\log|f(s)| \leq N\log|L(s+iT,\chi)|,$$

and the convexity bound [5, Thm 3]

$$|L(s,\chi)| \leq \left(\frac{k|s+1|}{2\pi}\right)^{(1+\eta-\sigma)/2} \zeta(1+\eta), \tag{4.8}$$

valid for $-\eta \leq \sigma \leq 1+\eta$, where $0 < \eta \leq \frac{1}{2}$, to show that

$$J_2 \leq \pi N\log\zeta(1+\eta) + N\frac{r(\frac{1}{2}+\eta)}{2}\int_{\pi/2}^{3\pi/2}(-\cos\phi)\log\left\{\frac{kTw(T,\phi,\eta,r)}{2\pi}\right\} d\phi, \tag{4.9}$$

where

$$w(T,\phi,\eta,r)^2 = 1 + \frac{2r(\frac{1}{2}+\eta)\sin\theta}{T} + \frac{r^2(\frac{1}{2}+\eta)^2 + (2+\eta)^2 + 2r(\frac{1}{2}+\eta)(2+\eta)\cos\theta}{T^2}. \tag{4.10}$$

For $\phi \in [\pi/2, \pi]$, the function $w(T, \phi, \eta, r)$ is decreasing in T; for $\phi \in [\pi, 3\pi/2]$ it is bounded above by $w^*(T, \phi, \eta, r)$ where

$$w^*(T, \phi, \eta, r)^2 = 1 + \frac{r^2(\frac{1}{2} + \eta)^2 + (2 + \eta)^2 + 2r(\frac{1}{2} + \eta)(2 + \eta)\cos\theta}{T^2}, \quad (4.11)$$

which is decreasing in T.

To bound n using (4.5) it remains to bound $-\log|f(1 + \eta)|$. This is done by using (4.4) and (4.6) to show that

$$-\log|f(1 + \eta)| \to -N\log|L(1 + \eta + iT)| \le -N\log[\zeta(2 + 2\eta)/\zeta(1 + \eta)].$$

This, together with (2.8), (3.6), (4.5), (4.7), (4.9) and sending $N \to \infty$, shows that, when $T \ge T_0$

$$\left|N(T, \chi) - \frac{T}{\pi}\log\frac{kT}{2\pi e}\right| \le \frac{r(\frac{1}{2} + \eta)}{2\pi\log r}\log kT + C_2, \quad (4.12)$$

where

$$\begin{aligned} C_2 = {} & \frac{2}{\pi}\left\{\log\zeta(\tfrac{1}{2} + \sqrt{2}(\tfrac{1}{2} + \eta)) + g(1, T) + \frac{E}{2}\right\} + \frac{3}{2\log r}\log\zeta(1 + \eta) \\ & - \frac{\log\zeta(2 + 2\eta)}{\log r} + \frac{1}{2\pi\log r}\int_{-\pi/2}^{\pi/2}\log\zeta(1 + \eta + r(\tfrac{1}{2} + \eta)\cos\phi)\, d\phi \\ & + \frac{r(\frac{1}{2} + \eta)}{4\pi\log r}\Bigg\{ -2\log 2\pi + \int_{\pi/2}^{\pi}(-\cos\phi)\log w(T_0, \phi, \eta, r)\, d\phi \\ & \qquad + \int_{\pi}^{3\pi/2}(-\cos\phi)\log w^*(T_0, \phi, \eta, r)\, d\phi\Bigg\}. \end{aligned}$$

4.3 A small improvement

Consider that what is really sought is a number p satisfying $-\eta \le p < 0$ for which one can bound $L(p + it, \chi)$, provided that $1 + \eta - r(\frac{1}{2} + \eta) \ge p$. Indeed the restriction that $p \ge -\eta$ can be relaxed by adapting the convexity bound, but, as will be shown soon, this is unnecessary.

The convexity bound (4.8) becomes the rather ungainly

$$|L(s, \chi)| \le \left\{\left(\frac{k|1 + s|}{2\pi}\right)^{(1/2 - p)(1 + \eta - \sigma)}\zeta(1 - p)^{1 + \eta - \sigma}\zeta(1 + \eta)^{\sigma - p}\right\}^{1/(1 + \eta - p)},$$

valid for $-\eta \le p \le \sigma \le 1 + \eta$. Such an alternation only changes J_2, whence the coefficient of $\log kT$ in (4.12) becomes

$$\frac{r(\frac{1}{2} + \eta)(\frac{1}{2} - p)}{\pi(1 + \eta - p)\log r}.$$

This is minimised when $r = (1+\eta-p)/(1/2+\eta)$, whence (4.12) becomes

$$\left|N(T,\chi) - \frac{T}{\pi}\log\frac{kT}{2\pi e}\right| \leq \frac{\frac{1}{2}-p}{\pi\log\left(\frac{1+\eta-p}{1/2+p}\right)}\log kT + C_2, \tag{4.13}$$

where

$$\begin{aligned} C_2 = \frac{2}{\pi}&\left\{\log\zeta(\tfrac{1}{2}+\sqrt{2}(\tfrac{1}{2}+\eta)) + g(1,T) + \frac{G(0,\sqrt{2}(\frac{1}{2}+\eta),T_0)}{2}\right\} \\ &+ \frac{1}{\log\left(\frac{1+\eta-p}{1/2+\eta}\right)}\left\{\frac{3}{2}\log\zeta(1+\eta) - \log\zeta(2+2\eta) + \frac{1}{\pi}\log\frac{\zeta(1-p)}{\zeta(1+\eta)}\right. \\ &+ \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\log\zeta(1+\eta+(1+\eta-p)\cos\phi)\,d\phi + \frac{\frac{1}{2}-p}{2\pi}\Bigg(-2\log 2\pi \\ &+ \left.\int_{\pi/2}^{\pi}(-\cos\phi)\log w(T_0,\phi,\eta,r)\,d\phi + \int_{\pi}^{3\pi/2}(-\cos\phi)\log w^*(T_0,\phi,\eta,r)\,d\phi\Bigg)\right\}, \end{aligned} \tag{4.14}$$

in which $g(1,T)$, $G(a,\delta,T_0)$, w and w^* are defined in (2.5), (3.10), (4.10) and (4.11).

The coefficient of $\log kT$ in (4.13) is minimal when $p = 0$ and $r = \frac{1+\eta}{1/2+\eta}$. One cannot choose $p = 0$ nor should one choose p to be too small a negative number lest the term $\log\zeta(1-p)/\zeta(1+\eta)$ become too large. Choosing $p = -\eta/7$ ensures that C_2 in (4.13) is always smaller than the corresponding term in McCurley's proof. Theorem 1 follows upon taking $T_0 = 1$ and $T_0 = 10$. One could prove different bounds were one interested in 'large' values of kT. In this instance the term C_2 is not so important, whence one could choose a smaller value of p.

5 The Dedekind zeta-function

This section employs the notation of §§2-3. Consider a number field K with degree $n_K = [K:\mathbb{Q}]$ and absolute discriminant d_K. In addition let r_1 and r_2 be the number of real and complex embeddings in K, whence $n_K = r_1 + 2r_2$. Define the Dedekind zeta-function to be

$$\zeta_K(s) = \sum_{\mathfrak{a}\subset\mathcal{O}_K}\frac{1}{(\mathbb{N}\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs over the non-zero ideals. The completed zeta-function

$$\xi_K(s) = s(s-1)\left(\frac{d_K}{\pi^{n_K}2^{2r_2}}\right)^{s/2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s) \tag{5.1}$$

satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{5.2}$$

Let $a(s) = (s-1)\zeta_K(s)$ and let

$$f(\sigma) = \frac{1}{2}\left\{a(s+iT)^N + a(s-iT)^N\right\}. \tag{5.3}$$

It follows from (5.1) and (5.2) that

$$\left|\Delta_+ \arg a(s) + \Delta_- \arg a(s)\right| \le F(\delta, t) + n_K G(0, \delta, t), \tag{5.4}$$

where $F(\delta, t) = 2\tan^{-1}\frac{1}{2t} - \tan^{-1}\frac{1/2+\delta}{t} - \tan^{-1}\frac{1/2-\delta}{t}$, and $G(0, \delta, t)$ is defined in (3.10).

Thus, following the arguments in §§2-4.2, one arrives at

$$\left|N_K(T) - \frac{T}{\pi}\log\left\{d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right\}\right| \le \frac{2(n+1)}{N} + \frac{2n_K}{\pi}\left\{|g(0,T)| + \log\zeta(\sigma_1)\right\} + 2, \tag{5.5}$$

where n is bounded above by (4.5), in which $f(s)$ is defined in (5.3). Using the right inequality in

$$\frac{\zeta_K(2\sigma)}{\zeta_K(\sigma)} \le |\zeta_K(s)| \le \{\zeta(\sigma)\}^{n_K}, \tag{5.6}$$

one can show that the corresponding estimate for J_1 is

$$J_1/N \le \pi\log T + \int_{-\pi/2}^{\pi/2}\left\{\log\tilde{w}(T,\phi,\eta,r) + n_K\log\zeta(1+\eta+r(\tfrac{1}{2}+\eta)\cos\phi)\right\}\, d\phi \tag{5.7}$$

where

$$\tilde{w}(T,\phi,\eta,r)^2 = 1 + \frac{2r(\frac{1}{2}+\eta)\sin\theta}{T} + \frac{r^2(\frac{1}{2}+\eta)^2 + \eta^2 + 2r\eta(\frac{1}{2}+\eta)\cos\theta}{T^2}. \tag{5.8}$$

For $\phi \in [0, \pi/2]$, the function $\tilde{w}(T,\phi,\eta,r)$ is decreasing in T; for $\phi \in [-\pi/2, 0]$ it is bounded above by $\tilde{w}^*(T,\phi,\eta,r)$ where

$$\tilde{w}^*(T,\phi,\eta,r)^2 = 1 + \frac{r^2(\frac{1}{2}+\eta)^2 + \eta^2 + 2r\eta(\frac{1}{2}+\eta)\cos\theta}{T^2}. \tag{5.9}$$

which is decreasing in T.

The integral J_2 is estimated using the following convexity result.

Lemma 3. *Let $-\eta \le p < 0$. For $p \le 1+\eta-r(\frac{1}{2}+\eta)$ the following bound holds*

$$|a(s)|^{1+\eta-p} \le \left(\frac{1+\eta}{1-\eta}\right)^{1+\eta-\sigma}\zeta_K(1+\eta)^{\sigma-p}\zeta_K(1-p)^{1+\eta-\sigma}|1+s|^{1+\eta-p} \times \left\{d\left(\frac{|1+s|}{2\pi}\right)^n\right\}^{(1+\eta-\sigma)(1/2-p)}.$$

Proof. See [5, §7]. When $p = -\eta$ the bound reduces to that in [5, Thm 4]. □

Using this it is straightforward to show that

$$J_2/N \le \frac{2r(\frac{1}{2}+\eta)}{1+\eta-p}\left\{\log\frac{\zeta_K(1-p)}{\zeta_k(1+\eta)} + \log\frac{1+\eta}{1-\eta} + (1/2-p)\log\frac{d}{(2\pi)^n}\right\}$$
$$+\pi\zeta_K(1+\eta) + \log T\left(\pi + \frac{2rn_K(\frac{1}{2}+\eta)(\frac{1}{2}-p)}{1+\eta-p}\right)$$
$$+\int_{\pi/2}^{3\pi/2}\log w(T_0,r,\eta,\phi)\,d\phi\left(1+\frac{n_K r(\frac{1}{2}+\eta)(\frac{1}{2}-p)(-\cos\phi)}{1+\eta-p}\right)d\phi \tag{5.10}$$

The quotient of Dedekind zeta-functions can be dispatched easily enough using

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) \le n_K\left\{-\frac{\zeta'}{\zeta}(\sigma)\right\}$$

to show that

$$\log\frac{\zeta_K(1-p)}{\zeta_K(1+\eta)} = \int_{1-p}^{1+\eta} -\frac{\zeta_K'}{\zeta_K}(\sigma)\,d\sigma \le n_K\int_{1-p}^{1+\eta} -\frac{\zeta'}{\zeta}(\sigma)\,d\sigma \le n_K\log\frac{\zeta(1-p)}{\zeta(1+\eta)}.$$

Finally the term $-\log|f(1+\eta)|$ is estimated as in the Dirichlet L-function case — cf. (4.4). This shows that

$$\log|f(1+\eta)| \ge N\log\frac{\zeta_K(2+2\eta)}{\zeta_K(1+\eta)} + \frac{N}{2}\log(\eta^2+T^2) + o(1).$$

This, together with (5.5), (5.7), (5.8), (5.9) and (5.10) and sending $N\to\infty$, shows that, when $T \ge T_0$,

$$\left|N_K(T) - \frac{T}{\pi}\log\left\{d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right\}\right| \le \frac{r(\frac{1}{2}+\eta)(\frac{1}{2}-p)}{\pi\log r(1+\eta-p)}\{\log d_K + n_K\log T\}$$
$$+\left(C_2 - \frac{2}{\pi}\left[g(1,T) - |g(0,T)|\right]\right)n_K + D_3, \tag{5.11}$$

where C_2 is given in (4.14) and

$$D_3 = 2 + \frac{r(\frac{1}{2}+\eta)}{\pi\log r(1+\eta-p)}\log\left(\frac{1+\eta}{1-\eta}\right) + \frac{1}{\pi}F(0,\sqrt{2}(\frac{1}{2}+\eta),T_0)$$
$$+\frac{1}{2\pi\log r}\left(\int_{-\pi/2}^{0}\log\tilde{w}^*(T_0,r,\eta,\phi)\,d\phi + \int_0^{\pi/2}\log\tilde{w}(T_0,r,\eta,\phi)\,d\phi\right.$$
$$\left.+\int_{\pi/2}^{\pi}\log w(T_0,r,\eta,\phi)\,d\phi + \int_{\pi}^{3\pi/2}\log w^*(T_0,r,\eta,\phi)\,d\phi\right) \tag{5.12}$$

Should one choose $p = -\eta/5$, to ensure that the lower order terms in (5.11) are smaller than those in [2], one arrives at Theorem 2. One may choose a smaller value of p if one is less concerned about the term D_2.

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