Involving copula functions in Conditional Tail Expectation

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Abstract

We discuss a new notion of risk measures that preserve the property of coherence called Copula Conditional Tail Expectation (CCTE). This measure describes the expected amount of risk that can be experienced given that a potential bivariate risk exceeds a bivariate threshold value, and provides an important measure for right-tail risk. Our goal is to propose an alternative risk measure which takes into account the fluctuations of losses and possible correlations between random variables.

keywords: Conditional tail expectation; Copulas; Dependence concepts; Risk measure; Capital requirement; Heavy-tailed distributions.

1 Introduction

Measuring risks is a very important element in the prescription of capital requirements. The axiomatic approach chosen in [3] allows, even requires, us to search for similarities and differences between the banking and insurance industries. From an internal viewpoint risk measurement is also important for allocation of capital and performance evaluation. Several risk measures have been proposed in actuarial science literature, namely: the Value-at-Risk (VaR), the expected shortfall or the conditional tail expectation (CTE), the distorted risk

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measures (DRM), and recently the copula distorted risk measure (CDRM) as an alternative risk measure which takes into account the fluctuations of losses and dependence between random variables (rv). See [5].

The concept of VaR has become the standard risk measure used to evaluate risk exposure. In general terms, VaR is the amount of capital required to ensure, with a high degree of certainty. In practice, it can be a high number such as 99.95% for the entire enterprise, or it can be much lower, such as 95%, for a single unit within the enterprise. That the enterprise doesn't become technically insolvent. This lower percentage may reflect the inter-unit diversification that exists. This concept has prompted the study of risk measures by numerous authors (e.g. [23], [24]). Specific desirable properties of risk measures were proposed as axioms in connection with risk pricing by [25] and more generally in risk measure by [2] and [3].

The concept of coherent risk measures and the axiomatic that captures the characteristics required for risk measurement was introduced by [3] in a finite probability space, and further extended by [9] to the general probability space framework. A coherent risk measure is a real functional ϑ defined on a space of rv's satisfying the following axioms:

H1. Subadditivity: For all random losses X and Y, $\vartheta(X+Y) \leq \vartheta(X) + \vartheta(Y)$.
H2. Monotonicity: If $X \leq Y$ for each outcome, then $\vartheta(X) \leq \vartheta(Y)$.
H3. Positive Homogeneity: For positive constant λ, $\vartheta(\lambda X) = \lambda \vartheta(X)$.
H4. Translation Invariance: For constant c, $\vartheta(X+c) = \vartheta(X) + c$.

The CTE in risk analysis represents the conditional expected loss given that the loss exceeds its VaR and provides an important measure for right-tail risk. In this paper we will always consider random variables with finite mean. For a real number s in $(0,1)$, the CTE of a risk X is given by

$$\mathbb{CTE}\,(s) := \mathbb{E}\left[X \mid X > VaR_X\,(s)\right], \tag{1.1}$$

where $VaR_X\,(s) := \inf\{x : F\,(x) \geq s\}$ is the quantile of order s pertaining to distribution function (df) F.

One of the strategy of an Insurance companies is to set aside amounts of capital from which it can draw from in the event that premium revenues become insufficient to pay out claims. Of course, determining these amounts is not a simple calculation. It has to determine the best risk measure that can be used to determine the amount of loss to cover with a high degree of confidence.

Suppose now that we deal with a couple of random losses (X, Y). It's clear that the CTE of X is unrelated withe Y. If we had to control the overflow of the two risks X and Y at the same time, CTE does not answer the problem, then we need another formulation of CTE which takes into account the excess of the two risks X and Y. Then we deal with the amount

$$\mathbb{E}\left[X \mid X > VaR_X(s), Y > VaR_Y(t)\right]. \tag{1.2}$$

If the couple of random losses (X, Y) are independents rv's then the amount (1.2) defined only the CTE of X. Therefore the case of independence is not important.

In the recent years dependence is beginning to play an important role in the world of risk. The increasing complexity of insurance and financial activities products has led to increased actuarial and financial interest in the modeling of dependent risks. While independence can be defined in only one way, but dependence can be formulated in an infinite ways. Therefore, the assumption of independence it makes the treatment easy. Nevertheless, in applications dependence is the rule and independence is the exception.

The copulas is a function that completely describes the dependence structure, it contains all the information to link the marginal distributions to their joint distribution. To obtain a valid multivariate distribution function, we combines several marginal distribution functions, or a different distributional families, with any copula function. Using Sklar's theorem, we can construct bivariate distributions with arbitrary marginal distributions. Thus, for the purposes of statistical modeling, it is desirable to have a large collection of copulas at one's disposal. A great many examples of copulas can be found in the literature, most are members of families with one or more real parameters. For a formal treatment of copulas and their properties, see the monographs by [14], [8], [15], the conference proceedings edited by [4], [7], [10], and [20].

The main idea of this paper is to use the information of dependence between risks to quantifying insurance losses and measuring financial risk assessments, we propose a risk measure defined by:

$$\mathbb{CCTE}_X(s;t) := \mathbb{E}\left[X \mid X > VaR_X(s), Y > VaR_Y(t)\right].$$

We will call this new risk measure the *Copula Conditional Tail Expectation* (CCTE), like a risk measure which measure the conditional expectation give the two dependents loss exceeds $VaR_X(s)$ and $VaR_Y(t)$ for $0 < s, t < 1$ and usually with $s, t > 0.9$. Again, CCTE satisfies all the desirable properties of a coherent risk measure ([3]). The notion of copula in risk measure filed has recently been considered by several authors, see for instance [11], [1], [17] and [5].

This risk measures can give a good quantifying of losses when we have a combined dependents risks, this dependence can influence in the losses of interested risks. Therefore, quantify the riskiness of our position is useful to decide if it acceptable or not. For this reason we use the all information a bout this interest risk and the dependence of our risk with other risks is one of important information that we must take it in consideration.

This paper is organized as follows. In Section 2, we give an explicit formulations of the new notion copula conditional tail expectation risk measure in bivariate case. In Section 3 we presents an illustration examples to explain how to use the new CCTE. Concluding notes are given in Section 4. Proofs are relegated to the Appendix.

2 Copula conditional tail expectation

A risk measure quantifies the risk exposure in a way that is meaningful for the problem at hand. The most commonly used risk measure in finance and insurance are a VaR and CTE (also known as Tail-VaR or expected shortfall). The risk measure is simply loss size for which there is a small (e.g. 1%) probability of exceeding. For some time, it has been recognized that this measure suffers from serious deficiencies if losses are not normally distributed.

According to [3] and [26], the conditional tail expectation of a random variable X at its $VaR_X(s)$ is defined by:

$$\mathbb{CTE}_X(s) = \frac{1}{1 - F(VaR_X(s))} \int_{VaR_X(s)}^{\infty} (1 - F(x))\, dx.$$

where

$$VaR_X(s) := \inf\{x : F(x) \geq s\}, \quad 0 < s < 1$$

Since X is continuous, then $F(VaR_X(s)) = s$, it follows that for all $0 < s < 1$

$$\mathbb{CTE}_X(s) = \frac{1}{1 - s} \int_s^1 VaR_X(s)\, ds. \tag{2.3}$$

The CTE can be larger that the VaR measure for the same value of level s described above since it can be thought of as the sum of the quantile $VaR_X(s)$ and the expected excess loss. TailVaR is a coherent measure in the sense of Artzner [3]. For the application of this kind of coherent risk measures we refer to the papers [3] and [26].

Application of the CTE in a multivariate context to elliptical distributions was considered by [16] and [13], under the notion of the iterated CTE. In univariate context [18] present an empirical estimator of the CTE as well as an estimator of its variance, [6] construct an estimators for the CTE functions withe the confidence intervals and bands for the functions in both of parametric and non-parametric approaches and [19] propose a new CTE estimator, which is applicable when losses have finite means and infinite variances.

Thus the CTE is nothing, see [21], but the mathematical transcription of the concept of 'average loss in the worst $100(1-s)\%$ case', defining by $\tau = VaR_X(s)$ a critical loss threshold corresponding to some confidence level s, $\mathbb{CTE}_X(s)$ provides a cushion against the mean value of losses exceeding the critical threshold τ.

Now, assume that X and Y are dependent with joint df H and continuous margins F and G, respectively. Through this paper we calls X the *target risk* and Y the *associated risk*. In this case, the problem becomes different and its resolution requires more than the usual background. Several authors discussed the risks measures, when applied to univariate and independent rv's.

Our contribution is to introduce the copula notion to provide more flexibility to the CTE of risk of rv's in terms of loss and dependence structure. For comprehensive details on copulas one may consult the textbook of [20].

According to Sklar's Theorem ([22]), there exists a unique copula $C : [0,1]^d \to [0,1]$ such that

$$H(x,y) = C(F(x), G(y)) \tag{2.4}$$

The CTE only focuses on the average of loss. Therefore one must take into account the dependence structure and the behavior of margin tails. These two aspects have an important influence when quantifying risks. If the correlation factor is neglected, the calculation of the CTE follows formula (2.3), which only focuses on the target risk X.

Now taking into account the dependence structure between the target and the associated risks, we define a new notion of CTE called *Copula Conditional Tail Expectation* (CCTE) given in (1.2), this notion led to give a risk measurement focused in the target risk and the link between target and associated risk.

Let's denote the survival functions by $\overline{F}(x) = \mathbb{P}(X > x)$, $\overline{G}(y) = \mathbb{P}(Y > y)$, and the joint survival function by $\overline{H}(x) = \mathbb{P}(X > x, Y > y)$. The function $\overline{C}$ which couples $\overline{H}$ to $\overline{F}$ and

$\overline{G}$ via $\overline{H}(x,y)=\overline{C}(\overline{F}(x),\overline{G}(y))$ is called the survival copula of (X,Y). Furthermore, $\overline{C}$ is a copula, and

$$\overline{C}(u,v)=u+v-1+C(1-u,1-v), \tag{2.5}$$

where C is the (ordinary) copula of X and Y. For more details on the survival copula function see, Section 2.6 in [20, page 32].

Corollary 2.1 *Let C be a copula absolutely continuous with density c, denote for all s and t in $(0,1)$*

$$J_t(u):=\int_t^1 c(u,v)\,dv, \tag{2.6}$$

then

$$\int_s^1 J_t(u)\,du=\overline{C}(1-s,1-t).$$

where $\overline{C}$ is the survival copula.

The CCTE of the target X with respect to the associated risk Y is given in the following proposition.

Proposition 2.1 *Let (X,Y) a bivariate r.v. with joints df represented by the copula C. Assume that X have a finite mean and df F. Then for all s and t in $(0,1)$ the copula conditional tail expectation of X with respect to the bivariate thresholds (s,t) is given by*

$$\mathbb{CCTE}_X(s;t)=\frac{\int_s^1 J_t(u)\,F^{-1}(u)\,du}{\int_s^1 J_t(u)\,du}, \tag{2.7}$$

where $J_t(u)$ is given in (2.6), and F^{-1} is the quantile function of F.

By this Proposition, we got a new risk measure that consists using the link between a couple of risks in the calculation of risk measurement. This notion *does not depend on the df of the associated risk*, but it depend only by the copula function and the df of target risk.

In the next theorem we will proved that the risk when we take into account the link between risks is greater than in the case of a single one.

Theorem 2.1 *Let (X,Y) be a bivariate non-negative r.v. with finite mean and joints df represented by the copula C. For all s and t in $(0,1)$ then*

$$\mathbb{CCTE}_X(s;t)\geq\mathbb{CTE}_X(s).$$

In Figure 3.1 and 3.2 we can see that the graph of $\mathbb{CCTE}$ is above the graph of $\mathbb{CTE}$.

Corollary 2.2 *The CCTE and CTE measures coincide in the non-dependence case.*

The case of non-dependent or independent risks means that the copula C is the product copula, denoted by $\Pi(u,v) := uv$, so the CCTE measure will be CTE one.

3 Illustration examples

3.1 CCTE via Farlie-Gumbel-Morgenstern Copulas

Consider three loss bivariate dependent random variables, (X_i, T), $i = 1, 2, 3$ with joint df H represented by Farlie-Gumbel-Morgenstern (FGM) copula of parameters $\theta = 0$, $\theta = 0.5$ and $\theta = 0.9$ respectively.

The choices of parameters $\theta_i, i = 1, 2, 3$ corresponding respectively to the independence, the medium dependence and the high dependence. The FGM copula is defined as

$$C_\theta^{FGM}(u,v) = uv + \theta uv(1-u)(1-v), \quad u, v \in [0,1],$$

where $\theta \in [-1,1]$, and the density function of FGM copulas is for any $u, v \in [0,1]$

$$\begin{aligned} c_\theta(u,v) &= \frac{\partial^2}{\partial u \partial v} C_\theta^{FGM}(u,v) \\ &= \theta - 2\theta u - 2\theta v + 4\theta uv + 1. \end{aligned}$$

Assume now that X_i, $i = 1, 2, 3$ and T with marginal Pareto df F and G of parameter α, such that

$$1 - G(x) = 1 - F(x) = x^{-\alpha}, \alpha > 1, x > 1. \tag{3.8}$$

In this example, the target risks are X_i and the associated risk is T. The $\mathbb{CTE}$'s of X_i are the same and are given by

$$\mathbb{CTE}_{X_i}(s) = \frac{\alpha (1-s)^{-1/\alpha}}{\alpha - 1}, \quad i = 1, 2, 3. \tag{3.9}$$

For $i = 1, 2, 3$, the $\mathbb{CCTE}$ of X_i is given in the following proposition.

Proposition 3.1 *Let (X_i, T) be a Pareto distributed bivariate random variables with joint df defined by a bivariate FGM copula as follows*

$$H_{X_i,T}(x, y) = C_{\theta_i}^{FGM}(F(x), G(y)), i = 1, 2, ...$$

with $\theta_i \in [-1, 1]$. Then, the CCTE's of X_i at levels $0 < s, t < 1$, are

$$\mathbb{CCTE}_{X_i}(s; t) = \frac{\alpha(2\alpha + t\theta_i - 2st\theta_i + 2st\alpha\theta_i - 1)}{(2\alpha^2 - 3\alpha + 1)(st\theta_i + 1)}(1 - s)^{-1/\alpha}.$$

We have in Table 3.1 and Figure 3.1 the comparison of the riskiness of X_1, X_2 and X_3. Recall that, the $\mathbb{CTE}$'s risk measure of X_i at level s are the same in all cases. But the $\mathbb{CCTE}$ of the loss X_3 is clearly considerably riskier than X_2 and X_1, and it's coincide with CTE in the independence case $\theta_1 = 0$.

s		0.90	0.95	0.95	0.95	0.99	0.99	0.990
t		0	0	0.95	0.99	0	0.99	0.995
$\mathbb{E}[X_i]$,		3	3	3	3	3	3	3
$VaR_{X_i}(s)$,		4.641	7.368	7.368	7.368	21.544	21.544	21.544
$\mathbb{CTE}_{X_i}(s)$,		13.925	22.104	22.104	22.104	64.633	64.633	64.633
$\mathbb{CCTE}_{X_1}(s,t)$	$\theta_1 = 0$	13.925	22.104	22.104	22.104	64.633	64.633	64.633
$\mathbb{CCTE}_{X_2}(s,t)$	$\theta_2 = 0.5$	13.925	22.104	22.285	22.290	64.633	64.740	64.741
$\mathbb{CCTE}_{X_3}(s,t)$	$\theta_3 = 0.95$	13.925	22.104	22.373	22.379	64.633	64.790	64.791

Table 3.1: Risk measures of dependent Pareto (1.5) rv's with FGM copula.

3.2 CCTE via Archimedean Copulas

A copula is said to be Archimedean (see, [12]) if it can be expressed by

$$C(\mathbf{u}) = \psi^{[-1]}\left(\sum_{i=1}^{d}\psi(u_i)\right),$$

where ψ, called the generator of C, is a continuous strictly decreasing convex function from $[0, 1]$ to $[0, \infty]$ such that $\psi(1) = 0$ with $\psi^{[-1]}$ denotes the *pseudo-inverse* of ψ, that is

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t) & \text{for} \quad t \in [0, \psi(0)] \\ 0 & \text{for} \quad t \geq \psi(0). \end{cases}$$

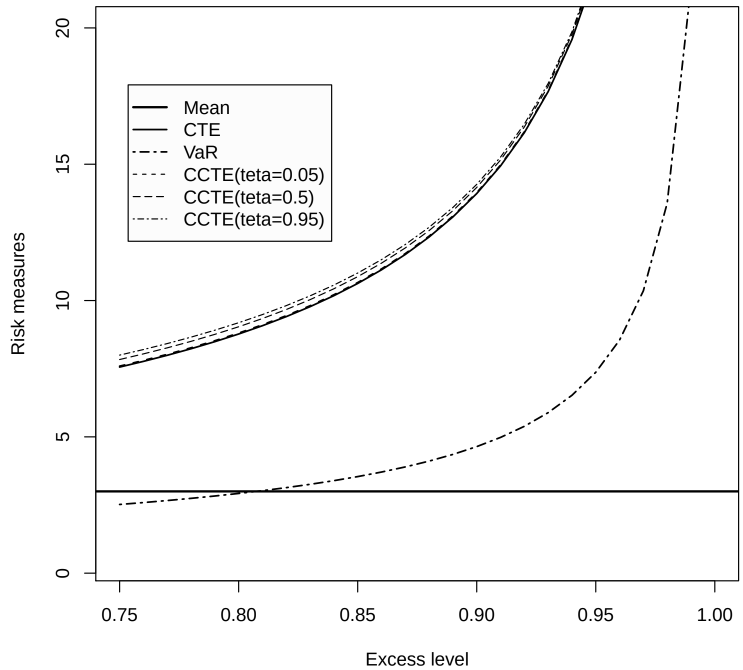

Figure 3.1: $\mathbb{CCTE}$, $\mathbb{CTE}$, and VaR risks measures of dependent Pareto (1.5) rv's with FGM copula for $s = t$.

When $\psi(0) = \infty$, the generator ψ and C are said to be *strict* and therefore $\psi^{[-1]} = \psi^{-1}$. Archimedean copulas are widely used in applications due to their simple form, a variety of dependence structures, and other "nice" properties. For example, most but not all extend to higher dimensions via the associativity property [C is associative if $C(C(u,v),w) = C(u, C(v,w))$].

For an Archimedean copula, the Kendall's tau can be evaluated directly from the generator of the copula, as shown by Corollary 5.1.4 in ([20])

$$\tau = 1 + 4 \int_0^1 \frac{\psi(t)}{\psi'(t)} dt. \tag{3.10}$$

A collection of twenty-two one-parameter families of Archimedean copulas can be found in Table 4.1 of [20].

Notice that in the case of Archimedean copula the copula conditional tail expectation has not an explicit formula, so we give by the following Corollary the expression of $J_t(\cdot)$ in term of generator.

Corollary 3.1 *Let C be an Archimedean copula absolutely continuous with generator ψ, then for all s and t in* $(0, 1)$

$$J_t(u) = 1 - \frac{\psi'(u)}{\psi'(C(u,t))}. \tag{3.11}$$

Then the CCTE of the target risk in term of generator of Archimedean copula with respect to the bivariate thresholds (s, t), $0 < s, t < 1$, is given by

$$\mathbb{CCTE}_X(s;t) = \frac{1}{\overline{C}(1-s,1-t)} \left((1-s)\,\mathbb{CTE}_X(s) - \int_s^1 \frac{\psi'(u)}{\psi'(C(u,t))} F^{-1}(u)\,du \right).$$

Not that in practice we can easily fit copula-based models with the maximum likelihood method or with estimate the dependence parameter by the relationship between Kendall's tau of the data and the generator of the Archimedean copula given in (3.10) under the specified copula model.

In the following section we give same examples to explain how to calculate and compare the copula conditional tail expectation with other risk measure such Value-at-Risk and conditional tail expectation.

3.2.1 Clayton CCTE

In the following example, we consider the bivariate Clayton copula which is a member of the class of Archimedean copula, with the dependence parameter θ restricted on the region $(0, \infty)$. The margins become independent as θ approaches to zero, the copula attains the Fréchet upper bound as θ approaches to infinity. The Clayton copula has been used to study correlated risks, it cannot account for negative dependence. It exhibits strong left tail dependence and relatively weak right tail dependence. It has the form

$$C_{\theta_i}^{Clayton}(u,v) = \left[\max\left(u^{-\theta_i} + v^{-\theta_i} - 1, 0\right)\right]^{-1/\theta_i}. \tag{3.12}$$

Let (X_i, T), $i = 1, 2, 3$ three bivariate dependent loss random variables, withe marginal Pareto df F and G, respectively, of parameter α and joint df H represented by the Clayton copula of parameters $\theta_1 = 2$, $\theta_2 = 5$ and $\theta_3 = 10$ respectively. The Clayton copula $C_{\theta_i}^{Clayton}$ is defined in (3.12) as where $\theta_i \geq 0$, $i = 1, 2, 3$.

The CTE's of X_i is same and it's given by (3.9), for $i = 1, 2, 3$. The CCTE of X_i with respect to the bivariate thresholds (s, t) is given by

$$\mathbb{CCTE}_X(s;t) = \frac{1}{\overline{C}_{\theta_i}(1-s, 1-t)} \left(\frac{\alpha (1-s)^{-1/\alpha+1}}{(\alpha - 1)} - \int_s^1 \frac{\left(t^{-\theta_i} + u^{-\theta_i} - 1\right)^{-1-1/\theta_i}}{(1-u)^{1/\alpha} u^{\theta_i+1}} du \right).$$

where

$$\overline{C}_{\theta_i}(1-s, 1-t) = 1 - s - t + C_{\theta_i}^{Clayton}(s, t).$$

The choices of $\theta_i, i = 1, 2, 3$ corresponding respectively to the Kendall's tau 0.50, 0.71 and 0.83.

In Table 3.2 and Figure 3.2 we shows that the loss X_3 is clearly considerably riskier than X_2 and X_1.

s		0.90	0.95	0.95	0.95	0.99	0.99	0.990
t		0	0	0.95	0.99	0	0.99	0.995
$\mathbb{E}[X_i]$,		3	3	3	3	3	3	3
$VaR_{X_i}(s)$,		4.641	7.368	7.368	7.368	21.544	21.544	21.544
$\mathbb{CTE}_{X_i}(s)$,		13.925	22.104	22.104	22.104	64.633	64.633	64.633
$\mathbb{CCTE}_{X_1}(s,t)$	$\theta_1 = 2$	13.925	22.104	22.607	22.660	64.633	64.950	64.954
$\mathbb{CCTE}_{X_2}(s,t)$	$\theta_2 = 5$	13.925	22.104	23.214	23.480	64.633	65.405	65.427
$\mathbb{CCTE}_{X_3}(s,t)$	$\theta_3 = 10$	13.925	22.104	23.937	24.817	64.633	66.113	66.192

Table 3.2: Risk measures of dependent Pareto (1.5) rv's with Clayton copula.

Clayton copula is the best suited for applications in which two outcomes are likely to experience low values together, since the dependence is strong in the left tail and weak in the right tail.

3.2.2 Gumbel CCTE

The Gumbel copula is an asymmetric Archimedean copula. This copula is given by

$$C_\theta^{Gumbel}(u, v) = \exp\left\{ -\left[(-\ln u)^\theta + (-\ln v)^\theta \right]^{1/\theta} \right\},$$

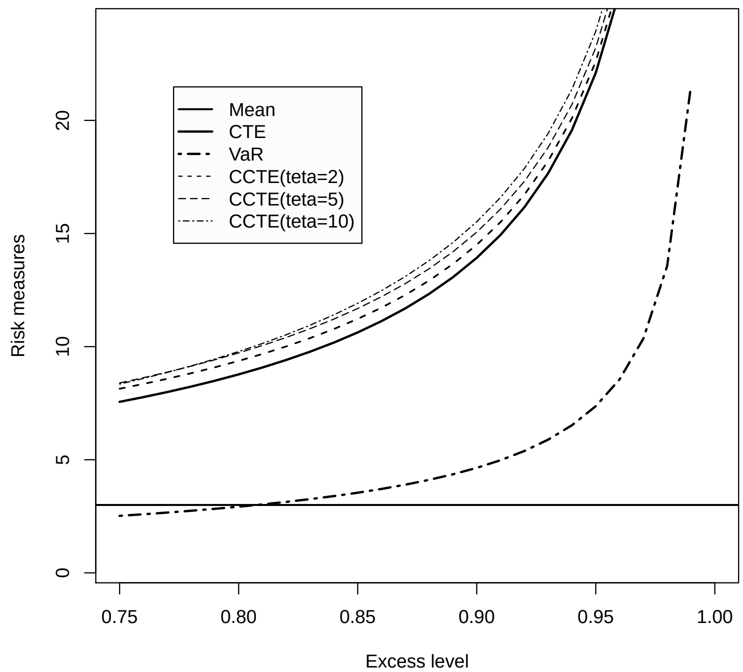

Figure 3.2: $\mathbb{CCTE}$, $\mathbb{CTE}$, and VaR risks measures of dependent pareto (1.5) rv's with Clayton copula for $s = t$.

and its generator is

$$\varphi_\theta (t) = (-\ln t)^\theta .$$

The dependence parameter is restricted to the interval $[1, \infty)$. Values of 1 and ∞ correspond, respectively, to independence and the Fréchet upper bound. This copula does not attain the Fréchet lower bound for any value of θ. Similar to the Clayton copula, Gumbel does not allow negative dependence. Gumbel exhibits greater dependence in the positive tail than in the negative.

We can now give the CCTE in term of Gumbel generator by

$$\mathbb{CCTE}_X (s; t) = \frac{1}{\overline{C}_{\theta_i} (1-s, 1-t)} \left(\frac{\alpha (1-s)^{-1/\alpha+1}}{(\alpha-1)} - \int_s^1 \frac{(-\ln u)^{\theta_i - 1} \exp\left(-\left((-\ln u)^{\theta_i} + (-\ln t)^{\theta_i} \right)^{1/\theta_i} \right)}{u (1-u)^{1/\alpha} \left((-\ln u)^{\theta_i} + (-\ln t)^{\theta_i} \right)^{1-1/\theta_i}} du \right),$$

where

$$\overline{C}_{\theta}(1-s,1-t)=1-s-t+C_{\theta_i}^{Gumbel}(s,t).$$

The Gumbel copula be the appropriate choice in modelisation of dependency if the outcomes are known to be strongly correlated at high values but less correlated at low values.

4 Conclusion notes

One of the most important strategy in investment is to divide the capital of investment in more then one market, but the most important question that if this markets are linked and if one of them is collapsed. Do the rest of the market interrelated collapse as well?

Therefore, to reduce the risk, in preference for this markets to be independent, or preferably for the investors to choose the independent markets or the less dependent one to invests their money.

In this paper we give a new risk measure called copula conditional tail expectation which preserve the property of coherence. This measure aid to understanding the relationships among multivariate assets and to help us greatly about how best to position our investments and enhance our financial risk protection.

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A Appendix

Throughout all this Proofs, we suppose that C is absolutely continuous with density c, and X and Y are continuous random variables with df's F and G, respectively.

A.1 Proof of Corollary 2.1

From (2.5) we have

$$\overline{C}(1-s,1-t)=1-s-t+C\left(s,t\right),$$

and we can write

$$\overline{C}(1-s,1-t) = 1 - \int_0^1 \int_0^s c(u,v)\,dudv \tag{1.13}$$
$$- \int_0^t \int_0^1 c(u,v)\,dudv + \int_0^t \int_0^s c(u,v)\,dudv.$$

Observe that

$$1 = \int_0^1 \int_0^1 c(u,v)\,dudv = \int_0^1 \int_0^s c(u,v)\,dudv + \int_0^1 \int_s^1 c(u,v)\,dudv. \tag{1.14}$$

Substituting the expression (1.14) in (1.13) leads to the following expression of

$$\begin{aligned}
\overline{C}(1-s,1-t) &= \int_0^1 \int_s^1 c(u,v)\,dudv - \int_0^t \int_0^1 c(u,v)\,dudv + \int_0^t \int_0^s c(u,v)\,dudv \\
&= \int_0^1 \int_s^1 c(u,v)\,dudv - \int_0^t \int_0^s c(u,v)\,dudv - \int_0^t \int_s^1 c(u,v)\,dudv \\
&+ \int_0^t \int_0^s c(u,v)\,dudv \\
&= \int_t^1 \int_s^1 c(u,v)\,dudv \\
&= \int_s^1 J_t(u)\,du.
\end{aligned}$$

□

A.2 Proof of Proposition 2.1

By calculating we have

$$\begin{aligned}
&\mathbb{P}\left(X \middle| X \geq VaR_X(s), Y \geq VaR_Y(t)\right) \\
&= \frac{\mathbb{P}\left(X \leq x, X > VaR_X(s), Y > VaR_Y(t)\right)}{\mathbb{P}\left(X > VaR_X(t), Y > VaR_Y(s)\right)} \\
&= \frac{\mathbb{P}\left(VaR_X(s) < X \leq x, Y \geq VaR_Y(t)\right)}{\mathbb{P}\left(X > VaR_X(s), Y > VaR_Y(t)\right)} \\
&= \frac{\mathbb{P}\left(VaR_X(s) < X \leq x, Y \geq VaR_Y(t)\right)}{1 - \mathbb{P}\{X \leq F^{-1}(s)\} - \mathbb{P}\{Y \leq G^{-1}(t)\} + \mathbb{P}\{X \leq F^{-1}(s), Y \leq G^{-1}(t)\}} \\
&= \frac{\mathbb{P}\left(VaR_X(s) < X \leq x, Y \geq VaR_Y(t)\right)}{1 - \mathbb{P}\{F(X) \leq s\} - \mathbb{P}\{G(Y) \leq t\} + \mathbb{P}\{F(X) \leq s, G(Y) \leq t\}}.
\end{aligned}$$

On the other hand, we have

$$\mathbb{P}\left(VaR_X(s) < X \leq x, Y \geq VaR_Y(t)\right) = \int_{VaR_X(t)}^{\infty} \int_{VaR_Y(s)}^{x} \frac{\partial^2 C\left(F(u), G(v)\right)}{\partial u \partial v} dudv,$$

and

$$1-\mathbb{P}\{F(X)\le s\}-\mathbb{P}\{G(Y)\le t\}+\mathbb{P}\{F(X)\le s,G(Y)\le t\}=1-s-t+C(s,t)$$
$$=\overline{C}(1-s,1-t).$$

Then

$$\begin{aligned}&\mathbb{P}(X|\,X\ge VaR_X(s),Y\ge VaR_Y(t))\\&=\frac{1}{\overline{C}(1-s,1-t)}\int_{VaR_X(t)}^{\infty}\int_{VaR_Y(s)}^{x}\frac{\partial^2 C(F(u),G(v))}{\partial u\partial v}dudv,\end{aligned}$$

Then the CCTE is given by

$$\mathbb{CCTE}_X(s,t)=\frac{1}{\overline{C}(1-s,1-t)}\int_{VaR_X(t)}^{\infty}\int_{VaR_Y(s)}^{\infty}x\frac{\partial^2 C(F(x),G(y))}{\partial x\partial y}dxdy.$$

We suppose that the densities of F and G are f and g, respectively, then

$$\mathbb{CCTE}_X(s,t)=\frac{1}{\overline{C}(1-s,1-t)}\int_{VaR_X(t)}^{\infty}\int_{VaR_Y(s)}^{\infty}xc(F(x),G(y))f(x)g(y)dxdy.$$

Transforming by $F(x)=u$ and $G(y)=v$, then

$$\begin{aligned}\mathbb{CCTE}_X(s,t)&=\frac{1}{\overline{C}(1-s,1-t)}\int_t^1\int_s^1 F^{-1}(u)c(u,v)dudv.\\&=\frac{1}{\overline{C}(1-s,1-t)}\int_s^1 F^{-1}(u)\left(\int_t^1 c(u,v)dv\right)du.\end{aligned}$$

By Corollary 2.1 it follow that

$$\int_s^1 J_t(u)du=\overline{C}(1-s,1-t),$$

where $J_t(u)$ is given in (2.6). Then

$$\mathbb{CCTE}_X(s,t)=\frac{\int_s^1 J_t(u)F^{-1}(u)du}{\int_s^1 J_t(u)du}.$$

This close the proof of proposition 2.1. □

A.3 Proof of Theorem 2.1

Let H be the joint df of bivariate random vector (X,Y) such that

$$H(x,y)=C(F(x),G(y)),$$

The conditional probability density of X and Y occurring given that $B_1 := \{X > VaR(s)\}$ and $B_2 := \{Y > VaR(t)\}$ occurs, and compute it using the formula

$$f_{B_1}(x) = \int_{-\infty}^{+\infty} h(x,y)\,\mathbb{1}_{B_1} dy$$

where

$$h(x,y) = c(F(x), G(y))\, f(x)\, g(y),$$

and

$$f(x) = \int_{-\infty}^{+\infty} h(x,y)\, dy \text{ and } g(y) = \int_{-\infty}^{+\infty} h(x,y)\, dx.$$

Then

$$\begin{aligned}
\mathbb{CTE}_X(s) &= \frac{1}{1-s}\int_{-\infty}^{+\infty} x f_{B_1}(x)\, dx \\
&= \frac{1}{1-s}\int_{VaR(s)}^{+\infty} x\left(\int_{-\infty}^{+\infty} h(x,y)\, dy\right) dx \\
&= \frac{1}{1-s}\int_{VaR(s)}^{+\infty}\int_{-\infty}^{+\infty} xc(F(x), G(y))\, f(x)\, g(y)\, dxdy,
\end{aligned}$$

by variables change, we have

$$\begin{aligned}
\mathbb{CTE}_X(s) &= \frac{1}{1-s}\int_s^1 F^{-1}(u)\, du \int_0^1 c(u,v)\, dv, \\
&= \frac{1}{1-s}\int_s^1 F^{-1}(u)\, du \int_0^t c(u,v)\, dv \\
&+ \frac{1}{1-s}\int_s^1 F^{-1}(u)\, du \int_t^1 c(u,v)\, dv \\
&:= \frac{1}{1-s}(A_0 + A_t)
\end{aligned}$$

we have that B_2 occur, then

$$\begin{aligned}
A_0 &= \frac{1}{1-s}\int_s^1 F^{-1}(u)\, du \int_0^t c(u,v)\, dv \\
&= \frac{1}{1-s}\int_{VaR(s)}^{+\infty} x\left(\int_{-\infty}^{+\infty} h(x,y)\,\mathbb{1}_{\{Y\leq VaR(t)\}} dy\right) dx \\
&= 0
\end{aligned}$$

then

$$\begin{aligned}
\mathbb{CTE}_X(s) &= \frac{1}{1-s}\int_s^1 F^{-1}(u)\, du \int_t^1 c(u,v)\, dv, \\
&= \frac{1}{1-s}\int_s^1 J_t(u)\, F^{-1}(u)\, du.
\end{aligned}$$

Thus

$$\begin{aligned}\overline{C}(1-s,1-t) &= 1-s-t+C(s,t)\\ &\leq 1-s-t+st\\ &\leq 1-s-t(1-s)\\ &\leq 1-s.\end{aligned}$$

Then

$$\frac{1}{1-s}\leq\frac{1}{\overline{C}(1-s,1-t)}.$$

Therefore

$$\begin{aligned}\mathbb{CTE}_X(s) &= \frac{1}{1-s}\int_s^1 J_t(u)\,F^{-1}(u)\,du\\ &\leq \frac{1}{\overline{C}(1-s,1-t)}\int_s^1 J_t(u)\,F^{-1}(u)\,du\\ &= \frac{\int_s^1 J_t(u)\,F^{-1}(u)\,du}{\int_s^1 J_t(u)\,du} = \mathbb{CCTE}_X(s;t)\end{aligned}$$

□

A.4 Proof of Corollary 2.2

In the non-dependency case the events B_1 and B_2 are independent, so

$$\begin{aligned}&\mathbb{P}(X|\,X\geq VaR_X(s),Y\geq VaR_Y(t))\\ &=\frac{\mathbb{P}(VaR_X(s)<X\leq x)\,\mathbb{P}(Y\geq VaR_Y(t))}{1-\mathbb{P}(F(X)\leq s)-\mathbb{P}(G(Y)\leq t)+\mathbb{P}(F(X)\leq s)\,\mathbb{P}(G(Y)\leq t)}\\ &=\frac{\mathbb{P}(VaR_X(s)<X\leq x)\,\mathbb{P}(Y\geq VaR_Y(t))}{(1-\mathbb{P}(F(X)\leq s))(1-\mathbb{P}(G(Y)\leq t))}\\ &=\frac{\mathbb{P}(VaR_X(s)<X\leq x)}{\mathbb{P}(X\geq VaR_X(s))}\\ &=\mathbb{CTE}_X(s)\end{aligned}$$

□

A.5 Proof of Proposition 3.1

Let $(X_i, T), i = 1, 2, 3$ be a bivariate r.v. with FGM copula df of parameter $\theta_i, i = 1, 2, 3$ respectively, then by Theorem (2.1) we have that

$$\int_s^1 J_t(u)\, du = \overline{C}(1-s, 1-t)$$
$$= 1 - s - t + st + \theta_i st(1-s)(1-t). \tag{1.15}$$

Now we calculate for $i = 1, 2, 3$

$$\begin{aligned}\int_s^1 J_t(u) F^{-1}(u)\, du &= \int_s^1 (1-u)^{-1/\alpha} (\theta_i - 2u\theta_i - 2v\theta_i + 4uv\theta_i + 1)\, dudv \\ &= \int_t^1 (\theta_i - 2\theta_i v + 1)\, dv \int_s^1 (1-u)^{-1/\alpha}\, du \\ &+ 2\theta_i \int_t^1 (2v-1)\, dv \int_s^1 u(1-u)^{-1/\alpha}\, du,\end{aligned}$$

then

$$\int_s^1 J_t(u) F^{-1}(u)\, du = \frac{\alpha(1-t)(2\alpha + t\theta_i - 2st\theta_i + 2st\alpha\theta_i - 1)}{2\alpha^2 - 3\alpha + 1}(1-s)^{1-\frac{1}{\alpha}}. \tag{1.16}$$

Finely, by substitution (1.15) and (1.16) in (2.7) we get:

$$\begin{aligned}\mathbb{CCTE}_X(s;t) &= \frac{\dfrac{\alpha(1-t)(2\alpha + t\theta_i - 2st\theta_i + 2st\alpha\theta_i - 1)}{(2\alpha^2 - 3\alpha + 1)}(1-s)^{1-\frac{1}{\alpha}}}{(1 - s - t + st + \theta_i st(1-s)(1-t))} \\ &= \frac{\alpha(2\alpha + t\theta_i - 2st\theta_i + 2st\alpha\theta_i - 1)}{(st\theta_i + 1)(2\alpha^2 - 3\alpha + 1)}(1-s)^{-1/\alpha}.\end{aligned}$$

This completes the proof of Proposition 3.1. □

A.6 Proof of Corollary 3.1

Let denote by

$$C_u(u, v) := \frac{\partial}{\partial u} C(u, v),$$

then by Corollary 2.1, we have

$$\begin{aligned}J_t(u) &= \int_t^1 c(u, v)\, dv = C_u(u, v)]_t^1 \\ &= C_u(u, 1) - C_u(u, t).\end{aligned}$$

So, C is Archimedean copula, then

$$C_u(u,v) = \frac{\psi'(u)}{\psi'(C(u,v))}.$$

Finely, we get (3.11) by the propriety of copula that $C(u,1) = u$. $\square$