Absolute continuity of the best Sobolev constant of a bounded domain

Grey Ercole *
Departamento de Matemática - ICEx, Universidade Federal de Minas Gerais,
Av. Antônio Carlos 6627, Caixa Postal 702, 30161-970, Belo Horizonte, MG, Brazil

June 4, 2012

Abstract

Let $\lambda_q := \inf\left\{\|\nabla u\|^p_{L^p(\Omega)} / \|u\|^p_{L^q(\Omega)} : u \in W_0^{1,p}(\Omega)\setminus\{0\}\right\}$, where Ω is a bounded and smooth domain of $\mathbb{R}^N$, $1 < p < N$ and $1 \le q \le p^\star := \frac{Np}{N-p}$. We prove that the function $q \mapsto \lambda_q$ is absolutely continuous in the closed interval $[1, p^\star]$.

2000 Mathematics Subject Classification. 46E35; 35J25; 35J70.
Keywords: Absolute continuity, Lipschitz continuity, p-Laplacian, Rayleigh quotient, Sobolev best constants.

1 Introduction.

Let Ω be a bounded and smooth domain of Euclidean space $\mathbb{R}^N$, $N \ge 2$, and let $1 < p < N$. For each $1 \le q \le p^\star := \dfrac{Np}{N-p}$, let $\mathcal{R}_q : W_0^{1,p}(\Omega)\setminus\{0\} \longrightarrow \mathbb{R}$ be the Rayleigh quotient associated with the Sobolev immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. That is,

$$\mathcal{R}_q(u) := \left(\int_\Omega |\nabla u|^p\,dx\right)\left(\int_\Omega |u|^q\,dx\right)^{-\frac{q}{p}} = \frac{\|\nabla u\|_p^p}{\|u\|_q^p}$$

where $\|\cdot\|_s := \left(\int_\Omega |\cdot|^s\,dx\right)^{\frac{1}{s}}$ denotes the usual norm of $L^s(\Omega)$.

It is well-known that the immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous if $1 \le q \le p^\star$ and compact if $1 \le q < p^\star$. Hence, there exist

$$\lambda_q := \inf\left\{\mathcal{R}_q(u) : u \in W_0^{1,p}(\Omega)\setminus\{0\}\right\},\ 1 \le q \le p^\star \tag{1}$$

and $w_q \in W_0^{1,p}(\Omega)\setminus\{0\}$ such that

$$\mathcal{R}_q(w_q) = \lambda_q,\ \ 1 \le q < p^\star. \tag{2}$$

*E-mail: grey@mat.ufmg.br. The author was supported by FAPEMIG and CNPq, Brazil.

Since $\mathcal{R}_q$ is homogeneous of degree zero the extremal function w_q for the Rayleigh quotient can be chosen such that $\|w_q\|_q = 1$.

It is straightforward to verify that such a normalized extremal w_q is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u &= \lambda_q |u|^{q-2} u \quad \text{in } \Omega \\ u &= 0 \quad\quad\quad\quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

for the p-Laplacian operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. Hence, classical results imply that w_q can still be chosen to be positive in Ω and that $w_q \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.

In the case $q = p$, the constant λ_p is the well-known first eigenvalue of the Dirichlet p-Laplacian and w_p is the correspondent eigenfunction L^p-normalized.

If $q = 1$ the pair (λ_1, w_1) is obtained from the *Torsional Creep Problem*:

$$\begin{cases} -\Delta_p u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4}$$

In fact, if ϕ_p is the *torsion function* of Ω, that is, the solution of (4), then it easy to check that the only positive weak solution of (3) with $q = 1$ is $\lambda_1^{\frac{1}{p-1}} \phi_p$. Thus, $w_1 = \lambda_1^{\frac{1}{p-1}} \phi_p$ and since $\|w_1\|_1 = 1$ one has

$$\lambda_1 = \frac{1}{\|\phi_p\|_1^{p-1}} \text{ and } w_1 = \frac{\phi_p}{\|\phi_p\|_1}. \tag{5}$$

In the particular case where $\Omega = B_R(x_0)$, the ball of radius $R > 0$ centered at $x_0 \in \mathbb{R}^N$, the torsion function is explicitly given by $\phi_p(x) = \Phi_p(|x - x_0|)$ where

$$\Phi_p(r) := \frac{p-1}{p} N^{-\frac{1}{p-1}} \left(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}} \right), \; 0 \le r \le R.$$

Hence, for $\Omega = B_R(x_0)$ one obtains

$$\lambda_1 = \left[\frac{p + N(p-1)}{\omega_N (p-1)} \right]^{p-1} \frac{N}{R^{(p^*-1)(N-p)}} \tag{6}$$

and

$$w_1(x) = \frac{p + N(p-1)}{p\omega_N R^N} \left(1 - (|x - x_0| / R)^{\frac{p}{p-1}} \right)$$

where ω_N is the N dimensional Lebesgue volume of the unit ball $B_1(0)$. (More properties of the torsion function and some of its applications are given in [4, 7].)

In the critical case $q = p^\star$ extremals for the Rayleigh quotient exist if the domain is the whole Euclidean space $\mathbb{R}^N$. In fact, in $\mathbb{R}^N$ one has the *Sobolev Inequality*

$$\mathcal{S}_{p,N} \|u\|_{L^{p^\star}(\mathbb{R}^N)} \le \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N) \tag{7}$$

where (see [2, 9]):

$$\mathcal{S}_{p,N} := \sqrt{\pi} N^{\frac{1}{p}} \left(\frac{N-p}{p-1} \right)^{\frac{p-1}{p}} \left(\frac{\Gamma(N/p)\Gamma(1+N-N/p)}{\Gamma(1+N/2)\Gamma(N)} \right)^{\frac{1}{N}} \tag{8}$$

and $\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}ds$ is the Gamma Function. The *Sobolev constant* $\mathcal{S}_{p,N}$ is optimal and achieved, necessarily, by radially symmetric functions of the form (see [9]):

$$w(x) = a\left(1 + b\,|x - x_0|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}} \tag{9}$$

for any $a \neq 0$, $b > 0$ and $x_0 \in \mathbb{R}^N$.

A remarkable fact is that for any domain Ω (open, but non-necessarily bounded) the Sobolev constant $\mathcal{S}_{p,N}$ is still sharp with respect to the inequality (7), that is:

$$\mathcal{S}_{p,N}^p = \lambda_{p^\star} := \inf\left\{\mathcal{R}_{p^\star}(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\}\right\}. \tag{10}$$

This property of the critical case $q = p^\star$ may be easily verified by using a simple scaling argument. As a consequence, in this critical case, the only domain Ω whose the Rayleigh quotient has an extremal is $\mathbb{R}^N$. Indeed, if $w \in W_0^{1,p}(\Omega)/\{0\}$ is an extremal for the Rayleigh quotient in Ω, then (by extending w to zero out of Ω) w is also an extremal for the Rayleigh quotient in $\mathbb{R}^N$. This implies that w must have an expression as in (9) and hence its support must be the whole space $\mathbb{R}^N$, forcing thus the equality $\Omega = \mathbb{R}^N$.

In this paper we are concerned with the behavior of λ_q with respect to $q \in [1, p^\star]$. Thus, we investigate the function $q \mapsto \lambda_q$ defined by (1). We prove that this function is of bounded variation in $[1, p^\star]$, Lipschitz continuous in any closed interval of the form $[1, p^\star - \epsilon]$ for $\epsilon > 0$, and left-continuous at $q = p^\star$. These combined results imply that λ_q is absolute continuous on $[1, p^\star]$.

Up to our knowledge, the only result about the continuity of the function $q \mapsto \lambda_q$ is given in [6, Thm 2.1], where the author proves the continuity of this function in the open interval $(1, p)$ and the lower semi-continuity in the open interval $(p, p^\star)$.

Besides the theoretical aspects, our results are also important for the computational approach of the Sobolev constants λ_q, since these constants or the correspondent extremals are not explicitly known in general, even for simple bounded domains. For recent numerical approaches related to Sobolev type constants we refer to [1, 5].

This paper is organized as follows. In Section 2 we derive a formula that describes the dependence of $\mathcal{R}_q$ with respect to q and obtain, in consequence, the bounded variation of the function $q \mapsto \lambda_q$ in the closed interval $[1, p^\star]$ and also the left-continuity of this function at $q = p^\star$. Still in Section 2 we obtain a upper bound for $\mathcal{S}_{p,N}$ (see (17)) and we also show that for $1 \leq q < p^\star$ the Sobolev constant λ_q of bounded domains Ω tends to zero when these domains tend to $\mathbb{R}^N$.

By applying set level techniques, we deduce in Section 3 some estimates for w_q and in Section 4 we combine these estimates with the formula derived in Section 2 to prove the Lipschitz continuity of the function $q \mapsto \lambda_q$ in each closed interval of the form $[1, p^\star - \epsilon]$. Our results are in fact proved for the function $q \mapsto |\Omega|^{\frac{p}{q}}\lambda_q$ where $|\Omega|$ denotes the N-dimensional Lebesgue volume of Ω. But, of course, they are automatically transferred to the function $q \mapsto \lambda_q$.

2 Bounded variation and left-continuity

We first describe the dependence of the Rayleigh quotient $\mathcal{R}_q(u)$ with respect to the parameter q.

Lemma 1 *Let* $0 \not\equiv u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. *Then, for each* $1 \leq s_1 < s_2 \leq p^*$ *one has*

$$|\Omega|^{\frac{p}{s_1}} \mathcal{R}_{s_1}(u) = |\Omega|^{\frac{p}{s_2}} \mathcal{R}_{s_2}(u) \exp\left(p \int_{s_1}^{s_2} \frac{K(t,u)}{t^2} dt\right) \tag{11}$$

where

$$K(t,u) := \frac{\int_\Omega |u|^t \ln |u|^t \, dx}{\|u\|_t^t} + \ln\left(|\Omega| \, \|u\|_t^{-t}\right) \geq 0. \tag{12}$$

Before proving Lemma 1 let us make a technical remark related to the assumptions of this lemma. If $u \in W_0^{1,p}(\Omega)$ and $1 \leq t < p^\star$, then

$$\int_\Omega |u|^t \left|\ln |u|^t\right| dx = \int_{|u|<1} |u|^t \left|\ln |u|^t\right| dx + t \int_{|u|\geq 1} |u|^t \ln |u| \, dx$$

$$\leq \frac{|\Omega|}{e} + t \int_{|u|\geq 1} |u|^t \frac{|u|^{p^\star - t}}{e(p^\star - t)} dx \leq \frac{|\Omega|}{e} + \frac{p^\star \|u\|_{p^\star}^{p^\star}}{e(p^\star - t)} < \infty.$$

However, we were not able to determine the finiteness of the integral $\int_\Omega |u|^{p^\star} |\ln |u|| \, dx$ without assuming that $u \in L^\infty(\Omega)$. Fortunately, the assumption $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ will be sufficient to our purposes in this paper.
Proof of Lemma 1. We firstly note that

$$\frac{d}{dq} \ln\left(\frac{|\Omega|^{\frac{1}{q}}}{\|u\|_q}\right) = \frac{d}{dq} \frac{\ln |\Omega|}{q} - \frac{d}{dq}\left(\frac{1}{q} \ln \int_\Omega |u|^q \, dx\right)$$

$$= -\frac{1}{q^2} \ln |\Omega| - \frac{1}{q^2}\left[- \ln \|u\|_q^q + \frac{\int_\Omega |u|^q \ln |u|^q \, dx}{\|u\|_q^q}\right] = -\frac{K(q,u)}{q^2}.$$

Thus, integration on the interval $[s_1, s_2]$ gives

$$\frac{|\Omega|^{\frac{1}{s_2}}}{\|u\|_{s_2}} = \frac{|\Omega|^{\frac{1}{s_1}}}{\|u\|_{s_1}} \exp\left(- \int_{s_1}^{s_2} \frac{K(t,u)}{t^2} dt\right)$$

from what (11) follows easily.

Since the continuous function $h : [0, +\infty) \longrightarrow \mathbb{R}$ defined by $h(\xi) := \xi \ln \xi$, if $\xi > 0$, and $h(0) = 0$ is convex, it follows from Jensen's inequality that

$$h\left(|\Omega|^{-1} \int_\Omega |u|^t \, dx\right) \leq |\Omega|^{-1} \int_\Omega h(|u|^t) dx,$$

thus yielding

$$\|u\|_t^t \ln\left(|\Omega|^{-1} \|u\|_t^t\right) \leq \int_\Omega |u|^t \ln |u|^t \, dx,$$

from what follows that $K(t,u)$ defined in (12) is nonnegative. □

Proposition 2 *The function* $q \mapsto |\Omega|^{\frac{p}{q}} \lambda_q$ *is strictly decreasing in* $[1, p^\star]$.

Proof. Let $1 \le s_1 < s_2 \le p^\star$ and $w_{s_1} \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ the positive and L^{s_1}-normalized extremal of the Rayleigh quotient $\mathcal{R}_{s_1}$. Note from the definition of w_{s_1} that

$$-\Delta_p w_{s_1} = \lambda_{s_1} w_{s_1}^{s_1 - 1} \text{ in } \Omega. \tag{13}$$

It follows from Lemma 1 that

$$|\Omega|^{\frac{p}{s_1}} \lambda_{s_1} = |\Omega|^{\frac{p}{s_2}} \mathcal{R}_{s_2}(w_{s_1}) \exp\left(p \int_{s_1}^{s_2} \frac{K(t, w_{s_1})}{t^2} dt \right) \ge |\Omega|^{\frac{p}{s_2}} \mathcal{R}_{s_2}(w_{s_1}) > |\Omega|^{\frac{p}{s_2}} \lambda_{s_2}$$

since $\mathcal{R}_{s_1}(w_{s_1}) = \lambda_{s_1}$, $K(t, w_{s_1}) \ge 0$ and $\mathcal{R}_{s_2}(w_{s_1}) > \lambda_{s_2}$. We need only to guarantee the strictness of the last inequality. Obviously, if $s_2 = p^\star$ the inequality is really strict because the Rayleigh quotient $\mathcal{R}_{p\star}$ does not reach a minimum value. Thus, let us suppose that $\lambda_{s_2} = \mathcal{R}_{s_2}(w_{s_1})$ for $s_2 < p^\star$. Then

$$-\Delta_p (w_{s_1} / \|w_{s_1}\|_{s_2}) = \lambda_{s_2} (w_{s_1} / \|w_{s_1}\|_{s_2})^{s_2 - 1}$$

and hence the $(p-1)$-homogeneity of the operator Δ_p yields

$$-\Delta_p w_{s_1} = \lambda_{s_2} \|w_{s_1}\|_{s_2}^{p - s_2} w_{s_1}^{s_2 - 1} \text{ in } \Omega. \tag{14}$$

The combining of (13) with (14) produces

$$w_{s_1} \equiv \left(\frac{\lambda_{s_1} \|w_{s_1}\|_{s_2}^{s_2 - p}}{\lambda_{s_2}} \right)^{\frac{1}{s_2 - s_1}} \text{ in } \Omega.$$

Since the only constant function in $W_0^{1,p}(\Omega)$ is the null function we arrive at the contradiction $0 \equiv w_{s_1} > 0$ in Ω.

Thus, we have concluded that $|\Omega|^{\frac{p}{s_1}} \lambda_{s_1} > |\Omega|^{\frac{p}{s_2}} \lambda_{s_2}$ for $1 \le s_1 < s_2 \le p^\star$. □

The following corollary is immediate after writing λ_q as a product of two monotonic functions: $\lambda_q = |\Omega|^{-\frac{p}{q}} (|\Omega|^{\frac{p}{q}} \lambda_q)$.

Corollary 3 *The function* $q \mapsto \lambda_q$ *is of bounded variation in* $[1, p^\star]$.

Another consequence of Proposition 2 is that for each $1 \le q < p^*$ the Sobolev constant λ_q of a bounded domain Ω tends to zero as $\Omega \nearrow \mathbb{R}^N$. In fact, this asymptotic behavior follows from the following corollary.

Corollary 4 *Let* $B_R(x_0) \subset \mathbb{R}^N$ *denote the ball centered at* x_0 *and with radius* R *and let*

$$\lambda_q(R) := \min \left\{ \frac{\|\nabla u\|_{L^p(B_R(x_0))}^p}{\|u\|_{L^q(B_R(x_0))}^p} : u \in W_0^{1,p}(B_R(x_0)) / \{0\} \right\}, \quad 1 \le q < p^\star.$$

Then

$$\lambda_q(R) \to 0 \quad \text{as} \quad R \to \infty. \tag{15}$$

Proof. It follows from Proposition 2 that

$$\lambda_q(R) \le \lambda_1(R)(\omega_N R^N)^{p(1-\frac{1}{q})}$$

where, as before, $\omega_N = |B_1(0)|$.

Now, replacing $\lambda_1(R)$ by its expression (6) we obtain

$$\lambda_q(R) \le \left[\frac{p+N(p-1)}{\omega_N(p-1)}\right]^{p-1} \frac{N(\omega_N)^{p(1-\frac{1}{q})}}{R^{(N-p)(\frac{p^\star}{q}-1)}}, \tag{16}$$

yielding (15). □

Remark 5 *Since* $\lambda_{p^*}(R) \equiv \mathcal{S}_{N,p}^p$, $\omega_N = \pi^{N/2}/\Gamma(1+N/2)$ *and* $\frac{1}{p} - \frac{1}{p^\star} = \frac{1}{N}$, *by making* $q = p^*$ *in (16) we obtain the following upper bound for* $\mathcal{S}_{N,p}$ *with is quite comparable with the expression (8):*

$$\mathcal{S}_{N,p} \le \sqrt{\pi} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}} \frac{(p^*-1)^{\frac{p-1}{p}}}{\Gamma(1+N/2)^{\frac{1}{N}}}. \tag{17}$$

We now prove the left-continuity of the function $q \mapsto \lambda_q$ in the interval $(1, p^\star]$. Hence, as a particular case we obtain

$$\lim_{q\to(p^\star)^-} \lambda_q = \lambda_{p^\star} \quad (= \mathcal{S}_{p,N}^p). \tag{18}$$

Theorem 6 *For each* $q \in (1, p^\star]$ *it holds* $\lim_{s\to q^-} \lambda_s = \lambda_q$.

Proof. Let us fix $s < q$ and $u \in C_c^\infty(\Omega) \setminus \{0\}$. If follows from Lemma 1 and Proposition 2 that

$$|\Omega|^{\frac{p}{q}} \lambda_q < |\Omega|^{\frac{p}{s}} \lambda_s \le |\Omega|^{\frac{p}{s}} \mathcal{R}_s(u) = |\Omega|^{\frac{p}{q}} \mathcal{R}_q(u) \exp\left(p \int_s^q \frac{K(t,u)}{t^2} dt\right). \tag{19}$$

For $s \le t \le q$ Hölder's inequality implies that

$$|\Omega|^{-\frac{1}{s}} \|u\|_s \le |\Omega|^{-\frac{1}{t}} \|u\|_t \le |\Omega|^{-\frac{1}{q}} \|u\|_q.$$

Hence, since $|\Omega|^{-\frac{1}{s}} \|u\|_s \to |\Omega|^{-\frac{1}{q}} \|u\|_q$ as $s \to q$ we obtain

$$\frac{|\Omega|^{-\frac{1}{q}} \|u\|_q}{2} \le |\Omega|^{-\frac{1}{s}} \|u\|_s \le |\Omega|^{-\frac{1}{t}} \|u\|_t \le |\Omega|^{-\frac{1}{q}} \|u\|_q$$

for $s \le t \le q$ with s sufficiently close to q.

It follows from these estimates that

$$\begin{aligned} K(t,u) &= \frac{t \int_\Omega |u|^t \ln |u| \, dx}{\|u\|_t^t} + t \ln\left(\frac{|\Omega|^{\frac{1}{t}}}{\|u\|_t}\right) \\ &\le \frac{t \ln \|u\|_\infty \int_\Omega |u|^t \, dx}{\|u\|_t^t} + t \ln\left(\frac{2\,|\Omega|^{\frac{1}{q}}}{\|u\|_q}\right) = t \ln\left(\frac{2\,|\Omega|^{\frac{1}{q}} \|u\|_\infty}{\|u\|_q}\right) =: t M_q(u). \end{aligned}$$

Therefore,

$$\exp\left(p\int_s^q \frac{K(t,u)}{t^2}dt\right) \le \exp\left(pM_q(u)\ln(\frac{q}{s})\right) = \left(\frac{q}{s}\right)^{pM_q(u)}$$

and (19) yields

$$|\Omega|^{\frac{p}{q}}\lambda_q < |\Omega|^{\frac{p}{s}}\lambda_s \le |\Omega|^{\frac{p}{q}}\mathcal{R}_q(u)\left(\frac{q}{s}\right)^{pM_q(u)}.$$

By making $s \to q^-$ we conclude that

$$\lambda_q \le \liminf_{s\to q^-}\lambda_s \le \limsup_{s\to q^-}\lambda_s \le \mathcal{R}_q(u)$$

for each $u \in C_c^\infty(\Omega)\setminus\{0\}$. Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ this clearly implies that

$$\lambda_q \le \liminf_{s\to q^-}\lambda_s \le \limsup_{s\to q^-}\lambda_s \le \mathcal{R}_q(u) \text{ for all } u \in W_0^{1,p}(\Omega)\setminus\{0\}.$$

Therefore,

$$\lambda_q \le \liminf_{s\to q^-}\lambda_s \le \limsup_{s\to q^-}\lambda_s \le \lambda_q$$

from what follows that $\lim_{s\to q^-}\lambda_s = \lambda_q$. □

3 Bounds for w_q

In this section we deduce some bounds for the extremal w_q defined by (2). Our results are based on level set techniques and inspired by [3] and [8].

Proposition 7 *Let $1 \le q < p^\star$ and $\sigma \ge 1$. Then, it holds*

$$2^{-\frac{N(p-1)+\sigma p}{p}} C_q \left\|w_q\right\|_\infty^{\frac{N(p-q)+\sigma p}{p}} \le \left\|w_q\right\|_\sigma^\sigma \tag{20}$$

where

$$C_q := \left(\frac{p}{p+N(p-1)}\right)^{N+1}\left(\frac{\mathcal{S}_{N,p}^p}{\lambda_q}\right)^{\frac{N}{p}}. \tag{21}$$

Proof. Since w_q is a positive weak solution of (3) we have that

$$\int_\Omega \left|\nabla w_q\right|^{q-2}\nabla w_q\cdot\nabla\phi dx = \lambda_q\int_\Omega w_q^{q-1}\phi dx \tag{22}$$

for all test function $\phi \in W_0^{1,p}(\Omega)$.

For each $0 < t < \left\|w_q\right\|_\infty$, define $A_t = \left\{x\in\Omega : w_q > t\right\}$. Since $w_q \in C^{1,\alpha}\left(\overline{\Omega}\right)$ for some $0<\alpha<1$ it follows that A_t is open and $\nabla(w_q-t)^+ = \nabla w_q$ in A_t.

Thus, the function

$$(w_q - t)^+ = \max\{w_q - t, 0\} = \begin{cases} w_q - t, & \text{if } w_q > t \\ 0, & \text{if } w_q \le t \end{cases}$$

belongs to $W_0^{1,p}(\Omega)$ and by using it as a test function in (22) we obtain

$$\int_{A_t} |\nabla w_q|^p \, dx = \lambda_q \int_{A_t} w_q^{q-1}(w_q - t)\, dx \le \lambda_q \|w_q\|_\infty^{q-1} (\|w_q\|_\infty - t) |A_t| . \tag{23}$$

Now, we estimate $\int_{A_t} |\nabla w_q|^p \, dx$ from below. Applying Hölder and Sobolev inequalities we obtain

$$\left(\int_{A_t} (w_q - t)\, dx \right)^p \le \left(\int_{A_t} (w_q - t)^{p^\star} dx \right)^{\frac{p}{p^\star}} |A_t|^{p - \frac{p}{p^\star}} \le \mathcal{S}_{N,p}^{-p} |A_t|^{p - \frac{p}{p^\star}} \int_{A_t} |\nabla w_q|^p \, dx$$

and thus,

$$\mathcal{S}_{N,p}^{p} |A_t|^{\frac{p}{p^\star} - p} \left(\int_{A_t} (w_q - t)\, dx \right)^p \le \int_{A_t} |\nabla w_q|^p \, dx.$$

By combining this inequality with (23) we obtain

$$\mathcal{S}_{N,p}^{p} |A_t|^{\frac{p}{p^\star} - p} \left(\int_{A_t} (w_q - t)\, dx \right)^p \le \lambda_q \|w_q\|_\infty^{q-1} (\|w_q\|_\infty - t) |A_t| .$$

Since $\frac{1}{p^\star} + \frac{1}{N} = \frac{1}{p}$ the previous inequality can be rewritten as

$$\left(\int_{A_t} (w_q - t)\, dx \right)^{\frac{N}{N+1}} \le \left[\lambda_q \mathcal{S}_{N,p}^{-p} \|w_q\|_\infty^{q-1} (\|w_q\|_\infty - t) \right]^{\frac{N}{p(N+1)}} |A_t| . \tag{24}$$

In the sequel we use twice the following Fubini's theorem: if $u \ge 0$ is measurable, $\sigma \ge 1$, and $E_\tau = \{x : u(x) > \tau\}$, then

$$\int_\Omega u(x)^\sigma dx = \sigma \int_0^\infty \tau^{\sigma - 1} |E_\tau|\, d\tau. \tag{25}$$

Let us define $g(t) := \int_{A_t} (w_q - t)\, dx$. It follows from (25) that

$$g(t) = \int_0^\infty \left| \{w_q - t > \tau\} \right| d\tau = \int_t^\infty \left| \{w_q > s\} \right| ds = \int_t^\infty |A_s|\, ds = \int_t^{\|w_q\|_\infty} |A_s|\, ds$$

and therefore $g'(t) = -|A_t| \le 0$. Thus, (24) can be written as

$$\left[\lambda_q \mathcal{S}_{N,p}^{-p} \|w_q\|_\infty^{q-1} (\|w_q\|_\infty - t) \right]^{-\frac{N}{p(N+1)}} \le -g(t)^{-\frac{N}{N+1}} g'(t)$$

and integration over the interval $[t, \|w_q\|_\infty]$ produces

$$C_q \|w_q\|_\infty^{-\frac{N(q-1)}{p}} (\|w_q\|_\infty - t)^{\frac{N(p-1)+p}{p}} \le g(t) \tag{26}$$

where C_q is given by (21).

By using the fact that $g(t) \le \left(\|w_q\|_\infty - t\right)|A_t|$ we obtain from (26) that

$$C_q \|w_q\|_\infty^{-\frac{N(q-1)}{p}} \left(\|w_q\|_\infty - t\right)^{\frac{N(p-1)}{p}} \le |A_t|.$$

If $\sigma \ge 1$, multiplying the previous inequality by $\sigma t^{\sigma-1}$ and integrating over the interval $[0, \|w_q\|_\infty]$, we get

$$C_q \|w_q\|_\infty^{-\frac{N(q-1)}{p}} \sigma \int_0^{\|w_q\|_\infty} \left(\|w_q\|_\infty - t\right)^{\frac{N(p-1)}{p}} t^{\sigma-1} dt \le \|w_q\|_\sigma^\sigma \tag{27}$$

since (25) gives

$$\|w_q\|_\sigma^\sigma = \int_\Omega w_q^\sigma dx = \sigma \int_0^{\|w_q\|_\infty} t^{\sigma-1} |A_t| \, dt.$$

The change of variable $t = \tau \|w_q\|_\infty$ produces

$$\int_0^{\|w_q\|_\infty} \left(\|w_q\|_\infty - t\right)^{\frac{N(p-1)}{p}} t^{\sigma-1} dt = \|w_q\|_\infty^{\frac{N(p-1)+\sigma p}{p}} \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\sigma-1} d\tau. \tag{28}$$

Since we have

$$\int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\sigma-1} d\tau \ge (1/2)^{\frac{N(p-1)}{p}} \int_0^{\frac{1}{2}} \tau^{\sigma-1} d\tau = \frac{2^{-\frac{N(p-1)+\sigma p}{p}}}{\sigma},$$

combining this with (27) and (28) produces (20). □

We remark from Theorem 6 that

$$\lim_{q \to p^\star} C_q = \left(\frac{p}{p + N(p-1)}\right)^{N+1} < 1. \tag{29}$$

Thus it follows from the monotonicity of the function $q \mapsto |\Omega|^{\frac{p}{q}} \lambda_q$ that both C_q and $(C_q)^{-1}$ are bounded.

Corollary 8 *If $1 \le q \le p^\star - \epsilon$, then*

$$|\Omega|^{-\frac{1}{q}} \le \|w_q\|_\infty \le \mathcal{C}_\epsilon \tag{30}$$

where $\mathcal{C}_\epsilon$ is a positive constant that depends on ϵ but not on q.

Proof. The first inequality is trivial, since $\|w_q\|_q = 1$. Let us suppose that $1 \le q \le p$. It follows from (20) with $\sigma = 1$ that

$$2^{-\frac{N(p-1)+p}{p}} C_q \|w_q\|_\infty^{\frac{N(p-q)+p}{p}} \le \|w_q\|_1 \le |\Omega|^{\frac{q-1}{q}} \|w_q\|_q = |\Omega|^{\frac{q-1}{q}}.$$

Thus,

$$\|w_q\|_\infty \le \widetilde{\mathcal{A}} := \max_{1 \le q \le p} \left(\frac{2^{\frac{N(p-1)+p}{p}} |\Omega|^{\frac{q-1}{q}}}{C_q}\right)^{\frac{p}{p+N(p-q)}}.$$

Now, let us consider $p \leq q \leq p^\star - \epsilon$. Then, by making $\sigma = q$ in (20) we obtain

$$2^{-\frac{N(p-1)+qp}{p}} C_q \left\| w_q \right\|_\infty^{\frac{N(p-q)+qp}{p}} \leq \left\| w_q \right\|_q^q = 1.$$

that is,

$$\left\| w_q \right\|_\infty \leq \widetilde{\mathcal{B}}_\epsilon := \max_{p \leq q \leq p^\star - \epsilon} \left[\frac{2^{\frac{N(p-1)+qp}{p}}}{C_q} \right]^{\frac{p}{(N-p)(p^\star - q)}},$$

since $N(p-q) + qp = (N-p)(p^\star - q)$.

Therefore, $\left\| w_q \right\|_\infty \leq \max \left\{ \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}_\epsilon \right\} := \mathcal{C}_\epsilon$. □

Note from (29) that $\widetilde{\mathcal{B}}_\epsilon \to \infty$ as $\epsilon \to 0^+$.

4 Absolute continuity

In this section we prove our main result: the absolute continuity of the function $q \mapsto \lambda_q$ in the closed interval $[1, p^\star]$. For this we first prove the Lipschitz continuity of the function $q \mapsto |\Omega|^{\frac{p}{q}} \lambda_q$ in each close interval of the form $[1, p^\star - \epsilon]$. Obviously, this is equivalent to the Lipschitz continuity of the function $q \mapsto \lambda_q$ in same interval.

Theorem 9 *For each $\epsilon > 0$, there exists a positive constant $\mathcal{L}_\epsilon$ such that*

$$\left| |\Omega|^{\frac{p}{s}} \lambda_s - |\Omega|^{\frac{p}{q}} \lambda_q \right| \leq \mathcal{L}_\epsilon |s - q|$$

for all $s, q \in [1, p^\star - \epsilon]$.

Proof. Without loss of generality let us suppose that $s < q$. Thus, the monotonicity of the function $\tau \mapsto |\Omega|^{\frac{p}{\tau}} \lambda_\tau$ implies

$$\left| |\Omega|^{\frac{p}{s}} \lambda_s - |\Omega|^{\frac{p}{q}} \lambda_q \right| = |\Omega|^{\frac{p}{s}} \lambda_s - |\Omega|^{\frac{p}{q}} \lambda_q.$$

Take $t \in \mathbb{R}$ so that $s \leq t \leq q$. It follows from (20) with $\sigma = 1$ that

$$2^{-\frac{N(p-1)+p}{p}} C_q \left\| w_q \right\|_\infty^{1+\frac{N(p-q)}{p}} \leq \left\| w_q \right\|_1 \leq |\Omega|^{1-\frac{1}{t}} \left\| w_q \right\|_t$$

and therefore

$$\frac{|\Omega|^{\frac{1}{t}} \left\| w_q \right\|_\infty}{\left\| w_q \right\|_t} \leq \frac{2^{\frac{N(p-1)+p}{p}} |\Omega|}{C_q \left\| w_q \right\|_\infty^{\frac{N(p-q)}{p}}}.$$

Hence, for $1 \leq q \leq p$ the first inequality in (30) gives

$$\frac{|\Omega|^{\frac{1}{t}} \left\| w_q \right\|_\infty}{\left\| w_q \right\|_t} \leq \frac{2^{\frac{N(p-1)+p}{p}} |\Omega|}{C_q \left\| w_q \right\|_\infty^{\frac{N(p-q)}{p}}} \leq \mathcal{A} := 2^{\frac{N(p-1)+p}{p}} \max_{1 \leq q \leq p} \frac{|\Omega|^{1+\frac{N(p-q)}{pq}}}{C_q}$$

while for $p \le q \le p^\star - \epsilon$ the second inequality in (30) gives

$$\frac{|\Omega|^{\frac{1}{t}} \left\|w_q\right\|_\infty}{\left\|w_q\right\|_t} \le \frac{2^{\frac{N(p-1)+p}{p}} |\Omega|}{C_q} \left\|w_q\right\|_\infty^{\frac{N(q-p)}{p}} \le \mathcal{B}_\epsilon := 2^{\frac{N(p-1)+p}{p}} |\Omega| \max_{p\le q\le p^\star-\epsilon} \frac{\mathcal{C}_\epsilon^{\frac{N(q-p)}{p}}}{C_q}.$$

Therefore,

$$\frac{|\Omega|^{\frac{1}{t}} \left\|w_q\right\|_\infty}{\left\|w_q\right\|_t} \le \mathcal{D}_\epsilon := \max\left\{\mathcal{A}, \mathcal{B}_\epsilon\right\}. \tag{31}$$

Thus,

$$K(t,w_q) = \frac{t\int_\Omega \left|w_q\right|^t \ln\left|w_q\right| dx}{\left\|w_q\right\|_t^t} + t\ln\left(\frac{|\Omega|^{\frac{1}{t}}}{\left\|w_q\right\|_t}\right)$$
$$\le \frac{t(\ln\left\|w_q\right\|_\infty)\int_\Omega \left|w_q\right|^t dx}{\left\|w_q\right\|_t^t} + t\ln\left(\frac{|\Omega|^{\frac{1}{t}}}{\left\|w_q\right\|_t}\right) = t\ln\left(\frac{|\Omega|^{\frac{1}{t}}\left\|w_q\right\|_\infty}{\left\|w_q\right\|_t}\right) \le t\mathcal{D}_\epsilon$$

and we obtain

$$\exp\left(p\int_s^q \frac{K(t,w_q)}{t^2}dt\right) \le \exp\left(p\mathcal{D}_\epsilon\int_s^q \frac{dt}{t}\right) = \left(\frac{q}{s}\right)^{p\mathcal{D}_\epsilon}.$$

But

$$|\Omega|^{\frac{p}{s}}\lambda_s \le |\Omega|^{\frac{p}{s}}\mathcal{R}(w_q)$$
$$= |\Omega|^{\frac{p}{q}}\mathcal{R}(w_q)\exp\left(p\int_s^q \frac{K(t,w_q)}{t^2}dt\right) = |\Omega|^{\frac{p}{q}}\lambda_q\exp\left(p\int_s^q \frac{K(t,w_q)}{t^2}dt\right)$$

yields

$$|\Omega|^{\frac{p}{s}}\lambda_s - |\Omega|^{\frac{p}{q}}\lambda_q \le |\Omega|^{\frac{p}{q}}\lambda_q\left[\exp\left(p\int_s^q \frac{K(t,w_q)}{t^2}dt\right) - 1\right]$$
$$\le |\Omega|^p\lambda_1\left[\exp\left(p\int_s^q \frac{K(t,w_q)}{t^2}dt\right) - 1\right] \le |\Omega|^p\lambda_1\left[\left(\frac{q}{s}\right)^{p\mathcal{D}_\epsilon} - 1\right].$$

Therefore,

$$0 < \frac{|\Omega|^{\frac{p}{s}}\lambda_s - |\Omega|^{\frac{p}{q}}\lambda_q}{q-s} \le \frac{|\Omega|^p\lambda_1}{s}\frac{\left(\frac{q}{s}\right)^{p\mathcal{D}_\epsilon} - 1}{\frac{q}{s} - 1} \le |\Omega|^p\lambda_1 H\left(\frac{q}{s}\right)$$

where

$$H(\xi) = \frac{\xi^{p\mathcal{D}_\epsilon} - 1}{\xi - 1}; \qquad 1 \le \xi \le p^\star - \epsilon.$$

Since $\lim_{\xi\to 1^+} H(\xi) = p\mathcal{D}_\epsilon$, we conclude that H is bounded in $[1, p^\star - \epsilon]$ and thus

$$\frac{\exp\left(p\int_s^q \frac{K(t,w_q)}{t^2}dt\right) - 1}{q-s} \le \mathcal{L}_\epsilon := \max_{1\le\xi\le p^\star-\epsilon} H(\xi).$$

Hence, for $1 \le s < q \le p^\star - \epsilon$ we have

$$\left| |\Omega|^{\frac{p}{s}}\lambda_s - |\Omega|^{\frac{p}{q}}\lambda_q \right| = |\Omega|^{\frac{p}{s}}\lambda_s - |\Omega|^{\frac{p}{q}}\lambda_q \le \mathcal{L}_\epsilon(q-s) = \mathcal{L}_\epsilon|s-q|. \qquad \square$$

Theorem 10 *The function $q \mapsto \lambda_q$ is absolutely continuous in* $[1, p^\star]$.

Proof. According to Corollary 3 the function $q \mapsto \lambda_q$ is of bounded variation. Therefore, its derivative $(\lambda_q)'$ exists almost everywhere in $[1, p^\star]$ and it is Lebesgue integrable in this interval. Thus, Lebesgue's dominated convergence theorem implies that

$$\lim_{q \to p^\star} \int_1^q (\lambda_s)' ds = \int_1^{p^\star} (\lambda_s)' ds. \tag{32}$$

On the other hand, since Lipschtiz continuity implies absolute continuity it follows from Theorem 9 that λ_q is absolutely continuous in each interval of the form $[1, p^\star - \epsilon]$. Therefore,

$$\lambda_q = \lambda_1 + \int_1^q (\lambda_s)' ds, \quad \text{for } 1 \le q < p^\star. \tag{33}$$

Hence, the left-continuity (18) combined with (32) imply that (33) is also valid for $q = p^\star$. We have concluded that λ_q is the indefinite integral of a Lebesgue integrable (its derivative) function what guarantees that λ_q is absolutely continuous. □

References

[1] P. F. Antonietti, A. Pratelli, *Finite element approximation of the Sobolev constant*, Numer. Math. **117** (2011) 37–64.

[2] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differ. Geom. **11** (1976), 573–598 .

[3] C. Bandle, *Rayleigh-Faber-Krahn inequalities and quasilinear elliptic boundary value problems*, Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. Vol. 1,2, Kluwer Acad. Publ., Dordrecht, 2003, p.p 227–240.

[4] H. Bueno and G. Ercole, *Solutions of the Cheeger problem via torsion functions*, J. Math. Anal. Appl. **381** (2011) 263–279

[5] A. Caboussat, R. Glowinski and A. Leornar, *Looking for the best constant in a Sobolev inequality: a numerical approach*, Calcolo **47** (2010) 211–238.

[6] Y. X. Huang, *A note on the asymptotic behavior of positive solutions for some elliptic equation*, Nonlinear Analysis TMA **29** (1997), 533–537.

[7] B. Kawohl, *On a family of torsional creep problems*, J. reine angew. Math. **410** (1990), 1–22.

[8] O. Ladyzhenskaya and N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York-London, 1968.

[9] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.