A SEMILINEAR HYPERBOLIC SYSTEM VIOLATING THE NULL CONDITION

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Dedicated to the memory of Professor Rentaro Agemi

Abstract. We consider a two-component system of semilinear wave equations in three space dimensions with quadratic nonlinear terms not satisfying the null condition. We prove small data global existence of the classical solution if some quantity defined from the nonlinearities is positive. It is also shown that only one component is dissipated and the other one behaves like a free solution in the large time.

1. Introduction and the main results

This paper is concerned with large time behavior of classical solutions to the Cauchy problem for

$$\begin{cases} \Box v_1 = -c_2(\partial_t v_1)(\partial_t v_2), \\ \Box v_2 = c_1(\partial_t v_1)^2 \end{cases} \quad \text{for} \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \tag{1.1}$$

where $v = (v_1, v_2) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^2$ is an unknown function, $\Box = \partial_t^2 - \Delta_x = \partial_t^2 - (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)$, and c_1, c_2 are non-zero real constants. In what follows we will use the notation $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$ for $j = 1, 2, 3$.

The system (1.1) has the following feature: It has a conserved quantity, that is to say,

$$I := \sum_{j=1}^{2} \frac{c_j}{2} \int_{\mathbb{R}^3} \left(|\partial_t v_j(t,x)|^2 + |\nabla_x v_j(t,x)|^2 \right) dx$$

is independent of t if $v = (v_1, v_2)$ satisfies (1.1). Based on this fact, it may not be surprising that (1.1) admits a global solution in a suitable weak sense if $c_1c_2 > 0$. However, it is not trivial at all whether (1.1) admits a global solution in the classical sense (even when $c_1c_2 > 0$) since the nonlinear terms appearing in (1.1) do not satisfy the null condition in the sense of Christodoulou [2] and Klainerman [13]. Moreover, in view of recent works on the global existence and the asymptotic behavior of solutions under some structural conditions related to the weak null condition (see [17], [1], [10], [8], etc.), it is quite natural to expect some long-range effects should appear because quadratic interaction is critical for the wave equations in the three space dimensions and

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the null condition is violated. So we are lead to the following questions: *Is there a unique global classical solution for* (1.1) *if the initial data are smooth? Moreover, what can we say about the asymptotic behavior of the solution as* $t \to \pm\infty$*?* The aim of this paper is to address these questions in the case where the Cauchy data are sufficiently small, smooth and compactly-supported.

In what follows, we suppose that the initial data are of the form

$$v_j(0,x) = \varepsilon f_j(x), \quad \partial_t v_j(0,x) = \varepsilon g_j(x) \quad \text{for} \quad x \in \mathbb{R}^3, \quad j = 1,2, \tag{1.2}$$

where ε is a small positive parameter and f_j, g_j belong to $C_0^\infty(\mathbb{R}^3)$. We introduce the energy norm $\|\cdot\|_E$ by

$$\|\phi(t)\|_E = \left(\frac{1}{2} \int_{\mathbb{R}^3} \left(|\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2 \right) dx \right)^{1/2}.$$

Note that the conservation law mentioned above is rewritten as

$$c_1 \|v_1(t)\|_E^2 + c_2 \|v_2(t)\|_E^2 = I. \tag{1.3}$$

Our main results are as follows:

Theorem 1.1 (Global existence)**.** *Suppose* $c_1 c_2 > 0$*. Then, for any* $f_j, g_j \in C_0^\infty(\mathbb{R}^3)$*, there exists* $\varepsilon_0 > 0$ *such that* (1.1)–(1.2) *admits a unique global* C^∞ *solution* v *for* $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ *if* $\varepsilon \in (0, \varepsilon_0]$*. On the other hand, if* $c_1 c_2 < 0$*, we can choose* $f_j, g_j \in C_0^\infty(\mathbb{R}^3)$ *such that the corresponding classical solution for* (1.1)–(1.2) *blows up in finite time, both in the future and the past, no matter how small* ε *is.*

Theorem 1.2 (Asymptotic behavior)**.** *Suppose* $c_1 c_2 > 0$*. Let* ε *be sufficiently small and* $v = (v_1, v_2)$ *be the global solution for* (1.1)–(1.2) *whose existence is guaranteed by Theorem* 1.1*. Then we have*

$$\lim_{t \to \pm\infty} \|v_1(t)\|_E = 0, \tag{1.4}$$

and there exist two pairs of functions $(f_2^\pm, g_2^\pm) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *such that*

$$\lim_{t \to \pm\infty} \|v_2(t) - v_2^\pm(t)\|_E = 0, \tag{1.5}$$

where two functions $v_2^\pm = v_2^\pm(t,x)$ *solve* $\Box v_2^\pm = 0$ *with* $(v^\pm, \partial_t v_2^\pm)(0) = (f_2^\pm, g_2^\pm)$*.*

Here $\dot{H}^1(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm given by $\|\phi\|_{\dot{H}^1} = \|\nabla_x \phi\|_{L^2}$.

Remark 1.1. From (1.3), (1.4), (1.5) and the energy conservation for the free wave equation, it follows that

$$\|\nabla_x f_2^\pm\|_{L^2}^2 + \|g_2^\pm\|_{L^2}^2 = \frac{2I}{c_2},$$

which implies $(f_2^\pm, g_2^\pm) \neq (0,0)$ unless the Cauchy data for the original problem vanish identically. Therefore Theorem 1.2 tells us that only v_1 is dissipated and v_2 behaves like a (non-trivial) free solution in the large time. As far as the authors know, there are no previous results on such decoupling in the context of nonlinear wave equations.

Remark 1.2. Our proof does not rely on (1.3) at all. In fact our approach can be applied to a bit more general system which does not have the (explicit) conservation law. For example, let us consider the system

$$\begin{cases} \Box v_1 = -c_2(\partial_t v_1)(\partial_t v_2) + N_1(\partial v), \\ \Box v_2 = c_1(\partial_t v_1)^2 + N_2(\partial v), \end{cases} \tag{1.6}$$

where

$$N_j(\partial v) = \sum_{k,l=1}^{2} A_j^{kl} Q_0(v_k, v_l) + \sum_{a,b=0}^{3} B_j^{ab} Q_{ab}(v_1, v_2)$$

with real constants A_j^{kl} and B_j^{ab}. Here Q_0 and Q_{ab} are the *null forms*

$$Q_0(\phi, \psi) := (\partial_t \phi)(\partial_t \psi) - (\nabla_x \phi) \cdot (\nabla_x \psi),$$
$$Q_{ab}(\phi, \psi) := (\partial_a \phi)(\partial_b \psi) - (\partial_b \phi)(\partial_a \psi), \quad a, b \in \{0, 1, 2, 3\}.$$

If $c_1 c_2 > 0$ then the global existence part of Theorem 1.1 and the conclusion of Theorem 1.2 remain true (see also Section 8 below). The null condition is satisfied if and only if $c_1 = c_2 = 0$. If the initial date are small, the null condition ensures the global existence of a unique solution v, whose components v_1 and v_2 in the large time behave like free solutions, which are non-trivial in general, differently from (1.4). On the other hand, in the case of $c_1 = 0$ and $c_2 \neq 0$, or the case of $c_1 \neq 0$ and $c_2 = 0$, (1.6) admits a global solution for small data, whose energy grows up to ∞ as $t \to \infty$ (at different growth rates for two cases; see [8] for the details).

For closely related works on the nonlinear Klein-Gordon systems in two space dimensions, see Kawahara-Sunagawa [12] (see also [3], [11], [19], etc.).

2. Reduction of the problem

In this section, we make some reduction of the problem and give a proof of the blow-up part of Theorem 1.1.

First we observe that the system (1.1) is invariant under the time-reversing $(t, x) \mapsto (-t, x)$. So we have only to consider the forward Cauchy problem (i.e., the problem for $t > 0$).

Proof of the blow-up part of Theorem 1.1. Let us remember the famous result by John [7]: For every f_0, $g_0 \in C_0^\infty(\mathbb{R}^3)$ with $(f_0, g_0) \not\equiv (0, 0)$, the classical solution $w(t, x)$ for $\Box w = (\partial_t w)^2$ with $(w, \partial_t w)(0) = (\varepsilon f_0, \varepsilon g_0)$ blows up in finite time no matter how small ε is. Now we assume $c_1 c_2 < 0$ and set

$$v_1(t, x) = \frac{1}{\sqrt{-c_1 c_2}} w(t, x), \quad v_2(t, x) = \frac{-1}{c_2} w(t, x)$$

with the above $w(t, x)$. Then (v_1, v_2) is a blow-up solution for (1.1), which yields the desired conclusion. □

Now we make further reduction for the proof of the global existence part. We assume $c_1 c_2 > 0$ from now on. Then we see that it is sufficient to consider the case of $c_1 = c_2 = 1$ through the scaling: If we put $u_1 = \sqrt{c_1 c_2} v_1$ and

$u_2 = c_2 v_2$, then we see that (u_1, u_2) satisfies (1.1) with $c_1 = c_2 = 1$. Moreover, if we put $u = u_1 + iu_2$, then u satisfies

$$\Box u = F(\partial_t u), \tag{2.1}$$

where

$$F(z) = i(\operatorname{Re} z)z.$$

Here and hereafter, the symbol i always stands for the imaginary unit $\sqrt{-1}$. Remark that

$$\operatorname{Re}(\overline{z}F(z)) = 0 \tag{2.2}$$

for all $z \in \mathbb{C}$. Eventually our problem is reduced to (2.1) for $t > 0$, $x \in \mathbb{R}^3$ with the initial data

$$u(0, x) = \varepsilon f(x), \quad \partial_t u(0, x) = \varepsilon g(x), \tag{2.3}$$

where $f,\ g \in C_0^\infty(\mathbb{R}^3; \mathbb{C})$. Since the local existence of the solution is well known, what we have to do for the proof of the global existence is to get a suitable *a priori* estimate for the solution to (2.1)–(2.3). This will be carried out in Section 5 after some preliminaries in Sections 3 and 4. The proof of Theorem 1.2 will be given in Sections 6 and 7.

3. Commuting vector fields

In this section, we recall basic properties of some vector fields associated with the wave equation. In what follows, we denote several positive constants by C which may vary from one line to another. For $y \in \mathbb{R}^N$ with a positive integer N, the notation $\langle y \rangle = (1 + |y|^2)^{1/2}$ will be often used. Also we will use the following convention on implicit constants: The expression $f = \sum'_{\lambda \in \Lambda} g_\lambda$ means that there exists a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of constants such that $f = \sum_{\lambda \in \Lambda} A_\lambda g_\lambda$.

Let us introduce

$$\begin{aligned}
&S = t\partial_t + \sum_{j=1}^{3} x_j \partial_j, \\
&L_j = t\partial_j + x_j \partial_t, && j \in \{1, 2, 3\}, \\
&\Omega_{jk} = x_j \partial_k - x_k \partial_j, && j, k \in \{1, 2, 3\}, \\
&\partial = (\partial_a)_{a=0,1,2,3} = (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}),
\end{aligned}$$

and we set

$$\Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_{10}) = (S, L_1, L_2, L_3, \Omega_{23}, \Omega_{31}, \Omega_{12}, \partial_0, \partial_1, \partial_2, \partial_3).$$

For a multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{10})$, we write $\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_{10}^{\alpha_{10}}$ and $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_{10}$. We define

$$|\phi(t, x)|_k = \left(\sum_{|\alpha| \le k} |\Gamma^\alpha \phi(t, x)|^2 \right)^{1/2}, \quad \|\phi(t, \cdot)\|_k = \left(\sum_{|\alpha| \le k} \|\Gamma^\alpha \phi(t, \cdot)\|_{L^2}^2 \right)^{1/2}$$

for a non-negative integer k and a smooth function $\phi = \phi(t, x)$. As is well known, these vector fields satisfy $[\Box, S] = 2\Box$ and $[\Box, L_j] = [\Box, \Omega_{jk}] = [\Box, \partial_a] = 0$, where $[A, B] = AB - BA$ for linear operators A and B. From them it follows that

$$\Box\Gamma^\alpha\phi = \widetilde{\Gamma}^\alpha\Box\phi, \tag{3.1}$$

where $\widetilde{\Gamma}^\alpha = (\Gamma_0 + 2)^{\alpha_0}\Gamma_1^{\alpha_1} \cdots \Gamma_{10}^{\alpha_{10}}$. We also note that

$$[\Gamma_j, \Gamma_k] = \sum_{l=0}^{10}{}' \Gamma_l, \quad [\Gamma_j, \partial_a] = \sum_{b=0}^{3}{}' \partial_b.$$

Hence we can check that the estimates

$$\begin{aligned} &|\Gamma^\alpha\Gamma^\beta\phi| \le C|\phi|_{|\alpha|+|\beta|}, \\ &C^{-1}|\partial\phi|_s \le \sum_{|\alpha|\le s} |\partial\Gamma^\alpha\phi| \le C|\partial\phi|_s \end{aligned} \tag{3.2}$$

are valid for any multi-indices α, β and any non-negative integer s.

Next we set $r = |x|$, $\omega_j = x_j/r$, $\partial_r = \sum_{j=1}^3 \omega_j\partial_j$, and $\partial_\pm = \partial_t \pm \partial_r$. We write $\omega = (\omega_j)_{j=1,2,3}$. For simplicity of exposition, we also introduce

$$D_\pm = \pm\frac{1}{2}\partial_\pm = \frac{1}{2}(\partial_r \pm \partial_t).$$

We summarize several useful inequalities related to Γ.

Lemma 3.1. *For a smooth function ϕ of $(t, x) \in [0, \infty) \times \mathbb{R}^3$, we have*

$$|D_+(r\phi(t, x))| \le C|\phi(t, x)|_1, \tag{3.3}$$

$$|r\partial_t\phi(t, x) + D_-(r\phi(t, x))| \le C|\phi(t, x)|_1, \tag{3.4}$$

and

$$|r\partial_j\phi(t, x) - \omega_j D_-(r\phi(t, x))| \le C|\phi(t, x)|_1 \tag{3.5}$$

for $j = 1, 2, 3$.

Proof. (3.3) and (3.4) are direct consequences of the following relations:

$$\begin{aligned} D_+(r\phi) &= \frac{r}{2(r+t)}(S\phi + L_r\phi) + \frac{\phi}{2}, \\ r\partial_t\phi &= -D_-(r\phi) + D_+(r\phi), \end{aligned}$$

where $L_r = r\partial_t + t\partial_r = \sum_{j=1}^3 \omega_j L_j$. (3.5) follows just from

$$r(\partial_j - \omega_j\partial_r)\phi = \sum_{k=1}^{3} \omega_k\Omega_{kj}\phi \tag{3.6}$$

and

$$r\partial_r\phi = D_-(r\phi) + D_+(r\phi) - \phi,$$

if we use (3.3) to estimate $D_+\phi$. $\Box$

Lemma 3.2. *For a smooth function ϕ of $(t,x) \in [0,\infty) \times \mathbb{R}^3$ and a non-negative integer s, we have*

$$|\partial\phi(t,x)|_s \le C\langle t-|x|\rangle^{-1}|\phi(t,x)|_{s+1}.$$

This lemma is due to Lindblad [16], which comes from the identities

$$(t-r)\partial_t\phi = \frac{1}{t+r}\left(tS - rL_r\right)\phi,$$

$$(t-r)\partial_r\phi = \frac{1}{t+r}\left(tL_r - rS\right)\phi,$$

and $t\Omega_{kj}\phi = x_k L_j\phi - x_j L_k\phi$, as well as (3.6) (see [16] for the detail of the proof).

We close this section with the following decay estimate for solutions to inhomogeneous wave equations.

Lemma 3.3 (Hörmander's L^1–L^∞ estimate)**.** *Let ϕ be a smooth solution to*

$$\Box\phi = G, \quad (t,x) \in (0,T) \times \mathbb{R}^3$$

with $\phi(0,x) = \partial_t\phi(0,x) = 0$. It holds that

$$\langle t+|x|\rangle|\phi(t,x)| \le C\sum_{|\alpha|\le 2}\int_0^t \|\Gamma^\alpha G(\tau,\cdot)\|_{L^1(\mathbb{R}^3)}\frac{d\tau}{\langle\tau\rangle}$$

for $0 \le t < T$. Here the constant C is independent of T.

See [5] for the proof (see also Lemma 6.6.8 of [6], or Section 2.1 of [18]).

4. The profile equation

Let $0 < T \le \infty$, and let u be the solution to (2.1) on $[0,T) \times \mathbb{R}^3$. We suppose that

$$\operatorname{supp} f \cup \operatorname{supp} g \subset B_R \tag{4.1}$$

for some $R > 0$, where $B_M = \{x \in \mathbb{R}^3; |x| \le M\}$ for $M > 0$. Then, from the finite propagation property, we have

$$\operatorname{supp} u(t,\cdot) \subset B_{t+R}, \quad 0 \le t < T. \tag{4.2}$$

Now we put $r = |x|$, $\omega = (\omega_1,\omega_2,\omega_3) = x/|x|$ and set

$$\Delta_{\mathbb{S}^2} = \sum_{1\le j<k\le 3}\Omega_{jk}^2$$

so that

$$r\Box\phi = \partial_+\partial_-(r\phi) - \frac{1}{r}\Delta_{\mathbb{S}^2}\phi. \tag{4.3}$$

We define

$$U(t,x) := D_-\big(ru(t,x)\big), \quad (t,x) \in [0,T) \times (\mathbb{R}^3 \setminus \{0\}) \tag{4.4}$$

for the solution u of (2.1). In view of (3.4) and (3.5), the asymptotic profiles as $t \to \infty$ of $\partial_t u$ and $\nabla_x u$ should be given by $-U/r$ and $\omega U/r$, respectively, because we can expect $|u(t,x)|_1 \to 0$ as $t \to \infty$. Also it follows from (4.3) that

$$\partial_+ U(t,x) = -\frac{1}{2t} F\big(-U(t,x)\big) + H(t,x), \tag{4.5}$$

where $H = H(t,x)$ is given by

$$H = -\frac{1}{2}\left(rF(\partial_t u) - \frac{1}{t}F(-U)\right) - \frac{1}{2r}\Delta_{\mathbb{S}^2} u.$$

As we will see in Lemma 4.1 below, H can be regarded as a remainder. For these reasons, we call (4.5) *the profile equation associated with* (2.1), which plays an important role in our analysis. We also need an analogous equation for $\Gamma^\alpha u$ with a multi-index α. For this purpose, we put

$$U^{(\alpha)}(t,x) := D_-\big(r\Gamma^\alpha u(t,x)\big).$$

Since $\Box(\Gamma^\alpha u) = \widetilde{\Gamma}^\alpha\left(F(\partial_t u)\right)$, we deduce from (4.3) that

$$\partial_+ U^{(\alpha)} = -\frac{1}{2t} G_\alpha + H_\alpha \tag{4.6}$$

for $|\alpha| \geq 1$, where

$$G_\alpha(t,x) = i\Big\{(\operatorname{Re} U^{(\alpha)})U + (\operatorname{Re} U)U^{(\alpha)}\Big\}$$

and

$$H_\alpha(t,x) = -\frac{1}{2}\left(r\widetilde{\Gamma}^\alpha F(\partial_t u) - \frac{1}{t}G_\alpha\right) - \frac{1}{2r}\Delta_{\mathbb{S}^2}\Gamma^\alpha u.$$

In the rest part of this section, we focus on preliminary estimates for H and H_α in terms of the solution u near the light cone. To be more specific, we put

$$\Lambda_{T,R} := \{(t,x) \in [0,T) \times \mathbb{R}^3;\ 1 \leq t/2 \leq |x| \leq t+R\}.$$

Note that we have

$$(1+t+|x|)^{-1} \leq |x|^{-1} \leq 2t^{-1} \leq 3(1+t)^{-1} \leq 3(R+2)(1+t+|x|)^{-1}$$

for $(t,x) \in \Lambda_{T,R}$. In other words, the weights $\langle t+|x|\rangle^{-1}$, $(1+t)^{-1}$, $|x|^{-1}$, and t^{-1} are equivalent to each other in $\Lambda_{T,R}$. For a non-negative integer s, we also introduce an auxiliary notation $|\cdot|_{\sharp,s}$ by

$$|\phi(t,x)|_{\sharp,s} := |\partial\phi(t,x)|_s + \langle t+|x|\rangle^{-1}|\phi(t,x)|_{s+1}.$$

Lemma 4.1. *We have*

$$|H(t,x)| \leq C\left(|u|_{\sharp,0}|u|_1 + t^{-1}|u|_2\right),$$

for $(t,x) \in \Lambda_{T,R}$. Here the constant C is independent of T. Also, in the case of $s \geq 1$, we have

$$\sum_{|\alpha|=s} |H_\alpha(t,x)| \leq C_s(|u|_{\sharp,s}|u|_{s+1} + t|\partial u|_{s-1}^2 + t^{-1}|u|_{s+2})$$

for $(t,x) \in \Lambda_{T,R}$, where C_s is a positive constant which does not depend on T.

Proof. Let $(t,x)=(t,r\omega)\in\Lambda_{T,R}$ and $s=|\alpha|\ge 0$. First we note that

$$\begin{aligned}|U^{(\alpha)}(t,x)| &\le r|D_-\Gamma^\alpha u|+\frac{1}{2}|\Gamma^\alpha u|\\ &\le Cr(|\partial u|_s+r^{-1}|u|_s)\le Ct|u(t,x)|_{\sharp,s}\end{aligned}\tag{4.7}$$

by the definition of $|\cdot|_{\sharp,s}$, and that

$$\begin{aligned}\langle t-r\rangle|U^{(\alpha)}(t,x)| &\le Ct\left(\langle t-r\rangle|\partial u(t,x)|_s+\frac{\langle t-r\rangle}{\langle t+r\rangle}|u(t,x)|_s\right)\\ &\le Ct|u(t,x)|_{s+1}\end{aligned}\tag{4.8}$$

by Lemma 3.2. Also (3.3) implies

$$|D_+(r\Gamma^\alpha u)|\le C|u|_{s+1}.\tag{4.9}$$

Now we consider the estimate for H. We decompose it as follows:

$$H=-\frac{1}{2r}\Big(r^2F(\partial_t u)-F(-U)\Big)-\frac{t-r}{2rt}F(-U)-\frac{1}{2r}\Delta_{\mathbb{S}^2}u.$$

It is easy to see that the third term can be dominated by $Ct^{-1}|u|_2$. As for the second term, we have

$$\frac{|t-r|}{rt}|F(-U)|\le Ct^{-1}\langle t-r\rangle|U|\cdot t^{-1}|U|\le C|u|_1|u|_{\sharp,0},$$

because of (4.7) and (4.8) with $s=0$. To estimate the first term, we use the relation

$$\begin{aligned}r^2F(\partial_t u)-F(-U)&=F\big(\partial_t(ru)\big)-F(-U)\\ &=F\big(-U+D_+(ru)\big)-F(-U)\\ &=ir(\partial_t u)\operatorname{Re}D_+(ru)-i(\operatorname{Re}U)D_+(ru)\end{aligned}$$

to obtain

$$\frac{1}{2r}|r^2F(\partial_t u)-F(-U)|\le C(|\partial u|+r^{-1}|U|)|D_+(ru)|\le C|u|_{\sharp,0}|u|_1$$

with the help of (4.7) and (4.9).

Next we turn to the estimate for H_α with $s=|\alpha|\ge 1$. For this purpose, we set

$$\widetilde{F}_\alpha=i\Big\{(\operatorname{Re}\partial_t\Gamma^\alpha u)\partial_t u+(\operatorname{Re}\partial_t u)\partial_t\Gamma^\alpha u\Big\}$$

to split H_α into the following form:

$$H_\alpha=-\frac{r}{2}\left(\widetilde{\Gamma}^\alpha F(\partial_t u)-\widetilde{F}_\alpha\right)-\frac{1}{2r}\left(r^2\widetilde{F}_\alpha-G_\alpha\right)-\frac{t-r}{2rt}G_\alpha-\frac{1}{2r}\Delta_{\mathbb{S}^2}\Gamma^\alpha u.$$

Since the first term consists of a linear combination of the terms in the form $r(\Gamma^\beta\partial_a u)(\Gamma^\gamma\partial_b u)$ with $|\beta|,|\gamma|\le s-1$ and $a,b\in\{0,1,2,3\}$, it can be estimated by $Ct|\partial u|_{s-1}^2$. Other terms can be treated in the same way as in the previous case. □

5. Proof of the global existence part of Theorem 1.1

Let $u(t,x)$ be a smooth solution to (2.1)–(2.3) on $[0,T_0)\times\mathbb{R}^3$ with some $T_0\in(0,\infty]$. For $0<T\le T_0$, we put

$$e[u](T)=\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\Big(\langle t+|x|\rangle\langle t-|x|\rangle^{1-\mu}|\partial u(t,x)| \\ +\langle t+|x|\rangle^{1-\nu}\langle t-|x|\rangle^{1-\mu}|\partial u(t,x)|_k\Big)$$

with some μ, $\nu>0$ and a positive integer k. We also put

$$e[u](0)=\lim_{T\to+0}e[u](T).$$

Observe that there is a positive constant ε_1 such that $0<\varepsilon\le\varepsilon_1$ implies $e[u](0)\le\sqrt{\varepsilon}/2$, because we have $e[u](0)=O(\varepsilon)$.

The main step toward global existence is to show the following.

Lemma 5.1 (*A priori* estimate). *Let $k\ge 3$, $0<\mu<1/2$, and $0<4(k+1)\nu\le\mu$. There exist positive constants ε_2 and m, which depend only on k, μ and ν, such that*

$$e[u](T)\le\sqrt{\varepsilon} \tag{5.1}$$

implies

$$e[u](T)\le m\varepsilon, \tag{5.2}$$

provided that $0<\varepsilon\le\varepsilon_2$ and $0<T\le T_0$.

Once the above lemma is obtained, we can show the small data global existence for (2.1)–(2.3) by the so-called continuity argument: Let T^* be the lifespan of the classical solution for (2.1)–(2.3) and assume $T^*<\infty$. Then, it follows from the standard blow-up criterion (see e.g., [18]) that

$$\lim_{t\to T^*-0}|\partial u(t,x)|=\infty. \tag{5.3}$$

On the other hand, by setting

$$T_*=\sup\left\{T\in[0,T^*)\,;e[u](T)\le\sqrt{\varepsilon}\right\},$$

we can see that Lemma 5.1 yields $T_*=T^*$, provided that ε is small enough. Indeed, if $T_*<T^*$, then we have $e[u](T_*)\le\sqrt{\varepsilon}$, and Lemma 5.1 implies that

$$e[u](T_*)\le m\varepsilon\le\sqrt{\varepsilon}/2$$

for $0<\varepsilon\le\min\{\varepsilon_1,\varepsilon_2,1/4m^2\}$ (note that we have $T_*>0$ for $\varepsilon\le\varepsilon_1$). Then, by the continuity of $[0,T^*)\ni T\mapsto e[u](T)$, we can take $\delta>0$ such that

$$e[u](T_*+\delta)\le\sqrt{\varepsilon},$$

which contradicts the definition of T_*, and we conclude that $T_*=T^*$.

In particular, we have

$$e[u](T^*)\le\sqrt{\varepsilon}.$$

This implies that (5.3) never occurs for small ε. In other words, we must have $T^* = \infty$, that is, the solution u exists globally for small data. We also note that

$$e[u](\infty) \le \sqrt{\varepsilon} \tag{5.4}$$

holds for this global solution u, and Lemma 5.1 again yields

$$e[u](\infty) \le m\varepsilon. \tag{5.5}$$

Now we turn to the proof of Lemma 5.1. It will be divided into several steps.

Proof of Lemma 5.1. In what follows, we always suppose $0 \le t < T$.

Step 1: Rough bounds for $|u(t,x)|_{k+2}$ and $|\partial u(t,x)|_{k+1}$.

First of all, we will establish the following energy estimates:

$$\|\partial u(t)\|_l \le C\varepsilon(1+t)^{C_*\sqrt{\varepsilon}+l\nu} \tag{5.6}$$

for $l \in \{0, 1, \dots, 2k+1\}$, where C_* is a positive constant to be fixed later.

In preparation for the proof of (5.6), we make some observations: Let $1 \le l \le 2k+1$. From (3.1), (3.2), and the standard energy inequality, we get

$$\|\partial u(t)\|_l \le C_{1,l}\left(\|\partial u(0)\|_l + \int_0^t \big\|F\big(\partial u(\tau)\big\|_l\,d\tau\right), \tag{5.7}$$

where $C_{1,l}$ is a positive constant depending only on l. From (5.1) we have $|\partial u(t,x)| \le \sqrt{2\varepsilon}(1+t)^{-1}$ and $|\partial u(t,x)|_k \le \sqrt{2\varepsilon}(1+t)^{\nu-1}$, since $\langle t+|x|\rangle^{-1} \le \sqrt{2}(1+t)^{-1}$. Hence we get

$$\begin{aligned}|F(\partial_t u)|_l \le& C_{2,l}\left(|\partial u|\,|\partial u|_l + |\partial u|_{[l/2]}|\partial u|_{l-1}\right)\\ \le& C_{2,l}\sqrt{2\varepsilon}\left((1+t)^{-1}|\partial u|_l + (1+t)^{\nu-1}|\partial u|_{l-1}\right)\end{aligned}$$

with a positive constant $C_{2,l}$ depending only on l, which leads to

$$\big\|F\big(\partial_t u(t)\big)\big\|_l \le \sqrt{2}C_{2,l}\sqrt{\varepsilon}\left((1+t)^{-1}\|\partial u(t)\|_l + (1+t)^{\nu-1}\|\partial u(t)\|_{l-1}\right). \tag{5.8}$$

Now we put $C_* = \sqrt{2} + \max_{1\le l\le 2k+1}\sqrt{2}C_{1,l}C_{2,l}$, and we shall prove (5.6) by induction on l. In the case of $l=0$, it follows from the standard energy inequality and (5.1) that

$$\begin{aligned}\|\partial u(t)\|_0 &\le C\varepsilon + \int_0^t \|F(\partial_t u(\tau))\|_0 d\tau\\ &\le C\varepsilon + \int_0^t \|\partial u(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)}\|\partial u(\tau)\|_0 d\tau\\ &\le C\varepsilon + \sqrt{2\varepsilon}\int_0^t (1+\tau)^{-1}\|\partial u(\tau)\|_0 d\tau,\end{aligned}$$

whence the Gronwall lemma implies

$$\|\partial u(t)\|_0 \le C\varepsilon(1+t)^{\sqrt{2\varepsilon}} \le C\varepsilon(1+t)^{C_*\sqrt{\varepsilon}}.$$

Next we assume that (5.6) holds for some $l \in \{0, 1, \dots, 2k\}$. Then it follows from (5.7) and (5.8) that

$$\begin{aligned}\|\partial u(t)\|_{l+1} \le & C\varepsilon + C_*\sqrt{\varepsilon}\int_0^t \left((1+\tau)^{-1}\|\partial u(\tau)\|_{l+1} + (1+\tau)^{-1+\nu}\|\partial u(\tau)\|_l\right) d\tau \\ \le & C\varepsilon + C_*\sqrt{\varepsilon}\int_0^t (1+\tau)^{-1}\|\partial u(\tau)\|_{l+1} d\tau \\ & + C\varepsilon^{3/2}\int_0^t (1+\tau)^{-1+C_*\sqrt{\varepsilon}+(l+1)\nu} d\tau \\ \le & C\varepsilon + C_*\sqrt{\varepsilon}\int_0^t (1+\tau)^{-1}\|\partial u(\tau)\|_{l+1} d\tau + C\varepsilon^{3/2}(1+\tau)^{C_*\sqrt{\varepsilon}+(l+1)\nu},\end{aligned}$$

which yields

$$\|\partial u(t)\|_{l+1} \le C\varepsilon(1+t)^{C_*\sqrt{\varepsilon}} + C\varepsilon^{3/2}(1+t)^{C_*\sqrt{\varepsilon}+(l+1)\nu} \le C\varepsilon(1+t)^{C_*\sqrt{\varepsilon}+(l+1)\nu}.$$

This means that (5.6) remains true when l is replaced by $l+1$, and (5.6) has been proved for all $l \in \{0, 1, \dots, 2k+1\}$.

From now on, we assume that $\varepsilon \le \nu^2/C_*^2$. Then, since $k \ge 3$ and $2(k+1)\nu \le \mu/2$, it follows from (5.6) with $l = 2k+1$ that

$$\|\partial u(t)\|_{k+4} \le \|\partial u(t)\|_{2k+1} \le C\varepsilon\langle t\rangle^{2(k+1)\nu} \le C\varepsilon\langle t\rangle^{\mu/2}$$

and

$$\big\||F(\partial_t u(t,\cdot))|_{k+4}\big\|_{L^1(\mathbb{R}^3)} \le C\|\partial u(t)\|_{k+4}^2 \le C\varepsilon^2\langle t\rangle^{\mu}.$$

Hence Lemma 3.3 yields

$$\langle t+|x|\rangle|u(t,x)|_{k+2} \le C\varepsilon + C\varepsilon^2\int_0^t \langle\tau\rangle^{\mu-1}d\tau \le C\varepsilon\langle t+|x|\rangle^{\mu},$$

that is,

$$|u(t,x)|_{k+2} \le C\varepsilon\langle t+|x|\rangle^{-1+\mu} \tag{5.9}$$

for $(t,x) \in [0,T)\times\mathbb{R}^3$. By Lemma 3.2, we also have

$$|\partial u(t,x)|_{k+1} \le C\varepsilon\langle t+|x|\rangle^{-1+\mu}\langle t-|x|\rangle^{-1} \tag{5.10}$$

for $(t,x) \in [0,T)\times\mathbb{R}^3$.

Step 2: Estimates for $|\partial u(t,x)|_k$ away from the light cone.

Now we put $\Lambda_{T,R}^{c} := \big([0,T)\times\mathbb{R}^3\big)\setminus\Lambda_{T,R}$, where R is the constant appearing in (4.1). In the case of $t/2 < 1$ or $|x| < t/2$, we see that

$$\langle t-|x|\rangle^{-1} \le C\langle t+|x|\rangle^{-1}.$$

On the other hand, it follows from (4.2) that $u(t,x) = 0$ if $|x| > t+R$. Hence (5.10) implies

$$\sup_{(t,x)\in\Lambda_{T,R}^{c}} \langle t+|x|\rangle\langle t-|x|\rangle^{1-\mu}|\partial u(t,x)|_k \le C\varepsilon. \tag{5.11}$$

Step 3: Estimates for $|\partial u(t,x)|$ near the light cone.

Let $(t,x)\in\Lambda_{T,R}$ throughout this step. Remember that t^{-1}, $|x|^{-1}$, $\langle t\rangle^{-1}$, and $\langle t+|x|\rangle^{-1}$ are equivalent to each other in $\Lambda_{T,R}$. We define U, $U^{(\alpha)}$, H, H_α and $|\cdot|_{\sharp,s}$ as in Section 4. We see from (5.9) and (5.10) that

$$|u(t,x)|_{\sharp,k}\le C\varepsilon t^{\mu-1}\langle t-|x|\rangle^{-1}. \tag{5.12}$$

By (3.2), (3.4), (3.5), and (5.9), we have

$$\begin{aligned} t|\partial u(t,x)|_l \le & C\sum_{|\alpha|\le l}\big|\,|x|\partial\Gamma^\alpha u(t,x)\big| \\ \le & C\sum_{|\alpha|\le l}|U^{(\alpha)}(t,x)|+C\varepsilon t^{\mu-1} \end{aligned} \tag{5.13}$$

for $l\le k$. Also, it follows from (5.9), (5.12), and Lemma 4.1 that

$$|H(t,x)|\le C\left(\varepsilon^2t^{2\mu-2}\langle t-|x|\rangle^{-1}+\varepsilon t^{\mu-2}\right)\le C\varepsilon t^{2\mu-2}\langle t-|x|\rangle^{-\mu}. \tag{5.14}$$

Next we put

$$\Sigma=\{(t,x)\in\Lambda_{T,R}\,;\, t/2=1\ \text{ or }\ t/2=|x|\}$$

and we define $t_{0,\sigma}=\max\{2,-2\sigma\}$ for $\sigma\le R$. What is important here is that the line segment $\{(t,(t+\sigma)\omega);\,0\le t<T\}$ meets Σ at the point $(t_{0,\sigma},(t_{0,\sigma}+\sigma)\omega)$ for each fixed $(\sigma,\omega)\in(-\infty,R]\times\mathbb{S}^2$. We also remark that

$$C^{-1}\langle\sigma\rangle\le t_{0,\sigma}\le C\langle\sigma\rangle,\quad \sigma\le R. \tag{5.15}$$

When $(t,x)\in\Sigma$, we have $t^\mu\le C\langle t-|x|\rangle^\mu$. So it follows from (4.7) and (5.12) that

$$\sum_{|\alpha|\le k}|U^{(\alpha)}(t,x)|\le C\varepsilon t^\mu\langle t-|x|\rangle^{-1}\le C\varepsilon\langle t-|x|\rangle^{\mu-1},\quad (t,x)\in\Sigma. \tag{5.16}$$

Now we define

$$V_{\sigma,\omega}(t)=U\big(t,(t+\sigma)\omega\big) \tag{5.17}$$

for $0\le t<T$, with $(\sigma,\omega)\in(-\infty,R]\times\mathbb{S}^2$ being fixed. Then, since the profile equation (4.5) is rewritten as

$$V'_{\sigma,\omega}(t)=-\frac{1}{2t}F(-V_{\sigma,\omega}(t))+H(t,(t+\sigma)\omega) \tag{5.18}$$

for $t_{0,\sigma}<t<T$, it follows from (2.2) that

$$\begin{aligned} \frac{d}{dt}(|V_{\sigma,\omega}(t)|^2) &= 2\,\mathrm{Re}\left(\overline{V_{\sigma,\omega}(t)}\frac{dV_{\sigma,\omega}}{dt}(t)\right) \\ &= 2\,\mathrm{Re}\left(\overline{V_{\sigma,\omega}(t)}H(t,(t+\sigma)\omega)\right) \\ &\le 2|V_{\sigma,\omega}(t)||H(t,(t+\sigma)\omega)| \end{aligned} \tag{5.19}$$

for $t_{0,\sigma}<t<T$. We also note that (5.16) for $k=0$ can be interpreted as

$$|V_{\sigma,\omega}(t_{0,\sigma})|\le C\varepsilon\langle\sigma\rangle^{\mu-1}. \tag{5.20}$$

From (5.14), (5.15), (5.19), and (5.20) we get

$$\begin{aligned}|V_{\sigma,\omega}(t)| &\le |V_{\sigma,\omega}(t_{0,\sigma})| + \int_{t_{0,\sigma}}^{t} |H(\tau,(\tau+\sigma)\omega)|d\tau \\ &\le C\varepsilon\langle\sigma\rangle^{\mu-1} + C\varepsilon\langle\sigma\rangle^{-\mu}\int_{t_{0,\sigma}}^{t}\tau^{2\mu-2}d\tau \\ &\le C\varepsilon\left(\langle\sigma\rangle^{\mu-1} + \langle\sigma\rangle^{-\mu}t_{0,\sigma}^{2\mu-1}\right) \\ &\le C\varepsilon\langle\sigma\rangle^{\mu-1} \qquad (5.21)\end{aligned}$$

for $t \ge t_{0,\sigma}$, where C is independent of ε, σ, and ω. (5.21) implies

$$|U(t,x)| = |V_{|x|-t,x/|x|}(t)| \le C\varepsilon\langle t-|x|\rangle^{\mu-1}, \quad (t,x)\in\Lambda_{T,R}.$$

Finally, in view of (5.13) with $l=0$, we obtain

$$\sup_{(t,x)\in\Lambda_{T,R}} \langle t+|x|\rangle\langle t-|x|\rangle^{1-\mu}|\partial u(t,x)| \le C\varepsilon. \qquad (5.22)$$

We remark that the derivation of (5.19) is the only point where we make use of the structure (2.2) (see also Section 8 below).

Step 4: Estimates for $|\partial u(t,x)|_k$ near the light cone.

We assume $(t,x)\in\Lambda_{T,R}$ also in this step. Let $1\le|\alpha|\le k$. For a non-negative integer s, we set

$$\mathcal{U}^{(s)}(t,x) := \sum_{|\beta|\le s}|U^{(\beta)}(t,x)|.$$

By (5.13) we get

$$|\partial u(t,x)|_{|\alpha|-1} \le C\left(t^{-1}\mathcal{U}^{(|\alpha|-1)}(t,x) + \varepsilon t^{\mu-2}\right). \qquad (5.23)$$

It follows from (5.9), (5.12), (5.23), and Lemma 4.1 that

$$\begin{aligned}|H_\alpha(t,x)| \le& C\left(\varepsilon^2 t^{2\mu-2}\langle t-|x|\rangle^{-1} + \varepsilon t^{\mu-2} + \varepsilon^2 t^{2\mu-3} + t^{-1}\big(\mathcal{U}^{(|\alpha|-1)}(t,x)\big)^2\right) \\ \le& C\varepsilon t^{2\mu-2}\langle t-|x|\rangle^{-\mu} + Ct^{-1}\big(\mathcal{U}^{(|\alpha|-1)}(t,x)\big)^2. \qquad (5.24)\end{aligned}$$

We put

$$V_{\sigma,\omega}^{(\alpha)}(t) = U^{(\alpha)}\big(t,(t+\sigma)\omega\big)$$

for $0\le t<T$ and $(\sigma,\omega)\in(-\infty,R]\times\mathbb{S}^2$. Then (4.6) is rewritten as

$$\big(V_{\sigma,\omega}^{(\alpha)}\big)'(t) = -\frac{i}{2t}\Big\{(\operatorname{Re}V_{\sigma,\omega}^{(\alpha)}(t))V_{\sigma,\omega}(t) + (\operatorname{Re}V_{\sigma,\omega}(t))V_{\sigma,\omega}^{(\alpha)}(t)\Big\} + H_\alpha(t,(t+\sigma)\omega)$$

for $t_{0,\sigma}<t<T$. Hence by (5.21) and (5.24) we obtain

$$\begin{aligned}\frac{d}{dt}|V_{\sigma,\omega}^{(\alpha)}(t)|^2 \le& \frac{2}{t}|V_{\sigma,\omega}(t)|\,|V_{\sigma,\omega}^{(\alpha)}(t)|^2 + 2|H_\alpha(t,(t+\sigma)\omega)|\,|V_{\sigma,\omega}^{(\alpha)}(t)| \\ \le& \frac{2C^*\varepsilon}{t}|V_{\sigma,\omega}^{(\alpha)}(t)|^2 + C\left(\varepsilon t^{2\mu-2}\langle\sigma\rangle^{-\mu} + t^{-1}\big(\mathcal{V}_{\sigma,\omega}^{(|\alpha|-1)}(t)\big)^2\right)|V_{\sigma,\omega}^{(\alpha)}(t)|,\end{aligned}$$

where

$$\mathcal{V}_{\sigma,\omega}^{(s)}(t) := \sum_{|\beta|\le s} |V_{\sigma,\omega}^{(\beta)}(t)|,$$

and C^* is a positive constant independent of α. Therefore it follows from (5.15) and (5.16) that

$$\begin{aligned} t^{-C^*\varepsilon}|V_{\sigma,\omega}^{(\alpha)}(t)| \le & t_{0,\sigma}^{-C^*\varepsilon}|V_{\sigma,\omega}^{(\alpha)}(t_{0,\sigma})| + C\varepsilon\langle\sigma\rangle^{-\mu}\int_{t_{0,\sigma}}^{t} \tau^{-C^*\varepsilon+2\mu-2}d\tau \\ & + C\int_{t_{0,\sigma}}^{t} \tau^{-C^*\varepsilon-1}\big(\mathcal{V}_{\sigma,\omega}^{(|\alpha|-1)}(\tau)\big)^2 d\tau \\ \le & C\varepsilon\langle\sigma\rangle^{\mu-1} + C\int_{2}^{t} \tau^{-C^*\varepsilon-1}\big(\mathcal{V}_{\sigma,\omega}^{(|\alpha|-1)}(\tau)\big)^2 d\tau. \end{aligned}$$

To sum up with respect to $|\alpha| \le l$, we have

$$t^{-C^*\varepsilon}\mathcal{V}_{\sigma,\omega}^{(l)}(t) \le C\varepsilon\langle\sigma\rangle^{\mu-1} + C\int_{2}^{t} \tau^{-C^*\varepsilon-1}\big(\mathcal{V}_{\sigma,\omega}^{(l-1)}(\tau)\big)^2 d\tau$$

for $l \in \{1,\dots,k\}$. Using this inequality, we can show inductively that

$$\mathcal{V}_{\sigma,\omega}^{(l)}(t) \le C\varepsilon\langle\sigma\rangle^{\mu-1}t^{2^{l-1}C^*\varepsilon} \tag{5.25}$$

for $t_{0,\sigma} \le t < T$ and $l \in \{1,\dots,k\}$. Indeed, we already know that

$$\mathcal{V}_{\sigma,\omega}^{(0)}(t) = |V_{\sigma,\omega}(t)| \le C\varepsilon\langle\sigma\rangle^{\mu-1}$$

by (5.21). Hence we have

$$t^{-C^*\varepsilon}\mathcal{V}_{\sigma,\omega}^{(1)}(t) \le C\varepsilon\langle\sigma\rangle^{\mu-1} + C\varepsilon^2\langle\sigma\rangle^{2\mu-2}\int_{2}^{\infty} \tau^{-C^*\varepsilon-1}d\tau \le C\varepsilon\langle\sigma\rangle^{\mu-1},$$

which implies (5.25) for $l = 1$. Next we suppose that (5.25) is true for some $l \in \{1,\dots,k-1\}$. Then we have

$$\begin{aligned} t^{-C^*\varepsilon}\mathcal{V}_{\sigma,\omega}^{(l+1)}(t) \le & C\varepsilon\langle\sigma\rangle^{\mu-1} + C\varepsilon^2\langle\sigma\rangle^{2\mu-2}\int_{2}^{t} \tau^{(2^l-1)C^*\varepsilon-1}d\tau \\ \le & C\varepsilon\langle\sigma\rangle^{\mu-1}t^{(2^l-1)C^*\varepsilon}, \end{aligned}$$

which yields (5.25) with l replaced by $l+1$. Hence (5.25) for $l \in \{1,\dots,k\}$ has been proved.

By (5.13) and (5.25) with $l = k$, we have

$$|\partial u(t,x)|_k \le C\varepsilon\langle t+|x|\rangle^{-1+2^{k-1}C^*\varepsilon}\langle t-|x|\rangle^{-1+\mu}, \quad (t,x) \in \Lambda_{T,R}.$$

Finally we take $\varepsilon \le 2^{1-k}\nu/C^*$ to obtain

$$\sup_{(t,x)\in\Lambda_{T,R}} \langle t+|x|\rangle^{1-\nu}\langle t-|x|\rangle^{1-\mu}|\partial u(t,x)|_k \le C\varepsilon. \tag{5.26}$$

The final step.

By (5.11), (5.22), and (5.26), we see that there exist two positive constants ε_2 and m such that (5.2) holds for $0 < \varepsilon \le \varepsilon_2$. This completes the proof of Lemma 5.1. □

6. Asymptotics for the solution to the profile equation

This section is devoted to preliminaries for the proof of Theorem 1.2. Let $t_0 \ge 1$. Keeping the application to the profile equation (or (5.18)) in mind, we consider the following ordinary differential equation for $t > t_0$:

$$i\frac{dz}{dt}(t) = \frac{\Phi(z(t))}{t} z(t) + J(t), \tag{6.1}$$

where $\Phi : \mathbb{C} \to \mathbb{R}$ satisfies

$$|\Phi(z) - \Phi(w)| \le C_0|z - w| \quad \text{for} \quad z, w \in \mathbb{C} \tag{6.2}$$

with a positive constant C_0, and $J : [t_0, \infty) \to \mathbb{C}$ satisfies

$$|J(t)| \le E_0 t^{-1-\lambda} \tag{6.3}$$

with positive constants E_0 and λ. Concerning the asymptotics for the solution $z(t)$ of (6.1), we have the following lemma.

Lemma 6.1. *Let $z(t)$ be the solution of (6.1), and suppose $C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda) < \lambda^2$. Then there is a function $p = p(s)$ on $[\log t_0, \infty)$ such that we have*

$$|z(t) - p(\log t)| \le \frac{E_0\lambda}{\{\lambda^2 - C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda)\}t^{\lambda}}, \quad t \ge t_0, \tag{6.4}$$

and

$$i\frac{dp}{ds}(s) = \Phi(p(s))p(s), \quad s \ge \log t_0. \tag{6.5}$$

To prove Lemma 6.1, we introduce some sequences. For the solution $z(t)$ of (6.1), we define sequences $\{z_n(t)\}_{n=0}^{\infty}$, $\{\Theta_n(t)\}_{n=0}^{\infty}$, and $\{\zeta_n\}_{n=0}^{\infty}$ in the following way: We set $z_0(t) = z(t)$, and inductively define

$$\Theta_n(t) = \int_{t_0}^{t} \Phi\big(z_n(\tau)\big)\frac{d\tau}{\tau}, \quad t \ge t_0, \tag{6.6}$$

$$\zeta_n = \lim_{\tau\to\infty} z_n(\tau)e^{i\Theta_n(\tau)},$$

$$z_{n+1}(t) = \zeta_n e^{-i\Theta_n(t)}, \qquad t \ge t_0 \tag{6.7}$$

for $n \in \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the set of non-negative integers. In order to see that this definition works well, we have only to check the convergence of $\lim_{\tau\to\infty} z_n(\tau)e^{i\Theta_n(\tau)}$ for each n.

Lemma 6.2. *The above sequences* $\{z_n(t)\}_{n=0}^\infty$, $\{\Theta_n(t)\}_{n=0}^\infty$, *and* $\{\zeta_n\}_{n=0}^\infty$ *are well-defined. Moreover we have*

$$\zeta_n = \left(z(t_0) - i \int_{t_0}^\infty J(\tau) e^{i\Theta_0(\tau)} d\tau \right) \times \exp\left(i \int_{t_0}^\infty \left\{ \Phi(z_n(\tau)) - \Phi(z_0(\tau)) \right\} \frac{d\tau}{\tau} \right) \tag{6.8}$$

and

$$|z_{n+1}(t) - z_n(t)| \le \frac{E_0}{\lambda t^\lambda} \left(\frac{C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda)}{\lambda^2} \right)^n \tag{6.9}$$

for $n \in \mathbb{N}_0$.

Proof. We prove Lemma 6.2 by the induction on n.

First we consider the case of $n = 0$. Since $z_0 = z$, it follows from (6.1) that

$$\left(z_0(t) e^{i\Theta_0(t)} \right)' = -iJ(t) e^{i\Theta_0(t)},$$

which yields

$$z_0(t) e^{i\Theta_0(t)} = z(t_0) - i \int_{t_0}^t J(\tau) e^{i\Theta_0(\tau)} d\tau.$$

This shows that $z_0(\tau) e^{i\Theta_0(\tau)}$ converges as $\tau \to \infty$, and that (6.8) for $n = 0$ holds, because (6.3) implies $J(\cdot) e^{i\Theta_0(\cdot)} \in L^1(t_0, \infty)$. As for (6.9) with $n = 0$, we have

$$\left(z_1(t) - z_0(t) \right) e^{i\Theta_0(t)} = \zeta_0 - z_0(t) e^{i\Theta_0(t)} = -i \int_t^\infty J(\tau) e^{i\Theta_0(\tau)} d\tau,$$

whence

$$|z_1(t) - z_0(t)| \le \int_t^\infty |J(\tau)| d\tau \le \frac{E_0}{\lambda t^\lambda}.$$

Note that by (6.3) we have

$$|\zeta_0| = \left| z(t_0) - i \int_{t_0}^\infty J(\tau) e^{i\Theta_0(\tau)} d\tau \right| \le |z(t_0)| + \frac{E_0}{\lambda t_0^\lambda}. \tag{6.10}$$

Next we consider the case of $n = n_0 + 1$ under the assumption that ζ_n for $n \le n_0$ are well-defined (thus $z_n(t)$ and $\Theta_n(t)$ for $n \le n_0 + 1$ are also well-defined), and that (6.8) and (6.9) are true for $n \le n_0$. We set $K = C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda)/\lambda^2$. By (6.2) and (6.9) for $n = n_0$, we get

$$\left| \Phi(z_{n_0+1}(t)) - \Phi(z_{n_0}(t)) \right| \le C_0 |z_{n_0+1}(t) - z_{n_0}(t)| \le \frac{C_0 E_0}{\lambda t^\lambda} K^{n_0}. \tag{6.11}$$

We put

$$\theta_{n_0} = \int_{t_0}^\infty \left\{ \Phi(z_{n_0+1}(\tau)) - \Phi(z_{n_0}(\tau)) \right\} \frac{d\tau}{\tau},$$

which is finite because of (6.11). It also follows from (6.11) that

$$\begin{aligned}\big|\Theta_{n_0+1}(t)-\Theta_{n_0}(t)-\theta_{n_0}\big| \le& \int_t^\infty \big|\Phi(z_{n_0+1}(\tau))-\Phi(z_{n_0}(\tau))\big|\frac{d\tau}{\tau}\\ \le& \frac{C_0E_0}{\lambda^2 t^\lambda}K^{n_0}. \end{aligned} \tag{6.12}$$

Now we obtain from (6.7) for $n = n_0$ and (6.12) that

$$\begin{aligned}\zeta_{n_0+1} =& \lim_{\tau\to\infty}\left(z_{n_0+1}(\tau)e^{i\Theta_{n_0+1}(\tau)}\right) = \zeta_{n_0}\exp\Big(i\lim_{\tau\to\infty}\left(\Theta_{n_0+1}(\tau)-\Theta_{n_0}(\tau)\right)\Big)\\ =&\zeta_{n_0}e^{i\theta_{n_0}},\end{aligned}$$

which immediately leads to (6.8) for $n = n_0 + 1$ if we replace ζ_{n_0} by the right-hand side of (6.8) for $n = n_0$. Since $|\zeta_{n_0}| = |\zeta_0|$, it follows from (6.7), (6.10), and (6.12) that

$$\begin{aligned}\big|z_{n_0+2}(t)-z_{n_0+1}(t)\big| &= \Big|\zeta_{n_0}e^{i\theta_{n_0}}e^{-i\Theta_{n_0+1}(t)} - \zeta_{n_0}e^{-i\Theta_{n_0}(t)}\Big|\\ &\le \big|\zeta_{n_0}\big|\big|\theta_{n_0}-\Theta_{n_0+1}(t)+\Theta_{n_0}(t)\big|\\ &\le \left(|z(t_0)|+\frac{E_0}{\lambda t_0^\lambda}\right)\frac{C_0E_0}{\lambda^2t^\lambda}K^{n_0}\\ &= \frac{E_0}{\lambda t^\lambda}K^{n_0+1},\end{aligned}$$

which is (6.9) for $n = n_0 + 1$. This completes the proof. □

Now we are in a position to prove Lemma 6.1.

Proof of Lemma 6.1. We put $K = C_0(E_0t_0^{-\lambda} + |z(t_0)|\lambda)/\lambda^2$. Then we have $0 < K < 1$ from the assumption. By (6.9) we can easily show that $\{z_n(\cdot)\}_{n=0}^\infty$ is a uniform Cauchy sequence on $[t_0,\infty)$, and $\{z_n(\cdot)\}_{n=0}^\infty$ converges uniformly on $[t_0,\infty)$ as $n \to \infty$. We put

$$p(s) := \lim_{n\to\infty} z_n(e^s), \quad s \ge \log t_0.$$

Note that we have $|p(s)| = |\zeta_0|$, because (6.7) and (6.8) imply $|z_n(t)| = |\zeta_{n-1}| = |\zeta_0|$ for any $n \in \mathbb{N}$. Since we have $p(\log t) = \lim_{n\to\infty} z_n(t)$ and $0 < K < 1$, it follows from (6.9) that

$$\begin{aligned}|z(t)-p(\log t)| =& \lim_{n\to\infty}|z_0(t)-z_n(t)|\\ \le&\sum_{n=0}^\infty |z_{n+1}(t)-z_n(t)| \le \sum_{n=0}^\infty \frac{E_0}{\lambda t^\lambda}K^n \le \frac{E_0}{\lambda(1-K)t^\lambda},\end{aligned}$$

which is (6.4).

To show (6.5), we set

$$\Theta_\infty(t) = \int_{t_0}^t \Phi\big(p(\log\tau)\big)\frac{d\tau}{\tau} = \int_{\log t_0}^{\log t}\Phi(p(\sigma))d\sigma,$$

which is well-defined because $|p(s)| = |\zeta_0|$ for $s \ge \log t_0$. Then it follows that

$$\begin{aligned}|\Theta_\infty(t) - \Theta_n(t)| &\le \int_{t_0}^{t} C_0 |p(\log\tau) - z_n(\tau)| \frac{d\tau}{\tau} \\ &\le \int_{t_0}^{\infty} C_0 \sum_{j=n}^{\infty} \frac{E_0}{\lambda \tau^\lambda} K^j \frac{d\tau}{\tau} \\ &\le \frac{C_0 E_0 K^n}{\lambda^2 (1-K) t_0^\lambda},\end{aligned}$$

whence $\lim_{n\to\infty} \Theta_n(t) = \Theta_\infty(t)$. Similarly we can show

$$\lim_{n\to\infty} \int_{t_0}^{\infty} \left\{ \Phi\big(z_n(\tau)\big) - \Phi\big(z_0(\tau)\big) \right\} \frac{d\tau}{\tau} = \int_{t_0}^{\infty} \left\{ \Phi\big(p(\log\tau)\big) - \Phi\big(z_0(\tau)\big) \right\} \frac{d\tau}{\tau},$$

which implies that $\{\zeta_n\}$ converges as $n \to \infty$ with the help of (6.8) (note that (6.4) shows the existence of the integral on the right-hand side of the identity above). Thus, by setting $\zeta_\infty = \lim_{n\to\infty} \zeta_n$, we have

$$p(s) = \lim_{n\to\infty} \zeta_{n-1} e^{-i\Theta_{n-1}(e^s)} = \zeta_\infty e^{-i\Theta_\infty(e^s)} = \zeta_\infty \exp\left(-i \int_{\log t_0}^{s} \Phi(p(\sigma)) d\sigma \right).$$

By differentiation, we see that $p(s)$ solves the desired equation (6.5). □

In the remaining part of this section, we will apply Lemma 6.1 to the profile equation. Let u be the global solution to (2.1)–(2.3) for small ε, let U be as in (4.4), and let R be the constant appearing in (4.1). From now on, we write $V(t;\sigma,\omega) = U(t,(t+\sigma)\omega)$, instead of $V_{\sigma,\omega}(t)$, for $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^2$ and $t > \max\{0, -\sigma\}$. It follows from (5.18) that $V(t;\sigma,\omega)$ satisfies

$$i\partial_t V(t;\sigma,\omega) = \frac{\mathrm{Re}\big(V(t;\sigma,\omega)\big)}{2t} V(t;\sigma,\omega) + iH(t,(t+\sigma)\omega) \tag{6.13}$$

for $t > t_{0,\sigma}$ and $\sigma \le R$. Note that all the estimates obtained in the proof of Lemma 5.1 are valid with $T = \infty$, because we have already shown that (5.4) is valid. On the other hand, for $\sigma > R$, we have

$$\lim_{t\to\infty} V(t;\sigma,\omega) = \lim_{t\to\infty} 0 = 0$$

because of the finite propagation property (4.2).

As an application of Lemma 6.1, we have the following.

Corollary 6.3. *Let ε be sufficiently small. Then $\lim_{t\to\infty} V(t;\sigma,\omega)$ exists for each $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^2$. If we put*

$$V^+(\sigma,\omega) := \lim_{t\to\infty} V(t;\sigma,\omega)$$

for each $(\sigma,\omega) \in \mathbb{R} \times \mathbb{S}^2$, then we have

$$\mathrm{Re}\, V^+(\sigma,\omega) = 0 \tag{6.14}$$

for $(\sigma,\omega)\in\mathbb{R}\times\mathbb{S}^2$*. Moreover we have* $V^+\in L^2(\mathbb{R}\times\mathbb{S}^2)$ *and*

$$\lim_{t\to\infty}\int_{\mathbb{R}\times\mathbb{S}^2}|\chi_t(\sigma)V(t;\sigma,\omega)-V^+(\sigma,\omega)|^2 d\sigma dS_\omega=0, \tag{6.15}$$

where $\chi_t(\sigma)=1$ *for* $\sigma>-t$*, and* $\chi_t(\sigma)=0$ *for* $\sigma\le -t$*.*

Proof. First we show the convergence of $V(t;\sigma,\omega)$ as $t\to\infty$, and (6.14). We have only to consider the case $\sigma\le R$, because the opposite case is trivial. By (5.14) and (5.20), we can apply Lemma 6.1 to (6.13) with $z(t)=V(t;\sigma,\omega)$, $J(t)=iH(t,(t+\sigma)\omega)$, $t_0=t_{0,\sigma}$ if ε is small enough, because we have

$$C_0(E_0t_0^{-\lambda}+|z(t_0)|\lambda)\le C_1\varepsilon<\lambda^2$$

for $0<\varepsilon<\lambda^2/C_1$, where we have taken $C_0=1/2$, $E_0=C\varepsilon\langle\sigma\rangle^{-\mu}$, and $\lambda=1-2\mu$, while C_1 is an appropriate positive constant independent of σ and ω. It follows from Lemma 6.1 that for any $(\sigma,\omega)\in(-\infty,R]\times\mathbb{S}^2$, there is $p(s)$ satisfying

$$i\frac{dp}{ds}(s)=\frac{\operatorname{Re}\big(p(s)\big)}{2}p(s)$$

and

$$\lim_{t\to\infty}|V(t;\sigma,\omega)-p(\log t)|=0.$$

So it is enough to show that $p(s)$ converges as $s\to\infty$, and that $\operatorname{Re}p(s)\to 0$ as $s\to\infty$. For this purpose, we set $X(s)=\operatorname{Re}p(s)/2$, $Y(s)=\operatorname{Im}p(s)/2$ to rewrite the above equation as

$$\frac{dX}{ds}(s)=X(s)Y(s),\quad \frac{dY}{ds}(s)=-X(s)^2. \tag{6.16}$$

We observe that

$$\frac{d}{ds}\Big(X(s)^2+Y(s)^2\Big)=0,$$

which implies that $X(s)^2+Y(s)^2$ is independent of s. We denote this conserved quantity by ρ^2, where $\rho\ge 0$. The case $\rho=0$ is trivial, because we have $X(s)=Y(s)\equiv 0$. Hence we assume $\rho>0$ from now on. From the second equation of (6.16) we have

$$\frac{dY}{ds}(s)=Y(s)^2-\rho^2.$$

This can be explicitly integrated as

$$Y(s)=\rho\frac{(\rho+\eta)e^{-\rho s}-(\rho-\eta)e^{\rho s}}{(\rho+\eta)e^{-\rho s}+(\rho-\eta)e^{\rho s}}$$

with some real constant η satisfying $|\eta|\le\rho$. We can also see that

$$X(s)=\frac{2\rho\xi}{(\rho+\eta)e^{-\rho s}+(\rho-\eta)e^{\rho s}}$$

with some real constant ξ satisfying $\xi^2 + \eta^2 = \rho^2$. If $\xi = 0$, then we have $X(s) \equiv 0$, and $Y(s) \equiv \pm\rho$. If $\xi \neq 0$, then $\eta^2 < \rho^2$. Especially we have $\eta \neq \rho$, and we get

$$\lim_{s\to\infty} X(s) = \lim_{s\to\infty} \frac{2\rho\xi e^{-\rho s}}{(\rho+\eta)e^{-2\rho s} + (\rho-\eta)} = 0,$$
$$\lim_{s\to\infty} Y(s) = \rho \lim_{s\to\infty} \frac{(\rho+\eta)e^{-2\rho s} - (\rho-\eta)}{(\rho+\eta)e^{-2\rho s} + (\rho-\eta)} = -\rho.$$

Now the existence of $\lim_{t\to\infty} V(t;\sigma,\omega)$ and (6.14) have been established.

It follows from (5.5) and (5.9) that

$$|U(t,r\omega)| = \big|D_-\big(ru(t,r\omega)\big)\big| \leq C\varepsilon\langle t-r\rangle^{-1+\mu}$$

for any $(t,r,\omega) \in [0,\infty) \times (0,\infty) \times \mathbb{S}^2$. Since $V(t;\sigma,\omega) = U\big(t,(t+\sigma)\omega\big)$, we obtain

$$|V(t;\sigma,\omega)| \leq C\varepsilon\langle\sigma\rangle^{-1+\mu}$$

for $(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^2$ and $t > \max\{0,-\sigma\}$. Hence, by taking the limit of this inequality as $t \to \infty$, we have

$$|V^+(\sigma,\omega)| \leq C\varepsilon\langle\sigma\rangle^{-1+\mu}, \quad (\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^2,$$

which shows $V^+ \in L^2(\mathbb{R}\times\mathbb{S}^2)$ since $\mu < 1/2$. Furthermore we have

$$|\chi_t(\sigma)V(t;\sigma,\omega) - V^+(\sigma,\omega)|^2 \leq C\varepsilon^2\langle\sigma\rangle^{-2+2\mu} \in L^1(\mathbb{R}\times\mathbb{S}^2)$$

for $t \geq 0$. Now, since $\lim_{t\to\infty} |\chi_t(\sigma)V(t;\sigma,\omega) - V^+(\sigma,\omega)|^2 = 0$ for each $(\sigma,\omega) \in \mathbb{R}\times\mathbb{S}^2$, Lebesgue's convergence theorem implies (6.15). This completes the proof. □

7. Proof of Theorem 1.2

In the following, we write

$$\hat{\omega}(x) = \big(\widehat{\omega}_a(x)\big)_{a=0,1,2,3} = (-1, x_1/|x|, x_2/|x|, x_3/|x|)$$

for $x \in \mathbb{R}^3 \setminus \{0\}$. For the proof of Theorem 1.2, we will use the following lemma:

Lemma 7.1. *Let $\phi \in C\big([0,\infty); \dot{H}^1(\mathbb{R}^3)\big) \cap C^1\big([0,\infty); L^2(\mathbb{R}^3)\big)$. There exists $(\phi_0^+, \phi_1^+) \in \dot{H}^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that*

$$\lim_{t\to\infty} \|\phi(t) - \phi^+(t)\|_E = 0,$$

if and only if there is a function $P = P(\sigma,\omega) \in L^2(\mathbb{R}\times\mathbb{S}^2)$ such that

$$\lim_{t\to\infty} \|\partial u(t,\cdot) - \hat{\omega}(\cdot)P^\sharp(t,\cdot)\|_{L^2(\mathbb{R}^3)} = 0,$$

where $\phi^+ \in C\big([0,\infty); \dot{H}^1(\mathbb{R}^3)\big) \cap C^1\big([0,\infty); L^2(\mathbb{R}^3)\big)$ is the unique solution to $\Box\phi^+ = 0$ with $(\phi^+, \partial_t\phi^+)(0) = (\phi_0^+, \phi_1^+)$, $P^\sharp$ is given by

$$P^\sharp(t,x) = \frac{1}{|x|}P(|x|-t, |x|^{-1}x), \quad x \neq 0,$$

and $\partial=(\partial_0,\partial_1,\partial_2,\partial_3)$.

See [9] for the proof (see also [8], where the above result was implicitly proved). We note that if (ϕ_0^+,ϕ_1^+) is given, P above is obtained as the translation representation of (ϕ_0^+,ϕ_1^+), which was introduced by Lax-Phillips [14, Chapter IV].

Proof of Theorem 1.2. Let u be the global solution to (2.1)–(2.3) for small ε. We put $U(t,x):=D_-\big(ru(t,x)\big)$ (with $r=|x|$), $V(t;\sigma,\omega)=U(t,(t+\sigma)\omega)$, and $V^+(\sigma,\omega)=\lim_{t\to\infty}V(t;\sigma,\omega)$ as in the previous section. Then, as we have mentioned above, all the estimates in the proof of Lemma 5.1 are valid.

Let

$$V^{+,\sharp}(t,x)=\frac{1}{|x|}V^+(|x|-t,|x|^{-1}x),\quad x\neq 0.$$

We define

$$J_1(t)=\left(\int_{\mathbb{S}^2}\left(\int_0^\infty |r\partial u(t,r\omega)-\widehat{\omega}(r\omega)V(t;r-t,\omega)|^2dr\right)dS_\omega\right)^{1/2},$$

$$J_2(t)=\left(\int_{\mathbb{S}^2}\left(\int_0^\infty |\widehat{\omega}(r\omega)V(t;r-t,\omega)-r\widehat{\omega}(r\omega)V^{+,\sharp}(t,r\omega)|^2dr\right)dS_\omega\right)^{1/2}.$$

By (3.4), (3.5), and (5.9) we get

$$\begin{aligned}J_1(t)^2\le& C\int_{\mathbb{S}^2}\left(\int_0^\infty |u(t,r\omega)|_1^2dr\right)dS_\omega\le C\varepsilon^2\int_0^\infty\langle t+r\rangle^{2\mu-2}dr\\ \le& C\varepsilon^2\langle t\rangle^{2\mu-1}\to 0\end{aligned}$$

as $t\to\infty$. It follows from (6.15) that

$$\begin{aligned}J_2(t)^2=&2\int_{\mathbb{S}^2}\left(\int_0^\infty |V(t;r-t,\omega)-V^+(r-t,\omega)|^2dr\right)dS_\omega\\ =&2\int_{\mathbb{S}^2}\left(\int_{-t}^\infty |\chi_t(\sigma)V(t;\sigma,\omega)-V^+(\sigma,\omega)|^2d\sigma\right)dS_\omega\\ \le&2\int_{\mathbb{S}^2}\left(\int_{\mathbb{R}} |\chi_t(\sigma)V(t;\sigma,\omega)-V^+(\sigma,\omega)|^2d\sigma\right)dS_\omega\to 0\end{aligned}$$

as $t\to\infty$, because $\chi_t(\sigma)=1$ for $\sigma>-t$. Therefore we get

$$\begin{aligned}&\big\|\partial u(t,\cdot)-\widehat{\omega}(\cdot)V^{+,\sharp}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\ &=\left(\int_{\mathbb{S}^2}\left(\int_0^\infty |r\partial u(t,r\omega)-r\widehat{\omega}(r\omega)V^{+,\sharp}(t,r\omega)|^2dr\right)dS_\omega\right)^{1/2}\\ &\le J_1(t)+J_2(t)\to 0 \qquad (7.1)\end{aligned}$$

as $t\to\infty$.

We write $u_1=\operatorname{Re}u$ and $u_2=\operatorname{Im}u$ as before. Similarly we put $V_1^+=\operatorname{Re}V^+$, $V_2^+=\operatorname{Im}V^+$, $V_1^{+,\sharp}=\operatorname{Re}V^{+,\sharp}$, and $V_2^{+,\sharp}=\operatorname{Im}V^{+,\sharp}$. (6.14) says that

$V_1^+(\sigma,\omega)=0$, and accordingly we have $V_1^{+,\sharp}(t,x)=0$. Hence from (7.1) we get

$$\lim_{t\to\infty}\|u_1(t,\cdot)\|_E=\lim_{t\to\infty}\frac{1}{\sqrt{2}}\|\partial u_1(t,\cdot)-\widehat{\omega}(\cdot)V_1^{+,\sharp}(t,\cdot)\|_{L^2(\mathbb{R}^3)}=0.$$

It also follows from (7.1) that

$$\lim_{t\to\infty}\|\partial u_2(t,\cdot)-\widehat{\omega}(\cdot)V_2^{+,\sharp}(t,\cdot)\|_{L^2(\mathbb{R}^3)}=0.$$

Hence recalling that $V_2^+\in L^2(\mathbb{R}\times\mathbb{S}^2)$, we can apply Lemma 7.1 to conclude the existence of $(f_2^+,g_2^+)\in\dot{H}^1\times L^2$ such that

$$\lim_{t\to\infty}\|u_2(t)-u_2^+(t)\|_E=0,$$

where u_2^+ solves $\Box u_2^+=0$ with $(u_2^+,\partial_t u_2^+)(0)=(f_2^+,g_2^+)$. This completes the proof. □

8. Concluding remarks

Our reduction of the original two-component system (1.1) to the single complex-valued equation (2.1) in Section 2 is just for simplicity of exposition and not essential in our proof. In fact, we can apply our method to show the small data global existence for an N-component system

$$\Box v_j=F_j(\partial v),\quad (t,x)\in\mathbb{R}\times\mathbb{R}^3 \tag{8.1}$$

for $1\le j\le N$ with $v=(v_j)_{1\le j\le N}:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^N$ under the following assumptions:

(i) F_j vanishes of quadratic order at the origin of $\mathbb{R}^{4N}$,

(ii) There are positive constants $\kappa_1,\dots,\kappa_N$ such that

$$\sum_{j=1}^N \kappa_j Y_j F_j^{\mathrm{red}}(\omega,Y)=0,\quad (\omega,Y)\in\mathbb{S}^2\times\mathbb{R}^N, \tag{8.2}$$

where, writing $F_j(\partial v)=F_j(\partial_0 v,\partial_1 v,\partial_2 v,\partial_3 v)$, we define the *reduced nonlinearity* F_j^{red} by

$$F_j^{\mathrm{red}}(\omega,Y):=\lim_{\lambda\to+0}\lambda^{-2}F_j(-\lambda Y,\omega_1\lambda Y,\omega_2\lambda Y,\omega_3\lambda Y)$$

for $\omega=(\omega_1,\omega_2,\omega_3)\in\mathbb{S}^2$ and $Y=(Y_j)_{1\le j\le N}\in\mathbb{R}^N$. Note that

$$\begin{cases} F_1^{\mathrm{red}}(\omega,Y)=-c_2Y_1Y_2,\\ F_2^{\mathrm{red}}(\omega,Y)=c_1Y_1^2\end{cases}$$

in the case of (1.1) or (1.6), whence (ii) is satisfied if and only if $c_1c_2>0$ or $c_1=c_2=0$.

For a solution $v=(v_j)_{1\le j\le N}$ to (8.1), we put $U_j(t,r\omega)=D_-\big(rv_j(t,r\omega)\big)$ and $U=(U_j)_{1\le j\le N}$. Then the associated system of profile equations becomes

$$\partial_+U_j(t,r\omega)=-\frac{1}{2t}F_j^{\mathrm{red}}\big(\omega,U(t,r\omega)\big)+H_j(t,r\omega),$$

where H_j is given by

$$H_j = -\frac{1}{2}\left(rF_j(\partial v) - \frac{1}{t}F_j^{\mathrm{red}}(\omega, U)\right) - \frac{1}{2r}\Delta_{\mathbb{S}^2} v_j.$$

We also put $U_j^{(\alpha)}(t, r\omega) = D_-\big(r\Gamma^\alpha v_j(t, r\omega)\big)$ for $|\alpha| \geq 1$. Then the system corresponding to (4.6) is

$$\partial_+ U_j^{(\alpha)} = -\frac{1}{2t}G_{j,\alpha} + H_{j,\alpha},$$

where

$$G_{j,\alpha} = \sum_{k=1}^{N}\left(\partial_{Y_k}F_j^{\mathrm{red}}\right)(\omega, U)U_k^{(\alpha)},$$

$$H_{j,\alpha} = -\frac{1}{2}\left(r\widetilde{\Gamma}^\alpha F_j(\partial v) - \frac{1}{t}G_{j,\alpha}\right) - \frac{1}{2r}\Delta_{\mathbb{S}^2}\Gamma^\alpha v_j.$$

The condition (ii) plays the role of (2.2) in the derivation of an estimate corresponding to (5.19), because (8.2) implies

$$\partial_+ \sum_{j=1}^{N} \kappa_j |U_j(t, r\omega)|^2 = 2\sum_{j=1}^{N}\kappa_j U_j(t, r\omega)H_j(t, r\omega).$$

We need only apparent modifications for the other parts of the arguments to obtain the small data global existence. We can also show that the global solution v to (8.1) satisfies

$$\sup_{t\in\mathbb{R}} \|v(t)\|_E < \infty$$

and

$$|\partial v(t, x)| \leq C\varepsilon\langle t + |x|\rangle^{-1}\langle t - |x|\rangle^{-1+\mu}, \quad (t, x) \in \mathbb{R}\times\mathbb{R}^3$$

under (i) and (ii), where $\mu \in (0, 1/2)$ can be arbitrarily fixed. However, it is difficult to specify the asymptotic profile of the solution as precisely as that stated in Theorem 1.2 because our argument heavily depends on the form of the profile equation and the explicit integrability of (6.5). For related results on the nonlinear Schrödinger systems, see [4], [15], etc.

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