Pretty good state transfer on double stars

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Abstract

Let A be the adjacency matrix of a graph X and suppose $U(t) = \exp(itA)$. We view A as acting on $\mathbb{C}^{V(X)}$ and take the standard basis of this space to be the vectors e_u for u in $V(X)$. Physicists say that we have *perfect state transfer* from vertex u to v at time τ if there is a scalar γ such that

$$U(\tau)e_u = \gamma e_v.$$

(Since $U(t)$ is unitary, $\|\gamma\| = 1$.) For example, if X is the d-cube and u and v are at distance d then we have perfect state transfer from u to v at time $\pi/2$. Despite the existence of this nice family, it has become clear that perfect state transfer is rare. Hence we consider a relaxation: we say that we have *pretty good* state transfer from u to v if there is a complex number γ and, for each positive real ϵ there is a time t such that

$$\|U(t)e_u - \gamma e_v\| < \epsilon.$$

Again we necessarily have $|\gamma| = 1$.

In Godsil, Kirkland, Severini and Smith [7] it is shown that we have have pretty good state transfer between the end vertices of the path P_n if and only $n+1$ is a power of two, a prime, or twice a prime.

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(There is perfect state transfer between the end vertices only for P_2 and P_3.) It is something of a surprise that the occurrence of pretty good state transfer is characterized by a number-theoretic condition. In this paper we study *double-star* graphs, which are trees with two vertices of degree $k+1$ and all other vertices with degree one. We prove that there is never perfect state transfer between the two vertices of degree $k+1$, and that there is pretty good state transfer between them if and only if $4k+1$ is a perfect square.

1 Introduction

Let X be a graph on n vertices with adjacency matrix A and let $U(t)$ denote the matrix-valued function $\exp(iAt)$. We note that $U(t)$ is both symmetric and unitary, and that it determines what is called a *continuous quantum walk*. Work in quantum computing has raised many questions about the relation between physically interesting properties of $U(t)$ and properties of the graph X. For recent surveys see [4], [8].

The physical properties of interest to us in this paper are *perfect state transfer* and *pretty good state transfer*. Assume $n = |V(X)|$ and identify the coordinates of $\mathbb{R}^n$ with $V(X)$. If $u \in V(X)$, we use e_u to denote the standard basis vector indexed by u. If u and v are distinct vertices of X we say we have *perfect state transfer* from u to v at time τ if there is a complex number γ such that

$$U(\tau)e_u = \gamma e_v.$$

Since $U(t)$ is unitary, $|\gamma| = 1$. The evidence is that perfect state transfer is uncommon, and we consider a relaxation of it. We say that we have *pretty good state transfer* from u to v if there is a complex number γ and, for each positive real ϵ there is a time t such that

$$\|U(t)e_u - \gamma e_v\| < \epsilon.$$

Again we necessarily have $|\gamma| = 1$. Pretty good state transfer was introduced in [4].

There is a considerable literature on perfect state transfer. In a seminal paper on the topic Christandl et al. [3] show that there is perfect state transfer between the end vertices of the paths P_2 and P_3, but perfect state transfer does not occur between the end vertices of any path on four or more vertices. From [4] we know that, for any integer k, there are only finitely many connected graphs with maximum valency at most k on which perfect state transfer occurs.

Much less is known about pretty good state transfer. In [4] it is shown that it takes place on P_4 and P_5. Vinet and Zhedanov [12] have studied pretty good state transfer on weighted paths with loops Godsil, Kirkman, Severini and Smith that pretty good state transfer occurs between the end-vertices of P_n if and only $n+1=2^m$, or if $n+1=p$ or $2p$ where p is an odd prime. It is surprising to see that the existence of pretty good state transfer depends so delicately on the prime divisors of $n+1$.

In this paper we provide a second class of graphs where pretty good state transfer occurs if and only if a number theoretic condition holds. Let $S_{k,k}$ denote the graph we get by taking two copies of $K_{1,k}$ and joining the two vertices of degree k by a new edge. We show that there is pretty good state transfer between the vertices of degree $k+1$ in $S_{k,k}$ if and only if $4k+1$ is not a perfect square. We also show that there is never perfect state transfer between these two vertices. We conclude the paper with some remarks that show that if pretty good state transfer does occur on a graph, then, in a sense, it must occur regularly.

2 Quotients

We introduce a useful tool. A more expansive treatment will be found in [6, Ch. 9].

Let X be a graph. A partition π of vertex set $V(X)$ with cells

$$C_1, C_2, \cdots, C_r$$

is *equitable*, if the number of neighbors in C_j of any vertex u in C_i is a constant b_{ij}. The directed graph with the r cells of π as its vertices and b_{ij} arcs from the i-th to the j-th cells of π is called the *quotient* of X over π, and denoted by X/π. The entries of the adjacency matrix of this quotient graph are given by $A(X/\pi)_{i,j} = b_{ij}$. We can symmetrize $A(X/\pi)$ to B by letting $B_{i,j} = \sqrt{b_{ij}b_{ji}}$. We call the (weighted) graph with adjacency matrix B the *symmetrized quotient graph*. In the following we always use B to denote the symmetrized form of the matrix $A(X/\pi)$.

If π is a partition of $V(X)$, its characteristic matrix, denoted by P, is the 01-matrix whose columns are the characteristic vectors of the cells of π, viewed as subsets of $V(X)$. If we normalize the characteristic matrix

P such that each column have length one, then we obtain the *normalized characteristic matrix* of π, denoted by Q. Note that $Q^TQ = I$ and QQ^T is a block diagonal matrix with diagonal blocks $\frac{1}{r}J_r$, where J_r is the all-ones matrix of order $r \times r$, and the size of the i-th block is the size of the i-th cell of π. The vertex u forms a singleton cell of π if and only if $QQ^Te_u = e_u$.

For the sake of convenience, in the following text, we always denote by $\{u\}$ the singleton cell $\{u\}$.

Since A is symmetric, it has a spectral decomposition

$$A = \sum_r \theta_r E_r$$

where θ_r runs over the distinct eigenvalues θ_r of A and E_r is the matrix that represents orthogonal projection onto the the eigenspace belonging to θ_r.

2.1 Lemma. *Let π be an equitable partition of X with normalized characteristic matrix Q. Let A be the adjacency matrix of X and let B be the adjacency matrix of the symmetrized quotient graph. Then the idempotents in the spectral decomposition of B are the non-zero matrices Q^TE_rQ, where E_r runs over the idempotents in the spectral decomposition of A.*

Proof. As π is equitable, $AQ = QB$. Hence $A^kQ = QB^k$ and so if $f(t)$ is a polynomial then $f(A)Q = Qf(B)$. There is a polynomial f_r such that $f_r(A) = E_r$, and hence

$$E_rQ = Qf_r(B).$$

Then $f_r(B) = Q^TE_rQ$ is symmetric, we show it is idempotent. We have

$$(Q^TE_rQ)^2 = Q^TE_rQQ^TE_rQ$$

and since QQ^T commutes with A, it commutes with E_r. Since $Q^TQ = I$ we then have

$$Q^TE_rQQ^TE_rQ = Q^TQQ^TE_rQ = Q^TE_rQ.$$

It follows that

$$B = Q^TAQ = \sum_r \theta_r Q^TE_rQ \tag{2.1}$$

and since

$$\sum_r Q^TE_rQ = Q^TQ = I$$

we conclude that (2.1) is the spectral decomposition of B. $\square$

If u and v are singletons in π, the 2×2 submatrix of $Q^T E_e Q$ indexed by $\{u\}$ and $\{v\}$ is equal to the 2×2 submatrix of E_r indexed by u and v. Since u and v are strongly cospectral if and only if for each idempotent E_r, we have $(E_r)_{u,u} = (E_r)_{v,v} = \pm(E_r)_{u,v}$.

In [1], R. Bachman et el. studied perfect state transfer of quantum walks on quotient graphs. They showed that the ab-entry of the transition function $\exp(iA(G)t)$ of original graph G is equal to the $\{a\}\{b\}$-entry of the transition function $\exp(iA(G/\pi)t)$ of the quotient graph with equitable distance partition with respect to vertices a and b. In fact, their result and their proof hold for an arbitrary equitable partition with vertices a and b as singleton cells. (This is because all they need is that QQ^T commutes with B and $Q^T Q = I$, and these conditions hold for any equitable partition. We state the general result without proof. For the details, see [1, Theorem 2].

2.2 Lemma. *Let X be a graph with an equitable partition π. $\{a\}$ and $\{b\}$ are singleton cells. B is the adjacency matrix of symmetric quotient graph. Then, for any time t,*

$$(e^{-itA_X})_{a,b} = (e^{-itB})_{\{a\},\{b\}}.$$

Therefore, G has perfect state transfer from a to b at time t if and only if the symmetrized quotient graph has perfect state transfer from $\{a\}$ to $\{b\}$. $\square$

3 Strongly Cospectral Vertices

Let u and v be vertices in X. We say that u and v are *cospectral vertices* if the characteristic polynomials $\phi(X \setminus u, t)$ and $\phi(X \setminus v, t)$ are equal. If $\theta_1, \ldots, \theta_m$ are the distinct eigenvalues of X and the matrices $E_1, \ldots, E_m$ are the orthogonal projections onto the corresponding eigenspaces we have the spectral decomposition

$$A = \sum_r \theta_r E_r.$$

From [4] we know that u and v are cospectral if and only if $(E_r)_{u,u} = (E_r)_{v,v}$, for all r. Since $(E_r)_{u,u} = \|E_r e_u\|^2$, we see that u and v are cospectral if and only if the projections $E_r e_u$ and $E_r e_v$ have the same length for each r. We say that u and v are *strongly cospectral* if, for each r,

$$E_r e_v = \pm E_r e_u.$$

[9] observed that if there is perfect state transfer from vertices u to v, then u and v are strongly cospectral. In [4], an argument due to Dave Witte is presented, which shows that if there is pretty good state transfer from vertex u to vertex v, then u and v are strongly cospectral. If the eigenvalues of A are simple then two vertices are strongly cospectral if and only if they are cospectral.

3.1 Lemma. *Vertices u and v are strongly cospectral if and only if for each idempotent E_r, we have $(E_r)_{u,u} = (E_r)_{v,v} = \pm(E_r)_{u,v}$.*

Proof. The vertices u and v are cospectral if and only if $(E_r)_{u,u} = (E_r)_{v,v}$ for each r. Set $y = E_r e_u$ and $z = E_r e_v$. By Cauchy-Schwarz

$$0 \leq \|y\|^2 \|z\|^2 - |y^T z|^2$$

and equality holds if and only if $\{y, z\}$ is linearly dependent. Since

$$\|y\|^2 \|z\|^2 - |y^T z|^2 = (E_r)_{u,u}(E_r)_{v,v} - (E_r)^2_{u,v}.$$

and since $\|E_r e_u\| = \|E_r e_v\|$ when u and v are cospectral, the lemma follows. □

We will show that for the symmetric double star graphs $S_{k,k}$, the two central vertices are strongly cospectral.

Now, we will give characterize the strongly cospectral using equitable partition.

3.2 Lemma. *Let X be a graph and let π be an equitable partition of X in which $\{u\}$ and $\{v\}$ are singleton cells. Then u and v are strongly cospectral in X if and only if $\{u\}$ and $\{v\}$ are strongly cospectral in the symmetrized quotient graph X/π.*

Proof. Let Q be the normalized characteristic matrix of π and let B be the symmetrized quotient matrix. If E_r is an idempotent in the spectral decomposition of A, then $E_r = p_r(A)$ for some polynomial p and so

$$E_r Q = p_r(A)Q = Qp_r(B).$$

Therefore $p_r(B) = Q^T E_r Q$ and

$$AE_r Q = \theta_r E_r Q = \theta_r AQp_r(B) = \theta_r QBp_r(B),$$

from which it follows that the non-zero matrices $Q^T E_r Q$ are the idempotents in the spectral decomposition of B. If $a, b \in \{u, v\}$, then since $\{u\}$ and $\{v\}$ are singleton cells of π, we have

$$(Q^T E_r Q)_{a,b} = (E_r)_{a,b}.$$

We conclude that u and v are strongly cospectral in X if an only if they are strongly cospectral in X/π. □

3.3 Lemma. *([4]) If X admits perfect transfer from u to v, then $E_\theta e_u = \pm\, E_\theta e_v$ for all θ, and u and v are cospectral.*

4 Double Star Graphs

Let S_k and S_ℓ be the two star graphs with k and ℓ edges respectively. Then the double star graph, which we denote by $S(k, \ell)$, is the graph obtained by joining the two vertices with degrees k and ℓ of S_k and S_ℓ, respectively. We call a double star graph a *symmetric double-star* if $k = \ell$.

In this section, we will show that the symmetric double star graphs do not have perfect state transfer. Furthermore, we will show that these graphs have pretty good state transfer between two central vertices if and only if $4k + 1$ is not a perfect square.

First, we will give some known results which we will use later. [4] For the first, note that if G is a group of automorphisms of the graph x, then G_x denotes the subgroup consisting of the automorphisms that fix x.

4.1 Lemma. *Let G be the automorphism group of X. If we have a perfect state transfer from vertex u to vertex v, then $G_u = G_v$, where G_u is the automorphism group which fixed vertex u.* □

Denoted by $\phi(X, x)$ the characteristic polynomial of $A(X)$. Recall that vertices u and v in the graph X are cospectral if

$$\phi(X \backslash u, t) = \phi(X \backslash v, t).$$

For double star graph $S(k, \ell)$, by symmetry we may always assume in the following that $k \leq \ell$.

4.2 Lemma. *If the double star graph $S(k, \ell)$ has perfect state transfer then either:*

(a) $k=2$, $\ell>2$ *and perfect state transfer occurs between two vertices with degree one and adjacent to the same vertices with degree* 3, *or*

(b) $k=\ell$ *and perfect state transfer can only occur between the two central vertices* u *and* v.

Proof. Assume that double star graph $S(k,\ell)$ has perfect state transfer.

If $k=\ell=1$, then $S(k,\ell)$ is P_4 and has no perfect state transfer between two end vertices.

If $k=2, l\neq 2$, by Lemma 3.3 and Lemma 4.1, perfect state transfer could occurs between two vertices with degree one and adjacent to the same vertices with degree three.

If $k,\ell>2$, then by Lemma 4.1 if $S(k,\ell)$ admits perfect state transfer, then perfect state transfer could only occurs from vertex u to v, where u and v are the only two vertices in X with degree at least two. And furthermore,if there is perfect state transfer from u to v, then u and v are cospectral by Lemma 3.3. Therefore the degree of vertices u and v are the same, that is, $k=l$. □

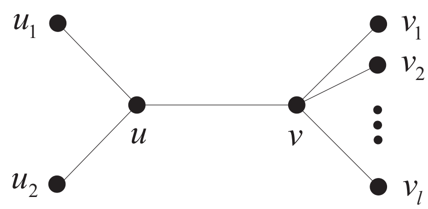

Figure 1: The double star graph $S_{2,\ell}$

4.3 Theorem. *The double star graph* $S(2,\ell)$ *has pretty good state transfer if* $\ell\neq 2$ *and* ℓ^2-2l+9 *or* $2l+6\pm\sqrt{l^2-2l+9}$ *is not a perfect square.*

Proof. Suppose u and v are the two central vertices of $S(2,\ell)$ with $d(u)=2$ and w_1,w_2 are the two neighbors of u. Then

$$\{\{w_1\},\{w_2\},\{u\},\{v\},N(v)\setminus\{u\}\}$$

is an equitable partition of $S(2,\ell)$. Let B be the adjacency matrix of the corresponding symmetrized quotient graph, then the eigenvalues of B are

$$0,\ \pm\frac{1}{2}\sqrt{2\ell+6+2\sqrt{\ell^2-2\ell+9}},\quad \pm\frac{1}{2}\sqrt{2\ell+6-2\sqrt{\ell^2-2\ell+9}}$$

Let

$$F=\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then by direct calculation, we have $F=-E_1+E_2+E_3+E_4+E_5$, where $E_1,\dots,E_5$ are the idempotents of the matrix B. If there is perfect state transfer, then

$$U(t)=\sum_r \exp(i\theta_r t)E_{\theta_r}=\gamma F.$$

Hence $\gamma=-1$ and $\exp(i\theta_i t)=-1$ for $i=2,3,4,5$. Therefore, there is no perfect state transfer between $[w_1]_B$ and $[w_2]_B$. However, by Kronecker's approximation theorem, if $\ell^2-2\ell+9$ or $2\ell+6\pm\sqrt{\ell^2-2\ell+9}$ is not a perfect square, then we can choose a sequence $\{t_k\}$ such that $\lim_{k\to\infty}\exp(i\theta_r t_k)=-1$ for $r=2,3,4,5$. Therefore, by Lemma 2.2, the result follows. □

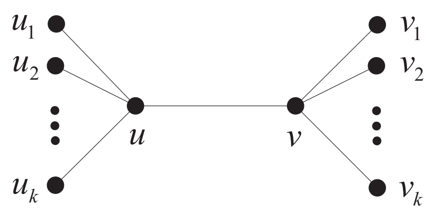

Figure 2: The double star graph $S_{k,k}$

4.4 Lemma. *Let X be a symmetric double star graph $S_{k,k}$. Then*

$$\exp(iAt)_{u,v}=\big((1-2\beta)\sin\alpha t+2\beta\sin(1-\alpha)t\big)i.$$

where $\alpha=\frac{1+\sqrt{1+4k}}{2}$ and $\beta=\frac{k}{1+4k+\sqrt{1+4k}}$.

Proof. Let π be an equitable partition with cells $\{N(u)\backslash v,\{u\},\{v\},N(v)\backslash u\}$. Then the adjacency matrix of the quotient graphX/π is in Fig. 1. Then

$$A_{X/\pi}=\begin{pmatrix} 0 & 1 & 0 & 0\\ k & 0 & 1 & 0\\ 0 & 1 & 0 & k\\ 0 & 0 & 1 & 0 \end{pmatrix}$$

After symmetrizing, we get

$$B = \begin{pmatrix} 0 & \sqrt{k} & 0 & 0 \\ \sqrt{k} & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{k} \\ 0 & 0 & \sqrt{k} & 0 \end{pmatrix}$$

The eigenvalues of B are $\alpha, 1-\alpha, \alpha-1, -\alpha$. The corresponding eigenvectors are the columns of the following matrix:

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ -\frac{\sqrt{k}}{\alpha} & \frac{\sqrt{k}}{\alpha-1} & \frac{\sqrt{k}}{\alpha-1} & -\frac{\sqrt{k}}{\alpha} \\ \frac{\sqrt{k}}{\alpha} & -\frac{\sqrt{k}}{\alpha-1} & \frac{\sqrt{k}}{\alpha-1} & \frac{\sqrt{k}}{\alpha} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned}\exp(iBt)_{\{u\},\{v\}} =& (\frac{1}{2} - \beta)\exp(i\alpha t) + \beta\exp(i(1-\alpha)t) \\ & - \beta\exp(-i(1-\alpha)t) - (\frac{1}{2} - \beta)\exp(-i\alpha t) \\ & = ((1-2\beta)\sin(\alpha t) + 2\beta\sin((1-\alpha)t))i.\end{aligned}$$

By Lemma 2.2, we have

$$\exp(iAt)_{u,v} = \exp(iBt)_{\{u\},\{v\}}.$$

The result follows. $\square$

4.5 Lemma. *Let $S_{k,k}$ be a double star graph. Then there is no perfect state transfer from one vertex of degree $k+1$ to the other.*

Proof. Note that $2\beta = \frac{2k}{1+4k+\sqrt{1+4k}}$ and so we have $0 < \beta < 1$. Hence

$$\begin{aligned}|\exp(iAt)_{u,v}| &= |(1-2\beta)\sin(\alpha t) + 2\beta\sin((1-\alpha)t)| \\ &\leq |1-2\beta| + |2\beta| \\ &= 1.\end{aligned}$$

Equality holds if and only $\sin \alpha t = \sin(1-\alpha)t = \pm 1$. Without loss of generality, assume that $\sin \alpha t = \sin(1-\alpha)t = 1$. Then

$$\alpha t = \frac{\pi}{2} + 2m\pi, \tag{4.1}$$

$$(1-\alpha)t = \frac{\pi}{2} + 2n\pi. \tag{4.2}$$

It follows that $\alpha = \frac{4m+1}{4(m+n)+2}$. This implies that α is not an integer.

On the other hand, suppose $\delta = 1 + 4k$ and $\alpha = \frac{1+\sqrt{\delta}}{2}$. Then we can rewrite equations (4.1) and (4.2) in the following form:

$$\frac{1+\sqrt{\delta}}{2} t = \frac{\pi}{2} + 2m\pi,$$

$$\frac{1-\sqrt{\delta}}{2} t = \frac{\pi}{2} + 2n\pi$$

and hence

$$\frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} = \frac{1+4m}{1+4n} \in \mathbb{Q}.$$

Note that on the other hand,

$$\frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} = \frac{1+\delta+2\sqrt{\delta}}{1-\delta},$$

which implies that δ is a perfect square. Since $\delta = 1 + 4k$ is odd, we can assume that $\delta = (2s+1)^2$, thus $\alpha = \frac{1+\sqrt{\delta}}{2} = s + 1$ is an integer. Contradiction. □

In the rest, we will investigate pretty good state transfer on the symmetric double star graph. We say we have pretty good state transfer from u to v if there is a sequence $\{t_k\}$ of real numbers and a scalar γ such that $\lim_{k\to\infty} U(t_k) = \gamma e_v$, where $\|\gamma\| = 1$.

4.6 Lemma. *Let X be a graph and π is an equitable partition with $\{u\}$ and $\{v\}$ are singletons. Then there is pretty good state transfer from vertex u to vertex v if and only if there is pretty good state transfer from $\{u\}$ to $\{v\}$ in the symmetrized quotient graph with adjacency matrix B.*

Proof. By Lemma 2.2, we have $\exp(iBt_k)_{\{u\},\{v\}} = \exp(iAt_k)_{u,v}$. Hence

$$\lim_{t\to\infty} \exp(iAt_k)_{u,v} = \gamma$$

if and only if

$$\lim_{t\to\infty} \exp(iBt_k)_{\{u\},\{v\}} = \gamma. \qquad \square$$

4.7 Theorem. *Let $S_{k,k}$ be a symmetric double star graph. Then either*

(a) $1+4k$ is not a perfect square and there is pretty good state transfer from vertex u to vertex v, or

(b) $1+4k$ is a perfect square and there is not pretty good state transfer from vertex u to vertex v.

Proof. If $4k+1$ is a perfect square then the eigenvalues of $S(k,k)$ are rational and hence integers. So in this case $S(k,k)$ is periodic and since we know that perfect state transfer does not occur, pretty good state transfer does not occur either.

Assume then that $4k+1$ is not a perfect square. By Lemma 4.4 we see that there is pretty good state transfer from u to v if and only if there is a sequence of times $(t_\ell)_{\ell\geq 0}$ such that

$$\lim_{\ell\to\infty} \sin \alpha t_\ell = \lim_{\ell\to\infty} \sin(1-\alpha)t_\ell = \pm 1. \tag{4.3}$$

Note that if $\lim_{\ell\to\infty} \sin \alpha t_\ell = \pm 1$ then $\lim_{\ell\to\infty} \cos \alpha t_\ell = 0$. Since

$$\cos t_\ell = \cos \alpha t_\ell \cos(1-\alpha)t_\ell - \sin \alpha t_\ell \sin(1-\alpha)t_\ell$$

we see that if (4.3) holds then $\lim_{\ell\to\infty} \cos t_\ell = -1$.

This implies that $t_\ell \approx (2m+1)\pi$ and $\alpha t_\ell \approx n\pi + \frac{\pi}{2}$ for $m,n \in \mathbb{Z}$. The question becomes whether we can choose integers m,n such that $(2m+1)\alpha - n \approx \frac{1}{2}$.

As $1+4k$ is not a perfect square, α is a irrational number. So α and $\frac{1}{2}$ are linearly independent over the rationals, and hence by Kronecker's approximation theorem the set

$$\{m\alpha - n : m,n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. Therefore we can chose $m,n \in \mathbb{Z}$, such that

$$m\alpha - n \approx \frac{1}{4} - \frac{1}{2}\alpha.$$

This implies we can choose a series $\{t_\ell\}$ such that both $\lim_{\ell\to\infty} \sin \alpha t_\ell = \pm 1$ and $\lim_{\ell\to\infty} \cos t_\ell = -1$. $\square$

5 Recurrence

If A is the adjacency matrix of a graph X then the set

$$\{U(t) : t \in \mathbb{R}\}$$

is an abelian group, which we denote by G. In fact G is a 1-parameter subgroup of the unitary group on $\mathbb{C}^n$, where $n = |V(X)|$, and so its closure $\overline{G}$ is an abelian Lie group. Since G is connected, so is its closure and therefore $\overline{G}$ is isomorphic to a direct product of some number of copies of $\mathbb{R}/\mathbb{Z}$. Asking whether there is perfect state transfer from u to v is equivalent to asking whether there is a matrix M in G such that

$$M_{u,u} = M_{v,v} = 0, \quad M_{u,v} = M_{v,u} = \gamma \tag{5.1}$$

where $|\gamma| = 1$. Asking whether there is pretty good state transfer is asking whether there is a matrix M in $\overline{G}$ such that these conditions hold.

If we prove that pretty good state transfer does occur, we have shown that there is a sequence of times t_ℓ such that $t_\ell \to \infty$ and, for each $\epsilon > 0$ there is time t_ℓ such that $U(t_\ell)$ is within ϵ of a solution to the conditions of (5.1). However we can say something more concrete, using the following.

5.1 Lemma. *If $\epsilon > 0$, there is a time T such that each element of $\overline{G}$ lies within ϵ of an element of $\{U(s+t) : 0 \le t \le T\}$, for any s.*

Proof. Define

$$S = \{M \in \overline{G} : \|M - I\| < \epsilon\}$$

Then S is open and its translates under the action of G cover $\overline{G}$. Since $\overline{G}$ is compact, some finite set of translates of S cover $\overline{G}$. Hence there is a time T such that all elements of $\overline{G}$ lie within ϵ of an element of $\{U(t) : 0 \le t \le T\}$. Since $U(t)$ is unitary it follows that each element of $\overline{G}$ lies within ϵ of an element of

$$\{U(t+s) : 0 \le t \le T\},$$

for any s. We conclude that any element of $\overline{G}$ lies within ϵ of an element of $\{U(t+s) : 0 \le t \le T\}$, for any s. $\square$

5.2 Corollary. *If we have pretty good state transfer from u to v in X then for each positive ϵ there is a real number T such that, for each s, the set $\{U(t+s) : 0 \le t \le T\}$ contains an element within ϵ of a solution to* (5.1).

Since $U(0) = I$, the arguments above also yield the conclusion that, if $\epsilon > 0$ then there is a time T such that in each real interval of length T there is a time t such that $\|U(t) - I\| < \epsilon$. Thus we can say that any graph is *approximately periodic*. (We recall that a graph is *periodic* if there is a time T such that $U(t) = \gamma I$, for some complex number γ with norm 1. Periodic graphs are studied, and characterized, in [5]; their eigenvalues must be square roots of integers, and consequently these graphs are rare.)

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References

[1] R. Bachman, E. Fredette, J. Fuller, M. Landry, M. Opperman, C. Tamon and A. Tollefson, *Perfect state transfer on quotient graphs*, Quantum Information and Computation, 12 (2012), pp. 0293–0313. [arXiv:1108.0339v2 [quant-ph]]

[2] M. Christandl, N. Datta, T. Dorlas, A. Ekert, A. Kay, and A. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Phys. Rev. A 71 (2005), p. 032312. [arXiv:quant-ph/0411020v2]

[3] M. Christandl, N. Datta, A. Ekert and A. Landahl, *Perfect state transfer in quantum spin networks*, Phys. Rev. Lett. 92 (2004), p. 187902. [arXiv:quant-ph/0309131v2]

[4] C. Godsil, *State transfer on graphs*, Discrete Mathematics 312 (2012), 129–147. [arXiv:1102.4898v2 [math.CO]]

[5] C. Godsil, *Periodic Graphs*, Electron. J. Comb. 18(1) (2011). [arXiv:0806.2074v2 [math.CO]]

[6] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

[7] C. Godsil, S. Kirkland, S. Severini, J. Smith, *Number-theoretic nature of communication in quantum spin chains*, arXiv:1201.4822v1 [quant.ph].

[8] A. Kay, *Perfect, efficient, state transfer and its application as a constructive tool*, International Journal of Quantum Information 8(4) (2010) 641–676. [arXiv: 0903.4274v3 [quant-ph]]

[9] A. Kay, *Basics of Perfect communication through quantum networks*, Phys. Rev. A 84 (2011), p. 022337. [arXiv:1102.2338v2 [quant-ph]]

[10] V. Kendon, *Quantum walks on general graphs*, International Journal of Quantum Information 4(5) (2006) 791-805. [arXiv:quant-ph/0306140v3]

[11] N. Saxena, S. Severini, and I. Shparlinski, *Parameters of integral circulant graphs and periodic quantum dynamics*, International Journal of Quantum Information 5(3) (2007) 417–430. [arXiv:quant-ph/0703236v1]

[12] Luc Vinet, Alexei Zhedanov, *Almost perfect state transfer in quantum spin chains*. arXiv:1205.4680v2 [quant-ph].