CONSTRUCTING ULTRAPOWERS FROM ELEMENTARY EXTENSIONS OF FULL CLONES

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Abstract. Let A be an infinite set. Let $\Omega(A)$ be the algebra over A where every constant is a fundamental constant and every finitary function is a fundamental operation. We shall give a method of representing any algebra $\mathcal{L}$ in the variety generated by $\Omega(A)$ as limit reduced powers and even direct limits of limit reduced powers of $\mathcal{L}$. If the algebra $\mathcal{L}$ is elementarily equivalent to $\Omega(A)$, then this construction represents $\Omega(A)$ as a limit ultrapower and also as direct limits of limit ultrapowers of $\Omega(A)$. This method therefore gives a method of representing Boolean ultrapowers and other generalizations of the ultrapower construction as limit ultrapowers and direct limits of limit ultrapowers.

1. Motivation

For this paper, let A be a fixed infinite set. If $a \in A$, then let $\hat{a}$ be a constant symbol, and if $f : A^n \rightarrow A$, then let $\hat{f}$ be an n-ary function symbol. Let

$$\mathcal{F} = \{\hat{f} | f : A^n \rightarrow A \,\text{for some}\, n \geq 1\} \cup \{\hat{a} | a \in A\}.$$

Let $\Omega(A)$ be the algebra of type $\mathcal{F}$ with universe A and where $\hat{a}^{\Omega(A)} = a$ for all $a \in A$ and where $\hat{f}^{\Omega(A)} = f$ for each function f of finite arity. We shall now study the variety $V(\Omega(A))$ generated by $\Omega(A)$.

It is well known that $V(\Omega(A)) = HP_S(\Omega(A)) = HSP(\Omega(A))$. Therefore every algebra in $V(\Omega(A))$ is isomorphic to a quotient of a subdirect power of $\Omega(A)$. We shall soon see that the quotients of the subdirect powers of $\Omega(A)$ are simply the limit reduced powers of $\Omega(A)$.

If I is a set, then we shall write $\Pi(I)$ for the lattice of partitions of I. If $f : I \rightarrow X$ is a function, then we shall write $\Pi(f)$ for the partition $\{f_{-1}(\{x\}) | x \in X\} \setminus \{\emptyset\}$.

Theorem 1.1. *Let I be a set, and let $\mathcal{B} \subseteq \Omega(A)^I$ be a subalgebra. Then there is a filter F on $\Pi(I)$ such that $f \in B$ if and only if $\Pi(f) \in F$.*

Proof. Let $F = \{\Pi(f) | f \in \mathcal{B}\}$. We shall now show that F is a filter. Let $f, g \in \mathcal{B}$, and let $i : A^2 \rightarrow A$ be an injective function. Then $\hat{i}^{\mathcal{B}}(f, g) : I \rightarrow A$ is a function with $\hat{i}^{\mathcal{B}}(f, g) \in B$ and $\Pi(\hat{i}^{\mathcal{B}}(f, g)) = \Pi(f) \wedge \Pi(g)$. If $f \in B$ and $\Pi(f) \preceq P$, then there is a function $L : A \rightarrow A$ such that $\Pi(\hat{L}^{\mathcal{B}}(f)) = \Pi(L \circ f) = P$. Therefore F is a filter.

We now claim that $\mathcal{B} = \{f \in \Omega(A)^I | \Pi(f) \in F\}$. If $\Pi(f) \in F$, then there is a function $g \in B$ with $\Pi(f) = \Pi(g)$. Therefore there is a function $i : A \rightarrow A$ such that $f = i \circ g = \hat{i}^{\mathcal{B}}(g)$. Therefore $f \in B$. $\square$

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In other words, every subalgebra of $\Omega(A)^I$ is of the form $\{f\in\Omega(A)^I|\Pi(f)\in F\}$ for some filter $F\subseteq\Pi(I)$. We shall write $\Omega(A)^F$ for the algebra $\{f\in\Omega(A)^I|\Pi(f)\in F\}$. One can easily show that $\{\emptyset\}\cup\bigcup F$ is a Boolean algebra. We shall now give a one-to-one correspondence between the filters on $\{\emptyset\}\cup\bigcup F$ and the congruences on $\Omega(A)^F$.

If $Z\subseteq\{\emptyset\}\cup\bigcup F$ is a filter, then let $\theta\subseteq\Omega(A)^F\times\Omega(A)^F$ be the relation where we have $(f,g)\in\theta$ if and only if $\{i\in I|f(i)=g(i)\}\in Z$. One can easily show that θ is a congruence on Z. We shall let $\Omega(A)^F/Z$ denote the quotient algebra $\Omega(A)^F/\theta$, and we shall call $\Omega(A)^F/Z$ a limit reduced power of $\Omega(A)$. If Z is an ultrafilter, then we shall call $\Omega(A)^F/Z$ a limit ultrapower of $\Omega(A)$. If Z is a filter on the set I, then we shall write $\Omega(A)^I/Z$ for $\Omega(A)^{\Pi(I)}/Z$, and we shall call $\Omega(A)^I/Z$ a reduced power of A, and if Z is an ultrafilter, then we shall simply call $\Omega(A)^I/Z$ an ultrapower of A. The following theorem shows that every quotient of $\Omega(A)^F$ is a limit reduced power of $\Omega(A)$.

Theorem 1.2. *Let $F\subseteq\Pi(I)$ be a filter. Let θ be a congruence on $\Omega(A)^F$. Then define $Z\subseteq\{\emptyset\}\cup\bigcup F$ to be the set where we have $R\in Z$ if and only if whenever $f,g\in\Omega(A)^F$ and $f|_R=g|_R$, we have $(f,g)\in\theta$. Then Z is a filter on $\{\emptyset\}\cup\bigcup F$. Furthermore, we have $(f,g)\in\theta$ if and only if $\{i\in I|f(i)=g(i)\}\in Z$.*

Proof. We shall first show that Z is a filter. If $R,S\in\{\emptyset\}\cup\bigcup F,R\subseteq S,R\in Z$, then whenever $f|_S=g|_S$, we have $f|_R=g|_R$, so $(f,g)\in\theta$. Therefore $S\in Z$ as well. We conclude that Z is an upper set. Now assume that $R,S\in Z$. Assume that $f|_{R\cap S}=g|_{R\cap S}$. Then there is a function $h\in\Omega(A)^F$ where $h|_R=f|_R$ and $h|_S=g|_S$. Therefore $(h,f)\in\theta,(h,g)\in\theta$, so $(f,g)\in\theta$. Therefore Z is a filter.

Now assume that $(f,g)\in\theta$. Then let $R=\{i\in I|f(i)=g(i)\}$. Now let $f^\sharp,g^\sharp$ be functions where $f^\sharp|_R=g^\sharp|_R$. Let $P=\Pi(f)\wedge\Pi(g)\wedge\Pi(f^\sharp)\wedge\Pi(g^\sharp)$ and let $h:I\rightarrow A$ be a function such that $\Pi(h)=P$. One can easily show that there is a function $\alpha:A^2\rightarrow A$ such that $\alpha(h(i),f(i))=f^\sharp(i)$ for $i\in I$ and $\alpha(h(i),g(i))=g^\sharp(i)$ for $i\in I$. In other words, there is a function α where $\hat{\alpha}^{\Omega(A)^F}(h,f)=f^\sharp$ and $\hat{\alpha}^{\Omega(A)^F}(h,g)=g^\sharp$. Therefore since $(f,g)\in\theta$, we have $(f^\sharp,g^\sharp)\in\theta$ as well. Therefore $R\in Z$. Similarly, if $\{i\in I|f(i)=g(i)\}\in Z$, then clearly $(f,g)\in\theta$. We conclude that $(f,g)\in\theta$ if and only if $\{i\in I|f(i)=g(i)\}\in Z$. □

It is now clear that the elements of the variety $V(\Omega(A))$ are simply the algebras isomorphic to the limit reduced powers of $\Omega(A)$. We also conclude that the lattice of congruences on $\Omega(A)^F$ is isomorphic to the lattice of filters on the Boolean algebra $\{\emptyset\}\cup\bigcup F$. Furthermore, if $\Omega(A)^F/Z$ is a limit reduced power, then the lattice of congruences on $\Omega(A)^F/Z$ is isomorphic to the lattice of filters on the Boolean algebra $(\{\emptyset\}\cup\bigcup F)/Z$.

Let $\mathcal{L}\in V(\Omega(A))$. Then define a function $e:\Omega(A)\rightarrow\mathcal{L}$ by letting $e(a)=\hat{a}^{\mathcal{L}}$ for $a\in A$. One can easily show that e is the only homomorphisms from $\Omega(A)$ to $\mathcal{L}$. The following theorem shows that every elementary extension $\mathcal{L}$ of $\Omega(A)$ is isomorphic to a limit ultrapower of $\Omega(A)$. In the following theorem, one needs to take note that the variety $V(\Omega(A))$ is congruence permutable(congruence permutable means that $\theta_1\circ\theta_2=\theta_2\circ\theta_1$ whenever θ_1 and θ_2 are congruences in some algebra $\mathcal{L}\in V(\Omega(A))$). Congruence permutability follows from the limit reduced power representation of algebras or from Mal'Cev's characterization of congruence permutable varieties [1][Sec. 2.12].

Theorem 1.3. *Let $\mathcal{L} \in V(\Omega(A))$ be an algebra with more than one element. Then the following are equivalent.*

1. *$\mathcal{L}$ is simple.*
2. *$\mathcal{L}$ is subdirectly irreducible.*
3. *$\mathcal{L}$ is directly indecomposable.*
4. *The mapping $e : \Omega(A) \to \mathcal{L}$ is an elementary embedding.*

If $\mathcal{L} = \Omega(A)^F/Z$ is a limit reduced power of $\Omega(A)$, then the above four statements are equivalent to the following statement.

5. *Z is an ultrafilter on the Boolean algebra $\{\emptyset\} \cup \bigcup F$.*

Proof. Since every algebra in $V(\Omega(A))$ is isomorphic to some limit reduced power of A, we may assume that $\mathcal{L} = \Omega(A)^F/Z$.

$1 \to 2, 2 \to 3$ These directions are trivial.

$5 \to 4$ This is a consequence of Los's theorem for limit ultrapowers [4][Sec. 6.4].

$5 \to 1$. If Z is an ultrafilter, then since $\mathrm{Con}(\Omega(A)^F/Z)$ is isomorphic to the lattice of congruences on $(\{\emptyset\} \cup \bigcup F)/Z$, there are only 2 congruences on $\Omega(A)^F/Z$.

$4 \to 3$ We shall prove this direction by contrapositive. Assume that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ where $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-trivial algebras. Let $a, b \in A$ be distinct elements, and let $i : A \to A$ be a function with $i''(A) = \{a, b\}$ and where $i(a) = a, i(b) = b$. Then $\Omega(A)$ satisfies the sentence $\forall x(\hat{i}(x) = \hat{a} \vee \hat{i}(x) = \hat{b})$. However, we have $\hat{i}^{\mathcal{L}}(\hat{a}^{\mathcal{L}_1}, \hat{b}^{\mathcal{L}_2}) = (\hat{i}^{\mathcal{L}_1}(\hat{a}^{\mathcal{L}_1}), \hat{i}^{\mathcal{L}_2}(\hat{b}^{\mathcal{L}_2})) = (\hat{a}^{\mathcal{L}_1}, \hat{b}^{\mathcal{L}_2})$, but $(\hat{a}^{\mathcal{L}_1}, \hat{b}^{\mathcal{L}_2}) \neq \hat{a}^{\mathcal{L}}$ and $(\hat{a}^{\mathcal{L}_1}, \hat{b}^{\mathcal{L}_2}) \neq \hat{b}^{\mathcal{L}}$. Therefore $\mathcal{L} \not\models \forall x(\hat{i}(x) = \hat{a} \vee \hat{i}(x) = \hat{b})$. Therefore the mapping e is not an elementary embedding.

$3 \to 5$ If Z is not an ultrafilter on $\{\emptyset\} \cup \bigcup F$, then since the lattice of congruences on $\Omega(A)^F/Z$ is isomorphic to $\mathrm{Con}((\{\emptyset\} \cup \bigcup F)/Z)$, there is a pair θ_1, θ_2 of non-trivial congruences such that $\theta_1 \cap \theta_2 = \{(x, x) | x \in X\}$ and $\theta_1 \vee \theta_2 = X^2$. Clearly, we have $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ since variety $V(\Omega(A)^F)/Z$ is congruence permutable. Therefore, we have

$$\Omega(A)^F/Z \simeq (\Omega(A)^F/Z)/\theta_1 \times (\Omega(A)^F/Z)/\theta_2$$

by [1][Sec. 2.7], so $\Omega(A)^F/Z$ is not direct indecomposable. □

See [4][Sec. 6.4] for a similar but more model theoretic proof that every elementary extension of $\Omega(A)$ is a limit ultrapower of $\Omega(A)$, and see [2] for an algebraic proof of this result. We shall now represent the free algebras in $V(\Omega(A))$ as algebras of the form $\Omega(A)^F$. Since every algebra in $V(\Omega(A))$ can easily be represented as a quotient of a free algebra, one can easily represent any algebra in $V(\Omega(A))$ as a quotient of $\Omega(A)^F$, so the algebras in $V(\Omega(A))$ are representable as limit reduced powers and limit ultrapowers of $\Omega(A)$.

If P is a partition of a set X, then we shall write $x = y(P)$ if x and y are contained in the same block of the partition P. Let I be a set. If $i_1, \dots, i_n \in I$, then let $\mathcal{P}_{i_1,\dots,i_n}$ be the partition of A^I where $f = g(\mathcal{P}_{i_1,\dots,i_n})$ if and only if $f(i_1) = g(i_1), \dots, f(i_n) = g(i_n)$. Clearly $\{\mathcal{P}_{i_1,\dots,i_n} | n \in \mathbb{N}, i_1, \dots, i_n \in I\}$ is a filterbase on $\Pi(A^I)$. We shall write $\mathcal{P}(A, I)$ for the filter generated by the filterbase $\{\mathcal{P}_{i_1,\dots,i_n} | n \in \mathbb{N}, i_1, \dots, i_n \in I\}$. Let $\mathbf{F}(A, I) = \Omega(A)^{\mathcal{P}(A,I)}$.

We shall now show that $\mathbf{F}(A, I)$ is a free algebra. Let $\pi_i : A^I \to A$ be the projection function where $\pi_i(f) = f(i)$ for each $f : I \to A$. Clearly $\Pi(\pi_i) = \mathcal{P}_i$ since $\pi_i(f) = \pi_i(g)$ if and only if $f(i) = g(i)$ if and only if $f = g(\mathcal{P}_i)$. Therefore $\pi_i \in \mathbf{F}(A, I)$ for all $i \in I$.

Theorem 1.4. *The functions $(\pi_i)_{i \in I}$ freely generate $\mathbf{F}(A, I)$.*

Proof. For each $i \in I$, we have $\Pi(\pi_i) = \mathcal{P}_i$. Therefore we have $\langle\{\pi_i | i \in I\}\rangle = \Omega(A)^{\mathcal{P}(A,I)} = \mathbf{F}(A, I)$, so $\{\pi_i | i \in I\}$ generates $\mathbf{F}(A, I)$.

We shall now show that $(\pi_i)_{i\in I}$ freely generates $\mathbf{F}(A, I)$. It suffices to show that whenever $f : A^n \to A, g : A^m \to A$ and $\hat{f}^{\mathbf{F}(A,I)}(\pi_{i_1}, \dots, \pi_{i_n}) = \hat{g}^{\mathbf{F}(A,I)}(\pi_{j_1}, \dots, \pi_{j_m})$, then the identity $\hat{f}(x_{i_1}, \dots, x_{i_n}) = \hat{g}(x_{j_1}, \dots, x_{j_m})$ holds. If $(a_i)_{i\in I} \in A^I$, then we have

$$f(a_{i_1}, \dots, a_{i_n}) = f(\pi_{i_1}(a_i)_{i\in I}, \dots, \pi_{i_n}(a_i)_{i\in I}) = \hat{f}^{\mathbf{F}(A,I)}(\pi_{i_1}, \dots, \pi_{i_n})(a_i)_{i\in I}$$

$$= \hat{g}^{\mathbf{F}(A,I)}(\pi_{j_1}, \dots, \pi_{j_n})(a_i)_{i\in I} = g(a_{j_1}, \dots, a_{j_m}).$$

Therefore the identity $\hat{f}(x_{i_1}, \dots, x_{i_n}) = \hat{g}(x_{j_1}, \dots, x_{j_m})$ holds in the variety $V(\Omega(A))$. □

If $\alpha : I \to \mathcal{L}$, then let $\phi_\alpha : \mathbf{F}(A, I) \to \mathcal{L}$ be the unique homomorphism where we have $\phi_\alpha(\pi_i) = \alpha(i)$ for $i \in I$. One can clearly see that

$$\phi_\alpha(\hat{f}^{\mathbf{F}(A,I)}(\pi_{i_1}, \dots, \pi_{i_n})) = \hat{f}^{\mathcal{L}}(\phi_\alpha(\pi_{i_1}), \dots, \phi_\alpha(\pi_{i_n})) = \hat{f}^{\mathcal{L}}(\alpha(i_1), \dots, \alpha(i_n)).$$

Let Z_α be the filter on $\{\emptyset\} \cup \bigcup \mathcal{P}(A, I)$ where

$$\Omega(A)^{\mathcal{P}(A,I)}/Z_\alpha = \Omega(A)^{\mathcal{P}(A,I)}/\ker(\phi_\alpha) \simeq \langle\alpha''(I)\rangle.$$

Clearly Z_α is an ultrafilter if and only if $\langle\alpha''(I)\rangle$ is simple. If $\mathcal{L}$ is simple, then Z_α is always an ultrafilter for each $\alpha : I \to \mathcal{L}$. Let

$$\iota_\alpha : \Omega(A)^{\mathcal{P}(A,I)}/Z_\alpha \to \langle\alpha''(I)\rangle$$

be the canonical isomorphism. In other words, we have $\iota_\alpha([\ell]) = \phi_\alpha(\ell)$ where $[\ell]$ denotes the equivalence class of ℓ. Take note that if $\alpha''(I)$ generates $\mathcal{L}$, then ι_α is an isomorphism from $\Omega(A)^{\mathcal{P}(A,I)}/Z_\alpha$ to $\mathcal{L}$. We therefore have a method of representing any algebra in $V(\Omega(A))$ as a limit reduced power of $\Omega(A)$. In particular, if the mapping $e : \Omega(A) \to \mathcal{L}$ is an elementary embedding, then we can construct a limit ultrapower of $\Omega(A)$ isomorphic to $\mathcal{L}$.

If $\mathcal{L}$ is finitely generated, then one can easily show that $\mathcal{L}$ is generated by a single element. Furthermore, if $\alpha : \{1, \dots, n\} \to \mathcal{L}$ is a function such that $\alpha(1), \dots, \alpha(n)$ generates $\mathcal{L}$, then since $\iota_\alpha : \Omega(A)^{A^n}/Z_\alpha = \mathbf{F}(A, \{1, \dots, n\})/Z_\alpha \to \mathcal{L}$ is an isomorphism, the algebra $\mathcal{L}$ is representable as a reduced power of $\Omega(A)$. In particular, if $\mathcal{L}$ is simple and finitely generated, then $\mathcal{L}$ is representable as an ultrapower of $\Omega(A)$. Conversely, if $|I| \leq |A|$, then every reduced power and ultrapower of $\Omega(A)$ of the form $\Omega(A)^I/Z$ is finitely generated.

In the remainder of this paper, we shall discuss a method of representing every algebra $\mathcal{L} \in V(\Omega(A))$ as a direct limit of limit reduced powers of $\Omega(A)$. By representing algebras $\mathcal{L}$ as direct limits of limit reduced powers of $\Omega(A)$, one may be able to represent $\mathcal{L}$ as a limit reduced power besides the quotients of the algebra $\mathbf{F}(A, I)$. Furthermore, one may also represent $\mathcal{L}$ in terms of Boolean reduced powers and other generalizations of the reduced power and ultrapower constructions.

If X is a set and F is a filter on $\Pi(X)$, then the covers F generate a uniformity on X, so we may shall regard (X, F) as a uniform space. We shall call the partitions in the filter F uniform partitions. One may refer to [3] for information about uniform spaces, but no prior knowledge of uniform spaces is necessary to finish reading this paper.

If $f : X \to Y$ is a function and P is a partition of Y, then we shall write $[f]_{-1}(P)$ for the partition $\{f_{-1}(R)|R \in P\} \setminus \{\emptyset\}$. One can easily show that the following properties hold.

1. $[f]_{-1}(P_1 \wedge \cdots \wedge P_n) = [f]_{-1}(P_1) \wedge \cdots \wedge [f]_{-1}(P_n)$, and
2. $[f \circ g]_{-1}(P) = [g]_{-1}[f]_{-1}(P)$.

If X, Y are sets and $F \subseteq \Pi(X), G \subseteq \Pi(Y)$ are filters, then a function $f : X \to Y$ is said to be uniformly continuous if whenever $P \in G$, then $[f]_{-1}(P) \in F$. If G is generated by a filterbase $\mathfrak{G}$, then f is uniformly continuous if and only if whenever $P \in \mathfrak{G}$ we have $[f]_{-1}(P) \in F$. Clearly the composition of uniformly continuous maps is uniformly continuous.

The sets A^I shall always be given the uniformity generated by the filter $\mathcal{P}(A, I)$. Furthermore, the set A shall always have the uniformity generated by $\Pi(A)$.

Theorem 1.5. *A function $f : A^I \to A^J$ is uniformly continuous if and only if for each projection $\pi_j : A^J \to A$, we have $\pi_j \circ f$ be uniformly continuous.*

Proof. $\to$ The projections π_j are all uniformly continuous, so the mappings $\pi_j \circ f$ are uniformly continuous as well being the composition of two uniformly continuous functions.

$\leftarrow$ Assume that each $\pi_j \circ f$ is uniformly continuous. If $j_1, \ldots, j_n \in J$ are distinct elements, then we have $\mathcal{P}_{j_1,\ldots,j_n} = \mathcal{P}_{j_1} \wedge \cdots \wedge \mathcal{P}_{j_n}$. However, if $P = \{\{a\}|a \in A\}$, then we have

$$\mathcal{P}_{j_1} = [\pi_{j_1}]_{-1}(P), \ldots, \mathcal{P}_{j_n} = [\pi_{j_n}]_{-1}(P).$$

Therefore

$$[f]_{-1}(\mathcal{P}_{j_1,\ldots,j_n}) = [f]_{-1}(\mathcal{P}_{j_1} \wedge \cdots \wedge \mathcal{P}_{j_n}) = [f]_{-1}(\mathcal{P}_{j_1}) \wedge \cdots \wedge [f]_{-1}(\mathcal{P}_{j_n})$$
$$= [f]_{-1}[\pi_{j_1}]_{-1}(P) \wedge \cdots \wedge [f]_{-1}[\pi_{j_n}]_{-1}(P) = [\pi_{j_1} \circ f]_{-1}(P) \wedge \cdots \wedge [\pi_{j_n} \circ f]_{-1}(P)$$

is a uniform partition since each $\pi_j \circ f$ is uniformly continuous. $\square$

Let $f \in \mathbf{F}(A, I)$, and let $\mathcal{L} \in V(\Omega(A))$. Then let $\overline{f}^{\mathcal{L}} : \mathcal{L}^I \to \mathcal{L}$ be the mapping defined as follows. If $f = \hat{g}^{\mathbf{F}(A,I)}(\pi_{i_1}, \ldots, \pi_{i_n})$, then let $\overline{f}^{\mathcal{L}}((\ell_i)_{i\in I}) = \hat{g}^{\mathcal{L}}(\ell_{i_1}, \ldots, \ell_{i_n})$. We now show that $\overline{f}^{\mathcal{L}}$ is well defined. Assume that $f = \hat{g}^{\mathbf{F}(A,I)}(\pi_{i_1}, \ldots, \pi_{i_n}) = \hat{h}^{\mathbf{F}(A,I)}(\pi_{j_1}, \ldots, \pi_{j_m})$. Then since $(\pi_i)_{i\in I}$ freely generates $\mathbf{F}(A, I)$, the identity $\hat{f}(x_{i_1}, \ldots, x_{i_n}) = \hat{g}(x_{j_1}, \ldots, x_{j_n})$ holds in the variety $V(\Omega(A))$, so $\hat{g}^{\mathcal{L}}(\ell_{i_1}, \ldots, \ell_{i_n}) = \hat{h}^{\mathcal{L}}(\ell_{j_1}, \ldots, \ell_{j_m})$. Therefore, the mapping $\overline{f}^{\mathcal{L}}$ is well defined. If $f : A^I \to A^J$ is uniformly continuous, then let $\overline{f}^{\mathcal{L}} : \mathcal{L}^I \to \mathcal{L}^J$ be the mapping where $\overline{f}^{\mathcal{L}}(\alpha) = (\overline{\pi_j \circ f}^{\mathcal{L}}(\alpha))_{j\in J}$.

Theorem 1.6. *Let $f : A^I \to A^J, g : A^J \to A^K$ be uniformly continuous. Then $\overline{g}^{\mathcal{L}} \circ \overline{f}^{\mathcal{L}} = \overline{g \circ f}^{\mathcal{L}}$.*

Proof. Assume $k \in K$. Then $\pi_k \circ g : A^J \to A$ is uniformly continuous. Therefore there are $j_1, \ldots, j_n \in J$ and some $r : A^n \to A$ such that $\pi_k \circ g = \hat{r}^{\mathbf{F}(A,J)}(\pi_{j_1}, \ldots, \pi_{j_n})$. Furthermore, since $\pi_{j_1} \circ f, \ldots, \pi_{j_n} \circ f : A^I \to A$ are uniformly continuous, there are indices $i_1, \ldots, i_m \in I$ and mappings $s_1, \ldots, s_n : A^m \to A$ such that

$$\pi_{j_1} \circ f = \hat{s_1}^{\mathbf{F}(A,I)}(\pi_{i_1}, \ldots, \pi_{i_m}), \ldots, \pi_{j_n} \circ f = \hat{s_n}^{\mathbf{F}(A,I)}(\pi_{i_1}, \ldots, \pi_{i_m}).$$

Now let $t : A^m \to A$ be the mapping where

$$t(a_1, \ldots, a_m) = r(s_1(a_{i_1}, \ldots, a_{i_m}), \ldots, s_n(a_{i_1}, \ldots, a_{i_m})).$$

Then we have

$$\pi_k \circ g \circ f(a_i)_{i\in I} = \hat{r}^{\mathbf{F}(A,I)}(\pi_{j_1},\dots,\pi_{j_n})(f(a_i)_{i\in I}) = r(\pi_{j_1}\circ f(a_i)_{i\in I},\dots,\pi_{j_n}\circ f(a_i)_{i\in I})$$

$$= r(\hat{s_1}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m})(a_i)_{i\in I},\dots,\hat{s_n}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m})(a_i)_{i\in I})$$

$$= r(s_1(a_{i_1},\dots,a_{i_m}),\dots,s_n(a_{i_1},\dots,a_{i_m})) = t(a_{i_1},\dots,a_{i_m}) = \hat{t}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m})(a_i)_{i\in I},$$

so $\pi_k \circ g \circ f = \hat{t}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m})$.

We also have

$$\pi_k \circ \overline{g}^{\mathcal{L}} \circ \overline{f}^{\mathcal{L}}(\ell_i)_{i\in I} = \pi_k \circ \overline{g}^{\mathcal{L}}(\overline{\pi_j \circ f}^{\mathcal{L}}(\ell_i)_{i\in I})_{j\in J}$$

$$= \overline{\pi_k \circ g}^{\mathcal{L}}(\overline{\pi_j \circ f}^{\mathcal{L}}(\ell_i)_{i\in I})_{j\in J} = \hat{r}^{\mathcal{L}}(\overline{\pi_{j_1} \circ f}^{\mathcal{L}}(\ell_i)_{i\in I},\dots,\overline{\pi_{j_n} \circ f}^{\mathcal{L}}(\ell_i)_{i\in I})$$

$$= \hat{r}^{\mathcal{L}}(\hat{s_1}^{\mathcal{L}}(\ell_{i_1},\dots,\ell_{i_m}),\dots,\hat{s_n}^{\mathcal{L}}(\ell_{i_1},\dots,\ell_{i_m})) = \hat{t}(\ell_{i_1},\dots,\ell_{i_m})$$

$$= \overline{\pi_k \circ g \circ f}^{\mathcal{L}}(\ell_i)_{i\in I} = \pi_k \circ \overline{g \circ f}^{\mathcal{L}}(\ell_i)_{i\in I}.$$

Therefore, we have $\pi_k \circ \overline{g}^{\mathcal{L}} \circ \overline{f}^{\mathcal{L}} = \pi_k \circ \overline{g \circ f}^{\mathcal{L}}$ for all k. We conclude that $\overline{g}^{\mathcal{L}} \circ \overline{f}^{\mathcal{L}} = \overline{g \circ f}^{\mathcal{L}}$. □

If $f : A^I \to A^J$ is uniformly continuous, then define a mapping $f^\star : \mathbf{F}(A,J) \to \mathbf{F}(A,I)$ by $f^\star(g) = g \circ f$ whenever $g \in \mathbf{F}(A,J)$.

Theorem 1.7. *If $f : A^I \to A^J$, $\alpha : I \to \mathcal{L}, \beta : J \to \mathcal{L}$ and $\beta = \overline{f}^{\mathcal{L}}(\alpha)$, then*
1. $\phi_\beta = \phi_\alpha f^\star$, and
2. Whenever $R \in \{\emptyset\} \cup \bigcup \mathcal{P}(A,J)$, we have $R \in Z_\beta$ if and only if $f_{-1}(R) \in Z_\alpha$

Proof. 1. Let $\ell \in \mathbf{F}(A,J)$. Then there are $j_1,\dots,j_n \in J$ along with some map $r : A^n \to A$ such that $\ell = \hat{r}^{\mathbf{F}(A,J)}(\pi_{j_1},\dots,\pi_{j_n})$. Since $\pi_{j_1} \circ f,\dots,\pi_{j_n} \circ f \in F(A,I)$ there are $i_1,\dots,i_m \in I$ and functions $s_1,\dots,s_n : A^m \to A$ where

$$\pi_{j_1} \circ f = \hat{s_1}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m}),\dots,\pi_{j_n} \circ f = \hat{s_n}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m}).$$

Let $t : A^m \to A$ be the function where

$$t(a_1,\dots,a_m) = r(s_1(a_1,\dots,a_m),\dots,s_n(a_1,\dots,a_m)).$$

Then we have

$$\phi_\beta(\ell) = \phi_\beta(\hat{r}^{\mathbf{F}(A,J)}(\pi_{j_1},\dots,\pi_{j_n}))$$

$$= \hat{r}^{\mathcal{L}}(\beta(j_1),\dots,\beta(j_n)) = \hat{r}^{\mathcal{L}}(\overline{f}^{\mathcal{L}}(\alpha)(j_1),\dots,\overline{f}^{\mathcal{L}}(\alpha)(j_n))$$

$$= \hat{r}^{\mathcal{L}}(\overline{\pi_{j_1} \circ f}^{\mathcal{L}}(\alpha),\dots,\overline{\pi_{j_n} \circ f}^{\mathcal{L}}(\alpha))$$

$$= \hat{r}^{\mathcal{L}}(\hat{s_1}^{\mathcal{L}}(\alpha(i_1),\dots,\alpha(i_m)),\dots,\hat{s_n}^{\mathcal{L}}(\alpha(i_1),\dots,\alpha(i_m)))$$

$$= \hat{t}^{\mathcal{L}}(\alpha(i_1),\dots,\alpha(i_m)).$$

Now assume that $(a_i)_{i\in I} \in A^I$. Then

$$f^*(\ell)(a_i)_{i\in I} = \ell \circ f(a_i)_{i\in I} = \hat{r}^{\mathbf{F}(A,J)}(\pi_{j_1},\dots,\pi_{j_n})(f(a_i)_{i\in I})$$

$$= r(\pi_{j_1} \circ f(a_i)_{i\in I},\dots,\pi_{j_n} \circ f(a_i)_{i\in I})$$

$$= r(\hat{s_1}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m})(a_i)_{i\in I},\dots,\hat{s_n}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots,\pi_{i_m})(a_i)_{i\in I})$$

$$= r(s_1(a_{i_1},\dots,a_{i_m}),\dots,s_n(a_{i_1},\dots,a_{i_m})) = t(a_{i_1},\dots,a_{i_m})$$

$$= \hat{t}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots\pi_{i_m})(a_i)_{i\in I}.$$

We conclude that

$$\phi_a(f^*(\ell)) = \phi_a(\hat{t}^{\mathbf{F}(A,I)}(\pi_{i_1},\dots\pi_{i_m})) = \hat{t}^{\mathcal{L}}(\alpha(i_1),\dots,\alpha(i_m)) = \phi_\beta(\ell).$$

2. Let $\ell_1, \ell_2 : A^J \to A$ be two functions such that $\{\mathbf{a} \in A^J | \ell_1(\mathbf{a}) = \ell_2(\mathbf{a})\} = R$. Then $\mathbf{a} \in f_{-1}(R)$ if and only if $f(\mathbf{a}) \in R$ if and only if $\ell_1 \circ f(\mathbf{a}) = \ell_2 \circ f(\mathbf{a})$. Thus

$$f_{-1}(R) = \{\mathbf{a} \in A^I | \ell_1 \circ f(\mathbf{a}) = \ell_2 \circ f(\mathbf{a})\} = \{\mathbf{a} \in A^I | f^\star(\ell_1)(\mathbf{a}) = f^\star(\ell_2)(\mathbf{a})\}.$$

Therefore, we have $R \in Z_\beta$ if and only if $\phi_\beta(\ell_1) = \phi_\beta(\ell_2)$ if and only if $\phi_\alpha f^*(\ell_1) = \phi_\alpha f^*(\ell_2)$ if and only if $(f^\star(\ell_1), f^\star(\ell_2)) \in \ker(\phi_\alpha)$ if and only if

$$f_{-1}(R) = \{\mathbf{a} \in A^I | f^\star(\ell_1)(\mathbf{a}) = f^\star(\ell_2)(\mathbf{a})\} \in Z_\alpha.$$

□

In particular, for $\ell, \mathfrak{m} \in \mathbf{F}(A, J)$, if $\{\mathbf{a} \in A^J | \ell(\mathbf{a}) = \mathfrak{m}(\mathbf{a})\} \in Z_\beta$, then

$$\{\mathbf{a} \in A^I | \ell(f(\mathbf{a})) = \mathfrak{m}(f(\mathbf{a}))\} = f_{-1}(\{\mathbf{a} \in A^J | \ell(\mathbf{a}) = \mathfrak{m}(\mathbf{a})\}) \in Z_\alpha.$$

Therefore define a mapping $f^{\beta,\alpha} : \mathbf{F}(A, J)/Z_\beta \to \mathbf{F}(A, I)/Z_\alpha$ by $f^{\beta,\alpha}([\ell]) = [\ell \circ f] = [f^\star(\ell)]$. Let $\iota_{\beta,\alpha} : \langle \beta''(J) \rangle \to \langle \alpha''(I) \rangle$ be the inclusion mapping.

Theorem 1.8. *We have* $\iota_{\beta,\alpha}\iota_\beta = \iota_\alpha f^{\beta,\alpha}$.

$$\begin{array}{ccc}
\mathbf{F}(A, J)/Z_\beta & \xrightarrow{f^{\beta,\alpha}} & \mathbf{F}(A, I)/Z_\alpha \\
\big\downarrow \iota_\beta & & \big\downarrow \iota_\alpha \\
\langle \beta''(J) \rangle & \xrightarrow{\iota_{\beta,\alpha}} & \langle \alpha''(I) \rangle
\end{array}$$

Proof. Let $\ell \in \mathbf{F}(A, J)$. Then $\iota_{\beta,\alpha}\iota_\beta[\ell] = \iota_\beta[\ell] = \phi_\beta(\ell) = \phi_\alpha(f^\star(\ell)) = \iota_\alpha[f^\star(\ell)] = \iota_\alpha f^{\beta,\alpha}[\ell]$. Therefore $\iota_{\beta,\alpha}\iota_\beta = \iota_\alpha f^{\beta,\alpha}$. □

Since ι_α is bijective, we have $f^{\beta,\alpha} = \iota_\alpha^{-1}\iota_{\beta,\alpha}\iota_\beta$, and in particular, the function $f^{\beta,\alpha}$ does not depend on f. We shall therefore write $E^{\beta,\alpha}$ for the mapping $f^{\beta,\alpha} = \iota_\alpha^{-1}\iota_{\beta,\alpha}\iota_\beta$.

If $\alpha : I \to \mathcal{L}, \beta : J \to \mathcal{L}$, then we shall write $\beta \leq \alpha$ if $\langle \beta''(J) \rangle \subseteq \langle \alpha''(I) \rangle$. Clearly, the relation $\leq$ is a preordering on the class of all functions with range $\mathcal{L}$. One can clearly see that $\beta \leq \alpha$ if and only if for each $j \in J$ there is a $f : A^n \to A$ and $i_1, \dots, i_n \in I$ such that $\beta(j) = \hat{f}^{\mathcal{L}}(\alpha(i_1), \dots, \alpha(i_n))$. Furthermore, using theorem 1.5, one may show that $\beta \leq \alpha$ if and only if there is a uniformly continuous mapping $f : A^I \to A^J$ such that $\beta = \overline{f}^{\mathcal{L}}(\alpha)$.

Assume D is a directed set, and for $d \in D$ there is a set I_d and a function $\alpha_d : I_d \to \mathcal{L}$, and also assume $\alpha_d \leq \alpha_e$ whenever $d \leq e$, and that $\mathcal{L} = \bigcup_{d \in D} \langle \alpha_d''(I_d) \rangle$. Then we have $\mathcal{L} = \xrightarrow{Lim} (\langle \alpha_d''(I_d) \rangle, \iota_{\alpha_d,\alpha_e})_{d \leq e}$. However, since each $\iota_{\alpha_d} : \Omega(A)^{\mathcal{P}(A,I_d)}/Z_d \to \langle \alpha_d''(I_d) \rangle$ is an isomorphism and $\iota_{\alpha_d,\alpha_e}\iota_{\alpha_d} = \iota_{\alpha_e}E^{\alpha_d,\alpha_e}$, we have

$$\mathcal{L} \simeq \xrightarrow{Lim} (\mathbf{F}(A, I_d)/Z_d, E_{d,e})_{d \leq e}.$$

In fact, if we can find a directed system of mappings $(f_{d,e})_{d,e \in D, d \leq e}$ such that $\overline{f}_{d,e}(\alpha_e) = \alpha_d$ whenever $d \leq e$, then we can represent $\mathcal{L}$ as a generalization of the Boolean reduced power construction called a Boolean partition algebra reduced power.

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