Analysis of Finite Dimensional Approximations to a Class of Partial Differential Equations

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Abstract

In this paper, a unified approach for analyzing finite dimensional approximations to a class of partial differential equations boundary value problems (second-kind Fredholm differential equations) is introduced. The approach is shown to be general despite of its extremely simple form. In particular, it is expected to be useful in the convergence analysis of finite element methods for solving PDE problems. Three specific examples are presented to illustrate the broad applicability of the approach.

1 Introduction

Let $(X, \|\cdot\|)$ be a Hilbert space equipped with an inner product $(\cdot, \cdot)$ and a norm $\|\cdot\|$, $\mathcal{K} : X \longrightarrow X$ be a compact operator. This paper is concerned with the error analysis of finite dimensional approximations to the following Fredholm equation: Given $f \in X$, find $u \in X$ such that

$$(I - \mathcal{K})u = f . \tag{1.1}$$

Here, our basic assumption is that Equation (1.1) is uniquely solvable for any $f \in X$, *i.e.*, 1 is not an eigenvalue of $\mathcal{K}$.

The equation (1.1) provides a general framework for a large class of PDE problems that arise in diverse applications. Our goal of this paper is to develop a unified approach for analyzing the convergence of the discretized problems.

Assume that $\{X_h\}$ is a sequence of finite dimensional subspaces of X satisfying

$$\lim_{h \to 0} \inf_{v \in X_h} \|u - v\| = 0, \quad \forall u \in X. \tag{1.2}$$

If $P_h : X \longrightarrow X_h$ is the projection operator defined by

$$(u - P_h u, v) = 0, \quad \forall v \in X_h,$$

then

$$\|u - P_h u\| = \inf_{v \in X_h} \|u - v\| \tag{1.3}$$

and hence

$$\lim_{h \to 0} \|u - P_h u\| = 0, \quad \forall u \in X. \tag{1.4}$$

The organization of the paper is as follows. Convergence results are established in Section 2 for solutions with limited regularity under this general framework. Three examples are presented to illustrate the generality and applicability of the approach in Section 3. For the first two examples, an elliptic boundary value problem and an electromagnetic scattering problem, our present approach provides straight-forward alternative proofs to the existing convergence results. The convergence result for Example 3, diffraction by a periodic structure, is new.

Throughout, the letter C denotes a generic positive constant whose value is independent of the parameter h and may vary at different occurrences. We also adopt the standard notations for Sobolev spaces $W^{s,p}(\Omega)$ and the associated norms and seminorms [1]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v \mid_{\partial\Omega} = 0\}$, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{W^{s,2}(\Omega)}$ and $\|\cdot\|_\Omega = \|\cdot\|_{W^{0,2}(\Omega)}$.

2 Convergence of the finite dimensional discretization

The finite dimensional discretization scheme is defined as follows: Find $u_h \in X_h$ such that

$$(I - P_h \mathcal{K}) u_h = P_h f. \tag{2.1}$$

The following result concerns the well-posedness of the problem (2.1).

Theorem 2.1 *If $h \ll 1$, then $\forall f \in X$, the equation (2.1) attains a unique solution $u_h \in X_h$ with*

$$\|u - u_h\| \leq C \inf_{v \in X_h} \|u - v\|. \tag{2.2}$$

Consequently

$$\lim_{h \to 0} \|u - u_h\| = 0. \tag{2.3}$$

Remark 2.1. The above result is known in literature concerning numerical solution of integral equations [2], [3] or [4]. However, this is not the case in literature of numerical solution of partial differential equations. In fact, to our best knowledge, it has not been applied to analyze the discretization of PDE problems. Our results in the present work indicate that Theorem 2.1 is very general with useful applications not only to integral equations but also to partial differential equations.

The error estimate (2.3) is based on the knowledge of the exact solution u. Indeed certain error estimates may be also obtained if only information about f is provided. In other words, error estimates may be established even in the situation where no more information about $\mathcal{K}$ is available. Consequently, a convergence rate of finite dimensional approximation may be obtained even when the exact solution is only with limited regularity.

Let $Y \subset X$ be a normed linear space equipped with norm $\|\cdot\|_Y$. Introduce a quantity

$$\rho_Y(h) = \sup_{f \in Y, \|f\|_Y \le 1} \inf_{v \in X_h} \|(I - \mathcal{K})^{-1}f - v\|, \tag{2.4}$$

or equivalently

$$\rho_Y(h) = \|(I - P_h)(I - \mathcal{K})^{-1}\|_{Y \to X}.$$

We are now ready to present the error estimate based on the information of f only.

Theorem 2.2 *For any $f \in Y$, there holds*

$$\|u - u_h\| \le C\rho_Y(h)\|f\|_Y. \tag{2.5}$$

Moreover, if $(Y, \|\cdot\|_Y)$ is compactly embedded into $(X, \|\cdot\|)$, then

$$\lim_{h \to 0} \rho_Y(h) = 0. \tag{2.6}$$

Proof. The estimate (2.5) follows from (2.2) and the definition of $\rho_Y(h)$.

If Y is compactly embedded into X, then $(I - \mathcal{K})^{-1} : Y \longrightarrow X$ is compact. Hence $P_h(I - \mathcal{K})^{-1} : Y \longrightarrow X$ is compact and continuous. Since $\forall f \in Y$,

$$\lim_{h \to 0} \|(I - P_h)(I - \mathcal{K})^{-1}f\| = 0,$$

we get

$$\lim_{h \to 0} \sup_{f \in Y, \|f\|_Y \le 1} \|(I - P_h)(I - \mathcal{K})^{-1}f\| = 0,$$

which implies (2.6). $\qquad\square$

Remark 2.2. The convergence result may be improved by using some iteration procedure, such as $u^h = f + \mathcal{K}u_h$ or $u^h = P_h f + P_h \mathcal{K}u_h$.

Sometimes, it is useful to replace the discretization (2.1) with

$$(I - P_h \mathcal{K}_h)u_h = P_h f, \tag{2.7}$$

where $\mathcal{K}_h : X \longrightarrow X$ is an approximation to $\mathcal{K}$. For the scheme (2.7), similarly, we have (see Corollary 13.10 of [4])

Theorem 2.3 *Assume that*

$$\lim_{h \to 0} \|\mathcal{K}_h - \mathcal{K}\| = 0. \tag{2.8}$$

If $h \ll 1$, then $\forall f \in X$, the equation (2.7) attains a unique solution $u_h \in X_h$ with

$$\|u - u_h\| \leq C \left(\inf_{v \in X_h} \|u - v\| + \|(\mathcal{K}_h - \mathcal{K})u\| \right). \tag{2.9}$$

Consequently,

$$\|u - u_h\| \leq C \Big(\rho_Y(h) + \|\mathcal{K}_h - \mathcal{K}\| \Big) \|f\|_Y, \quad \forall f \in Y. \tag{2.10}$$

3 Applications

A large class of partial differential equations (as well as integral equations) may be formulated as (1.1). In this section, we apply the general convergence results of Section 2 to three different types of PDE problems.

3.1 An elliptic boundary value problem

Let $\Omega \subset R^d (d \geq 1)$ be a bounded domain. Consider the homogeneous boundary value problem

$$\begin{cases} \mathcal{L}u &= g, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here $g \in H^{-1}(\Omega)$ is given and $\mathcal{L}$ is a general linear second order elliptic operator:

$$\mathcal{L}u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu,$$

satisfying $a_{ij}, b_i, c \in L^\infty(\Omega)$, and (a_{ij}) is uniformly positive on Ω.

The variational form of (3.1) is as follows: Find $u \equiv \mathcal{L}^{-1}g \in H_0^1(\Omega)$ such that

$$a(u, v) \equiv \int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{j=1}^{d} b_j \frac{\partial u}{\partial x_j} v + cuv = \int_\Omega gv, \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

Set

$$(w, v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j}, \quad \forall w, v \in H_0^1(\Omega) \tag{3.3}$$

and define $\mathcal{K} : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ by

$$(\mathcal{K}w, v) = -\int_\Omega \sum_{j=1}^{d} b_j \frac{\partial w}{\partial x_j} v + cwv, \quad \forall w, v \in H_0^1(\Omega).$$

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Then (3.2) can be written as

$$(I - \mathcal{K})u = f, \tag{3.4}$$

where $f \in H_0^1(\Omega)$ satisfying

$$(f, v) = \int_\Omega gv, \ \forall v \in H_0^1(\Omega).$$

Clearly the operator $\mathcal{K} : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ is compact and the problem (3.2) is uniquely solvable for any $g \in H^{-1}(\Omega)$ if and only if the problem (3.4) is uniquely solvable for any $f \in H_0^1(\Omega)$. Note that $(\cdot, \cdot)$ defined by (3.3) is an inner product of $H_0^1(\Omega)$, hence the main results in [5] and [6] and the related result in [7] follow immediately from Theorem 2.1 and Theorem 2.2. More precisely, we have

Theorem 3.1 *Assume that the problem (3.2) is uniquely solvable and $\{X_h\}$ is a sequence of finite dimensional subspaces of $H_0^1(\Omega)$ such that*

$$\lim_{h \to 0} \inf_{v \in X_h} \|u - v\|_{1,\Omega} = 0, \quad \forall u \in H_0^1(\Omega).$$

If $h \ll 1$ and $g \in H^{-1}(\Omega)$, then the finite dimensional discretized problem

$$a(u_h, v) = \int_\Omega gv, \quad \forall v \in X_h$$

attains a unique solution $u_h \in X_h$ with

$$\lim_{h \to 0} \|u - u_h\|_{1,\Omega} = 0.$$

In addition, for $g \in L^2(\Omega)$, the following estimate holds

$$\|u - u_h\|_{1,\Omega} \le C\rho(h)\|g\|_{0,\Omega}$$

with

$$\rho(h) = \sup_{\phi \in L^2(\Omega), \|\phi\|_0 \le 1} \inf_{v \in X_h} \|\mathcal{L}^{-1}\phi - v\|_{1,\Omega} \to 0 \quad as \ \ h \to 0 \ .$$

Proof. Set $X = H_0^1(\Omega)$ and $\|\cdot\| = \|\cdot\|_{1,\Omega}$ in Theorem 2.1, we immediately obtain the first conclusion of Theorem 3.1.

Now we derive the second statement of Theorem 3.1 from Theorem 2.2. Define $\mathcal{K}_0 : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ by

$$(\mathcal{K}_0 w, v) = \int_\Omega wv, \ \forall w \in L^2(\Omega), \forall v \in H_0^1(\Omega),$$

then $\mathcal{K}_0 : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ is compact. As a result, for $Y = \mathcal{K}_0 L^2(\Omega)$, we have that $(Y, \|\cdot\|)$ can be compactly embedded into $(X, \|\cdot\|)$. Note that

$$\mathcal{L}^{-1}\phi = (I - \mathcal{K})^{-1}\mathcal{K}_0\phi, \quad \forall \phi \in L^2(\Omega),$$

we can estimate the quantity $\rho_Y(h)$, defined by (2.4), as follows

$$\rho_Y(h) \le C \sup_{\phi \in L^2(\Omega), \|\phi\|_0 \le 1} \inf_{v \in X_h} \|\mathcal{L}^{-1}\phi - v\|_{1,\Omega},$$

which together with Theorem 2.2 completes the proof. $\square$

3.2 An electromagnetic scattering problem

Consider the scattering of electromagnetic waves from an infinitely long cylinder containing an anisotropic inhomogeneous medium [8]. The electric and magnetic fields, denoted by $\hat{E}$ and $\hat{H}$, satisfy the following Maxwell equations:

$$\varepsilon \frac{\partial \hat{E}}{\partial t} + \sigma \hat{E} - \nabla \times \hat{H} = 0, \quad \mu \frac{\partial \hat{H}}{\partial t} + \nabla \times \hat{E} = 0,$$

where ε and μ are the electric permittivity and magnetic permeability, respectively.

For a fixed frequency ω, let ε_0 and μ_0 denote the electric permittivity and magnetic permeability of free space. Define the wave number $k = \omega\sqrt{\varepsilon_0\mu_0}$ and the index of refraction

$$N(x_1, x_2) = \frac{1}{\varepsilon}\left(\varepsilon(x_1, x_2) + i\frac{\sigma(x_1, x_2)}{\omega}\right).$$

Consider the special case of an anisotropic medium: an orthotropic medium

$$\varepsilon(x_1, x_2) = \begin{pmatrix} \varepsilon_{11}(x_1, x_2) & \varepsilon_{12}(x_1, x_2) & 0 \\ \varepsilon_{21}(x_1, x_2) & \varepsilon_{22}(x_1, x_2) & 0 \\ 0 & 0 & \varepsilon_{33}(x_1, x_2) \end{pmatrix}.$$

Assume further that $\sigma(x_1, x_2)$ and $\mu(x_1, x_2)$ are of the same form as ε and independent of x_3. The time-harmonic electric and magnetic fields can be written as

$$E(x_1, x_2, t) = \varepsilon_0^{-1/2} E(x_1, x_2)\exp(-i\omega t) \quad \text{and} \quad H(x_1, x_2, t) = \mu_0^{-1/2} H(x_1, x_2)\exp(-i\omega t)$$

so that

$$\nabla \times E - iknH = 0, \quad \vec{\nabla} \times H + ikNE = 0, \tag{3.5}$$

where $n(x_1, x_2) = \mu_{33}/\mu_0$ and $\vec{\nabla}\times$ is the vector *curl*. Also, E and H are assumed to be independent of x_3 with H perpendicular to the x_1x_2-plane,

$$E = \begin{pmatrix} E_1(x_1, x_2) \\ E_2(x_1, x_2) \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ 0 \\ H_3(x_1, x_2) \end{pmatrix}.$$

Under these assumptions, the Maxwell system (3.5) reduces to the following general Helmholtz equation for $u = H_3(x_1, x_2)$:

$$\nabla \cdot \mathcal{A}\nabla u + k^2 n u = 0,$$

where

$$\mathcal{A} = \frac{1}{N_{11}N_{22} - N_{12}N_{21}} \begin{pmatrix} N_{11} & N_{21} \\ N_{12} & N_{22} \end{pmatrix}. \tag{3.6}$$

Consider a bounded impenetrable scatterer, D, with smooth boundary ∂D contained in a bounded region outside of which $\mathcal{A} = I$ and $n = 1$. This corresponds to the cross section of the cylinder. Let $\Gamma \subset R^3 \setminus D$ be a closed uniformly Lipschitz curve surrounding D and Σ a closed uniformly Lipschitz curve surrounding Γ which does not intersect Γ. Denote Ω the bounded part with boundary $\partial D \cup \Sigma$.

Let $\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|)$, where $x = (x_1, x_2), y = (y_1, y_2)$ and $H_0^{(1)}(k|x - y|)$ is the Hankel function of the first kind and order zero.

Let $\psi(x)$ be a cut-off function in $C_0^\infty(\Omega)$ such that $\psi = 0$ in a neighborhood of Σ and $\psi = 1$ in a neighborhood of Γ. Let $(\ell\Phi)(x, y) = \psi(y)\Phi(x, y)$.

Introduce

$$
\begin{aligned}
(G_\Gamma u)(x) &= -k^2 \int_{\Omega_0} \ell\Phi(x, y)u(y)dy + \int_\Gamma u(y)\frac{\partial \Phi}{\partial \nu_y}(x, y)ds_y \\
&\quad + \int_{\Omega_0} \nabla_y u(y) \cdot \nabla_y \ell\Phi(x, y)dy, \quad \text{in } \Omega
\end{aligned}
$$

$$
(Lu)(x) = \frac{\partial u(x)}{\partial \nu_{\mathbf{x}}}, \quad \text{on } \Sigma,
$$

where Ω_0 is the domain surrounded by Γ and Σ.

Define the space

$$
X = \{v \in H^1(\Omega) : v \mid_{\partial D} = 0\}
$$

equipped with the usual $H^1(\Omega)$-norm.

Use the coupling technique suggested by Jami and Lenoir [9], we may write the model problem in this case as: Find $u \in X$ such that

$$
\begin{cases}
\nabla \cdot \mathcal{A}\nabla u + k^2 n u &= 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial D, \\
L(u - G_\Gamma u) &= Lu^i, \quad \text{on } \Sigma,
\end{cases} \tag{3.7}
$$

where u^i is the incident field, either a plane wave or a point source. The associated variational formulation is : Find $u \in X$ such that

$$
\int_\Omega (\overline{\nabla v} \cdot \mathcal{A}\nabla u - k^2 n \bar{v}u) - \int_\Sigma \bar{v}L(G_\Gamma u) = \int_\Sigma \bar{v}Lu^i, \quad \forall u, v \in X. \tag{3.8}
$$

The problem has recently been studied by Coyle and Monk [8]. See [8] for additional discussions and references. In the following, we show that Theorem 2.1 yields convergence results similar to those in [8].

Set

$$
(u, v) = \int_\Omega (\overline{\nabla v} \cdot \mathcal{A}\nabla u + k^2 \bar{v}u), \quad \forall u, v \in X.
$$

Under certain condition on $\mathcal{A}$, for example one of the conditions stated in [8] (Page 1593), $(\cdot, \cdot)$ becomes an inner product of X (Lemma 4.3 of [8]). Let $\mathcal{K} : X \longrightarrow X, f \in X$ satisfy

$$
(\mathcal{K}u, v) = k^2 \int_\Omega (n + 1)\bar{v}u + \int_\Sigma \bar{v}L(G_\Gamma u), \quad \forall u, v \in X,
$$

$$
(f, v) = \int_\Sigma \bar{v}Lu^i, \quad \forall v \in X.
$$

By Lemma 4.4 of [8], the operator $\mathcal{K}$ is compact. In addition, the variational formulation (3.8) amounts to the Fredholm equation (1.1)

$$(I - \mathcal{K})u = f, \tag{3.9}$$

which is well-posed. From Theorem 2.1, we obtain the following results similar to Theorem 5.8 in [8].

Theorem 3.2 *Assume that $\{X_h\}$ is a sequence of finite dimensional subspaces of X consisting of piecewise linear functions defined on a shape-regular mesh, where h denotes the mesh size. If $h \ll 1$, then the finite dimensional discretization: Find $u_h \in X_h$ such that*

$$\int_\Omega (\overline{\nabla v} \cdot \mathcal{A}\nabla u_h - k^2 \bar{v} u_h) - \int_\Sigma \bar{v} L(G_\Gamma u_h) = \int_\Sigma \bar{v} L u^i, \quad \forall v \in X_h \tag{3.10}$$

is uniquely solvable in X_h and for $u \in X \cap H^2(\Omega)$ there holds

$$\|u - u_h\|_{1,\Omega} \le Ch\|u\|_{2,\Omega}. \tag{3.11}$$

Proof. Under the assumption for $\{X_h\}$, we have (see, e.g., [10])

$$\lim_{h \to 0} \inf_{v \in X_h} \|u - v\|_{1,\Omega} = 0, \quad \forall u \in X \tag{3.12}$$

and

$$\inf_{v \in X_h} \|u - v\|_{1,\Omega} \le Ch\|u\|_{2,\Omega}, \quad \forall u \in X \cap H^2(\Omega).$$

Hence we obtain that (3.10) is uniquely solvable in X_h and (3.11) is true if $u \in X \cap H^2(\Omega)$.
□

Remark 3.1. In [8], the discretized problem may be viewed as

$$(I - P_h \mathcal{K}_h)u_h = P_h f,$$

where $\mathcal{K}_h$ is an approximation of $\mathcal{K}$ and satisfies (cf. Lemma 5.3 of [8])

$$\|\mathcal{K}_h - \mathcal{K}\| \le Ch, \tag{3.13}$$

hence Theorem 2.3 implies Theorem 5.8 of [8].

3.3 Diffraction by a biperiodic structure

Consider a plane wave incident on a biperiodic structure, *i.e.*, the structure is periodic in two orthogonal directions. The diffraction problem is to study the electromagnetic energy distributions away from the structure, which has many fundamental applications in micro-optics. The electromagnetic fields are governed by the time harmonic Maxwell equations (time dependence $\exp(-i\omega t)$) [11], [12]:

$$\nabla \times E - i\omega \mu H = 0, \quad \nabla \times H + i\omega \varepsilon E = 0, \tag{3.14}$$

8

where E and H denote the electric and magnetic fields in R^3, respectively. The magnetic permeability μ is assumed to be one everywhere. There are two constants Λ_1 and Λ_2 such that the dielectric coefficient ε satisfies, for any $n_1, n_2 \in Z \equiv \{0, \pm 1, \pm 2, \cdots\}$, and for almost all $x = (x_1, x_2, x_3) \in R^3$,

$$\varepsilon(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2, x_3) = \varepsilon(x_1, x_2, x_3).$$

Further, it is assumed that, for some fixed positive constant b and sufficiently small $\delta > 0$,

$$\varepsilon(x_1, x_2, x_3) = \varepsilon_1, \text{ for } x_3 > b - \delta,$$
$$\varepsilon(x_1, x_2, x_3) = \varepsilon_2, \text{ for } x_3 < -b + \delta,$$

where $\varepsilon(x) \in L^\infty$, $Re\,(\varepsilon(x)) \geq \varepsilon_0$, $Im\,(\varepsilon(x)) \geq 0$, $\varepsilon_0, \varepsilon_1$ and ε_2 are constants, ε_0 and ε_1 are real and positive, and $Re\,\varepsilon_2 > 0$, $Im\,\varepsilon_2 \geq 0$. The case $Im\,\varepsilon > 0$ accounts for materials which absorb energy.

Let $\Omega = \{x \in R^3 : -b < x_3 < b\}$, $\Omega_1 = \{x \in R^3 : x_3 > b\}$, $\Omega_2 = \{x \in R^3 : x_3 < b\}$, $\Gamma_1 = \{x \in R^3 : x_3 = b\}$, and $\Gamma_2 = \{x \in R^3 : x_3 = -b\}$.

Consider a plane wave in Ω_1 in the form of $(E_I, H_I) = (s, p)\exp(iq \cdot x)$ incident on Ω. Here $q = (\alpha_1, \alpha_2, -\beta) = \omega\sqrt{\varepsilon}(\cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2, -\sin\theta_1)$ is the incident wave vector whose direction is specified by θ_1 and θ_2 with $0 < \theta_1 < \pi$ and $0 < \theta_2 \leq 2\pi$. The vectors s and p satisfy

$$s = \frac{1}{\omega\varepsilon}(p \times q), \quad q \cdot p = \omega^2\varepsilon_1, \quad p \cdot q = 0.$$

As usual, we are interested in quasiperiodic solutions, i.e., solutions E and H such that the fields E_α, H_α defined by, for $(\alpha_1, \alpha_2, 0)$,

$$E_\alpha = \exp(-\alpha \cdot x)E(x), \quad H_\alpha = \exp(-\alpha \cdot x)H(x)$$

are periodic, with period Λ_1 in the x_1 direction, and with period Λ_2 in the x_2 direction.

Define the lattice
$$\Lambda = \Lambda_1 Z \times \Lambda_2 Z \times \{0\} \subset R^3.$$

Since the fields H_α are $\Lambda-$periodic, we can move the problem from R^3 to the quotient space R^3/Λ. For the remainder of the paper, we shall identify Ω with the cylinder Ω/Λ, and similarly for the boundaries $\Gamma_j \equiv \Gamma_j/\Lambda$. Thus from now on, all functions defined on Ω and Γ_j are implicitly $\Lambda-$periodic.

As shown in Bao and Dobson [12], the scattering problem can be formulated as follows

$$\begin{cases} \nabla_\alpha \times \left(\frac{1}{\varepsilon}\nabla_\alpha \times H_\alpha\right) - \nabla_\alpha\left(\frac{1}{\varepsilon_c}\nabla_\alpha \cdot H_\alpha\right) - \omega^2 H_\alpha &= 0, \text{ in } \Omega, \\ \nu \times \left(\nabla_\alpha \times (H_\alpha - H_{I,\alpha})\right) - B_1\left(P(H_\alpha - H_{I,\alpha})\right) &= 0, \text{ on } \Gamma_1, \\ \nu \times (\nabla_\alpha \times H_\alpha) - B_2\left(P(H_\alpha)\right) &= 0, \text{ on } \Gamma_2, \\ (T_1^\alpha - \frac{\partial}{\partial\nu})H_{\alpha,3} - 2i\beta_1 p_3\exp(-i\beta_1 b) &= 0, \text{ on } \Gamma_1, \\ (T_2^\alpha - \frac{\partial}{\partial\nu})H_{\alpha,3} &= 0, \text{ on } \Gamma_2, \end{cases} \quad (3.15)$$

where ε_c is some fixed constant (penalty), $H_{I,\alpha} = H_I \exp(-i\alpha \cdot x), \nabla_\alpha = \nabla + i\alpha = \nabla + i(\alpha_1, \alpha_2, 0)$, P is the projection onto the plane orthogonal to ν, i.e., $Pf = -\nu \times (\nu \times f)$. Here the operators B_1, B_2, T_1^α, and T_2^α are nonlocal pseudo-differential operators of order one whose explicit forms are given in [12].

The weak form of the scattering problem takes form

$$a(H, F) = \ell(F), \quad \forall F \in \left(H^1(\Omega)\right)^3, \tag{3.16}$$

where

$$
\begin{aligned}
a(H, F) &= \int_\Omega \frac{1}{\varepsilon}(\nabla \times H) \cdot (\overline{\nabla_\alpha \times F}) + \int_\Omega \frac{1}{\varepsilon_c}(\nabla_\alpha \cdot H)(\overline{\nabla_\alpha \cdot F}) + \int_{\Gamma_1} \frac{1}{\varepsilon_1} B_1\big(P(H)\big) \cdot \bar{F} \\
&\quad - \int_{\Gamma_1} \frac{1}{\varepsilon_c} T_1(H_3)\bar{F}_3 + \int_{\Gamma_2} \frac{1}{\varepsilon_1} B_2\big(P(H)\big) \cdot \bar{F} - \int_{\Gamma_2} \frac{1}{\varepsilon_c} T_2(H_3)\bar{F}_3 \\
&\quad - \int_{\partial\Omega} \frac{1}{\varepsilon_c}(\nabla_{\alpha t} \cdot H)(\overline{\nu \cdot F}) - \omega^2 \int_\Omega H \cdot \bar{F}, \\
\ell(F) &= \int_{\Gamma_1} \left(\nu \times \nabla_\alpha \times H_I - B_1 P\big(H_I\big)\right) \cdot \bar{F} + 2i \int_{\Gamma_1} \beta_1 \frac{1}{\varepsilon_c} p_3 \exp(-i\beta_1 b)\bar{F},
\end{aligned}
$$

where $\nabla_{\alpha t} = (\partial_{x_1} + i\alpha_1, \partial_{x_2} + i\alpha_2, 0)$.

Well-posedness of the continuous model has been established in [12]. We also refer the paper for additional references. Here, by using Theorem 2.2, we prove a new well-posedness and convergence result for the discretized problem.

It is shown in Theorem 4.3 of [12] that if the real constant ε_c is chosen as

$$\inf_{x \in \Omega} Re\, \frac{1}{\varepsilon} \geq \frac{3}{4} \frac{1}{\varepsilon_c},$$

then there exist an inner product $(\cdot, \cdot)$ of $X = \left(H^1(\Omega)\right)^3$ and a compact operator $\mathcal{K}$ on $\left(H^1(\Omega)\right)^3$ such that

$$((I - \mathcal{K})w, F) = a(w, F), \quad \forall F \in \left(H^1(\Omega)\right)^3,$$

which is equivalent to

$$(I - \mathcal{K})H = f, \tag{3.17}$$

where $f \in X$ satisfies

$$(f, F) = \ell(F), \quad \forall F \in \left(H^1(\Omega)\right)^3.$$

Consider a finite dimensional approximation scheme: Find $H_h \in X_h$ such that

$$a(H_h, F) = \ell(F), \quad \forall F \in X_h, \tag{3.18}$$

where $X_h \subset \left(H^1(\Omega)\right)^3$ is a finite dimensional subspace.

It is easy to see that (3.18) is equivalent to an approximate operator equation: Find $H_h \in X_h$ such that

$$(I - P_h\mathcal{K})H_h = P_hf.$$

The following convergence result of the finite dimensional approximation is a direct consequence of Theorem 2.1.

Theorem 3.3 *Assume that $\{X_h\}$ is a sequence of finite dimensional subspaces of $\left(H^1(\Omega)\right)^3$ such that*

$$\lim_{h\to 0} \inf_{v\in X_h} \|u - v\|_{1,\Omega} = 0, \quad \forall u \in \left(H^1(\Omega)\right)^3.$$

If $h \ll 1$, then the equation (3.18) is uniquely solvable in X_h and there holds

$$\lim_{h\to 0} \|H - H_h\|_{1,\Omega} = 0. \tag{3.19}$$

Remark 3.2. Note that B_1, B_2, T_1^α, and T_2^α are nonlocal operators. In practice, these operators are often approximated by some local operators. Because of Theorem 2.3, we can also obtain well-posedness and convergence in this situation.

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