Multi-component generalizations of four integrable differential-difference equations: soliton solutions and bilinear Bäcklund transformations

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Abstract

Bilinear approach is applied to derive integrable multi-component generalizations of the so-called 1+1 dimensional special Toda lattice, the Volterra lattice, a simple differential-difference equation found by Adler, Moser, Weiss, Veselov and Shabat and another integrable lattice reduced from the discrete BKP equation. Their soliton solutions expressed by pfaffians and the corresponding bilinear Bäcklund transformations are obtained.

1 Introduction

Integrable multi-component generalization of soliton equations is one of the most exciting topics in soliton theory. Much research on this subject has been conducted. For example, for the celebrated KdV equation, many coupled extensions of the KdV equation have been proposed in the literature (see, e.g., [1]-[10]). Several approaches have been developed to search for various integrable coupled versions of soliton equations. One of them is the bilinear approach. Very recently, a vector potential KdV equation and vector Ito equation have been proposed based on their bilinear forms [11].

The purpose of this paper is to apply the bilinear approach for multi-component generalizations of soliton equations to differential-difference case. We will consider integrable multi-component generalizations of the following four bilinear differential-difference equations:

$$D_t(D_t - \sinh(D_n))f(n) \cdot f(n) = 0, \tag{1.1}$$

$$\sinh(\frac{1}{2}D_n)(D_t + 2\sinh(D_n))f(n) \cdot f(n) = 0, \tag{1.2}$$

$$(D_t^2 \cosh(D_n) - \frac{1}{2}D_t \sinh(D_n))f(n) \cdot f(n) = 0, \tag{1.3}$$

$$\sinh(\frac{1}{2}D_n)(D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n))f(n) \cdot f(n) = 0, \tag{1.4}$$

where the bilinear operators D_t^k and $\exp(D_n)$ are defined by [12, 13, 14]

$$D_t^k\ a \cdot b \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(t)b(t')|_{t'=t}, \quad \exp(\delta D_n)a(n) \cdot b(n) = a(n+\delta)b(n-\delta).$$

As a result, new multi-component generalizations of these four integrable differential-difference equations (1.1)-(1.4) are proposed.

This paper is organized as follows. Section 2 is devoted to integrable multi-component generalization of equation (1.1). Soliton solutions expressed by pfaffians and the corresponding bilinear Bäcklund transformation are obtained for the multi-component version of equation (1.1). In section 3, we will consider an integrable multi-component generalization of equation (1.2). Soliton solutions expressed by pfaffians and the corresponding bilinear Bäcklund transformations are presented for the multi-component version of equation (1.2). Next in section 4, an integrable multi-component generalization of equation (1.3) is found. We also give its soliton solutions expressed by pfaffians and the corresponding bilinear Bäcklund transformation. Furthermore, we derive an integrable multi-component generalization of equation (1.4) in section 5. Soliton solutions expressed by pfaffians and the corresponding bilinear Bäcklund transformation are deduced for the multi-component version of equation (1.4). Section 6 summarizes the obtained results and gives some discussions. Finally, in Appendix A, some essential elements of Pfaffians (such as definition of Pfaffian and Pfaffian identities) are introduced, while some bilinear operator identities are listed in Appendix B.

2 An integrable multi-component generalization of equation (1.1), its soliton solutions and Bäcklund transformation

Equation (1.1) can be obtained from a (2+1)-dimensional bilinear equation by reduction [15]. By the dependent variable transformation $u(n) = \ln \frac{f(n+1)}{f(n)}$, equation (1.1) is transformed into

$$u_{tt}(n) = \frac{1}{2}(u_t(n+1) + u_t(n))e^{u(n+1)-u(n)} - \frac{1}{2}(u_t(n) + u_t(n-1))e^{u(n)-u(n-1)}, \tag{2.1}$$

which is a reduced equation of the so-called special 2+1 dimensional Toda lattice [16]. Therefore, we may call equation (1.1) the 1+1 dimensional special Toda lattice. Some results concerning (1.1) or (2.1) have been achieved. For example, a Lax pair for equation (2.1) was given in [17]. In this section, we will consider an integrable multi-component generalization of equation (1.1).

Based on the fact that equation (1.1) can be rewritten as

$$(D_t - \sinh(D_n))g(n) \cdot f(n) = 0, \quad g(n) = f_t(n), \tag{2.2}$$

we now propose a natural coupled form

$$(D_t - \sinh(D_n))g_\mu(n) \cdot f(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{2.3}$$

$$\sum_{\mu=1}^{M} g_\mu(n) = \frac{\partial f(n)}{\partial t} \tag{2.4}$$

for equation (1.1). By the dependent variable transformation

$$u(n) = \ln \frac{f(n+1)}{f(n)}, \quad v_\mu(n) = \frac{g_\mu(n+1)}{f(n+1)} - \frac{g_\mu(n)}{f(n)},$$

equations (2.3) and (2.4) are transformed into

$$v_{\mu,t}(n) - \frac{1}{2}[v_\mu(n+1) + v_\mu(n)]e^{u(n+1)-u(n)} + \frac{1}{2}[v_\mu(n) + v_\mu(n-1)]e^{u(n)-u(n-1)} = 0, \tag{2.5}$$

for $\mu = 1, 2, \cdots, M$,

$$\sum_{\mu=1}^{M} v_\mu(n) = u_t(n). \tag{2.6}$$

In the following, we will show that equations (2.3) and (2.4) are an integrable multi-component generalization of equation (1.1) in the sense of having soliton solutions and bilinear Bäcklund transformation.

Using a perturbational method we obtain a 3-soliton solution to the coupled 1+1 dimensional special Toda lattice (2.3) and (2.4), which is expressed as follows,

$$\begin{aligned} f(n) &= 1+\exp[\eta_1]+\exp[\eta_2]+\exp[\eta_3] \\ &\quad +a_{1,2}b_{1,2}\exp[\eta_1+\eta_2]+a_{1,3}b_{1,3}\exp[\eta_1+\eta_3]+a_{2,3}b_{2,3}\exp[\eta_2+\eta_3] \\ &\quad +a_{1,2}a_{1,3}a_{2,3}b_{1,2}b_{1,3}b_{2,3}\exp[\eta_1+\eta_2+\eta_3], \end{aligned} \tag{2.7}$$

$$\begin{aligned} g_\mu(n) &= c_\mu(1)\exp[\eta_1]+c_\mu(2)\exp[\eta_2]+c_\mu(3)\exp[\eta_3] \\ &\quad +c_\mu(1,2)\exp[\eta_1+\eta_2]+c_\mu(1,3)\exp[\eta_1+\eta_3]+c_\mu(2,3)\exp[\eta_2+\eta_3] \\ &\quad +c_\mu(1,2,3)\exp[\eta_1+\eta_2+\eta_3], \quad \text{for } \mu=1,2,\cdots,M, \end{aligned} \tag{2.8}$$

with

$$\exp(\eta_j)=P_j^{n+n_{j,0}}\exp(q_j t), \quad q_j=\frac{P_j^2-1}{2P_j} \tag{2.9}$$

$$a_{j,k}=\frac{P_j-P_k}{P_jP_k-1}, \quad b_{j,k}=\frac{(P_j-P_k)(P_jP_k+1)}{(P_j+P_k)(P_jP_k-1)}, \tag{2.10}$$

$$\begin{aligned} &c_\mu(j,k)=a_{j,k}(c_\mu(j)-c_\mu(k)), \\ &c_\mu(1,2,3)=a_{1,2}a_{1,3}a_{2,3}[b_{2,3}c_\mu(1)+b_{3,1}c_\mu(2)+b_{1,2}c_\mu(3)] \\ &\qquad \text{for } j,k=1,2,3 \text{ and } \mu=1,2,\cdots,M, \end{aligned} \tag{2.11}$$

where $n_{j,0}, P_j$, $(j=1,2,3)$, are constants and $c_\mu(1)$, $c_\mu(2)$ and $c_\mu(3)$ are parameters that satisfy $\sum_{\mu=1}^{M} c_\mu(j)=\frac{P_j^2-1}{2P_j}$.

These expressions suggest that N-soliton solutions to equations (2.3) and (2.4) may be expressed by pfaffians. In fact, we find that the pfaffian solutions are

$$f(n)=\mathrm{pf}(d_0,a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N,\beta_0), \tag{2.12}$$

$$g_\mu(n)=\mathrm{pf}(d_0,a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N,\beta_\mu), \quad \text{for } \mu=1,2,\cdots,M, \tag{2.13}$$

where the entries of these pfaffians are defined as follows,

$$\mathrm{pf}(d_0,a_j)=\exp(\eta_j), \quad \mathrm{pf}(d_0,b_j)=-1, \quad \mathrm{pf}(d_0,\beta_0)=1, \quad \mathrm{pf}(a_j,\beta_\mu)=0, \tag{2.14}$$

$$\mathrm{pf}(a_j,a_k)=-a_{j,k}\exp(\eta_j+\eta_k), \quad \mathrm{pf}(a_j,b_k)=\delta_{j,k}, \quad \mathrm{pf}(d_0,\beta_\mu)=0, \tag{2.15}$$

$$\mathrm{pf}(b_j,b_k)=b_{j,k}, \quad \mathrm{pf}(b_j,\beta_0)=1, \quad \mathrm{pf}(b_j,\beta_\mu)=c_\mu(j), \quad \mathrm{pf}(a_j,\beta_0)=0, \tag{2.16}$$

$$\text{for } j,k=1,2,\cdots,N \text{ and for } \mu=1,2,\cdots,M,$$

with

$$\exp(\eta_j)=P_j^{n+n_{j,0}}\exp\left(\frac{P_j^2-1}{2P_j}t\right), \quad a_{j,k}=\frac{P_j-P_k}{P_jP_k-1},$$

$$b_{j,k}=\frac{(P_j-P_k)(P_jP_k+1)}{(P_j+P_k)(P_jP_k-1)}, \quad \delta_{j,k}=\begin{cases} 1 & \text{for } j=k, \\ 0 & \text{for } j\neq k. \end{cases}$$

Here P_j and $n_{j,0}$ are constants and $c_\mu(j)$,$(j=1,2,\cdots,N)$, are parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j)=\frac{P_j^2-1}{2P_j}$.

In what follows, we will show that $f(n)$ and $g_\mu(n)$ given by (2.12) and (2.13) satisfy equations (2.3) and (2.4). Firstly, it is noted that the linear equation (2.4) and the structure of soliton solutions expressed by (2.12) and (2.13) are similar to those of the vector Ito equation considered in [11]. Therefore, we can show that $f(n)$ and $g_\mu(n)$ given by (2.12) and (2.13) satisfy the linear equation (2.4) following the deduction procedure for the vector Ito equation in [11]. Next, we will show that $f(n)$ and $g_\mu(n)$ satisfy the bilinear equation (2.3) , which is equivalent to

$$\frac{\partial g_\mu(n)}{\partial t}f(n)-g_\mu(n)\frac{\partial f(n)}{\partial t}-\frac{1}{2}g_\mu(n+1)f(n-1)+\frac{1}{2}g_\mu(n-1)f(n+1)=0. \tag{2.17}$$

With the help of the new characters $\tilde{d}_1$ and $\tilde{d}_{-1}$ defined by the following entries,

$$\begin{aligned}
&\mathrm{pf}(\tilde{d}_1,a_j)=(P_j-1)\exp(\eta_j),\quad \mathrm{pf}(\tilde{d}_1,b_j)=0,\quad \mathrm{pf}(\tilde{d}_1,\beta_0)=0,\\
&\mathrm{pf}(\tilde{d}_{-1},a_j)=(\frac{1}{P_j}-1)\exp(\eta_j),\quad \mathrm{pf}(\tilde{d}_{-1},b_j)=0,\quad \mathrm{pf}(\tilde{d}_{-1},\beta_0)=0,\\
&\mathrm{pf}(\tilde{d}_1,\beta_\mu)=0,\quad \mathrm{pf}(\tilde{d}_{-1},\beta_\mu)=0,\quad \mathrm{pf}(d_0,\tilde{d}_1)=\mathrm{pf}(d_0,\tilde{d}_{-1})=\mathrm{pf}(\tilde{d}_1,\tilde{d}_{-1})=0,\\
&\quad \text{for } j=1,2,\cdots,N,\quad \mu=1,2,\cdots M,
\end{aligned}$$

and simplified notation $\{\cdots\cdots\}$ for $\{a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N\}$, we obtain the following difference formulae for $f(n)$ and $g_\mu(n)$,

$$f(n)=\mathrm{pf}(d_0,\beta_0,\cdots\cdots), \tag{2.18}$$
$$f(n+1)=\mathrm{pf}(d_0,\beta_0,\cdots\cdots)-\mathrm{pf}(d_0,\tilde{d}_1,\cdots\cdots), \tag{2.19}$$
$$f(n-1)=\mathrm{pf}(d_0,\beta_0,\cdots\cdots)-\mathrm{pf}(d_0,\tilde{d}_{-1},\cdots\cdots), \tag{2.20}$$
$$g_\mu(n)=\mathrm{pf}(d_0,\beta_\mu,\cdots\cdots), \tag{2.21}$$
$$g_\mu(n+1)=\mathrm{pf}(d_0,\beta_\mu,\cdots\cdots)+\mathrm{pf}(d_0,\tilde{d}_1,\beta_\mu,\beta_0,\cdots\cdots), \tag{2.22}$$
$$g_\mu(n-1)=\mathrm{pf}(d_0,\beta_\mu,\cdots\cdots)+\mathrm{pf}(d_0,\tilde{d}_{-1},\beta_\mu,\beta_0,\cdots\cdots), \tag{2.23}$$

and the following differential formulae [11],

$$\frac{\partial f(n)}{\partial t}=-\frac{1}{2}\left[\mathrm{pf}(d_0,\tilde{d}_1,\cdots\cdots)-\mathrm{pf}(d_0,\tilde{d}_{-1}\cdots\cdots)-\mathrm{pf}(d_0,\tilde{d}_1,\tilde{d}_{-1},\beta_0,\cdots\cdots)\right], \tag{2.24}$$

$$\begin{aligned}\frac{\partial g_\mu(n)}{\partial t}=\frac{1}{2}\Big[&\mathrm{pf}(d_0,\tilde{d}_1,\beta_\mu,\beta_0,\cdots\cdots)-\mathrm{pf}(d_0,\tilde{d}_{-1},\beta_\mu,\beta_0,\cdots\cdots)\\
&+\mathrm{pf}(d_0,\tilde{d}_1,\tilde{d}_{-1},\beta_\mu,\cdots\cdots)\Big].\end{aligned} \tag{2.25}$$

Substituting these relations (2.18)-(2.25) into eq.(2.17), we find that the bilinear equation is reduced to the pfaffian identity (see,[14, 11] or Appendix A),

$$\begin{aligned}&\mathrm{pf}(\tilde{d}_{-1},d_0,\cdots\cdots)\mathrm{pf}(\tilde{d}_1,\beta_\mu,\beta_0,d_0,\cdots\cdots)-\mathrm{pf}(\tilde{d}_1,d_0,\cdots\cdots)\mathrm{pf}(\tilde{d}_{-1},\beta_\mu,\beta_0,d_0,\cdots\cdots)\\
&+\mathrm{pf}(\beta_\mu,d_0,\cdots\cdots)\mathrm{pf}(\tilde{d}_{-1},\tilde{d}_1,\beta_0,d_0,\cdots\cdots)-\mathrm{pf}(\beta_0,d_0,\cdots\cdots)\mathrm{pf}(\tilde{d}_{-1},\tilde{d}_1,\beta_\mu,d_0,\cdots\cdots)=0.\end{aligned}$$

Therefore, $f(n)$ and $g_\mu(n)$ satisfy the bilinear equation (2.3). Here, the key point is to introduce the new character $\tilde{d}_1$ and $\tilde{d}_{-1}$ to deduce the difference formulae for $f(n)$ and $g_\mu(n)$, then by the dispersion relation (2.9) we can deduce the differential formulae.

In the following, we will present a bilinear Bäcklund transformation for Eqs. (2.3) and (2.4). In fact, we obtain the following Bäcklund transformation

$$e^{\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu) = \lambda e^{-\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu) + (k_\mu e^{\frac{1}{2}D_n} - k_\mu \lambda e^{-\frac{1}{2}D_n}) f \cdot f', \quad (2.26)$$

$$(2D_t - \frac{1}{\lambda}e^{D_n} + \lambda e^{-D_n} + \gamma) f \cdot f' = 0, \quad (2.27)$$

$$(2D_t - \frac{1}{\lambda}e^{D_n} + \lambda e^{-D_n} + \gamma)(g_\mu \cdot f' + f \cdot g'_\mu) = 0, \quad (2.28)$$

$$\text{for } \mu = 1, 2, \cdots, M$$

between equations (2.3), (2.4) and

$$(D_t - \sinh(D_n)) g'_\mu(n) \cdot f'(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \quad (2.29)$$

$$\sum_{\mu=1}^{M} g'_\mu(n) = \frac{\partial f'(n)}{\partial t}, \quad (2.30)$$

where we have assumed that

$$g_1(n) + g_2(n) + \cdots + g_M(n) = f_t(n), \quad g'_1(n) + g'_2(n) + \cdots + g'_M(n) = f'_t(n) \quad (2.31)$$

such that Eqs. (2.4) and (2.30) are satisfied automatically. Here λ, γ, k_μ ($\mu = 1, 2, \cdots, M$) are arbitrary constants. From the assumption (2.31), we know that, in order to show that (2.26)-(2.28) constitute a BT, it suffices to prove that

$$P_\mu \equiv 2[(D_t - \sinh(D_n)) g_\mu(n) \cdot f(n)] f'(n)^2 - 2f^2(n)(D_t - \sinh(D_n)) g'_\mu(n) \cdot f'(n) = 0.$$

For this purpose, we have, by using (2.26)-(2.28) and (B1)-(B4),

$$\begin{aligned}
P_\mu = \; & -2(D_t f \cdot f')(g_\mu f' + f g'_\mu) + 2ff' D_t(g_\mu \cdot f' + f \cdot g'_\mu) \\
& -2\sinh(\frac{1}{2}D_n)\{[e^{\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu)] \cdot (e^{-\frac{1}{2}D_n} f \cdot f') - (e^{\frac{1}{2}D_n} f \cdot f') \cdot [e^{-\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu)]\} \\
= \; & -2(D_t f \cdot f')(g_\mu f' + f g'_\mu) + 2ff' D_t(g_\mu \cdot f' + f \cdot g'_\mu) \\
& -2\sinh(\frac{1}{2}D_n)[\lambda e^{-\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu) + k_\mu e^{\frac{1}{2}D_n} f \cdot f'] \cdot (e^{-\frac{1}{2}D_n} f \cdot f') \\
& +2\sinh(\frac{1}{2}D_n)(e^{\frac{1}{2}D_n} f \cdot f') \cdot \left[\frac{1}{\lambda} e^{\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu) + k_\mu e^{-\frac{1}{2}D_n} f \cdot f'\right] \\
= \; & -2(D_t f \cdot f')(g_\mu f' + f g'_\mu) + 2ff' D_t(g_\mu \cdot f' + f \cdot g'_\mu) \\
& -\lambda[g_\mu f'(e^{-D_n} f \cdot f') - (e^{-D_n} g_\mu \cdot f') ff' - ff'(e^{-D_n} f \cdot g'_\mu) + (e^{-D_n} f \cdot f') f g'_\mu] \\
& +\frac{1}{\lambda}[(e^{D_n} f \cdot f') g_\mu f' - ff'(e^{D_n} g_\mu \cdot f') - (e^{D_n} f \cdot g'_\mu) ff' + f g'_\mu (e^{D_n} f \cdot f')] \\
= \; & [(-2D_t + \frac{1}{\lambda} e^{D_n} - \lambda e^{-D_n}) f \cdot f'](g_\mu f' + f g'_\mu) + ff'[(2D_t - \frac{1}{\lambda} e^{D_n} + \lambda e^{-D_n})(g_\mu \cdot f' + f \cdot g'_\mu)] \\
= \; & 0.
\end{aligned}$$

Therefore, we have proved that (2.26)-(2.28) constitute a bilinear BT between equations (2.3), (2.4) and equations (2.29),(2.30) under the assumption (2.31).

3 An integrable multi-component generalization of equation (1.2), its soliton solutions and Bäcklund transformation

It is known that equation (1.2) may serve as the bilinear form for the Volterra lattice or the differential-difference analogue of the KdV equation. By the dependent variable transformations $u(n) = \frac{f(n+1)f(n-2)}{f(n)f(n-1)}$ or $v(n) = \frac{f(n+1)f(n-1)}{f^2(n)}$, equation (1.2) can be transformed into

$$u_t(n) + u(n)(u(n+1) - u(n-1)) = 0 \tag{3.1}$$

or

$$v_t(n) + v^2(n)(v(n+1) - v(n-1)) = 0. \tag{3.2}$$

Obviously, $u(n) = v(n)v(n-1)$ is a Miura-type transformation between equation (3.1) and equation (3.2). In [18, 19], integrable multi-component generalizations of equation (3.2) were proposed and studied. In this section, we will give another integrable multi-component generalization of the lattice (1.2).

Based on the fact that equation (1.2) can be rewritten as

$$(D_t + 2\sinh(D_n))g(n) \cdot f(n) = 0, \quad g(n) = f(n+1) - f(n),$$

we now propose a natural coupled form

$$(D_t + 2\sinh(D_n))g_\mu(n) \cdot f(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{3.3}$$

$$\sum_{\mu=1}^{M} g_\mu(n) = f(n+1) - f(n). \tag{3.4}$$

By the dependent variable transformation

$$U(n) = \ln\frac{f(n+1)}{f(n)}, \quad V_\mu(n) = \frac{g_\mu(n+1)}{f(n+1)} - \frac{g_\mu(n)}{f(n)},$$

equations (3.3) and (3.4) are transformed into

$$V_{\mu,t}(n) + [V_\mu(n+1) + V_\mu(n)]e^{U(n+1)-U(n)} - [V_\mu(n) + V_\mu(n-1)]e^{U(n)-U(n-1)} = 0,$$
$$\text{for } \mu = 1, 2, \cdots, M,$$
$$\sum_{\mu=1}^{M} V_\mu(n) = e^{U(n+1)} - e^{U(n)}.$$

In the following, we will show that equations (3.3) and (3.4) are an integrable multi-component generalization of equation (1.2) in the sense of having soliton solutions and bilinear Bäcklund transformation.

First of all, we construct soliton solutions of the coupled Volterra lattice (3.3) and (3.4). Using a perturbation method, we obtain a 3-soliton solution to Eqs. (3.3) and (3.4) that takes the same form of (2.7) and (2.8) but with different coefficients and different dispersion relations. In this case,

$$\exp(\eta_j) = P_j^{n+n_{j,0}} \exp(q_j t), \quad q_j = (1 - P_j^2)/P_j,$$

and the values of of $a_{j,k}, b_{j,k}, c_\mu(j), c_\mu(j,k)$ and $c_\mu(1,2,3)$ are

$$a_{j,k} = \frac{P_j - P_k}{P_jP_k - 1}, \quad b_{j,k} = \frac{P_j - P_k}{P_jP_k - 1},$$
$$c_\mu(j,k) = a_{j,k}(c_\mu(j) - c_\mu(k)), \quad c_\mu(1,2,3) = a_{1,2}a_{1,3}a_{2,3}[b_{2,3}c_\mu(1) + b_{3,1}c_\mu(2) + b_{1,2}c_\mu(3)],$$
$$\text{for } j,k = 1,2,3 \text{ and } \mu = 1, 2, \cdots, M,$$

where $P_j, n_{j,0}$ are constants and $c_\mu(j)$,$(j = 1, 2, 3)$, are free parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j) = P_j - 1$. The above expressions suggest that the N-soliton solutions expressed by pfaffians to the equations (3.3) and (3.4) may also take the following form

$$f(n) = \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_0), \tag{3.5}$$

$$g_\mu(n) = \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_\mu), \quad \text{for } \mu = 1, 2, \cdots, M, \tag{3.6}$$

and the entries of pfaffians (3.5) and (3.6) are defined in the same form as (2.14)-(2.16), but with

$$\exp(\eta_j) = P_j^{n+n_{j,0}} \exp\left(\frac{1-P_j^2}{P_j} t\right), \quad b_{j,k} = a_{j,k} = \frac{P_j - P_k}{P_j P_k - 1}, \quad \delta_{j,k} = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}$$

and P_j and $n_{j,0}$ are constants and $c_\mu(j)$ are parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j) = P_j - 1$.

What we want to do next is to show that $f(n)$ and $g_\mu(n)$ given by (3.5) and (3.6) do satisfy equations (3.3) and (3.4). Based on the similarity between (3.3) and (2.3), we can show that $f(n)$ and $g_\mu(n)$ satisfy the bilinear equation (3.3) in the same way as done in section 2. Besides, we have

$$\begin{aligned} f(n+1) = {} & \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_0) \\ & -\mathrm{pf}(d_0, \tilde{d}_1, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N), \end{aligned} \tag{3.7}$$

where a new character $\tilde{d}_1$ is defined by

$$\mathrm{pf}(\tilde{d}_1, a_j) = (P_j - 1)\exp(\eta_j), \ \mathrm{pf}(\tilde{d}_1, b_j) = 0, \ \mathrm{pf}(\tilde{d}_1, d_0) = 0, \ \mathrm{pf}(\tilde{d}_1, \beta_0) = 0, \ \mathrm{pf}(\tilde{d}_1, \beta_\mu) = 0,$$
$$\text{for } j = 1, 2, \cdots, N, \quad \mu = 1, 2, \cdots M.$$

Therefore, in the following, we only prove that $f(n)$ and $g_\mu(n)$ satisfy the linear equation (3.4). Following the procedure described in [11], we first expand $g_\mu(n)$ with respect to the final character β_μ, and obtain

$$\begin{aligned} g_\mu(n) &= \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_\mu), \\ &= \sum_{j=1}^{N} c_\mu(j)(-1)^{N+j} \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, \hat{b}_j, \cdots, b_N). \end{aligned}$$

where the notion $\hat{\alpha}$ indicates the letter α is missing. The sum of $g_\mu(n)$ over μ gives

$$\begin{aligned} \sum_{\mu=1}^{M} g_\mu(n) &= \sum_{j=1}^{N} (-1)^{N+j}(P_j - 1)\mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, \hat{b}_j, \cdots, b_N), \\ &= \mathrm{pf}(d_0, \beta_0^*, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N), \end{aligned}$$

where the entries with the new character β_0^* are defined by

$$\mathrm{pf}(d_0, \beta_0^*) = 0, \quad \mathrm{pf}(\beta_0^*, a_j) = 0, \quad \mathrm{pf}(\beta_0^*, b_j) = 1 - P_j. \quad \text{for } j, k = 1, 2, \cdots, N.$$

On the other hand, we know from the difference formulae (3.5) and (3.7) that

$$f(n+1) - f(n) = -\mathrm{pf}(d_0, \tilde{d}_1, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N). \tag{3.8}$$

Exactly following the deduction procedure for the vector KdV equation given in [11], we can show that

$$\mathrm{pf}(d_0, \beta_0^*, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N) = -\mathrm{pf}(d_0, \tilde{d}_1, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N).$$

That means that $f(n)$ and $g_\mu(n)$ satisfy the linear equation

$$\sum_{\mu=1}^{M} g_\mu(n) = f(n+1) - f(n).$$

In what follows, we will present Bäcklund transformations for Eqs. (3.3) and (3.4). Firstly, based on the similarity between (3.3) and (2.3), we can easily work out a Bäcklund transformation

$$e^{\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu) = \lambda e^{-\frac{1}{2}D_n}(g_\mu \cdot f' - f \cdot g'_\mu) + (k_\mu e^{\frac{1}{2}D_n} - k_\mu \lambda e^{-\frac{1}{2}D_n}) f \cdot f', \tag{3.9}$$

$$(-D_t - \frac{1}{\lambda} e^{D_n} + \lambda e^{-D_n} + \gamma) f \cdot f' = 0, \tag{3.10}$$

$$(-D_t - \frac{1}{\lambda} e^{D_n} + \lambda e^{-D_n} + \gamma)(g_\mu \cdot f' + f \cdot g'_\mu) = 0, \tag{3.11}$$

$$\mu = 1, 2, \cdots, M$$

between equations (3.3), (3.4) and

$$(D_t + 2\sinh(D_n)) g'_\mu(n) \cdot f'(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{3.12}$$

$$\sum_{\mu=1}^{M} g'_\mu(n) = f'(n+1) - f'(n), \tag{3.13}$$

where we have assumed that

$$g_1(n) + g_2(n) + \cdots + g_M(n) = f(n+1) - f(n), \quad g'_1(n) + g'_2(n) + \cdots + g'_M(n) = f'(n+1) - f'(n)$$

such that Eqs.(3.4) and (3.13) are satisfied automatically. Here λ, γ, k_μ $(\mu = 1, 2, \cdots, M)$ are arbitrary constants.

Next, we give another Bäcklund transformation:

$$(e^{\frac{1}{2}D_n} - \lambda e^{-\frac{1}{2}D_n}) g_\mu \cdot f' = (k_\mu e^{-\frac{1}{2}D_n} - \frac{k_\mu}{\lambda} e^{\frac{1}{2}D_n}) f \cdot g'_\mu, \ \mu = 1, 2, \cdots, M \tag{3.14}$$

$$(-D_t - \frac{1}{\lambda} e^{D_n} + \lambda e^{-D_n} + \theta) f \cdot f' = 0, \tag{3.15}$$

$$(-D_t - \frac{1}{\lambda} e^{D_n} + \lambda e^{-D_n} + \theta) g_\mu \cdot g'_\mu = 0, \ \mu = 1, 2, \cdots, M \tag{3.16}$$

between equations (3.3),(3.4) and

$$(D_t + 2\sinh(D_n)) g'_\mu(n) \cdot f'(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{3.17}$$

$$\sum_{\mu=1}^{M} g'_\mu(n) = f'(n+1) - f'(n), \tag{3.18}$$

where we have also assumed that

$$g_1(n) + g_2(n) + \cdots + g_M(n) = f(n+1) - f(n), \quad g'_1(n) + g'_2(n) + \cdots + g'_M(n) = f'(n+1) - f'(n) \tag{3.19}$$

such that Eqs.(3.4) and (3.18) are satisfied automatically and λ, θ, k_μ $(\mu = 1, 2, \cdots, M)$ are arbitrary constants. In fact, in order to show that (3.14)-(3.16) constitute a BT, it suffices to prove that

$$P_\mu \equiv [(D_t + 2\sinh(D_n)) g_\mu(n) \cdot f(n)] g'_\mu(n) f'(n) - g_\mu(n) f(n) (D_t + 2\sinh(D_n)) g'_\mu(n) \cdot f'(n) = 0.$$

For this purpose, we have, by using (3.14)-(3.16) and (B3)-(B6),

$$
\begin{aligned}
P_\mu = \quad & (D_t g_\mu \cdot g'_\mu) f f' - (D_t f \cdot f') g_\mu g'_\mu + 2\sinh(\frac{1}{2}D_n)(e^{\frac{1}{2}D_n} g_\mu \cdot f') \cdot (e^{-\frac{1}{2}D_n} f \cdot g'_\mu) \\
& -2\sinh(\frac{1}{2}D_n)(e^{\frac{1}{2}D_n} f \cdot g'_\mu) \cdot (e^{-\frac{1}{2}D_n} g_\mu \cdot f') \\
= \quad & (D_t g_\mu \cdot g'_\mu) f f' - (D_t f \cdot f') g_\mu g'_\mu - 2\sinh(\frac{1}{2}D_n)(e^{\frac{1}{2}D_n} f \cdot g'_\mu) \cdot (e^{-\frac{1}{2}D_n} g_\mu \cdot f') \\
& +2\lambda \sinh(\frac{1}{2}D_n)(e^{-\frac{1}{2}D_n} g_\mu \cdot f') \cdot (e^{-\frac{1}{2}D_n} f \cdot g'_\mu) - 2\frac{k_\mu}{\lambda} \sinh(\frac{1}{2}D_n)(e^{\frac{1}{2}D_n} f \cdot g'_\mu) \cdot (e^{-\frac{1}{2}D_n} f \cdot g'_\mu) \\
= \quad & (D_t g_\mu \cdot g'_\mu) f f' - (D_t f \cdot f') g_\mu g'_\mu + \lambda[g_\mu g'_\mu e^{-D_n} f \cdot f' - (e^{-D_n} g_\mu \cdot g'_\mu) f f'] \\
& -2\sinh(\frac{1}{2}D_n)(e^{\frac{1}{2}D_n} f \cdot g'_\mu) \cdot [e^{-\frac{1}{2}D_n} g_\mu \cdot f' + \frac{k_\mu}{\lambda} e^{-\frac{1}{2}D_n} f \cdot g'_\mu] \\
= \quad & [(D_t - \lambda e^{-D_n}) g_\mu \cdot g'_\mu] f f' - [(D_t - \lambda e^{-D_n}) f \cdot f'] g_\mu g'_\mu - \frac{1}{\lambda}(e^{D_n} f \cdot f') g_\mu g'_\mu + \frac{1}{\lambda} f f' e^{D_n} g_\mu \cdot g'_\mu \\
= \quad & 0.
\end{aligned}
$$

Therefore, we have proved that (3.14)-(3.16) are another bilinear BT between (3.3),(3.4) and (3.17),(3.18) under the assumption (3.19).

4 An integrable multi-component generalization of equation (1.3), its soliton solutions and Bäcklund transformation

Equation (1.3) has been introduced in [20] (up to some changes). By the dependent variable transformation

$$
u(n) = \frac{1}{4} + \left(\ln \frac{f(n-1)}{f(n+1)}\right)_t,
$$

equation (1.3) can be transformed into the following simple differential-difference equation

$$
(u(n+1) + u(n-1))_t + (u(n+1) + u(n-1))(u(n-1) - u(n+1)) = 0, \tag{4.1}
$$

which has appeared in the literature as a Bäcklund transformation for the KdV equation and has been studied by several authors (see, e.g. [20, 21, 22]). Some coupled extensions of (4.1) have been proposed (see, e.g. [23, 24]). In this section, we will give another integrable multi-component generalization of the lattice (1.3).

Based on the fact that equation (1.3) can be rewritten as

$$
(D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n)) g(n) \cdot f(n) = 0, \quad g(n) = f_t(n),
$$

we now propose a natural coupled form

$$
(D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n)) g_\mu(n) \cdot f(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{4.2}
$$

$$
\sum_{\mu=1}^{M} g_\mu(n) = \frac{\partial f(n)}{\partial t}. \tag{4.3}
$$

By the dependent variable transformation $v_\mu(n) = g_\mu(n)/f(n), \quad U(n) = (\ln f(n))_t$, equations

(4.2) and (4.3) are transformed into

$$(v_\mu(n+1)+v_\mu(n-1))_t+(v_\mu(n+1)-v_\mu(n-1))(U(n+1)-U(n-1)-\frac{1}{2})=0,$$
$$\sum_{\mu=1}^{M} v_\mu(n)=U(n).$$

Direct calculations tell us that 3-soliton solutions to the coupled semi-discrete equations (4.2) and (4.3) are also in the form of (2.7) and (2.8) but with

$$\exp(\eta_j)=P_j^{n+n_{j,0}}\exp(q_j t),\quad q_j=\frac{P_j^2-1}{2(P_j^2+1)},\quad b_{j,k}=a_{j,k}=\frac{(P_j-P_k)(P_j+P_k)}{(P_jP_k+1)(P_jP_k-1)},$$
$$c_\mu(j,k)=a_{j,k}(c_\mu(j)-c_\mu(k)),\quad c_\mu(1,2,3)=a_{1,2}a_{1,3}a_{2,3}[b_{2,3}c_\mu(1)+b_{3,1}c_\mu(2)+b_{1,2}c_\mu(3)]$$
$$\text{for } j,k=1,2,3 \text{ and } \mu=1,2,\cdots,M,$$

where $P_j, n_{j,0}$ are constants and $c_\mu(j)$ are parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j)=\frac{P_j^2-1}{2(P_j+1)}$ for $j=1,2,3$.

These expressions suggest that the N-soliton solutions expressed by pfaffians to the coupled equations (4.2) and (4.3) may be as follows,

$$f(n)=\mathrm{pf}(d_0,a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N,\beta_0), \tag{4.4}$$
$$g_\mu(n)=\mathrm{pf}(d_0,a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N,\beta_\mu),\quad \text{for } \mu=1,2,\cdots,M, \tag{4.5}$$

and the entries of (4.4) and (4.5) are of the same form as (2.14)-(2.16) but with

$$\exp(\eta_j)=P_j^{n+n_{j,0}}\exp\left(\frac{P_j^2-1}{2(P_j^2+1)}t\right),$$
$$b_{j,k}=a_{j,k}=\frac{(P_j-P_k)(P_j+P_k)}{(P_jP_k-1)(P_jP_k+1)},\quad \delta_{j,k}=\begin{cases}1 & \text{for } j=k,\\ 0 & \text{for } j\neq k,\end{cases}$$

where P_j and $n_{j,0}$ are arbitratry constants and $c_\mu(j)$ are parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j)=\frac{P_j^2-1}{2(P_j^2+1)}$.

Let us now verify that (4.4) and (4.5) are indeed solutions to equation (4.2) and (4.3). Based on the fact that equation (4.3) is similar to equation (2.4), one can prove that $f(n)$ and $g_\mu(n)$ given by (4.4) and (4.5) satisfy (4.3) following the similar procedure in [11]. Now we begin to prove that $f(n)$ and $g_\mu(n)$ satisfy (4.2), which can be rewritten as

$$\frac{\partial g_\mu(n+1)}{\partial t}f(n-1)-g_\mu(n+1)\frac{\partial f(n-1)}{\partial t}+\frac{\partial g_\mu(n-1)}{\partial t}f(n+1)-g_\mu(n-1)\frac{\partial f(n+1)}{\partial t}$$
$$-\frac{1}{2}g_\mu(n+1)f(n-1)+\frac{1}{2}g_\mu(n-1)f(n+1)=0. \tag{4.6}$$

Using a method described in [11], we find the following differential formulae,

$$\frac{\partial}{\partial t}f(n)=-\mathrm{pf}(d_0,d_t,a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N), \tag{4.7}$$
$$\frac{\partial}{\partial t}g_\mu(n)=\mathrm{pf}(d_0,d_t,a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N,\beta_\mu,\beta_0), \tag{4.8}$$

where the entries are defined as follows,

$$\mathrm{pf}(d_0,d_t)=0,\quad \mathrm{pf}(d_t,b_j)=0,\quad \mathrm{pf}(d_t,\beta_0)=0,\quad \mathrm{pf}(d_t,\beta_\mu)=0,\quad \mathrm{pf}(d_t,a_j)=\frac{P_j^2-1}{2(P_j^2+1)}\exp(\eta_j),$$
$$\text{for } j=1,2,\cdots,N.$$

It is noted that (4.6) is equivalent to

$$\frac{\partial g_\mu(n+2)}{\partial t}f(n)-g_\mu(n+2)\frac{\partial f(n)}{\partial t}+\frac{\partial g_\mu(n)}{\partial t}f(n+2)-g_\mu(n)\frac{\partial f(n+2)}{\partial t}$$
$$-\frac{1}{2}g_\mu(n+2)f(n)+\frac{1}{2}g_\mu(n)f(n+2)=0. \tag{4.9}$$

So it suffices to show that $f(n)$ and $g_\mu(n)$ satisfy equation (4.9). Now we introduce a character $\tilde{d}_2$ defined by

$$\mathrm{pf}(\tilde{d}_2,a_j)=(P_j^2-1)\exp(\eta_j),\quad \mathrm{pf}(\tilde{d}_2,b_j)=\mathrm{pf}(\tilde{d}_2,\beta_0)=\mathrm{pf}(\tilde{d}_2,\beta_\mu)=\mathrm{pf}(\tilde{d}_2,d_0)=\mathrm{pf}(\tilde{d}_2,d_t)=0.$$

By doing so and denoting $\{a_1,a_2,\cdots,a_N,b_1,b_2,\cdots,b_N\}=\{\cdots\cdots\}$, we obtain the following difference formulae

$$f(n)=\mathrm{pf}(d_0,\beta_0,\cdots\cdots),\quad f(n+2)=\mathrm{pf}(d_0,\beta_0,\cdots\cdots)-\mathrm{pf}(d_0,\tilde{d}_2,\cdots\cdots), \tag{4.10}$$
$$g_\mu(n)=\mathrm{pf}(d_0,\beta_\mu,\cdots\cdots),\quad g_\mu(n+2)=\mathrm{pf}(d_0,\beta_\mu,\cdots\cdots)+\mathrm{pf}(d_0,\tilde{d}_2,\beta_\mu,\beta_0,\cdots\cdots) \tag{4.11}$$

and the following differential formulae [11],

$$\frac{\partial f(n+2)}{\partial t}=\mathrm{pf}(d_0,d_t,\cdots\cdots)-\frac{1}{2}\mathrm{pf}(d_0,\tilde{d}_2,\cdots\cdots)+\mathrm{pf}(d_0,\tilde{d}_2,d_t,\beta_0,\cdots\cdots), \tag{4.12}$$
$$\frac{\partial g(n+2)}{\partial t}=-\mathrm{pf}(d_0,d_t,\beta_\mu,\beta_0,\cdots\cdots)+\frac{1}{2}\mathrm{pf}(d_0,\tilde{d}_2,\beta_\mu,\beta_0,\cdots\cdots)+\mathrm{pf}(d_0,\tilde{d}_2,d_t,\beta_\mu,\cdots \tag{4.13}$$

Substituting these relations (4.7)-(4.8) and (4.10)-(4.13) into eq.(4.9), we find that this bilinear equation is reduced to the pfaffian identity [11, 14],

$$\mathrm{pf}(\beta_0,d_0,\cdots\cdots)\mathrm{pf}(\beta_\mu,\tilde{d}_2,d_t,d_0,\cdots\cdots)-\mathrm{pf}(\beta_\mu,d_0,\cdots\cdots)\mathrm{pf}(\beta_0,\tilde{d}_2,d_t,d_0,\cdots\cdots)$$
$$+\mathrm{pf}(\tilde{d}_2,d_0,\cdots\cdots)\mathrm{pf}(\beta_0,\beta_\mu,d_t,d_0,\cdots\cdots)\quad -\mathrm{pf}(d_t,d_0,\cdots\cdots)\mathrm{pf}(\beta_0,\beta_\mu,\tilde{d}_2,d_0,\cdots\cdots)=0.$$

Therefore, $f(n)$ and $g_\mu(n)$ satisfy the bilinear equation (3.3). It is noted that we introduced a character d_t acting as the normal differential operator to deduce the difference and differential formulae. In this aspect, it is different from the other three coupled systems.

In what follows, we will present a bilinear Bäcklund transformation for Eqs. (4.2) and (4.3). In fact, we have the following Bäcklund transformation

$$(e^{D_n}-\lambda^{-1}e^{-D_n})(g_\mu\cdot f'-f\cdot g'_\mu)+(\lambda\omega_\mu e^{D_n}-\omega_\mu e^{-D_n})f\cdot f'=0, \tag{4.14}$$
$$(D_te^{-D_n}+\lambda D_te^{D_n}-(\lambda-\lambda\gamma)e^{D_n}+\gamma e^{-D_n})f\cdot f'=0, \tag{4.15}$$
$$(D_te^{-D_n}+\lambda D_te^{D_n}-(\lambda-\lambda\gamma)e^{D_n}+\gamma e^{-D_n})(g_\mu\cdot f'+f\cdot g'_\mu)=0, \tag{4.16}$$
$$\mu=1,2,\cdots,M$$

between equations (4.2),(4.3) and

$$(D_t\cosh(D_n)-\frac{1}{2}\sinh(D_n))g'_\mu(n)\cdot f'(n)=0,\quad \text{for } \mu=1,2,\cdots,M, \tag{4.17}$$
$$\sum_{\mu=1}^{M}g'_\mu(n)=f'_t(n) \tag{4.18}$$

where we have assumed that

$$g_1(n) + g_2(n) + \cdots + g_M(n) = f_t(n), \quad g'_1(n) + g'_2(n) + \cdots + g'_M(n) = f'_t(n) \tag{4.19}$$

such that Eqs. (4.3) and (4.18) are satisfied automatically and $\lambda, \gamma, \omega_\mu$ $(\mu = 1, 2, \cdots, M)$ are arbitrary constants. In fact, in order to show (4.14)-(4.16) constitute a BT, it suffices to prove that

$$\begin{aligned} P_\mu \equiv 2\{&[D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n)]g_\mu(n) \cdot f(n)][e^{D_n} f'(n) \cdot f'(n)] \\ &-2[e^{D_n} f(n) \cdot f(n)][D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n)]g'_\mu(n) \cdot f'(n)\} = 0. \end{aligned}$$

For this purpose, we have, by using (4.14)-(4.16) and (B7)-(B9),

$$\begin{aligned} P_\mu &= \frac{1}{2}D_t\{[e^{D_n}(g_\mu \cdot f' - f \cdot g'_\mu)] \cdot (e^{-D_n} f \cdot f') - (e^{D_n} f \cdot f') \cdot [e^{-D_n}(g_\mu \cdot f' - f \cdot g'_\mu)]\} \\ &\quad +\frac{1}{2}[D_t e^{D_n}(g_\mu \cdot f' + f \cdot g'_\mu)](e^{-D_n} f \cdot f') - \frac{1}{2}(D_t e^{D_n} f \cdot f')[e^{-D_n}(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &\quad -\frac{1}{2}[e^{D_n}(g_\mu \cdot f' + f \cdot g'_\mu)](D_t e^{-D_n} f \cdot f') + \frac{1}{2}(e^{D_n} f \cdot f')[D_t e^{-D_n}(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &\quad -\frac{1}{2}[e^{D_n}(g_\mu \cdot f' + f \cdot g'_\mu)](e^{-D_n} f \cdot f') + \frac{1}{2}(e^{D_n} f \cdot f')[e^{-D_n}(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &= -\frac{1}{2}\omega_\mu^{-1} D_t[(e^{D_n} - \lambda^{-1}e^{-D_n})(g_\mu \cdot f' - f \cdot g'_\mu)] \cdot [(\lambda\omega_\mu e^{D_n} - \omega_\mu e^{-D_n}) f \cdot f'] \\ &\quad +[(D_t e^{-D_n} + \lambda D_t e^{D_n})(g_\mu \cdot f' + f \cdot g'_\mu)][(\frac{1}{2}e^{D_n} + \frac{1}{2}\lambda^{-1}e^{-D_n}) f \cdot f'] \\ &\quad -[(D_t e^{-D_n} + \lambda D_t e^{D_n}) f \cdot f'][(\frac{1}{2}e^{D_n} + \frac{1}{2}\lambda^{-1}e^{-D_n})(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &\quad -\frac{1}{2}[e^{D_n}(g_\mu \cdot f' + f \cdot g'_\mu)](e^{-D_n} f \cdot f') + \frac{1}{2}(e^{D_n} f \cdot f')[e^{-D_n}(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &= [((\lambda - \lambda\gamma)e^{D_n} - \gamma e^{-D_n})(g_\mu \cdot f' + f \cdot g'_\mu)][(\frac{1}{2}e^{D_n} + \frac{1}{2}\lambda^{-1}e^{-D_n}) f \cdot f'] \\ &\quad +[-(\lambda - \lambda\gamma)e^{D_n} + \gamma e^{-D_n}) f \cdot f'][(\frac{1}{2}e^{D_n} + \frac{1}{2}\lambda^{-1}e^{-D_n})(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &\quad -\frac{1}{2}[e^{D_n}(g_\mu \cdot f' + f \cdot g'_\mu)](e^{-D_n} f \cdot f') + \frac{1}{2}(e^{D_n} f \cdot f')[e^{-D_n}(g_\mu \cdot f' + f \cdot g'_\mu)] \\ &= 0. \end{aligned}$$

Therefore, we have proved that (4.14)-(4.16) constitute a bilinear BT between equations (4.2),(4.3) and (4.17),(4.18) under the assumption (4.19).

5 An integrable multi-component generalization of equation (1.4), its soliton solutions and Bäcklund transformation

It is noted that equation (1.4) can be derived by reduction from the discrete BKP equation

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_4 \exp(D_4)] f \cdot f = 0, \tag{5.1}$$

where D_1, D_2, D_3, D_4 and z_1, z_2, z_3, z_4 are bilinear operators and constants, respectively, satisfying

$$D_1 + D_2 + D_3 + D_4 = 0, \quad z_1 + z_2 + z_3 + z_4 = 0.$$

In fact, if we choose

$$
\begin{aligned}
&z_1 = \frac{1}{\delta} - 1,\ z_2 = \frac{1}{\delta},\ z_3 = -\frac{1}{\delta},\ z_4 = 1 - \frac{1}{\delta},\\
&D_1 = D_m + D_n + \delta D_t,\ D_2 = D_n - D_m - \delta D_t,\\
&D_3 = -D_m - D_n + \delta D_t,\ D_4 = D_m - D_n - \delta D_t,
\end{aligned}
$$

the discrete BKP equation (5.1) can be rewritten as

$$
\sinh(D_m)\left[\frac{1}{\delta}\sinh(\delta D_t)\cosh(D_n) - \frac{1}{2}\sinh(D_n + \delta D_t)\right] f \cdot f = 0, \tag{5.2}
$$

from which we can deduce equation (1.4) by choosing $D_m = \frac{1}{2}D_n$ and taking the limit as $\delta \to 0$. Therefore, equation (1.4) is an integrable differential-difference equation. In this section, we will give an integrable multi-component generalization of equation (1.4).

Based on the fact that equation (1.4) can be rewritten as

$$
(D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n))g(n) \cdot f(n) = 0, \quad g(n) = f(n+1) - f(n),
$$

we now propose a natural coupled form

$$
(D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n))g_\mu(n) \cdot f(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{5.3}
$$

$$
\sum_{\mu=1}^{M} g_\mu(n) = f(n+1) - f(n). \tag{5.4}
$$

By the dependent variable transformation $v_\mu(n) = g_\mu(n)/f(n), \quad u(n) = \ln(f(n+1)/f(n))$, equations (5.3) and (5.4) are transformed into

$$
\begin{aligned}
&(v_\mu(n+1) + v_\mu(n-1))_t + (v_\mu(n+1) - v_\mu(n-1))(u_t(n) + u_t(n-1) - \frac{1}{2}) = 0,\\
&\sum_{\mu=1}^{M} v_\mu(n) = e^{u(n)} - 1.
\end{aligned}
$$

In what follows, we shall firstly derive the N-soliton solutions to the coupled differential-difference system (5.3) and (5.4). Using the perturbational method, we find that the 3-soliton solutions to the above equations (5.3) and (5.4) are also in the same form of (2.7) and (2.8) but with

$$
\begin{aligned}
&\exp(\eta_j) = P_j^{n+n_{j,0}} \exp(q_j t), \quad q_j = \frac{P_j^2 - 1}{2(P_j^2 + 1)}, \quad a_{j,k} = \frac{(P_j - P_k)(P_j + P_k)}{(P_j P_k - 1)(P_j P_k + 1)}, \quad b_{j,k} = \frac{P_j - P_k}{P_j P_k - 1},\\
&c_\mu(j,k) = a_{j,k}(c_\mu(j) - c_\mu(k)), \quad c_\mu(1,2,3) = a_{1,2}a_{1,3}a_{2,3}[b_{2,3}c_\mu(1) + b_{3,1}c_\mu(2) + b_{1,2}c_\mu],\\
&\qquad \text{for } j, k = 1, 2, 3 \text{ and } \mu = 1, 2, \cdots, M,
\end{aligned}
$$

where $P_j, n_{j,0}$, are constants and $c_\mu(j)$ are free parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j) = P_j - 1$ for $j = 1, 2, 3$. Similar to the previous three coupled equations, the N-soliton solutions expressed by pfaffians to the equations (5.3) and (5.4) may also take the following form

$$
f(n) = \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_0), \tag{5.5}
$$

$$
g_\mu(n) = \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_\mu), \quad \text{for } \mu = 1, 2, \cdots, M, \tag{5.6}
$$

and the entries of (5.5) and (5.6) are defined in the same form as (2.14)-(2.16) but with

$$\exp(\eta_j) = P_j^{n+n_{j,0}} \exp\left(\frac{P_j^2-1}{2(P_j^2+1)}t\right), \quad \delta_{j,k} = \begin{cases} 1 & \text{for } j=k, \\ 0 & \text{for } j \neq k, \end{cases}$$
$$a_{j,k} = \frac{(P_j-P_k)(P_j+P_k)}{(P_jP_k-1)(P_jP_k+1)}, \quad b_{j,k} = \frac{P_j-P_k}{P_jP_k-1},$$

here P_j, $n_{j,0}$ are constants and $c_\mu(j)$ are free parameters satisfying $\sum_{\mu=1}^{M} c_\mu(j) = P_j - 1$.

Noticing that the bilinear equation (5.3) is the same as equation (4.2), so we can prove that (5.3) holds following the similar procedure in section 4. Now we turn to verify equation (5.4). It should be pointed out that due to the coefficient difference in the terms $\mathrm{pf}(b_j, b_k)$ between (3.5,3.6) and (5.5,5.6), we can not prove that $f(n)$ and $g_\mu(n)$ given by (5.5) and (5.6) satisfy (5.4) just following the procedure in section 3, although equation (5.4) is the same as (3.4) in the form. Instead, we will follow the similar procedure for the vector Ito equation given in [11] to verify equation (5.4) by replacing differential operator with difference operator. The first step is to rewrite $f(n)$ as

$$f(n) = f_0(n) + f'(n),$$
$$f_0(n) = \mathrm{pf}(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \quad f'(n) = \mathrm{pf}(d_0, \beta_0', a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N) \quad (5.7)$$

where β_0' is a new character defined by

$$\mathrm{pf}(d_0, \beta_0') = 0, \quad (\mathrm{pf}(d_0, \beta_0) = 1), \quad (5.8)$$
$$\mathrm{pf}(a_j, \beta_0') = 0, \quad \mathrm{pf}(b_j, \beta_0') = 1, \quad \text{for } j = 1, 2, \cdots, N. \quad (5.9)$$

It is known that $f_0(n)$ and $f'(n)$ are invariant under the following transformations

$$a_j \to a_j' (= a_j \exp(-\eta_j)), \quad b_j \to b_j' (= b_j \exp(\eta_j)),$$

So,

$$f_0(n) = \mathrm{pf}(a_1', a_2', \ldots, a_N', b_1', b_2', \ldots, b_N'), \quad f'(n) = \mathrm{pf}(d_0, \beta_0', a_1', a_2', \ldots, a_N', b_1', b_2', \ldots, b_N'),$$

where the entries are defined by,

$$\mathrm{pf}(a_j', a_k') = -a_{j,k}, \quad \mathrm{pf}(a_j', b_k') = \delta_{j,k}, \quad \mathrm{pf}(b_j', b_k') = b_{j,k} \exp(\eta_j + \eta_k), \quad \mathrm{pf}(\beta_0', a_j') = 0,$$
$$\mathrm{pf}(d_0, \beta_0') = 0, \quad \mathrm{pf}(d_0, a_j') = 1, \quad \mathrm{pf}(d_0, b_j') = -\exp(\eta_j), \quad \mathrm{pf}(\beta_0', b_j') = -\exp(\eta_j).$$

Now we introduce the second new character d_1' defined by

$$\mathrm{pf}(d_1', \beta_0') = 0, \quad \mathrm{pf}(d_1', b_j') = (P_j - 1)\exp(\eta_j), \quad \mathrm{pf}(d_1', d_0) = 0, \quad \mathrm{pf}(d_1', a_j') = 0,$$
$$\text{for} j = 1, 2, \cdots, N,$$

such that

$$\mathrm{pf}(a_j', a_k')_{n+1} = \mathrm{pf}(a_j', a_k')_n + \mathrm{pf}(d_1', \beta_0', a_j', a_k')_n,$$
$$\mathrm{pf}(a_j', b_k')_{n+1} = \mathrm{pf}(a_j, b_k')_n + \mathrm{pf}(d_1', \beta_0', a_j', b_k')_n,$$
$$\mathrm{pf}(b_j', b_k')_{n+1} = \mathrm{pf}(b_j', b_k')_n + \mathrm{pf}(d_1', \beta_0', b_j', b_k')_n,$$
$$\text{for } j, k = 1, 2, \cdots, N,$$

from which it follows by induction that

$$\begin{aligned}&\mathrm{pf}(a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_{n+1}\\ &\quad = \mathrm{pf}(a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_n + \mathrm{pf}(d'_1, \beta'_0, , a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_n, \end{aligned} \tag{5.10}$$

i.e.,

$$f_0(n+1) - f_0(n) = \mathrm{pf}(d'_1, \beta'_0, a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N), \tag{5.11}$$

where we have denoted $\mathrm{pf}(a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)$ to be $\mathrm{pf}(a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_n$ and so on.

Now let us introduce another character $d'_0 (= d_0 - \beta'_0)$ defined by

$$\begin{aligned}&\mathrm{pf}(d'_0, a'_j) = \mathrm{pf}(d_0, a'_j) - \mathrm{pf}(\beta'_0, a'_j) = 1,\\ &\mathrm{pf}(d'_0, b'_j) = \mathrm{pf}(d_0, b'_j) - \mathrm{pf}(\beta'_0, b'_j) = 0,\\ &\mathrm{pf}(d'_0, d'_1) = \mathrm{pf}(d_0, d'_1) - \mathrm{pf}(\beta'_0, d'_1) = 0,\\ &\qquad \text{for } j = 1, 2, \cdots, N,\end{aligned}$$

such that

$$\begin{aligned}&\mathrm{pf}(d_0, \beta'_0, a'_j, a'_k)_{n+1} = \mathrm{pf}(d_0, \beta'_0, a'_j, a'_k)_n + \mathrm{pf}(d'_1, d'_0, a'_j, a'_k)_n,\\ &\mathrm{pf}(d_0, \beta'_0, a'_j, b'_k)_{n+1} = \mathrm{pf}(d_0, \beta'_0, a'_j, b'_k)_n + \mathrm{pf}(d'_1, d'_0, a'_j, b'_k)_n,\\ &\mathrm{pf}(d_0, \beta'_0, b'_j, b'_k)_{n+1} = \mathrm{pf}(d_0, \beta'_0, b'_j, b'_k)_n + \mathrm{pf}(d'_1, d'_0, b'_j, b'_k)_n,\\ &\qquad \text{for} j, k = 1, 2, \cdots, N.\end{aligned}$$

Then by induction, we obtain

$$\begin{aligned}&\mathrm{pf}(d_0, \beta'_0, a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_{n+1}\\ &\quad = \mathrm{pf}(d_0, \beta'_0, a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_n + \mathrm{pf}(d'_1, d'_0, , a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N)_n,\end{aligned} \tag{5.12}$$

i.e.,

$$f'(n+1) - f'(n) = \mathrm{pf}(d'_1, d'_0, a'_1, a'_2, \cdots, a'_N, b'_1, b'_2, \cdots, b'_N). \tag{5.13}$$

Accordingly we obtain

$$\begin{aligned} f(n+1) - f(n) &= f_0(n+1) + f'(n+1) - (f_0(n) + f'(n))\\ &= \mathrm{pf}(d'_1, d_0, a'_1, a'_2, \ldots, a'_N, b'_1, \cdots, b'_N). \end{aligned} \tag{5.14}$$

Now we expand eq.(5.14) with respect to the first character d'_1,

$$\begin{aligned} f(n+1) - f(n) &= \sum_{j=1}^{N} (P_j - 1)(-1)^{N+j} \exp(\eta_j) \mathrm{pf}(d_0, a'_1, a'_2, \cdots, a'_N, b'_1, \cdots, \hat{b}'_j, \cdots, b'_N)\\ &= \sum_{j=1}^{N} (P_j - 1)(-1)^{N+j} \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, \hat{b}_j, \cdots, b_N), \end{aligned} \tag{5.15}$$

where the denotes $\hat{\alpha}$ indicates the letter α is missing. On the other hand, we expand $g_\mu(n)$ with respect to the final character β_μ, and obtain

$$g_\mu(n) = \mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, \beta_\mu), \tag{5.16}$$

$$= \sum_{j=1}^{N} c_\mu(j)(-1)^{N+j}\mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, \hat{b}_j, \cdots, b_N). \tag{5.17}$$

The sum of $g_\mu(n)$ over μ gives

$$\sum_{\mu=1}^{M} g_\mu(n) = \sum_{j=1}^{N}(-1)^{N+j}(P_j - 1)\mathrm{pf}(d_0, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, \hat{b}_j, \cdots, b_N). \tag{5.18}$$

Therefore, $f(n)$ and $g_\mu(n)$ satisfy the linear equation

$$\sum_{\mu=1}^{M} g_\mu = f(n+1) - f(n). \tag{5.19}$$

Based on the fact that (5.3) has the same form as (4.2), we can easily work out the following BT

$$(e^{D_n} - \lambda^{-1}e^{-D_n})(g_\mu \cdot f' - f \cdot g'_\mu) + (\lambda\omega_\mu e^{D_n} - \omega_\mu e^{-D_n})f \cdot f' = 0, \tag{5.20}$$

$$(D_t e^{-D_n} + \lambda D_t e^{D_n} - (\lambda - \lambda\gamma)e^{D_n} + \gamma e^{-D_n})f \cdot f' = 0, \tag{5.21}$$

$$(D_t e^{-D_n} + \lambda D_t e^{D_n} - (\lambda - \lambda\gamma)e^{D_n} + \gamma e^{-D_n})(g_\mu \cdot f' + f \cdot g'_\mu) = 0, \tag{5.22}$$

$$\mu = 1, 2, \cdots, M$$

between equations (5.3),(5.4) and

$$(D_t \cosh(D_n) - \frac{1}{2}\sinh(D_n))g'_\mu(n) \cdot f'(n) = 0, \quad \text{for } \mu = 1, 2, \cdots, M, \tag{5.23}$$

$$\sum_{\mu=1}^{M} g'_\mu(n) = f'(n+1) - f'(n), \tag{5.24}$$

where we have assumed that

$$g_1(n)+g_2(n)+\cdots+g_M(n) = f(n+1)-f(n), \quad g'_1(n)+g'_2(n)+\cdots+g'_M(n) = f'(n+1)-f'(n) \tag{5.25}$$

such that Eqs. (5.4) and (5.24) are satisfied automatically, and $\lambda, \gamma, \omega_\mu$ ($\mu = 1, 2, \cdots, M$) are arbitrary constants.

6 Conclusion and discussions

In this paper, Hirota's bilinear formalism has been utilized to generate integrable multi-component generalizations of the so-called 1+1 dimensional special Toda lattice, the Volterra lattice, a simple differential-difference equation found by Adler, Moser, Weiss, Veselov and Shabat and another integrable lattice deduced from the discrete BKP equation. Their soliton solutions expressed by pfaffians and the corresponding bilinear Bäcklund transformations are obtained. It is noted that these pfaffian solution can also be derived by means of the corresponding Bäcklund transformations. For example, consider the so-called 1+1 dimensional special Toda lattice. By applying the Bäcklund transformations (2.26)-(2.28) to $f(n) = \mathrm{pf}(d_0, a_1, b_1, \beta_0) = 1 + \exp(\eta_1),\ g_\mu(n) = \mathrm{pf}(d_0, a_1, b_1, \beta_\mu) = c_\mu(1)\exp(\eta_1)$, we can obtain

$$f'(n) = 1 - \exp(\eta_1)/b_{12} + \exp(\eta_2) - a_{12}\exp(\eta_1 + \eta_2), \tag{6.1}$$
$$g'_\mu(n) = -c_\mu(1)\exp(\eta_1)/b_{12} + c_\mu(2)\exp(\eta_2) - c_\mu(12)\exp(\eta_1 + \eta_2)/b_{12}, \tag{6.2}$$

for the parameters $\lambda = 1,\ \gamma = 0,\ k_\mu = -c_\mu(2) = \frac{b_{12}-1}{b_{12}+1}c_\mu(1)$; or

$$f'(n) = 1 - \exp(\eta_1)/a_{12} + \exp(\eta_2) - b_{12}\exp(\eta_1 + \eta_2), \tag{6.3}$$
$$g'_\mu(n) = -c_\mu(1)\exp(\eta_1)/a_{12} + c_\mu(2)\exp(\eta_2) - c_\mu(12)\exp(\eta_1 + \eta_2)/a_{12}, \tag{6.4}$$

for the parameters $\lambda = \frac{1}{P_2},\ \gamma = \frac{P_2^2-1}{P_2}, k_\mu = 0$. Obviously, by simple transformations, solutions (6.1)-(6.2) and (6.3)-(6.4) are respectively equivalent to

$$f'(n) = 1 + \exp(\eta_1) + \exp(\eta_2) + a_{12}b_{12}\exp(\eta_1 + \eta_2) = \mathrm{pf}(d_0, a_1, a_2, b_1, b_2, \beta_0), \tag{6.5}$$
$$g'_\mu(n) = c_\mu(1)\exp(\eta_1) + c_\mu(2)\exp(\eta_2) + c_\mu(12)\exp(\eta_1 + \eta_2) = \mathrm{pf}(d_0, a_1, a_2, b_1, b_2, \beta_\mu), \tag{6.6}$$

where $a_{12},\ b_{12},\ c_\mu(1),\ c_\mu(2),\ c_\mu(12)$ and $\exp(\eta_j),\ (j = 1, 2)$, are given by (2.9)-(2.11). Furthermore, starting from the bilinear Bäcklund transformations obtained in the paper, we may derive the corresponding Lax pairs for the multi-component systems under consideration. Again we take the Bäcklund transformation (2.26)-(2.28) as an example, by setting $f(n) = \phi(n)f'(n),\ g_\mu(n) = \psi_\mu(n)f'(n) + \phi(n)g'_\mu(n),\ u(n) = \ln\frac{f'(n+1)}{f'(n)},\ v_\mu(n) = \frac{g'_\mu(n+1)}{f'(n+1)} - \frac{g'_\mu(n)}{f'(n)}$, we derive from (2.26)-(2.28) a Lax pair for (2.5)-(2.6):

$$\begin{pmatrix} \phi(n) \\ \psi_\mu(n) \end{pmatrix}_t = U \begin{pmatrix} \phi(n) \\ \psi_\mu(n) \end{pmatrix}, \qquad \Delta_+ \begin{pmatrix} \phi(n) \\ \psi_\mu(n) \end{pmatrix} = V \begin{pmatrix} \phi(n) \\ \psi_\mu(n) \end{pmatrix},$$
$$U = \begin{pmatrix} e^{u(n)-u(n-1)}(E_+^2 - 1)E_- - \frac{1}{2}\gamma & 0 \\ e^{u(n)-u(n-1)}[(k_\mu - v_\mu(n-1))E_+ - (k_\mu + v_\mu(n))E_- - (v_\mu(n) + v_\mu(n-1))] & \frac{1}{2}\gamma \end{pmatrix}$$
$$V = \frac{1}{2}\begin{pmatrix} \Delta_+ & 0 \\ k_\mu\Delta_+ - v_\mu(n)(E_+ + 1) & 0 \end{pmatrix},$$

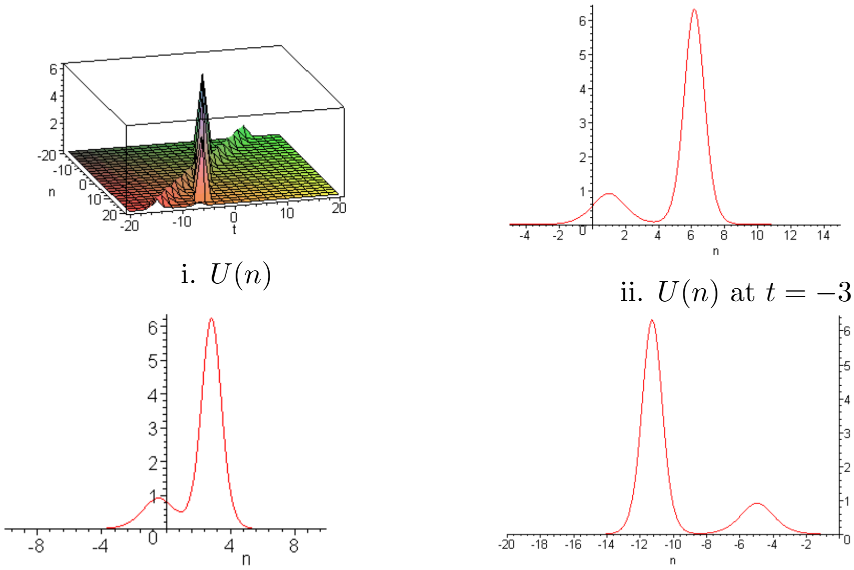
where $E_\pm f(n) = f(n\pm 1)$, $\Delta_\pm = E_\pm - 1$. In fact, we have checked that the compatibility condition of above spectral problem yields (2.5)-(2.6). Besides, based on the fact that the multi-component systems (2.3)-(2.4),(3.3)-(3.4),(4.2)-(4.3) and (5.3)-(5.4) consist of one linear equation and M bilinear equations with respect to f and g_μ respectively, it would be interesting to explain the meaning of these variables g_μ and reasons for investigation of linear equations for them. Actually, linear equations (2.4) and (4.3) may be viewed as a decomposition of f_t into sum of g_μ, while (3.4) and (5.4) may be thought of as a decomposition of $f(n+1) - f(n)$ into sum of g_μ. Bilinear equations (2.3) or (3.3) or (4.2) or (5.3) satisfied by f and g_μ may be respectively considered as components of bilinear equations (1.1) or (1.2) or (1.3) or (1.4) under such decompositions. Finally, it is remarked that the bilinear procedure to generate integrable multi-component generalizations of soliton equations may be utilized to derive integrable coupled versions for some fully discrete integrable equations, say the discrete KdV equation etc.. The work in this direction is in progress. Moreover, it would be of interest to investigate whether the Pfaffian forms of solutions obtained in the paper have soliton-like behaviours and how to clarify the asymptotics. These remain to be done further. Here we just give some plots for the physical quantities, corresponding to the 2-soliton solution of (2.5)-(2.6):

$$U(n) = \frac{\partial}{\partial t}\ln\frac{f(n+1)}{f(n)}, \quad v_\mu(n) = \frac{g_\mu(n+1)}{f(n+1)} - \frac{g_\mu(n)}{f(n)}, \quad \mu = 1, 2, \tag{6.7}$$

with $f(n)$, $g_\mu(n)$ given by (6.5)-(6.6), which are shown in Fig.1 and 2 where we choose $M = 2$, $n_{i,0} = 0$, $(i = 1, 2)$, $P_1 = 0.2$, $P_2 = 0.05$, $c_1(1) = -1.0$, $c_1(2) = -4.6432$, $c_2(1) = -1.4$, $c_2(2) = -4.3443$.

i. $v_1(n)$ ii. $v_2(n)$

Figure 1: The plots of $v_j(n)$, $j = 1, 2$



i. $U(n)$ ii. $U(n)$ at $t = -3$

iii. $U(n)$ at $t = -1$ iv. $U(n)$ at $t = 3$

Figure 2: The plots of $U(n)$

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Appendix A. Properties of Pfaffian

A1. Definition

For the sake of self-containedness, we review some properties of Pfaffian. Pfaffians are antisymmetric functions with respect to their characters,

$$\mathrm{pf}(a,b) = -\mathrm{pf}(b,a), \quad \text{for any } a \text{ and } b.$$

A 2n-th degree pfaffian can be defined by the following expansion rule,

$$\mathrm{pf}(1,2,\cdots,2n) = \sum_{j=2}^{2n}(-1)^j \mathrm{pf}(1,j)\mathrm{pf}(2,3,\cdots,\hat{j},\cdots,2n), \tag{A1}$$

where $\hat{\alpha}$ denotes the missing of the letter α. For example, if $n=2$, we have

$$\mathrm{pf}(1,2,3,4) = \mathrm{pf}(1,2)\mathrm{pf}(3,4) - \mathrm{pf}(1,3)\mathrm{pf}(2,4) + \mathrm{pf}(1,4)\mathrm{pf}(2,3).$$

It is noted that Pfaffians are closely related to determinants. One interesting fact is that a determinant of n-degree,

$$B = \det|b_{j,k}|_{1\le j,k\le n}$$

can be expressed by means of a pfaffian of $2n$-th degree

$$\det|b_{j,k}|_{1\le j,k\le n} = \mathrm{pf}(1,2,\cdots,n,n^*,\cdots,2^*,1^*),$$

whose entries are defined by

$$\mathrm{pf}(j,k) = 0, \quad \mathrm{pf}(j^*,k^*) = 0, \quad \mathrm{pf}(j,k^*) = b_{j,k}.$$

A2. Pfaffian identities

From the definition of Pfaffian, we know that its properties are closely related to those of determinants. One of these properties is that Pfaffian satisfies various kinds of Pfaffian identities. Here we introduce some fundamental identities in which bilinear soliton equations often result. Based on (A1), the following pfaffian identities are obtained [25]

$$\begin{aligned}&\mathrm{pf}(a_1,a_2,\cdots,a_{2m},1,2,\cdots,2n)\mathrm{pf}(1,2,\cdots,2n)\\ &\quad= \sum_{j=2}^{2m}(-1)^j \mathrm{pf}(a_1,a_j,1,2,\cdots,2n)\mathrm{pf}(a_2,\cdots,\hat{a}_j,\cdots,a_{2m},1,2,\cdots,2n)\end{aligned} \tag{A2}$$

$$\begin{aligned}&\mathrm{pf}(a_1,a_2,\cdots,a_{2m-1},1,2,\cdots,2n-1)\mathrm{pf}(1,2,\cdots,2n-1)\\ &= \sum_{j=1}^{2m-1}(-1)^{j-1}\mathrm{pf}(a_j,1,2,\cdots,2n-1)\mathrm{pf}(a_1,\cdots,\hat{a}_j,\cdots,a_{2m-1},1,2,\cdots,2n-1)\end{aligned} \tag{A3}$$

with $\mathrm{pf}(a_j,a_k)=0$, $(j,k=1,2,\cdots 2m)$. For example, in the case of $m=2$, (A2) and (A3) are respectively rewritten as

$$\begin{aligned}&(a_1,a_2,a_3,a_4,1,\cdots,2n)(1,\cdots,2n) = (a_1,a_2,1,\cdots,2n)(a_3,a_4,1,\cdots,2n)\\ &\quad-(a_1,a_3,1,\cdots,2n)(a_2,a_4,1,\cdots,2n) + (a_1,a_4,1,\cdots,2n)(a_2,a_3,1,\cdots,2n),\end{aligned} \tag{A4}$$

$$\begin{aligned}&(a_1,a_2,a_3,1,\cdots,2n-1)(1,\cdots,2n-1) = (a_1,1,\cdots,2n-1)(a_2,a_3,1,\cdots,2n-1)\\ &\quad-(a_2,1,\cdots,2n-1)(a_1,a_3,1,\cdots,2n-1) + (a_3,1,\cdots,2n-1)(a_1,a_2,1,\cdots,2n-1)\end{aligned} \tag{A5}$$

where $(a_i,a_j)=0$, $(i,j=1,2,3,4)$.

Appendix B. Hirota bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions a, b, c and d.

$$(D_t a \cdot b)c^2 - (D_t d \cdot c)b^2 = -(D_t b \cdot c)(ac + bd) + bcD_t(a \cdot c + b \cdot d). \tag{B1}$$

$$[\sinh(D_n)a \cdot b]c^2 = \sinh(\frac{1}{2}D_n)[(e^{\frac{1}{2}D_n}a \cdot c) \cdot (e^{-\frac{1}{2}D_n}b \cdot c) - (e^{\frac{1}{2}D_n}b \cdot c) \cdot (e^{-\frac{1}{2}D_n}a \cdot c)]. \tag{B2}$$

$$\sinh(\frac{\delta}{2}D_n)(e^{\frac{\delta}{2}D_n}a \cdot b) \cdot (e^{\frac{\delta}{2}D_n}c \cdot d) = \frac{1}{2}(e^{\delta D_n}a \cdot d)cb - \frac{1}{2}ad(e^{\delta D_n}c \cdot b). \tag{B3}$$

$$\sinh(\frac{1}{2}D_n)a \cdot a = 0. \tag{B4}$$

$$(D_t a \cdot b)cd - abD_t c \cdot d = (D_t a \cdot c)bd - acD_t b \cdot d. \tag{B5}$$

$$\begin{aligned}&[\sinh(D_n)a \cdot b]cd - ab\sinh(D_n)c \cdot d \\ &\quad = \sinh(\frac{1}{2}D_n)[(e^{\frac{1}{2}D_n}a \cdot d) \cdot (e^{-\frac{1}{2}D_n}b \cdot c) - (e^{\frac{1}{2}D_n}b \cdot c) \cdot (e^{-\frac{1}{2}D_n}a \cdot d)].\end{aligned} \tag{B6}$$

$$\begin{aligned}&2[D_t\cosh(D_n)a \cdot b][e^{D_n}c \cdot c] \\ &\quad = \frac{1}{2}D_t[e^{D_n}a \cdot c] \cdot [e^{-D_n}b \cdot c] + \frac{1}{2}[D_t e^{D_n}a \cdot c][e^{-D_n}b \cdot c] - \frac{1}{2}[e^{D_n}a \cdot c][D_t e^{-D_n}b \cdot c] \\ &\qquad + \frac{1}{2}D_t[e^{-D_n}a \cdot c] \cdot [e^{D_n}b \cdot c] + \frac{1}{2}[D_t e^{-D_n}a \cdot c][e^{D_n}b \cdot c] - \frac{1}{2}[e^{-D_n}a \cdot c][D_t e^{D_n}b \cdot c].\end{aligned} \tag{B7}$$

$$\begin{aligned}&2[\sinh(D_n)a \cdot b][e^{D_n}c \cdot c] - 2[\sinh(D_n)d \cdot c][e^{D_n}b \cdot b] \\ &\quad = [e^{D_n}(a \cdot c + b \cdot d)](e^{-D_n}b \cdot c) - (e^{D_n}b \cdot c)[e^{-D_n}(a \cdot c + b \cdot d)].\end{aligned} \tag{B8}$$

$$\begin{aligned}&D_t[e^{\delta D_n}(a \cdot c - b \cdot d)] \cdot [e^{\delta D_n}b \cdot c] \\ &\quad -[D_t e^{\delta D_n}(a \cdot c + b \cdot d)][e^{\delta D_n}b \cdot c] + [D_t e^{\delta D_n}b \cdot c][e^{\delta D_n}(a \cdot c + b \cdot d)] = 0.\end{aligned} \tag{B9}$$

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