Expansions of Step-transition Operators of Multi-step Methods and Order Barriers for Dahlquist Pairs[1]

Quan-Dong Feng[†‡] & Yi-Fa Tang[†2]

[†]LSEC, ICMSEC
Academy of Mathematics & Systems Science
Chinese Academy of Sciences
P.O. Box 2719, Beijing 100080, P.R. China

[‡]Graduate School of the Chinese Academy of Sciences
Beijing 100080, P.R. China

ABSTRACT

*Using least parameters, we expand the step-transition operator of any linear multi-step method (***LMSM***) up to $O(\tau^{s+5})$ with order $s = 1$ and rewrite the expansion of the step-transition operator for $s = 2$ (obtained by the second author in a former paper). We prove that in the Dahlquist relation $G_3^{\lambda\tau} \circ G_1^{\tau} = G_2^{\tau} \circ G_3^{\lambda\tau}$ with G_1 being an* **LMSM***, (1) the order of G_2 can not be higher than that of G_1; (2) if G_3 is also an* **LMSM** *and G_2 is a symplectic B-series, then the order of both G_1 and G_2 must be 2.*

Keywords: *Linear Multi-Step Method; Step-Transition Operator; B-series; Dahlquist (Conjugate) Relation; Symplecticity*

1. Introduction

For an ordinarily differential equation (**ODE**)

$$\frac{d}{dt}Z = f(Z), \quad Z \in R^p, \tag{1}$$

any compatible linear m-step difference scheme (**DS**)

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k f(Z_k) \quad \left(\sum_{k=0}^{m} \beta_k \neq 0 \right) \tag{2}$$

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[2]Corresponding author. E-mail: tyf@lsec.cc.ac.cn

can be characterized by a step-transition operator (**STO**) G (also denoted by G^τ): $R^p \to R^p$ satisfying

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k f \circ G^k, \tag{3}$$

where G^k stands for k-time composition of G: $G \circ G \cdots \circ G$ (refer to [2,3,5,6,7]). This operator G^τ can be represented as a power series in τ with first term equal to *identity* I. More precisely, one can expand[9] the **STO** $G^\tau(Z)$ of any linear multi-step method (**LMSM**)[3] of form (2) with order $s \geq 2$ up to $O(\tau^{s+5})$:

$$G^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1}A(Z) + \tau^{s+2}B(Z) + \tau^{s+3}C(Z) + \tau^{s+4}D(Z) + O(\tau^{s+5}) \tag{4}$$

(where $Z^{[0]} = Z$, $Z^{[1]} = f(Z)$, $Z^{[k+1]} = \frac{\partial Z^{[k]}}{\partial Z} Z^{[1]} = Z_z^{[k]} Z^{[1]}$ for $k = 1, 2, \cdots$) with complete formulae for calculation of $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$.

Thus, the **STO** G^τ satisfying equation (3) completely characterizes the **LMSM** (2) as: $Z_1 = G^\tau(Z_0), \cdots, Z_m = G^\tau(Z_{m-1}) = [G^\tau]^m(Z_0), \cdots$.

When equation (1) is a hamiltonian system, i.e., $p = 2n$ and $f(Z) = J\nabla H(Z)$, where $J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$, ∇ stands for the gradient operator, and $H : \mathbb{R}^{2n} \to \mathbb{R}^1$ is a smooth function, (1), (2) and (3) become

$$\frac{dZ}{dt} = J\nabla H(Z), \quad Z \in \mathbb{R}^{2n}, \tag{5}$$

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k J\nabla H(Z_k) \qquad \left(\sum_{k=0}^{m} \beta_k \neq 0\right), \tag{6}$$

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k J\nabla H \circ G^k, \tag{7}$$

and we can rewrite

$$\begin{aligned} Z^{[0]} &= Z, \\ Z^{[1]} &= J\nabla H, \\ Z^{[2]} &= JH_{zz}J\nabla H = Z_z^{[1]} Z^{[1]}, \\ Z^{[3]} &= Z_{z^2}^{[1]} \left(Z^{[1]}\right)^2 + Z_z^{[1]} Z^{[2]}, \\ Z^{[4]} &= Z_{z^3}^{[1]} \left(Z^{[1]}\right)^3 + 3Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} + Z_z^{[1]} Z^{[3]}, \end{aligned} \tag{8}$$

[3] More generally, one can use an **STO** to characterize any **DS** compatible with (1), and obviously the **STO** can be written in form (4).

$$Z^{[5]} = Z^{[1]}_{z^4}\left(Z^{[1]}\right)^4 + 6Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2 Z^{[2]} + 3Z^{[1]}_{z^2}\left(Z^{[2]}\right)^2 + 4Z^{[1]}_{z^2} Z^{[1]} Z^{[3]} + Z^{[1]}_{z} Z^{[4]},$$

and generally,

$$Z^{[r+1]} = \sum_{j=1}^{r} \sum_{i_1+i_2+\cdots+i_j=r; i_u\geq 1} \frac{r!}{j!} \frac{\Omega(i_1, i_2, \cdots, i_j)}{i_1! i_2! \cdots i_j!} J(\nabla H)_{z^j} Z^{[i_1]} Z^{[i_2]} \cdots Z^{[i_j]}$$

where $i_1 \leq i_2 \leq \cdots \leq i_j$, $\Omega(i_1, i_2, \cdots, i_j)$ is the number of all different permutations of $\{i_1, i_2, \cdots, i_j\}$, and $(\nabla H)_{z^j} Z^{[i_1]} Z^{[i_2]} \cdots Z^{[i_j]}$ stands for the multi-linear form

$$\sum_{1\leq t_1,\cdots,t_j\leq 2n} \frac{\partial^j (\nabla H)}{\partial Z_{(t_1)} \cdots \partial Z_{(t_j)}} Z^{[i_1]}_{(t_1)} \cdots Z^{[i_j]}_{(t_j)},$$

$Z^{[i_u]}_{(t_u)}$ stands for the t_u-th component of the 2n-dim vector $Z^{[i_u]}$.

The expansion of **STO** (4) has been used to study the *symplecticity* of **LMSM** (refer to [3], [7]), and also the *symplecticity* of Dahlquist pair (refer to [8]).

Definition 1. (due to K. Feng[2],[7]) *An* **LMSM** *is said to be symplectic for Hamiltonian system (5) iff its* **STO** *G defined by (7) is symplectic, i.e.,*

$$\left[\frac{\partial G(Z)}{\partial Z}\right]^\top J \left[\frac{\partial G(Z)}{\partial Z}\right] = J \tag{9}$$

for any hamiltonian function H and any sufficiently small step-size τ.

Definition 2. *If three B-serieses*[4] *G_1^τ, G_2^τ and G_3^τ in form (4) compatible with equation (1) satisfy*

$$G_3^{\lambda\tau} \circ G_1^\tau = G_2^\tau \circ G_3^{\lambda\tau} \tag{10}$$

for some real number λ and for any smooth function f and any sufficiently small step-size τ, then G_1^τ and G_2^τ are said to be a Dahlquist[5] *pair or a conjugate pair via G_3^τ, and we call equation (10) a Dahlquist relation or a conjugate relation. A Dahlquist pair G_1^τ and G_2^τ is said to be symplectic if G_1^τ or G_2^τ is symplectic. In this case when one of G_1^τ and G_2^τ is symplectic, we also call the other conjugate-symplectic.*

In the present paper, for any linear multi-step method (**LMSM**) with order $s = 1$, using 6 parameters we obtain the expansion of its step-transition operator in form (4) up to $O(\tau^6)$; and using 5 parameters we rewrite the expansion of the step-transition operator for $s = 2$ (obtained by Tang in a former paper [9] where 9 parameters are used) (in Section

[4]For the details about B-series, one can refer to [4]. We would like to thank Ernst Hairer for the suggestion that the case when G_3^τ is a B-series should be considered in the conjugate relation.

[5]It was G. Dahlquist[1] who found that the trapezoid rule and the mid-point rule are a conjugate pair via the Euler-forward scheme.

2). We prove that in Dahlquist relation (10) with G_1 being an **LMSM**, (1) the order of G_2 can not be higher than that of G_1 (that means, conjugation will not improve the order of any **LMSM**); (2) if G_3 is also an **LMSM** and G_2 is a symplectic B-series, then the order of both G_1 and G_2 must be 2 (in Section 3).

2. Expansion of Step-Transition Operator

Theorem 1. *If scheme (2) is of order $s = 1$, then the corresponding step-transition operator defined by (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^2 C(Z) + \tau^3 D(Z) + \tau^4 E(Z) + \tau^5 F(Z) + O(\tau^6), \tag{11}$$

where $C(Z), D(Z), E(Z), F(Z)$ can be determined by 6 *parameters ω, ρ, δ, σ, η, ν:*

$$C = \omega Z^{[2]}, \quad \omega = \frac{\sum_{k=0}^{m}\left(k\beta_k - \frac{k^2}{2}\alpha_k\right)}{\sum_{k=0}^{m} k\alpha_k}; \tag{12.1}$$

$$D = \left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) Z^{[3]} + \omega(\omega + \frac{1}{2}) Z_z^{[1]} Z^{[2]}, \tag{12.2}$$

$$\rho = \frac{\sum_{k=0}^{m}\left[k^2\beta_k - \frac{k^3}{3}\alpha_k\right]}{\sum_{k=0}^{m} k\alpha_k}, \quad \delta = \frac{\sum_{k=0}^{m} k^2\alpha_k}{\sum_{k=0}^{m} k\alpha_k};$$

$$\begin{aligned} E = &\left(\frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4}\right) Z^{[4]} \\ &+ \left(\rho\omega - \frac{3\delta\omega^2}{4} + \frac{3\omega^2}{4} + \frac{\omega}{3}\right) Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &+ \left(-\delta\omega^2 + \frac{\omega^2}{2} + \rho\omega - \frac{\omega\delta}{4} + \frac{\omega}{6} + \frac{\rho}{4}\right) Z_z^{[1]} Z^{[3]} \\ &+ \left(\omega^3 + \frac{3\omega^2}{4} - \frac{\omega^2\delta}{4} + \frac{\omega\rho}{2} + \frac{\omega}{6}\right) Z_z^{[1]} Z_z^{[1]} Z^{[2]}, \end{aligned} \tag{12.3}$$

$$\sigma = \frac{\sum_{k=0}^{m}\left(\frac{k^3}{2}\beta_k - \frac{k^4}{8}\alpha_k\right)}{\sum_{k=0}^{m} k\alpha_k}\omega, \quad \eta = \frac{\sum_{k=0}^{m} k^3\alpha_k}{\sum_{k=0}^{m} k\alpha_k};$$

$$\begin{aligned} F =& \nu Z^{[5]} \\ &+ \left\{-\frac{3\omega^2\delta}{4} + \frac{5\omega^2\delta^2}{8} - \frac{5\eta\omega^2}{12} + \frac{13\omega^2}{24} - \frac{3\omega\rho\delta}{4} + \frac{3\omega\rho}{4} + \sigma\omega + \frac{\omega}{8}\right\} Z_{z^3}^{[1]} \left(Z^{[1]}\right)^2 Z^{[2]} \\ &+ \left\{\frac{\omega^3}{2} - \frac{\delta\omega^3}{2} + \frac{5\delta^2\omega^2}{8} - \frac{3\delta\omega^2}{4} + \frac{\rho\omega^2}{2} - \frac{5\eta\omega^2}{12} + \frac{17\omega^2}{24} - \frac{3\delta\rho\omega}{4} + \frac{3\omega\rho}{4} + \sigma\omega + \frac{\omega}{8}\right\} \\ &Z_{z^2}^{[1]} \left(Z^{[2]}\right)^2 \end{aligned} \tag{12.4}$$

$$+\left\{\frac{9\delta^2\omega^2}{8}-\frac{5\delta\omega^2}{4}-\frac{5\eta\omega^2}{12}+\frac{13\omega^2}{24}-\frac{7\delta\rho\omega}{4}+\frac{5\rho\omega}{4}+\sigma\omega-\frac{\omega\delta}{6}+\frac{\omega}{8}+\frac{\rho^2}{2}+\frac{\rho}{6}\right\}$$
$$Z^{[1]}_{z^2}Z^{[1]}Z^{[3]}$$
$$+\left\{-\delta\omega^3+\omega^3+\frac{5\omega^2\delta^2}{8}-\frac{3\omega^2\delta}{4}+\rho\omega^2-\frac{5\eta\omega^2}{12}+\frac{7\omega^2}{8}-\frac{3\delta\rho\omega}{4}+\frac{3\rho\omega}{4}+\sigma\omega+\frac{\omega}{8}\right\}$$
$$Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[2]}$$
$$+\left\{-\frac{\eta\omega^2}{3}+\frac{3\omega^2\delta^2}{4}-\frac{\omega^2\delta}{2}+\frac{\omega^2}{6}+\frac{\delta^2\omega}{8}-\delta\rho\omega+\frac{2\sigma\omega}{3}+\frac{\rho\omega}{2}-\frac{\omega\delta}{12}-\frac{\eta\omega}{12}+\frac{\omega}{24}\right.$$
$$\left.+\frac{\rho^2}{4}+\frac{\rho}{12}+\frac{\sigma}{6}-\frac{\delta\rho}{8}\right\}Z^{[1]}_{z}Z^{[4]}$$
$$+\left\{-\frac{7\delta\omega^3}{4}+\omega^3-\frac{3\delta\omega^2}{4}+\frac{3\delta^2\omega^2}{8}+\frac{7\rho\omega^2}{4}-\frac{\eta\omega^2}{3}+\frac{7\omega^2}{12}-\frac{\delta\rho\omega}{2}+\frac{2\sigma\omega}{3}+\rho\omega+\frac{\omega}{12}\right\}$$
$$Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]}$$
$$+\left\{-\frac{3\delta\omega^3}{2}+\frac{\omega^3}{2}+\frac{3\delta^2\omega^2}{8}-\delta\omega^2+\frac{3\rho\omega^2}{2}-\frac{\omega^2\eta}{12}+\frac{3\omega^2}{8}-\frac{3\delta\rho\omega}{4}+\frac{\sigma\omega}{3}+\rho\omega+\frac{\omega}{24}\right.$$
$$\left.\frac{\rho^2}{4}+\frac{\rho}{12}-\frac{\omega\delta}{12}\right\}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[3]}$$
$$+\left\{\omega^4-\frac{5\delta\omega^3}{4}+\frac{3\omega^3}{2}+\frac{\delta^2\omega^2}{4}-\frac{\eta\omega^2}{4}+\frac{5\rho\omega^2}{4}-\frac{\delta\omega^2}{2}+\frac{5\omega^2}{8}-\frac{\delta\rho\omega}{4}+\frac{\rho\omega}{2}+\frac{\delta\omega}{8}\right.$$
$$\left.-\frac{\eta\omega}{12}+\frac{\sigma\omega}{3}\right\}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]},$$

$$\nu=\frac{1}{\sum\limits_{k=0}^{m}k\alpha_k}\sum_{k=0}^{m}\left\{\frac{k^4}{4!}\beta_k-\left[\frac{k^5}{5!}+\frac{k^4-2k^3+k^2}{24}\omega+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.\right.$$
$$\left.\left.+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right]\alpha_k\right\}.$$

Here we use the notation for example,

$$Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2Z^{[2]}=\sum_{i,j,k=1}^{p}\frac{\partial^3Z^{[1]}}{\partial z_i\partial z_j\partial z_k}\left[Z^{[1]}\right]_{(i)}\left[Z^{[1]}\right]_{(j)}\left[Z^{[2]}\right]_{(k)}$$

where z_i is the i-th component of p-dim vector Z, and $\left[Z^{[r]}\right]_{(j)}$ stands for the j-th component of p-dim vector $Z^{[r]}$.

The proof of Theorem 1 is tedious but straightforward calculation, and similar to that for $s\geq 2$ given in [9]. A difference is that we here try to use least parameters in expressing $C(Z)$, $D(Z)$, $E(Z)$ and $F(Z)$. We give the complete proof of Theorem 1 later in **Appendix 1**.

Similar result for $s\geq 2$ is already given in [9], where 9 parameters λ, μ, ν, ρ, ξ, σ, χ, η and ζ are used for expressing $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$ in (4). Using 5 parameters

ω_2, ρ_2, δ_2, σ_2 and ν_2, we rewrite the result for $s=2$ as follows:

Theorem 2. *If scheme (2) is of order $s=2$, then the step-transition operator decided by equation (3) has the following expansion:*

$$G(Z)=\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}+\tau^3C(Z)+\tau^4D(Z)+\tau^5E(Z)+\tau^6F(Z)+O(\tau^7), \tag{13}$$

where $C(Z),D(Z),E(Z),F(Z)$ can be expressed by 5 parameters ω_2, ρ_2, δ_2, σ_2 and ν_2:

$$C=\omega_2Z^{[3]},\quad \omega_2=\frac{\sum_{k=0}^m\left\{\frac{k^2}{2}\beta_k-\frac{k^3}{6}\alpha_k\right\}}{\sum_{k=0}^m k\alpha_k}; \tag{14.1}$$

$$D=\left(\rho_2-\frac{1}{2}\delta_2\omega_2+\frac{\omega_2}{2}\right)Z^{[4]}+\frac{\omega_2}{2}Z_z^{[1]}Z^{[3]}, \tag{14.2}$$

$$\rho_2=\frac{\sum_{k=0}^m\left[\frac{k^3}{6}\beta_k-\frac{k^4}{24}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k},\quad \delta_2=\frac{\sum_{k=0}^m k^2\alpha_k}{\sum_{k=0}^m k\alpha_k};$$

$$\begin{aligned}E=&\sigma_2Z^{[5]}+\left(\omega_2^2+\frac{\omega_2}{6}\right)Z_z^{[1]}Z_z^{[1]}Z^{[3]}\\&+\left(\omega_2^2-\frac{1}{4}\delta_2\omega_2+\frac{\omega_2}{6}+\frac{\rho_2}{2}\right)Z_z^{[1]}Z^{[4]}+\left(2\omega_2^2+\frac{\omega_2}{3}\right)Z_{z^2}^{[1]}Z^{[1]}Z^{[3]},\end{aligned} \tag{14.3}$$

$$\sigma_2=\frac{\sum_{k=0}^m\left[\frac{k^4}{24}\beta_k-\frac{k^5}{120}\alpha_k-\left\{\frac{2k^3-3k^2+k}{12}\omega_2+\frac{k^2-k}{2}\left(\rho_2-\frac{1}{2}\delta_2\omega_2+\frac{\omega_2}{2}\right)\right\}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k};$$

$$\begin{aligned}F=&\nu_2Z^{[6]}\\&+\left\{-\delta_2\omega_2^2+\frac{\omega_2^2}{2}+2\rho_2\omega_2+\frac{1}{24}\delta_2\omega_2-\frac{\omega_2}{24}-\frac{\rho_2}{12}+\frac{\sigma_2}{2}\right\}Z_z^{[1]}Z^{[5]}\\&+\left\{-\delta_2\omega_2^2+\omega_2^2+2\rho_2\omega_2-\frac{1}{12}\delta_2\omega_2+\frac{\omega_2}{24}+\frac{\rho_2}{6}\right\}Z_z^{[1]}Z_z^{[1]}Z^{[4]}\\&+\left\{-\frac{5}{2}\delta_2\omega_2^2+\frac{3}{2}\omega_2^2+5\rho_2\omega_2-\frac{1}{6}\delta_2\omega_2+\frac{\omega_2}{8}+\frac{\rho_2}{3}\right\}Z_{z^2}^{[1]}Z^{[1]}Z^{[4]}\\&+\left\{-\frac{1}{2}\delta_2\omega_2^2+\omega_2^2+\rho_2\omega_2+\frac{\omega_2}{24}\right\}Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[3]}\\&+\left\{-\delta_2\omega_2^2+2\omega_2^2+2\rho_2\omega_2+\frac{\omega_2}{12}\right\}Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[3]}\\&+\left\{-\frac{3}{2}\delta_2\omega_2^2+\frac{3}{2}\omega_2^2+3\rho_2\omega_2+\frac{\omega_2}{8}\right\}Z_{z^2}^{[1]}Z^{[1]}\left(Z_z^{[1]}Z^{[3]}\right)\\&+\left\{-\frac{3}{2}\delta_2\omega_2^2+\frac{3}{2}\omega_2^2+3\rho_2\omega_2+\frac{\omega_2}{8}\right\}Z_{z^2}^{[1]}Z^{[2]}Z^{[3]}\\&+\left\{-\frac{3}{2}\delta_2\omega_2^2+\frac{3}{2}\omega_2^2+3\rho_2\omega_2+\frac{\omega_2}{8}\right\}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[3]},\end{aligned} \tag{14.4}$$

$$\nu_2 = \frac{\sum_{k=0}^{m} \left\{ \frac{k^5}{5!}\beta_k - \left[\frac{k^6}{6!} + \frac{k^4-2k^3+k^2}{24}\omega_2 + \frac{2k^3-3k^2+k}{12}\left(\rho_2 - \frac{\delta_2\omega_2}{2} + \frac{\omega_2}{2}\right) + \frac{k^2-k}{2}\sigma_2 \right] \alpha_k \right\}}{\sum_{k=0}^{m} k\alpha_k}.$$

3. Order Barriers for STOs in Dahlquist Relation

Theorem 3. *In Dahlquist relation (10), if B-series G_1 stands for an* **LMSM***, then the order of G_2 can not be higher than that of G_1.*

Proof. Supposing the order of G_1^τ, G_1^τ and G_3^τ are u, v and $w-1$ respectively, we write their expansions as follows:

$$G_1^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{u+1}A(Z) + O(\tau^{u+2}) \tag{15}$$

$$G_2^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{v+1}M(Z) + O(\tau^{v+2}) \tag{16}$$

$$G_3^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{w}B(Z) + O(\tau^{w+1}) \tag{17}$$

where $A(Z) \neq 0$, $B(Z) \neq 0$ and $M(Z) \neq 0$.

Provided $v > u$, there are three cases:

Case 1. $w > u$, expanding both sides of (10) and comparing the terms in τ^{u+1} we have

$$A(Z) = 0. \tag{18}$$

Case 2. $w = u$, expanding both sides of (10) and comparing the terms in τ^{u+1} we have

$$\lambda^w B_z Z^{[1]} + A(Z) = \lambda^w Z_z^{[1]} B. \tag{19}$$

Case 3. $w < u$, expanding both sides of (10) and comparing the terms in τ^{w+1} we have

$$\lambda^w B_z Z^{[1]} = \lambda^w Z_z^{[1]} B. \tag{20}$$

From Theorem 1, Theorem 2 in Section 2 above, and Lemma 1 in [7], we know that in fact $A(Z) = aZ^{[u+1]}$ for some $a \neq 0$. Since B-series G_3^τ is compatible with (3), $w \geq 2$. When $\lambda \neq 0$, it's easy to check that any of the cases (18), (19) and (20) is impossible; when $\lambda = 0$, equation (10) becomes into $G_1^\tau(Z) = G_2^\tau(Z)$ which contradicts $v > u$.

So the only possible case should be $v \leq u$. □

Theorem 4. *In Dahlquist relation (10), if both G_1 and G_3 stand for* **LMSM***s, and G_2 is a symplectic B-series*[6]*, then the orders of G_1^τ, G_2^τ and G_3^τ are* 2, 2 *and* 1 *respectively.*

The result in Theorem 4 is a little different from that in Theorem 1 in [8], and the proof of the former will also based on the latter.

Proof of Theorem 4. Supposing the order of G_1^τ, G_1^τ and G_3^τ are u, v and $w-1$ respectively. Since they are compatible with (3), $u \geq 1$, $v \geq 1$ and $w \geq 2$. We write their expansions as (15-17). According to Theorem 1 in [8], if G_2 is symplectic, then the order of G_1^τ can not be greater than 2. So $1 \leq u \leq 2$. And according to Theorem 3 above, we know $v \leq u$. Let's discuss all the cases as follows:

Case 1. If $u = 1$, then $v = 1$. Expanding both sides of (10) and comparing the terms in τ^2 we have

$$A(Z) = M(Z). \tag{21}$$

Case 2. If $u = 2$, $v = 1$. Expanding both sides of (10) and comparing the terms in τ^2 we have

$$0 = M(Z). \tag{22}$$

Case 3. If $u = 2$, $v = 2$ and $w = 2$. Expanding both sides of (10) and comparing the terms in τ^3 we have

$$\lambda^w B_z Z^{[1]} + A(Z) = M(Z) + \lambda^w Z_z^{[1]} B. \tag{23}$$

Case 4. If $u = 2$, $v = 2$ and $w > 2$. Expanding both sides of (10) and comparing the terms in τ^3 we have

$$A(Z) = M(Z). \tag{24}$$

Since $A(Z) = aZ^{[u+1]}$ for some $a \neq 0$, and G_2 is a symplectic B-series with order v, cases (21), (22) and (24) are impossible. So the only possible case is (23), i.e., $u = v = w = 2$. □

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[6]For the details about symplectic B-series, one can refer to [4].

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Appendix 1. Proof of Theorem 1

When we set

$$G^k(Z) = \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} Z^{[i]} + \tau^2 C_k(Z) + \tau^3 D_k(Z) + \tau^4 E_k(Z) + \tau^5 F_k(Z) + O(\tau^6), \tag{25}$$

then

$$\begin{aligned} &\sum_{i=0}^{+\infty} \frac{(k+1)^i \tau^i}{i!} Z^{[i]} + \tau^2 C_{k+1}(Z) + \tau^3 D_{k+1}(Z) \\ &\qquad + \tau^4 E_{k+1}(Z) + \tau^5 F_{k+1}(Z) + O(\tau^6) \\ =& G^{k+1}(Z) = G^k\left[G(Z)\right] \qquad (26) \\ =& \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[G(Z)\right]^{[i]} + \tau^2 C_k\left[G(Z)\right] + \tau^3 D_k\left[G(Z)\right] \\ &\quad + \tau^4 E_k\left[G(Z)\right] + \tau^5 F_k\left[G(Z)\right] + O(\tau^6) \\ \equiv& \widetilde{I} + \widetilde{II} + \widetilde{III} + \widetilde{IV} + \widetilde{V} + O(\tau^6), \end{aligned}$$

and

$$\begin{aligned} \widetilde{I} =& \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right. \qquad (26.1) \\ &\qquad \left. + \tau^2 C_1(Z) + \tau^3 D_1(Z) + \tau^4 E_1(Z) + \tau^5 F_1(Z) + O(\tau^6)\right]^{[i]} \\ =& \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right]^{[i]} + \tau^2 C_1 + \tau^3 D_1 + \tau^4 E_1 + \tau^5 F_1 \\ &+ \frac{k\tau}{1!}\left\{Z_z^{[1]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1 + \tau^3 D_1 + \tau^4 E_1\right) + \frac{1}{2!} Z_{z^2}^{[1]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1\right)^2\right\} \\ &+ \frac{k^2\tau^2}{2!}\left\{Z_z^{[2]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1 + \tau^3 D_1\right)\right\} \\ &+ \frac{k^3\tau^3}{3!}\left\{Z_z^{[3]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1\right)\right\} + O(\tau^6) \\ =& \sum_{l=0}^{+\infty} \frac{(k+1)^l \tau^l}{l!} Z^{[l]} + \tau^2 C_1 + \tau^3\left\{D_1 + k Z_z^{[1]} C_1\right\} \\ &+ \tau^4\left\{E_1 + k Z_z^{[1]} D_1 + k Z_{z^2}^{[1]} Z^{[1]} C_1 + \frac{k^2}{2} Z_z^{[2]} C_1\right\} \\ &+ \tau^5\left\{F_1 + k Z_z^{[1]} E_1 + k Z_{z^2}^{[1]} Z^{[1]} D_1 + \frac{k}{2} Z_{z^2}^{[1]} Z^{[2]} C_1 + \frac{k}{2} Z_{z^3}^{[1]} \left(Z^{[1]}\right)^2 C_1 \right. \\ &\qquad \left. + \frac{k}{2} Z_{z^2}^{[1]} \left(C_1\right)^2 + \frac{k^2}{2} Z_z^{[2]} D_1 + \frac{k^2}{2} Z_{z^2}^{[2]} Z^{[1]} C_1 + \frac{k^3}{6} Z_z^{[3]} C_1\right\} + O(\tau^6); \end{aligned}$$

$$
\begin{aligned}
\widetilde{II} =& \tau^2 C_k \circ \left(Z + \tau Z^{[1]} + \frac{\tau^2}{2} Z^{[2]} + \frac{\tau^3}{6} Z^{[3]} + \tau^2 C_1 + \tau^3 D_1 \right) + O(\tau^6) \\
=& \tau^2 C_k + \tau^3 \left(C_k\right)_z Z^{[1]} + \tau^4 \left\{ \frac{1}{2} \left(C_k\right)_z Z^{[2]} + \left(C_k\right)_z C_1 + \frac{1}{2} \left(C_k\right)_{z^2} \left[Z^{[1]}\right]^2 \right\} \\
&+ \tau^5 \left\{ \frac{1}{6} \left(C_k\right)_z Z^{[3]} + \left(C_k\right)_z D_1 + \left(C_k\right)_{z^2} Z^{[1]} C_1 + \frac{1}{2} \left(C_k\right)_{z^2} \left[Z^{[1]} Z^{[2]}\right] \right. \\
&\left. + \frac{1}{6} \left(C_k\right)_{z^3} \left[Z^{[1]}\right]^3 \right\} + O(\tau^6);
\end{aligned} \tag{26.2}
$$

$$
\begin{aligned}
\widetilde{III} =& \tau^3 D_k \circ \left(Z + \tau Z^{[1]} + \frac{\tau^2}{2} Z^{[2]} + \tau^2 C_1 \right) + O(\tau^6) \\
=& \tau^3 D_k + \tau^4 \left(D_k\right)_z Z^{[1]} \\
&+ \tau^5 \left\{ \frac{1}{2} \left(D_k\right)_z Z^{[2]} + \left(D_k\right)_z C_z + \frac{1}{2} \left(D_k\right)_{z^2} \left[Z^{[1]}\right]^2 \right\} + O(\tau^6);
\end{aligned} \tag{26.3}
$$

$$
\begin{aligned}
\widetilde{IV} =& \tau^4 E_k \circ \left(Z + \tau Z^{[1]} \right) + O(\tau^6) \\
&= \tau^4 E_k + \tau^5 \left(E_k\right)_z Z^{[1]} + O(\tau^6);
\end{aligned} \tag{26.4}
$$

$$
\widetilde{V} = \tau^5 F_k + O(\tau^6). \tag{26.5}
$$

From (26), (26.1)–(26.5), we obtain

$$
C_{k+1} = C_1 + C_k; \tag{27.1}
$$

$$
D_{k+1} = D_1 + k Z_z^{[1]} C_1 + \left(C_k\right)_z Z^{[1]} + D_k; \tag{27.2}
$$

$$
\begin{aligned}
E_{k+1} =& E_1 + k Z_z^{[1]} D_1 + k Z_{z^2}^{[1]} \left[Z^{[1]} C_1\right] + \frac{k^2}{2} Z_z^{[2]} C_1 + \frac{1}{2} \left(C_k\right)_z Z^{[2]} \\
&+ \left(C_k\right)_z C_1 + \frac{1}{2} \left(C_k\right)_{z^2} \left[Z^{[1]}\right]^2 + \left(D_k\right)_z Z^{[1]} + E_k;
\end{aligned} \tag{27.3}
$$

$$
\begin{aligned}
F_{k+1} =& F_1 + k Z_z^{[1]} E_1 + k Z_{z^2}^{[1]} \left[Z^{[1]} D_1\right] + \frac{k}{2} Z_{z^2}^{[1]} \left[Z^{[2]} C_1\right] \\
&+ \frac{k}{2} Z_{z^3}^{[1]} \left[\left(Z^{[1]}\right)^2 C_1\right] + \frac{k}{2} Z_{z^2}^{[1]} \left(C_1\right)^2 + \frac{k^2}{2} Z_z^{[2]} D_1 \\
&+ \frac{k^2}{2} Z_{z^2}^{[2]} \left[Z^{[1]} C_1\right] + \frac{k^3}{6} Z_z^{[3]} C_1 + \frac{1}{6} \left(C_k\right)_z Z^{[3]} + \left(C_k\right)_z D_1 \\
&+ \frac{1}{2} \left(C_k\right)_{z^2} \left[Z^{[1]} Z^{[2]}\right] + \left(C_k\right)_{z^2} Z^{[1]} C_1 + \frac{1}{6} \left(C_k\right)_{z^3} \left[Z^{[1]}\right]^3 \\
&+ \frac{1}{2} \left(D_k\right)_z Z^{[2]} + \left(D_k\right)_z C_1 + \frac{1}{2} \left(D_k\right)_{z^2} \left[Z^{[1]}\right]^2 + \left(E_k\right)_z Z^{[1]} \\
&+ F_k.
\end{aligned} \tag{27.4}
$$

From (3), we have

$$
\begin{aligned}
&\sum_{k=0}^{m}\alpha_k\left[\sum_{i=0}^{+\infty}\frac{k^i\tau^i}{i!}Z^{[i]}+\tau^2C_k(Z)+\tau^3D_k(Z)+\tau^4E_k(Z)+\tau^5F_k(Z)+O(\tau^6)\right]\\
=&\tau\sum_{k=0}^{m}\beta_k f\left(\sum_{i=0}^{+\infty}\frac{k^i\tau^i}{i!}Z^{[i]}+\tau^2C_k(Z)+\tau^3D_k(Z)+\tau^4E_k(Z)+O(\tau^5)\right)\\
=&\tau\sum_{k=0}^{m}\beta_k\left\{f\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]+f_z\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]*\left(\tau^2C_k+\tau^3D_k+\tau^4E_k\right)\right.\\
&\left.+\frac{1}{2}f_{z^2}\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]*\left(\tau^2C_k\right)^2\right\}+O(\tau^6)\\
=&\sum_{l=0}^{+\infty}\sum_{k=0}^{m}\beta_k\frac{k^l\tau^{l+1}}{l!}Z^{[l+1]}+\tau^3\sum_{k=0}^{m}\beta_kZ_z^{[1]}C_k+\tau^4\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}D_k+kZ_{z^2}^{[1]}Z^{[1]}C_k\right\}\\
&+\tau^5\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}E_k+kZ_{z^2}^{[1]}Z^{[1]}D_k+\frac{k^2}{2}Z_{z^2}^{[1]}Z^{[2]}C_k+\frac{k^2}{2}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2C_k\right.\\
&\left.+\frac{1}{2}Z_{z^2}^{[1]}\left(C_k\right)^2\right\}+O(\tau^6),
\end{aligned}
\tag{28}
$$

comparing the coefficients of τ^2, τ^3, τ^4 and τ^5 respectively on both sides of (28) we obtain

$$
\sum_{k=0}^{m}\alpha_kC_k=\sum_{k=0}^{m}\left\{k\beta_k-\frac{k^2}{2!}\alpha_k\right\}Z^{[2]};\tag{29.1}
$$

$$
\sum_{k=0}^{m}\alpha_kD_k=\sum_{k=0}^{m}\left\{\frac{k^2}{2!}\beta_k-\frac{k^3}{3!}\alpha_k\right\}Z^{[3]}+\sum_{k=0}^{m}\beta_kZ_z^{[1]}C_k;\tag{29.2}
$$

$$
\begin{aligned}
\sum_{k=0}^{m}\alpha_kE_k=&\sum_{k=0}^{m}\left\{\frac{k^3}{3!}\beta_k-\frac{k^4}{4!}\alpha_k\right\}Z^{[4]}\\
&+\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}D_k+kZ_{z^2}^{[1]}Z^{[1]}C_k\right\};
\end{aligned}
\tag{29.3}
$$

$$
\begin{aligned}
\sum_{k=0}^{m}\alpha_kF_k=&\sum_{k=0}^{m}\left\{\frac{k^4}{4!}\beta_k-\frac{k^5}{5!}\alpha_k\right\}Z^{[5]}+\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}E_k+kZ_{z^2}^{[1]}Z^{[1]}D_k\right.\\
&\left.+\frac{k^2}{2}Z_{z^2}^{[1]}Z^{[2]}C_k+\frac{k^2}{2}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2C_k+\frac{1}{2}Z_{z^2}^{[1]}\left(C_k\right)^2\right\}.
\end{aligned}
\tag{29.4}
$$

From relations (27.1) and (29.1) we deduce directly

$$
C_k=kC_1\equiv kC,\tag{30}
$$

and (12.1). Substituting (30) into (27.2), we obtain

$$
D_k=kD_1+\frac{k^2-k}{2}\omega\left(Z_z^{[1]}Z^{[s+1]}+Z^{[3]}\right),\tag{31}
$$

substituting (31) and (30) into (29.2), we obtain

$$\left(\sum_{k=0}^{m} k\alpha_k\right) D_1 = \sum_{k=0}^{m}\left\{\frac{k^2}{2!}\beta_k - \frac{k^3}{3!}\alpha_k - \frac{k^2-k}{2}\omega\alpha_k\right\} Z^{[3]} + \sum_{k=0}^{m}\left\{k\omega\beta_k - \frac{k^2-k}{2}\omega\alpha_k\right\} Z_z^{[1]}Z^{[2]}.$$

Then we get (12.2), and

$$D_k = \left[\frac{k^2\omega}{2} + k\omega^2\right] Z_z^{[1]}Z^{[2]} + \left[\frac{k^2-k}{2}\omega + k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right] Z^{[3]}. \tag{32}$$

Substituting (30) and (32) into (27.3), we have

$$\begin{aligned} E_{k+1} =& E_1 + \frac{k^2\omega + 3k\omega^2 + k\omega}{2} Z_z^{[1]}Z_z^{[1]}Z^{[2]} \\ &+ \left[k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + k\omega^2 + \frac{k^2\omega}{2}\right] Z_z^{[1]}Z^{[3]} \\ &+ \frac{2k^2\omega + 3k\omega^2 + k\omega}{2} Z_{z^2}^{[1]}Z^{[1]}Z^{[2]} \\ &+ \left[k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{k^2\omega}{2}\right] Z^{[4]} + E_k, \end{aligned}$$

and then

$$\begin{aligned} E_k =& kE_1 + \left[\frac{k^2-k}{4}(3\omega^2 + \omega) + \frac{2k^3 - 3k^2 + k}{12}\omega\right] Z_{z^2}^{[1]}Z^{[1]}Z^{[2]} \\ &+ \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{k^2-k}{2}\omega^2 + \frac{2k^3 - 3k^2 + k}{12}\omega\right] Z_z^{[1]}Z^{[3]} \\ &+ \left[\frac{2k^3 - 3k^2 + k}{6}\omega + \frac{k^2-k}{2}\left(\frac{3\omega^2}{2} + \omega\right)\right] Z_{z^2}^{[1]}Z^{[1]}Z^{[2]} \\ &+ \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{2k^3 - 3k^2 + k}{12}\omega\right] Z^{[4]} \end{aligned} \tag{33}$$

Substituting (33), (30) and (32) into (29.3), we obtain

$$\begin{aligned} &\left(\sum_{k=0}^{m} k\alpha_k\right) E_1 = \\ &\sum_{k=0}^{m}\left\{\frac{k^3}{3!}\beta_k - \frac{k^4}{4!}\alpha_k - \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{2k^3 - 3k^2 + k}{12}\omega\right]\alpha_k\right\} Z^{[4]} \\ &+ \sum_{k=0}^{m}\left\{k^2\omega\beta_k - \frac{2k^3 - 3k^2 + k}{6}\omega\alpha_k - \frac{k^2-k}{2}\left(\frac{3\omega^2}{2} + \omega\right)\alpha_k\right\} Z_{z^2}^{[1]}Z^{[1]}Z^{[2]} \\ &+ \sum_{k=0}^{m}\left\{\left[k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{k^2-k}{2}\omega\right]\beta_k - \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right.\right. \end{aligned}$$

$$+\frac{k^2-k}{2}\omega^2+\frac{2k^3-3k^2+k}{12}\omega\Big]\alpha_k\Big\}Z_z^{[1]}Z^{[3]}$$

$$+\sum_{k=0}^{m}\left\{\left(k\omega^2+\frac{k^2\omega}{2}\right)\beta_k-\left[\frac{k^2-k}{4}\left(3\omega^2+\omega\right)+\frac{2k^3-3k^2+k}{12}\omega\right]\alpha_k\right\}Z_z^{[1]}Z_z^{[1]}Z^{[2]},$$

and we have (12.3), and

$$\begin{aligned}E_k=&\left[\frac{2k^3-3k^2+k}{12}\omega+\frac{k^2-k}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}\right.\right.\\&\left.\left.-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right]Z^{[4]}\\&+\left[\frac{k^3\omega}{3}+\frac{3k^2\omega^2}{4}+k\rho\omega-\frac{3k\omega^2\delta}{4}\right]Z_{z^2}^{[1]}Z^{[1]}Z^{[2]}\\&+\left[\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{2}\omega^2+\frac{2k^3-3k^2}{2}\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.\\&\left.+\frac{k\rho\omega}{2}-\frac{k\delta\omega^2}{2}\right]Z_z^{[1]}Z^{[3]}\\&+\left[\frac{k^3\omega}{6}+\frac{3k^2\omega^2}{4}+\frac{k\rho\omega}{2}+k\omega^3-\frac{k\omega^2\delta}{4}\right]Z_z^{[1]}Z_z^{[1]}Z^{[2]}.\end{aligned}\tag{34}$$

Substituting (30), (32) and (34) into (27.4), we obtain

$$\begin{aligned}&F_{k+1}=F_1+F_k\\&+\left\{\frac{k^2\omega^2}{2}+\frac{k^3}{6}\omega+\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right.\\&\quad\left.+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k\omega\rho}{2}-\frac{k\delta\omega^2}{2}\right\}Z_z^{[1]}Z^{[4]}\\&+\left\{k\omega^3+\frac{3k^2}{4}\omega^2+\frac{2k^3-k}{12}\omega+2k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\rho\omega+\frac{k^2+k}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.\\&\quad\left.-\frac{3k\omega^2\delta}{4}\right\}Z_z^{[1]}Z_z^{[1]}Z^{[3]}\\&+\left\{\frac{5k^2+2k}{4}\omega^2+\frac{2k^3+k^2}{4}\omega+2k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3}{2}k\rho\omega-\frac{5k\delta\omega^2}{4}\right.\\&\quad\left.+\left(k^2+k\right)\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right\}Z_{z^2}^{[1]}Z^{[1]}Z^{[3]}\\&+\left\{3k\omega^3+\frac{6k^2+5k}{4}\omega^2+\frac{k^3+3k^2+k}{6}\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k\omega\rho}{2}-\frac{k\delta\omega^2}{4}\right\}\\&\quad Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[2]}\\&+\left\{2k\omega^3+\frac{7k^2+5k}{4}\omega^2+\frac{2k^3+3k^2+2k}{6}\omega+\frac{3}{2}k\rho\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)-k\delta\omega^2\right\}\\&\quad Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[2]}\\&+\left\{2k\omega^3+\frac{9k^2+6k}{4}\omega^2+\frac{2k^3+3k^2+2k}{4}\omega+2k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k\rho\omega}{2}-\frac{k\delta\omega^2}{4}\right\}\end{aligned}$$

$$
\begin{aligned}
&Z^{[1]}_{z^2} Z^{[1]} \left[Z^{[1]}_z Z^{[2]}\right] \\
&+ \left\{ k\omega^3 + \frac{7k^2+6k}{4}\omega^2 + \frac{2k^3+3k^2+2k}{4}\omega + k\rho\omega + k\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) - \frac{3k\delta\omega^2}{4}\right\} \\
&Z^{[1]}_{z^2} \left(Z^{[2]}\right)^2 \\
&+ \left\{ \frac{5k^2+4k}{4}\omega^2 + \frac{2k^3+3k^2+2k}{4}\omega - \frac{3k\delta\omega^2}{4} + k\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + k\rho\omega\right\} \\
&Z^{[1]}_{z^3} \left(Z^{[1]}\right)^2 Z^{[2]} \\
&+ \left\{ \frac{k^3}{6}\omega + \frac{k^2}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + k\left(\frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4}\right)\right\} Z^{[5]},
\end{aligned}
$$

and then

$$
\begin{aligned}
F_k &= kF_1 \\
&+ \left\{ \frac{2k^3-3k^2+k}{12}\omega^2 + \frac{k^4-2k^3+k^2}{24}\omega + \frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right. \\
&\quad + \frac{k^2-k}{2}\left(\frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4}\right) + \frac{k^2-k}{2}\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) \\
&\quad \left. + \frac{k^2-k}{4}\rho\omega - \frac{k^2-k}{4}\delta\omega^2\right\} Z^{[1]}_z Z^{[4]} \\
&+ \left\{ \frac{k^2-k}{2}\omega^3 + \frac{2k^3-3k^2+k}{8}\omega^2 + \frac{k^4-2k^3+k}{24}\omega + \left(k^2-k\right)\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right. \\
&\quad \left. + \frac{k^2-k}{2}\rho\omega + \frac{k^3-k}{6}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) - \frac{3\left(k^2-k\right)}{8}\delta\omega^2\right\} Z^{[1]}_z Z^{[1]}_z Z^{[3]} \\
&+ \left\{ \frac{10k^3-9k^2-k}{24}\omega^2 + \frac{3k^4-4k^3+k}{24}\omega + \left(k^2-k\right)\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{3\left(k^2-k\right)}{4}\rho\omega\right. \\
&\quad \left. - \frac{5\left(k^2-k\right)}{8}\delta\omega^2 + \frac{k^3-k}{3}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right\} Z^{[1]}_{z^2} Z^{[1]} Z^{[3]} \\
&+ \left\{ \frac{3\left(k^2-k\right)}{2}\omega^3 + \frac{4k^3-k^2-3k}{8}\omega^2 + \frac{k^4+2k^3-3k^2}{24}\omega + \frac{k^2-k}{2}\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right. \\
&\quad \left. + \frac{k^2-k}{4}\rho\omega - \frac{k^2-k}{8}\delta\omega^2\right\} Z^{[1]}_z Z^{[1]}_z Z^{[1]}_z Z^{[2]} \\
&+ \left\{ \left(k^2-k\right)\omega^3 + \frac{7k^3-3k^2-4k}{12}\omega^2 + \frac{k^4-k}{12}\omega + \frac{3\left(k^2-k\right)}{4}\rho\omega - \frac{k^2-k}{2}\delta\omega^2\right. \\
&\quad \left. + \frac{k^2-k}{2}\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right\} Z^{[1]}_z Z^{[1]}_{z^2} Z^{[1]} Z^{[2]} \\
&+ \left\{ \left(k^2-k\right)\omega^3 + \frac{6k^3-3k^2-3k}{8}\omega^2 + \frac{k^4-k}{8}\omega + \left(k^2-k\right)\omega\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right. \\
&\quad \left. + \frac{k^2-k}{4}\rho\omega - \frac{k^2-k}{8}\delta\omega^2\right\} Z^{[1]}_{z^2} Z^{[1]} \left[Z^{[1]}_z Z^{[2]}\right] \\
&+ \left\{ \frac{k^2-k}{2}\omega^3 + \frac{14k^3-3k^2-11k}{24}\omega^2 + \frac{k^4-k}{8}\omega + \frac{k^2-k}{2}\rho\omega - \frac{3\left(k^2-k\right)}{8}\delta\omega^2\right.
\end{aligned}
\tag{35}
$$

$$+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg\} Z^{[1]}_{z^2}\left(Z^{[2]}\right)^2$$

$$+\Bigg\{\frac{10k^3-3k^2-7k}{24}\omega^2+\frac{k^4-k}{8}\omega-\frac{3\left(k^2-k\right)}{8}\delta\omega^2+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$

$$+\frac{k^2-k}{2}\rho\omega\Bigg\} Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2 Z^{[2]}$$

$$+\Bigg\{\frac{k^4-2k^3+k^2}{24}\omega+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)$$

$$+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg\} Z^{[5]}.$$

Substituting (30), (32), (34) and (35) into (29.4) we have

$$\left(\sum_{k=0}^{m} k\alpha_k\right) F_1 =$$

$$\sum_{k=0}^{m}\Bigg\{\frac{k^4}{4!}\beta_k-\frac{k^5}{5!}\alpha_k-\frac{k^4-2k^3+k^2}{24}\alpha_k\omega-\frac{2k^3-3k^2+k}{12}\alpha_k\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$

$$-\frac{k^2-k}{2}\alpha_k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\Bigg\} Z^{[5]}$$

$$+\sum_{k=0}^{m}\Bigg\{\Bigg[\frac{2k^3-3k^2+k}{12}\omega+\frac{k^2-k}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\Bigg(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}$$

$$+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\Bigg)\Bigg]\beta_k-\Bigg[\frac{2k^3-3k^2+k}{12}\omega^2+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$

$$+\frac{k^4-2k^3+k^2}{24}\omega+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)$$

$$+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{4}\delta\omega^2\Bigg]\alpha_k\Bigg\} Z^{[1]}_z Z^{[4]}$$

$$+\sum_{k=0}^{m}\Bigg\{\Bigg[\frac{k^3\omega}{6}+\frac{3k^2\omega^2}{4}+\frac{k\rho\omega}{2}+k\omega^3-\frac{k\delta\omega^2}{4}\Bigg]\beta_k-\Bigg[\frac{3\left(k^2-k\right)}{2}\omega^3$$

$$+\frac{4k^3-k^2-3k}{8}\omega^2+\frac{k^4+2k^3-3k^2}{24}\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$

$$+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{8}\delta\omega^2\Bigg]\alpha_k\Bigg\} Z^{[1]}_z Z^{[1]}_z Z^{[1]}_z Z^{[2]}$$

$$+\sum_{k=0}^{m}\Bigg\{\Bigg[\frac{k^3\omega}{3}+\frac{3k^2}{4}\omega^2+k\rho\omega-\frac{3k}{4}\delta\omega^2\Bigg]\beta_k-\Bigg[\left(k^2-k\right)\omega^3+\frac{7k^3-3k^2-4k}{12}\omega^2$$

$$+\frac{k^4-k}{12}\omega+\frac{3\left(k^2-k\right)}{4}\rho\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)-\frac{k^2-k}{2}\delta\omega^2\Bigg]\alpha_k\Bigg\}$$

$$Z^{[1]}_z Z^{[1]}_{z^2} Z^{[1]} Z^{[2]}$$

$$+\sum_{k=0}^{m}\Bigg\{\Bigg[\frac{k^2-k}{2}\omega^2+\frac{2k^3-3k^2}{12}\omega+\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$

$$
\begin{aligned}
&+\frac{k\rho\omega}{2}-\frac{k\delta\omega^2}{2}\Bigg]\beta_k-\Bigg[\frac{k^2-k}{2}\omega^3+\frac{2k^3-3k^2+k}{8}\omega^2+\frac{k^4-2k^3+k}{24}\omega \\
&+\left(k^2-k\right)\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{2}\rho\omega+\frac{k^3-k}{6}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right) \\
&-\frac{3\left(k^2-k\right)}{8}\delta\omega^2\Bigg]\alpha_k\Bigg\} Z_z^{[1]}Z_z^{[1]}Z^{[3]} \\
&+\sum_{k=0}^{m}\Bigg\{\Bigg[k^2\omega^2+\frac{k^2}{2}\omega+\frac{k^3-k^2}{2}\omega\Bigg]\beta_k-\Bigg[\left(k^2-k\right)\omega^3+\frac{6k^3-3k^2-3k}{8}\omega^2 \\
&+\frac{k^4-k}{8}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{8}\delta\omega^2\Bigg]\alpha_k\Bigg\} \\
&Z_{z^2}^{[1]}Z^{[1]}\left(Z_z^{[1]}Z^{[2]}\right) \\
&+\sum_{k=0}^{m}\Bigg\{\Bigg[\frac{k^3-k^2}{2}\omega+k^2\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg]\beta_k-\Bigg[\frac{10k^3-9k^2-k}{24}\omega^2 \\
&+\frac{3k^4-4k^3+k}{24}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3\left(k^2-k\right)}{4}\rho\omega \\
&-\frac{5\left(k^2-k\right)}{8}\delta\omega^2+\frac{k^3-k}{3}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg]\alpha_k\Bigg\} Z_{z^2}^{[1]}Z^{[1]}Z^{[3]} \\
&+\sum_{k=0}^{m}\Bigg\{\Bigg[\frac{k^3}{2}\omega+\frac{k^2}{2}\omega^2\Bigg]\beta_k-\Bigg[\frac{k^2-k}{2}\omega^3+\frac{14k^3-3k^2-11k}{24}\omega^2+\frac{k^4-k}{8}\omega \\
&+\frac{k^2-k}{2}\rho\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3\left(k^2-k\right)}{8}\delta\omega^2\Bigg]\alpha_k\Bigg\} Z_{z^2}^{[1]}\left(Z^{[2]}\right)^2 \\
&+\sum_{k=0}^{m}\Bigg\{\frac{k^3}{2}\beta_k\omega-\Bigg[\frac{k^4-k}{8}\omega+\frac{10k^3-3k^2-7k}{24}\omega^2-\frac{3\left(k^2-k\right)}{8}\delta\omega^2 \\
&+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{2}\rho\omega\Bigg]\alpha_k\Bigg\} Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[2]},
\end{aligned}
$$

and we obtain (12.4). $\square$