The nonconforming finite element method for Signorini problem

Lieheng Wang[1] **Dongying Hua**[2]

[1,2] *Institute of Computational Mathematics and Scientific/Engineering Computing*
Academy of Mathematics and System Sciences, Chinese Academy of Science
[2] *Graduate School of the Chinese Academy of Science*
Beijing 100080, P. R. China
E-mail: wlh@lsec.cc.ac.cn huady@amss.ac.cn

Abstract We present the linear nonconforming finite element approximation of the variational inequality resulting from Signorini problem. We show if the displacement field is of H^2 regularity, the convergence rate behaves like $\mathcal{O}(h^{3/4})$ with respect to the energy norm and can be improved to quasi-optimal $\mathcal{O}(h|logh|^{1/4})$. It is of the same convergence rate as that of the continuous linear finite element method. If stronger but reasonable regularity is available, the convergence rate can be optimal $\mathcal{O}(h)$ as expected by the linear approximation.

Key words: nonconforming finite element, Signorini problem
AMS subject classification: 65N30.

1. Introduction

Signorini problem is one of the model problems considered in the theory of variational inequality(see [7],[11]). The continuous linear finite element approximations of this problem have been studied in many works(see [2],[8],[12],[3]). As far as we have known that Scarpini and Vivaldi (see [12]) first gave the $\mathcal{O}(h^{3/4})$ convergence rate under the condition that the displacement field u is of H^2 regularity. Then, Brezzi, Hager and Raviart(see [2]) presented $\mathcal{O}(h)$ convergence rate by detailed analysis under the additional assumptions that $u|_{\partial\Omega} \in W^{1,\infty}(\partial\Omega)$ and that the number of points in the free boundary set where the constraint changes from binding to nonbinding is finite. For simplicity, we call these points "the critical points". Later, Belgacem (see [3]) proved that under weaker assumption, i.e., $u \in H^2(\Omega)$ and the number of critical points is finite, $\mathcal{O}(h|logh|^{1/2})$ convergence order can be obtained. Recently, Belgacem (see [4]) has established an improved result of $\mathcal{O}(h|logh|^{1/4})$ convergence order under the same assumptions as in his previous paper. However, the convergence rate is not optimal if stronger regularity and finite number of the critical points are not assumed. In this paper, we work with Crouzeix-Raviart linear finite element (see [6]) to approximate the Signorini problem and achieve the same results as those of the continuous linear finite elements. The whole process of analysis is more complicated and probably more skillful. Moreover, if the displacement field $u \in W^{2,p}$ with $p > 2$, we can obtain the optimal convergence rate without the assumption of the finite number of the critical points on the contact region.

Throughout this paper all the notation about Sobolev spaces can be found in [1]. In addition, the frequently used constant C is a generic positive constant whose value may be different under different context. The paper is organized as follows: In section 2, we establish some notation and lemmas. The main results are described in section 3, and in section 4, the proofs are given. Next, we state the framework of the Signorini problem.

For the sake of simplicity, we only consider the Signorini problem for the Poisson equation. The general continuous setting of this problem in $\mathbb{R}^2$ can be illustrated (a mathematical model) as follows. Suppose $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain, and it consists of three non-overlapping

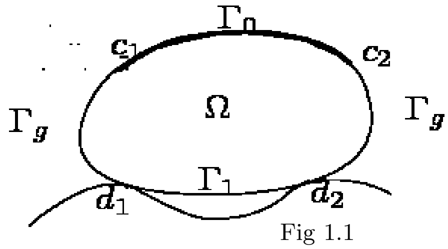

Fig 1.1

parts Γ_0, Γ_1 and Γ_g. Γ_0 is the fixed boundary (Dirichlet condition) with the end points c_1, c_2 while Γ_1 is the contact region subjected to a rigid fundation with d_1, d_2 as its endpoints, besides, Γ_g is the "glacis" with Neumann condition.

Now the Signorini problem can be restated as the following mathematical model:

$$\textit{to find} \;\; u \in K = \{u \in H^1_{\Gamma_0}(\Omega) : u \geq 0 \quad \text{on } \Gamma_1\}, \;\; \textit{such that} \tag{1.1}$$

$$a(u, v-u) \geq \chi(v-u), \qquad \forall v \in K, \tag{1.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad \chi(v) = \int_\Omega f v dx + \int_{\Gamma_g} g v ds. \tag{1.3}$$

The notation $H^1_{\Gamma_0}$ stands for the set $\{v \in H^1(\Omega) : \;\; v = 0 \text{ on } \;\; \Gamma_0\}$, $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_g$, and $int(\Gamma_0) \cap int(\Gamma_g) = \emptyset$, $int(\Gamma_1) \cap int(\Gamma_g) = \emptyset$. (see Fig 1.1). Here for concision, suppose the domain Ω is polygonal in $\mathbb{R}^2$, and we only consider $u \geq 0$ instead of $u \geq \alpha$ on Γ_1 in the closed convex set K, since the whole subsequent analysis can be carried out to the case where α does not vanish. It is easy to check that the equivalent differential form of (1.1) is the following

$$\left\{ \begin{array}{lll} -\triangle u = f & \text{in} & \Omega, \\ u = 0 & \text{on} & \Gamma_0, \\ \partial_\nu u = g & \text{on} & \Gamma_g, \\ u \geq 0, \quad \partial_\nu u \geq 0, \quad \partial_\nu u \cdot u = 0 & \text{on} & \Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+, \end{array} \right. \tag{1.4}$$

where ν is the unit outward normal to $\partial\Omega$ and $\Gamma_1^0 = \{x \in \Gamma_1 : u(x) = 0\}$, $\Gamma_1^+ = \{x \in \Gamma_1 : u(x) > 0\}$. The existence and uniqueness of the solution of the above problem can be easily verified by the ellipticity of $a(\cdot,\cdot)$ and the continuity of χ on $H^1_{\Gamma_0}$.

Suppose $\mathcal{J}_h$ is the regular triangulation of Ω, and $T \in \mathcal{J}_h$ is the triangular element. Let V_h be the Crouzeix-Raviart linear finite element space corresponding to $\mathcal{J}_h$, (which is nonconforming, i.e., $V_h \not\subseteq H^1(\Omega)$), that is to say,

$$V_h = \left\{ \begin{array}{c} v_h \in L^2(\Omega) : v_h|_T \in P_1(T), \, v_h \text{ is continuous at the midpoints of the edges of } T, \\ \text{for all } \; T \in \mathcal{J}_h, \quad \text{and } \; v_h(a'_{ij}) = 0, \text{where } a'_{ij} \text{ is the midpoint of } \;\; \overline{a'_i a'_j} \subset \Gamma_0. \end{array} \right\} \tag{1.5}$$

and let

$$\|v_h\|_h = (\sum_T |v_h|^2_{1,T})^{1/2}, \qquad \forall v_h \in V_h, \tag{1.6}$$

be the norm on V_h. Moreover, assume K_h is the following closed convex subset of V_h,

$$K_h = \{v_h \in V_h : v_h(a_{ij}) \geq 0, \text{where } a_{ij} \text{ is the midpoint of } \;\; \overline{a_i a_j} \subset \Gamma_1.\} \tag{1.7}$$

And we always consider that the triangulation $\mathcal{J}_h$ is built in such a way that the end points of Γ_0 and Γ_1 are always chosen as the vertices of triangular elements. Then the finite element approximation of problem (1.1)-(1.3) leads to : to find $u_h \in K_h$, such that

$$a_h(u_h, v_h - u_h) \geq \chi(v_h - u_h), \qquad \forall v_h \in K_h, \tag{1.8}$$

where

$$a_h(u_h, v_h) = \sum_T \int_T \nabla u_h \cdot \nabla v_h dx, \tag{1.9}$$

$$\chi(v_h) = \int_\Omega f v_h dx + \int_{\Gamma_g} g v_h ds. \tag{1.10}$$

As $\|v_h\|_h$ in (1.6) is a norm in V_h, the solution of the discrete problem (1.8)-(1.10) uniquely exists, and the following abstract error estimate holds:

Theorem 1.1 *Suppose $u \in K$ is the solution of the variational Signorini problem (1.1)-(1.3) and $u_h \in K_h$ the solution of the discrete one (1.8)-(1.10) respectively, then*

$$\|u - u_h\|_h \leq C \inf_{v_h \in K_h} \{\|u - v_h\|_h^2 + a_h(u, v_h - u_h) - \chi(v_h - u_h)\}^{1/2}. \tag{1.11}$$

The proof is similar to that of the second Strang lemma(see [5]), so we omit it here.
Remark: In the following sections, we often use the subscript h to denote something related to the finite element discretization.

2. Notation and lemmas

In this section, we introduce some notation and lemmas, which will be used in the later context. Let $F \subset \partial\Omega$ be the line element with respect to the triangulation $\mathcal{J}_h$, and let

$$\Gamma_{1h} = \{F : F \subset \Gamma_1\} \tag{2.1}$$

then Γ_{1h} can be divided into the following three non-overlapping sets:

$$\begin{cases} \Gamma_{1h}^0 = \{F \in \Gamma_{1h} : F \subset \Gamma_1^0\}, \\ \Gamma_{1h}^+ = \{F \in \Gamma_{1h} : F \subset \Gamma_1^+\}, \\ \Gamma_{1h}^- = \{F \in \Gamma_{1h} : F \cap \Gamma_1^0 \neq \emptyset, F \cap \Gamma_1^+ \neq \emptyset\}, \end{cases} \tag{2.2}$$

and

$$\Gamma_{1h} = \Gamma_{1h}^0 \cup \Gamma_{1h}^+ \cup \Gamma_{1h}^-. \tag{2.3}$$

Lemma 2.1 *The following discrete trace inequality holds, for $1 < p < \infty$,*

$$\|v\|_{0,p,\partial T} \leq C\{h_T^{-1}\|v\|_{0,p,T}^p + h_T^{p-1}|v|_{1,p,T}^p\}^{1/p}, \qquad \forall v \in W^{1,p}(T), \quad T \in \mathcal{J}_h. \tag{2.4}$$

where C is a constant independent of v and h_T.
The proof is the same as Stummel(see [13]).

Lemma 2.2 *Suppose $F \subset \partial T$ is an edge of the triangular element $T \in \mathcal{J}_h$, and $v \in H^1(F)$, moreover, there exists some $Q^F \in F$ such that $v(Q^F) = 0$, then*

$$\|v\|_{0,F} \leq Ch\|\frac{dv}{ds}\|_{0,F} \leq Ch|v|_{1,F} \tag{2.5}$$

$$\|v\|_{0,F} \leq Ch^{1/2}\|v\|_{1/2,F} \tag{2.6}$$

where $C = constant > 0$ *is independent of* h *and* v, *and* $\frac{dv}{ds}$ *denotes the derivative of* v *along* F.
Proof. Since $C^1(F) \subset H^1(F)$ densely, it is sufficient to prove the lemma for smooth function $v \in C^1(F)$. Firstly, we have

$$\begin{aligned}
\|v\|_{0,F}^2 &= \int_F |v^2(s) - v^2(Q^F)|ds \\
&= \int_F |\int_{Q^F}^s \frac{dv^2(t)}{dt} dt|ds \leq 2\int_F \{\int_{Q^F}^s |v(t)||\frac{dv(t)}{dt}|dt\}ds \\
&\leq 2|F|\int_F |v(t)||\frac{dv(t)}{dt}|dt \leq 2h\|v\|_{0,F}\|\frac{dv}{ds}\|_{0,F} \\
&\leq 2h\|v\|_{0,F}|v|_{1,F}
\end{aligned}$$

from which the estimate (2.5) is proved. Next, we also have

$$\begin{aligned}
\|v\|_{0,F}^2 &\leq 2\int_F |\int_{Q^F}^s v(t)\frac{dv(t)}{dt}dt|ds \\
&\leq 2|F||\int_F v(t)\frac{dv(t)}{dt}dt| \\
&\leq 2h\|v\|_{1/2,F}\|\frac{dv}{dt}\|_{-1/2,F} \leq Ch\|v\|_{1/2,F}^2
\end{aligned}$$

from which the estimate (2.6) can be obtained.

Lemma 2.3 *Let* u *and* u_h *be the solutions of the problems (1.1)-(1.3) and (1.8)-(1.10) respectively, and assume that* $u \in H^2(\Omega)$, *then*

$$-\sum_{F\in\Gamma_{1h}^-}\int_F \partial_\nu u \cdot u_h ds \leq Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^{3/2}\|u\|_{2,\Omega}^2, \tag{2.7}$$

where $C = constant > 0$ *is independent of* h.
Proof. For given $F \in \Gamma_{1h}^-$, if $u_h \geq 0$ on F, then $-\int_F \partial_\nu u \cdot u_h ds \leq 0$, since $\partial_\nu u \geq 0$ on Γ_1. Thus we need only consider such $F \in \Gamma_{1h}^-$, that $u_h \geq 0$ does not identically hold on F. Then, for those F, because we have $u_h(m^F) \geq 0$, with m^F the midpoint of F, and by the linearity of u_h on F, there exists some $Q^F \in F$, such that $u_h(Q^F) = 0$. Let

$$P_0^F(v) = \frac{1}{|F|}\int_F v ds, \qquad R_0^F(v) = v - P_0^F(v) \tag{2.8}$$

then,

$$\begin{aligned}
-\textstyle\int_F \partial_\nu u \cdot u_h ds &= -\int_F R_0^F(\partial_\nu u)u_h ds - P_0^F(\partial_\nu u)\int_F u_h ds \\
&\leq -\int_F R_0^F(\partial_\nu u)u_h ds \leq \|R_0^F(\partial_\nu u)\|_{0,F}\|u_h\|_{0,F}
\end{aligned} \tag{2.9}$$

since $\partial_\nu u \geq 0$ on F, $P_0^F(\partial_\nu u) \geq 0$ and $\int_F u_h ds = |F|u_h(m^F) \geq 0$. By lemma 2.1 for $p = 2$ and the interpolation error estimate (see [5]), it can be seen that, for $F \subset \partial T$,

$$\|R_0^F(\partial_\nu u)\|_{0,F} \leq Ch^{1/2}|u|_{2,T}, \tag{2.10}$$

where $C = constant > 0$ is independent of h. Thus again by lemma 2.1 for $p = 2$, (2.5) and (2.10), one yields

$$\begin{aligned}
-\int_F \partial_\nu u \cdot u_h ds &\leq Ch^{3/2}|u|_{2,T}|u_h|_{1,F} \\
&\leq Ch^{3/2}|u|_{2,T}(|u-u_h|_{1,F} + |u|_{1,F}) \\
&\leq Ch^{3/2}|u|_{2,T}(h^{-1}|u-u_h|_{1,T}^2 + h|u|_{2,T}^2)^{1/2} + Ch^{3/2}|u|_{2,T}|u|_{1,F} \\
&\leq Ch|u|_{2,T}|u-u_h|_{1,T} + Ch^2|u|_{2,T}^2 + Ch^{3/2}|u|_{2,T}|u|_{1,F}
\end{aligned}$$

where $F \subset \partial T$. Then, it comes out that

$$-\sum_{F\in\Gamma_{1h}^-}\int_F \partial_\nu u \cdot u_h ds \leq Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2|u|_{2,\Omega}^2 + Ch^{3/2}|u|_{2,\Omega}\|u\|_{1,\partial\Omega}$$

from which, and by the trace theorem, the proof is completed.

Lemma 2.4 *Under the assumptions of lemma 2.3, assume that the number of the critical points on Γ_1 is finite, then*

$$-\sum_{F\in\Gamma_{1h}^-}\int_F \partial_\nu u \cdot u_h ds \leq Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2\|u\|_{2,\Omega}^2 \tag{2.11}$$

where $C = constant > 0$ is independent of h.

Proof. To begin with, following the same analysis of lemma 2.3, we only need to consider those F, such that u_h has at least one zero point Q^F on F. Then, for those F, by (2.5) we have

$$\|u_h\|_{0,F} \leq Ch\|\frac{du_h}{ds}\|_{0,F} \tag{2.12}$$

Furthermore, for all $F \in \Gamma_{1h}^-$, we claim that there exists either some line segment $F' \subset F \in \Gamma_{1h}^-$ with $\text{meas}(F') > 0$ and $u|_{F'} = 0$ or $\text{card}\{x \in F : u(x) = 0\}$ is finite, since that the number of the critical points on Γ_1 is finite. For the latter case, which means $\partial_\nu u = 0$ almost everywhere on F, then $-\int_F \partial_\nu u \cdot u_h ds = 0$ by the definition of the Lebesgue integral. So we only need to consider the former case, which implies that there must be some $P^F \in F' \subset F$ such that $\frac{d}{ds}u(P^F) = 0$. Let $\frac{du}{ds}(s) = v(s)$, from (2.9),(2.10) and (2.12), it can be seen that

$$\begin{aligned}
-\int_F \partial_\nu u \cdot u_h ds &\leq Ch^{3/2}|u|_{2,T}\|\frac{du_h}{ds}\|_{0,F} \\
&\leq Ch^{3/2}|u|_{2,T}(\|\frac{du_h}{ds} - \frac{du}{ds}\|_{0,F} + \|\frac{du}{ds}\|_{0,F}) \\
&\leq Ch^{3/2}|u|_{2,T}(|u-u_h|_{1,F} + \|\frac{du}{ds}\|_{0,F}) \\
&\leq Ch^{3/2}|u|_{2,T}(h^{-1}|u-u_h|_{1,T}^2 + h|u|_{2,T}^2)^{1/2} + Ch^{3/2}|u|_{2,T}\|\frac{du}{ds}\|_{0,F} \\
&\leq Ch|u|_{2,T}|u-u_h|_{1,T} + Ch^2|u|_{2,T}^2 + Ch^{3/2}|u|_{2,T}\|\frac{du}{ds}\|_{0,F}
\end{aligned}$$

and by (2.6)

$$\|\frac{du}{ds}\|_{0,F} \leq Ch^{1/2}\|\frac{du}{ds}\|_{1/2,F} \leq Ch^{1/2}\|u\|_{3/2,F} \tag{2.13}$$

As a result, by the trace theorem,

$$\begin{aligned}
-\sum_{F\in\Gamma_{1h}^{-}}\partial_\nu u\cdot u_h ds &\le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2|u|_{2,\Omega}^2 + Ch^2\sum_{\substack{F\in\Gamma_{1h}^{-}\\ F\subset\partial T}}|u|_{2,T}\|u\|_{3/2,F}\\
&\le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2\|u\|_{2,\Omega}^2 \qquad (2.14)
\end{aligned}$$

and the proof is completed.

Remark: In the following sections, we often use the subscript h to denote something related to the finite element discretization.

3. Main results

In this section, we present the main results of the error estimates for Crouzeix-Raviart linear element approximation to the Signorini problem stated in (1.1)-(1.3) in section 1. We first provide the $\mathcal{O}(h^{3/4})$ convergence rate as that of the continuous finite element approximation(Theorem 3.1), then, we show the quasi-optimal error of the nonconforming method under the reasonable assumption(Theorem 3.2). Furthermore, if additional regularity is assumed, optimality can be achieved(Theorem 3.3). Finally, if the displacement field u is in $W^{2,p}(\Omega)(p>2)$, even without the assumption of the finite number of the critical points on the contact region, optimal convergence rate is available.

Theorem 3.1 *Suppose $\Omega\subset\mathbb{R}^2$ is a polygonal domain, u, u_h are the solutions of (1.1)-(1.3) and (1.8)-(1.10) respectively, and $u\in H^2(\Omega)$, then we have*

$$\|u-u_h\|_h \le Ch^{3/4}\|u\|_{2,\Omega}. \qquad (3.1)$$

Theorem 3.2 *Under the assumptions of theorem 3.1, moreover, assume that the number of the critical points in Γ_1 is finite, then we have*

$$\|u-u_h\|_h \le Ch|logh|^{1/4}\|u\|_{2,\Omega}. \qquad (3.2)$$

Theorem 3.3 *Under the assumptions of theorem 3.2, and in addition, assume $u|_{\partial\Omega}\in W^{1,\infty}(\partial\Omega)$, then*

$$\|u-u_h\|_h \le Ch|u|_{2,\Omega}(\|u\|_{2,\Omega}+|u|_{1,\infty,\partial\Omega}). \qquad (3.3)$$

Theorem 3.4 *Suppose $\Omega\subset\mathbb{R}^2$ is a polygonal domain, u, u_h are the solutions of (1.1)-(1.3) and (1.8)-(1.10) respectively, and $u\in W^{2,p}(\Omega)$ with $p>2$, then we have*

$$\|u-u_h\|_h \le Ch\|u\|_{2,p,\Omega}. \qquad (3.4)$$

We should note that for the continuous linear element approximation to the Signorini problme (1.1)-(1.3) in section 1, the above results (3.1),(3.2) and (3.3) have been obtained in [12],[4], and [2] respectively. In the proof of Theorem 3.2 in the next section, our method differs from that of the Belgacem's, but the result is same.

4. The proofs of the main results

Before verifying the main results, we present the following lemma.

Lemma 4.1 *Suppose u, u_h are the solutions of (1.1)-(1.3) and (1.8)-(1.10) respectively, and $u \in H^2(\Omega)$, then we have*

$$\|u - u_h\|_h^2 \le C\{h^2|u|_{2,\Omega}^2 + \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u(\Pi_h u - u_h)ds\} \tag{4.1}$$

Proof. By the abstract error estimate (1.11), we set

$$\begin{aligned} E_h(u, v_h - u_h) &= a_h(u, v_h - u_h) - \chi(v_h - u_h) \\ &= \sum_T \int_T \nabla u \cdot \nabla(v_h - u_h)dx - \int_\Omega f(v_h - u_h)dx - \int_{\Gamma_g} g(v_h - u_h)ds \\ &= -\int_\Omega (\triangle u + f)(v_h - u_h)dx + \sum_T \int_{\partial T} \partial_\nu u(v_h - u_h)ds - \int_{\Gamma_g} g(v_h - u_h)ds \\ &= \sum_T \sum_{\substack{F\subset\partial T\\ F\not\subseteq\partial\Omega}} \int_F \partial_\nu u(v_h - u_h)ds + \sum_{F\in\Gamma_{0h}} \int_F \partial_\nu u(v_h - u_h)ds \\ &+ \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u(v_h - u_h)ds \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{4.2}$$

Let $w_h = v_h - u_h$, by the standard error estimates of Crouzeix-Raviart linear finite element(see [13]), we have

$$I_1 = \sum_T \sum_{\substack{F\subset\partial T\\ F\not\subseteq\partial\Omega}} \int_F \partial_\nu u \cdot w_h ds \le Ch|u|_{2,\Omega}\|w_h\|_h \tag{4.3}$$

and

$$I_2 = \sum_{F\in\Gamma_{0h}} \int_F \partial_\nu u \cdot w_h ds \le Ch|u|_{2,\Omega}\|w_h\|_h \tag{4.4}$$

By (4.2)-(4.4), one gets

$$E_h(u, v_h - u_h) \le Ch|u|_{2,\Omega}\|v_h - u_h\|_h + I_3 \tag{4.5}$$

Thus, by(1.11),

$$\|u - u_h\|_h^2 \le C \inf_{v_h\in K_h}\{\|u - v_h\|_h^2 + Ch|u|_{2,\Omega}(\|u - v_h\|_h + \|u - u_h\|_h) + I_3\}$$

Using the Young's inequality

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \qquad \forall\varepsilon > 0$$

we obtain

$$\|u - u_h\|_h^2 \le C \inf_{v_h\in K_h}\{\|u - v_h\|_h^2 + I_3\} + Ch^2|u|_{2,\Omega}^2$$

Let Π_h be the linear interpolation operator of Crouzeix-Raviart linear finite element, then $\Pi_h v \in K_h$ for all $v \in K$ and choose $\Pi_h u = v_h$ in the above inequality and by the standard interpolation error estimates of Crouzeix-Raviart linear finite element in [6], we derive,

$$\|u - u_h\|_h^2 \le C(h^2|u|_{2,\Omega}^2 + I_3) = C\{h^2|u|_{2,\Omega}^2 + \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u(\Pi_h u - u_h)ds\} \tag{4.6}$$

which completes our proof.

With lemma 4.1 at hand, in order to prove the theorems in section 3, we only need to handle the last term of the right-hand side of (4.1), i.e., I_3.
Proof of Theorem 3.1.

$$I_3 = \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u(\Pi_h u - u)ds - \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u \cdot u_h ds = A + B \tag{4.7}$$

since $\partial_\nu u \cdot u = 0$ on Γ_1. By lemma 2.1 for $p = 2$ and the interpolation error estimates,

$$\begin{aligned} A &= \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u(\Pi_h u - u)ds \leq \sum_{F\in\Gamma_{1h}} \|\partial_\nu u\|_{0,F} \|\Pi_h u - u\|_{0,F} \\ &\leq \sum_{F\in\Gamma_{1h}} \|\partial_\nu u\|_{0,F} Ch^{3/2}|u|_{2,T} \leq Ch^{3/2}\|\partial_\nu u\|_{0,\Gamma_1}|u|_{2,\Omega} \leq Ch^{3/2}|u|^2_{2,\Omega} \end{aligned} \tag{4.8}$$

where $F \subset \partial T$. By the differential information (1.4), we know $\partial_\nu u = 0$ on Γ^+_{1h}, which results in

$$B = -\sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u \cdot u_h ds = -\sum_{F\in\Gamma^0_{1h}} \int_F \partial_\nu u \cdot u_h ds - \sum_{F\in\Gamma^-_{1h}} \int_F \partial_\nu u \cdot u_h ds = B_1 + B_2 \tag{4.9}$$

Consider for all $F \in \Gamma^0_{1h}, u|_F = 0$, so

$$\begin{aligned} -\textstyle\int_F \partial_\nu u \cdot u_h ds &= -\int_F R^F_0(\partial_\nu u)u_h ds - P^F_0(\partial_\nu u)\int_F u_h ds \\ &\leq -\int_F R^F_0(\partial_\nu u)u_h ds = -\int_F R^F_0(\partial_\nu u)R^F_0(u_h)ds \\ &= -\int_F R^F_0(\partial_\nu u)R^F_0(u_h - u)ds \\ &\leq \|R^F_0(\partial_\nu u)\|_{0,F}\|R^F_0(u_h - u)\|_{0,F} \end{aligned}$$

Note that

$$\begin{aligned} \|R^F_0(\partial_\nu u)\|^2_{0,F} &\leq 2\{\int_F |R^F_0(\partial_1 u)|^2 ds + \int_F |R^F_0(\partial_2 u)|^2 ds\} \\ &\leq 2\{\int_F |R^T_0(\partial_1 u)|^2 ds + \int_F |R^T_0(\partial_2 u)|^2 ds\} \leq Ch|u|^2_{2,T} \end{aligned} \tag{4.10}$$

The last inequality is obtained by the discrete trace inequality (2.4) in lemma 2.1 for $p = 2$. Now it follows again from (2.4) with $p = 2$ that

$$B_1 = -\sum_{F\in\Gamma^0_{1h}} \int_F \partial_\nu u \cdot u_h ds \leq \sum_{F\in\Gamma^0_{1h}} Ch^{1/2}|u|_{2,T}h^{1/2}|u - u_h|_{1,T} \leq Ch|u|_{2,\Omega}\|u - u_h\|_h \tag{4.11}$$

From (4.6)-(4.9), and (4.11), one gets

$$\|u - u_h\|^2_h \leq C(h^2|u|^2_{2,\Omega} + h^{3/2}|u|^2_{2,\Omega} + h|u|_{2,\Omega}\|u - u_h\|_h) + B_2 \tag{4.12}$$

Now using lemma 2.3, we know

$$B_2 \leq Ch|u|_{2,\Omega}\|u - u_h\|_h + Ch^{3/2}\|u\|^2_{2,\Omega} \tag{4.13}$$

Combining (4.12)-(4.13) together with the Young's inequality, the proof is completed.

Proof of Theorem 3.2. In order to obtain the quasi-optimal convergence rate, we should only re-estimate the term I_3. Note that $\partial_\nu u = 0$ on Γ_{1h}^+ and $\partial_\nu u \cdot u = 0$ on Γ_1,

$$\begin{aligned} I_3 &= \sum_{F\in\Gamma_{1h}} \int_F \partial_\nu u(\Pi_h u - u_h)ds = \sum_{F\in\Gamma_{1h}^0} \int_F \partial_\nu u(\Pi_h u - u_h)ds + \sum_{F\in\Gamma_{1h}^-} \int_F \partial_\nu u(\Pi_h u - u_h)ds \\ &= \sum_{F\in\Gamma_{1h}^0} \int_F \partial_\nu u(\Pi_h u - u_h)ds + \sum_{F\in\Gamma_{1h}^-} \int_F \partial_\nu u(\Pi_h u - u)ds - \sum_{F\in\Gamma_{1h}^-} \int_F \partial_\nu u \cdot u_h ds \\ &= D_1 + D_2 + B_2 \end{aligned} \tag{4.14}$$

Moreover, for any $F \in \Gamma_{1h}^0$, by lemma 2.1 for $p = 2$ and

$$\int_F (\Pi_h u - u_h)ds = |F|(\Pi_h u - u_h)(m^F) = -u_h(m^F)|F| \le 0$$

with m^F the midpoint of F, it follows easily that

$$\begin{aligned} \textstyle\int_F \partial_\nu u(\Pi_h u - u_h)ds &\le \int_F R_0^F(\partial_\nu u)(\Pi_h u - u_h)ds \\ &= \int_F R_0^F(\partial_\nu u)(\Pi_h u - u)ds + \int_F R_0^F(\partial_\nu u)R_0^F(u - u_h)ds \\ &\le \|R_0^F(\partial_\nu u)\|_{0,F}(\|\Pi_h u - u\|_{0,F} + \|R_0^F(u - u_h)\|_{0,F}) \\ &\le \|Ch^{1/2}|u|_{2,T}(h^{3/2}|u|_{2,T} + h^{1/2}|u - u_h|_{1,T}) \\ &\le Ch^2|u|_{2,T}^2 + Ch|u|_{2,T}|u - u_h|_{1,T} \end{aligned}$$

hence,

$$D_1 \le Ch^2|u|_{2,\Omega}^2 + Ch|u|_{2,\Omega}\|u - u_h\|_h \tag{4.15}$$

Now we turn to estimate D_2. For any $F \in \Gamma_{1h}^-$, by discrete trace inequality lemma 2.1 and the interpolation error estimates, we have

$$\begin{aligned} \textstyle\int_F \partial_\nu u(\Pi_h u - u)ds &\le \|\partial_\nu u\|_{L^{p'}(F)}\|\Pi_h u - u\|_{L^p(F)} \\ &\le C\|\partial_\nu u\|_{L^{p'}(F)}(h^{-1}\|\Pi_h u - u\|_{0,p,T}^p + h^{p-1}|\Pi_h u - u|_{1,p,T}^p)^{1/p} \\ &\le C\|\partial_\nu u\|_{L^{p'}(F)}(h^{p+1}|u|_{2,T}^p)^{1/p} \\ &\le Ch^{1+1/p}|u|_{2,T}\|\partial_\nu u\|_{L^{p'}(F)} \end{aligned}$$

Note that $H^{1/2}(\Gamma_1) \hookrightarrow L^{p'}(\Gamma_1)$, $(1 \le p' < +\infty)$ and $\|v\|_{L^{p'}(\Gamma_1)} \le C\sqrt{p'}\|v\|_{H^{1/2}(\Gamma_1)}$ (see [3]), then

$$\begin{aligned} D_2 &= \sum_{F\in\Gamma_{1h}^-} \int_F \partial_\nu u(\Pi_h u - u)ds \le Ch^{1+1/p}\|\partial_\nu u\|_{L^{p'}(\Gamma_1)} \sum_{F\in\Gamma_{1h}^-} |u|_{2,T} \\ &\le Ch^{1+1/p}\|\partial_\nu u\|_{L^{p'}(\Gamma_1)}|u|_{2,\Omega} \le C\sqrt{p'}h^{-1/p'}h^2\|u\|_{2,\Omega}^2 \end{aligned}$$

Choosing $p' = |logh|$, we obtain

$$D_2 \le C|logh|^{1/2}h^2\|u\|_{2,\Omega}^2 \tag{4.16}$$

Finally, using lemma 2.4,

$$B_2 \le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2\|u\|_{2,\Omega}^2 \tag{4.17}$$

we can finish the proof of Theorem 3.2 by (4.14)-(4.17) together with (4.6) and the Young's inequality.

Proof of Theorem 3.3. Following the proof of Theorem 3.2, to improve the convergence rate from $\mathcal{O}(h|logh|^{1/4})$ to $\mathcal{O}(h)$, it is sufficient to re-estimate the term D_2. For all $F \in \Gamma_{1h}^-$, by lemma 2.1 with $p = 2$ and the interpolation error,

$$\begin{aligned}
\textstyle\int_F \partial_\nu u(\Pi_h u - u)ds &\le \|\partial_\nu u\|_{0,\infty,F}\int_F (\Pi_h u - u)ds \\
&\le Ch^{1/2}\|\partial_\nu u\|_{0,\infty,F}\|u-\Pi_h u\|_{0,F} \\
&\le Ch^2|u|_{1,\infty,F}|u|_{2,T}
\end{aligned}$$

From which we deduce that

$$D_2 = \sum_{F\in\Gamma_{1h}^-}\int_F \partial_\nu u(\Pi_h u - u)ds \le Ch^2|u|_{1,\infty,\Gamma_1}\sum_{F\in\Gamma_{1h}^-}|u|_{2,T} \le Ch^2|u|_{1,\infty,\Gamma_1}|u|_{2,\Omega} \tag{4.18}$$

here we have used the assumption that the number of the critical points is finite. Then, as a consequence of (4.6),(4.14),(4.15),(4.17) and (4.18) as well as the Young's inequality, the proof is complete.

Proof of Theorem 3.4. Now observe that $u \in W^{2,p}(\Omega)$, $p > 2$, by the trace theorem we have $u|_{\partial\Omega} \in W^{2-1/p,p}(\partial\Omega)$, and then by the Sobolev imbedding theorem, $Du|_{\partial\Omega} \in W^{1-1/p,p}(\partial\Omega) \hookrightarrow C^0(\partial\Omega)$, as $p > 2$. We still write

$$I_3 = D_1 + D_2 + B_2 \tag{4.19}$$

It is easy to see that (4.15), (4.18) still hold. To be exactly, as a direct application of the above imbedding theorem, now (4.18) can be re-written as

$$D_2 \le Ch^2\|u\|_{2,p,\Omega}|u|_{2,\Omega} \tag{4.20}$$

Next, we need to bound B_2. As before, we only need consider those F on which u_h has at least one zero point and then (2.12) follows. Since $F \in \Gamma_{1h}^-$, u has at least one zero point which we denote by Q^F, i.e., $u(Q^F) = 0$. If furthermore there is a neighborhood $W \subset F$ of the point Q^F such that $u(x)|_W = 0$, which implies that there must be $\frac{du}{ds}(Q^F) = 0$. Otherwise, there exists a neighborhood $W \subset F$ of the point Q^F such that $u(x) > 0$ in W except on one point Q^F. In this case, it is easy to see that Q^F is the minimum point on W which implies $\frac{du}{ds}(Q^F) = 0$ since $\frac{du}{ds} \in C^0(\partial\Omega)$. In short, on those F, the fact that both u and $\frac{du}{ds}$ have zero points is crucial to our relaxation of the finite number of critical points. Subsequently, following the same line of the proof of lemma 2.4, (2.11) holds, i.e.,

$$B_2 \le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2\|u\|_{2,\Omega}^2 \tag{4.21}$$

Finally, By (4.6),(4.15),(4.19)-(4.21) as well as the Young's inequality we complete our proof.

References

[1] Adams, D.A., Sobolev spaces, Academic press, New York, 1975.

[2] Brezzi,F., Hager,W.W., Raviart,P.A. Error estimates for the finite element solution of variational inequalities, Part I, Primal theory, Numer.Math. 28, 431-443(1977).

[3] Ben Belgacem,F., Numerical simulation of some variational inequalityies arisen from unilateral contact problems by the finite element method, SIAM J.Numer.Anal.,37,1198-1216(2000).

[4] Ben Belgacem,F., Renard.Y., Hybrid finite element methods for the Signorini problem, Math.Comput., vol 72,no.243, 1117-1145(2003).

[5] Ciarlet,P.G., The finite element method for elliptic problems, North Holland, 1978.

[6] Crouzeix,M., Raviart,P.A., Conforming and non-conforming finite element methods for solving the stationary Stokes equations, R.A.I.R.O. Anal. Num.7, 33-76(1973).

[7] Duvaut,G., Lions,J.L., Les in*é*quations en M*é*canique et en physique, Dunod, Paris, 1972.

[8] Falk,R.S., Error estimates for the approximation of a class of variational inequalities, Math. Comput.,28, 963-971(1974).

[9] Falk,R.S., Mercier,B., Error estimates for elastoplastic problems, R.A.I.R.O. Anal. Num.11, 117-134(1977).

[10] Lions,J.L., Magenes,E., Non-homogeneous Boundary Value Problems and Applications, vol.I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[11] Lions,J.L., Stampacchia,G., Variational inequalityies, Comm. Pure Appl. Math. 20,493-519(1967).

[12] Scarpini,F., Vivaldi,M.A., Error estimates for the approximation of some unilateral problems, R.A.I.R.O. Anal. Num.11, 197-208(1977).

[13] Stummel,F., The generalized patch test, SIAM,J.Numer.Anal., 16, 449-471(1979).